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ON COMPACTIFICATIONS OF CHARACTER VARIETIES OF n-PUNCTURED PROJECTIVE LINE

by Arata KOMYO (*)

ABSTRACT. — In this paper, we construct compactifications of $SL_2(\mathbb{C})$-character varieties of n-punctured projective line and study the boundary divisors of the compactifications. This study is motivated by a conjecture for the configurations of the boundary divisors, due to C. Simpson. We verify the conjecture for a few examples.

RÉSUMÉ. — Dans cet article, nous construisons des compactifications de $SL_2(\mathbb{C})$-variétés de caractères d'une droite projective moins n points et étudions les diviseurs au bord des compactifications. Cette étude est motivée par une conjecture, due à C. Simpson, sur les configurations des diviseurs au bord. Nous vérifions quelques cas de la conjecture.

1. Introduction

Let C be a compact Riemann surface of genus g, and let $\{t_1, \ldots, t_n\}$ be the set of n-distinct points on C. For a positive integer $r > 0$, denote by $\mathcal{P}_r$ the set of partitions of r, and fix $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^n) \in (\mathcal{P}_r)^n$ where $\mu^i = (\mu^i_1, \ldots, \mu^i_{r_i}) \in \mathcal{P}_r$. For each partition $\mu^i \in \mathcal{P}_r$, let us fix semisimple conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_n \subset SL_r(\mathbb{C})$ which is generic in the sense of [4, Definition 2.1.1] and type $\mu^1, \ldots, \mu^n$, that is, the multiplicities of eigenvalues of matrices in $\mathcal{C}_i$ are given by $\mu^i = (\mu^i_1, \mu^i_2, \ldots)$. We consider a monodoromy $SL_r(\mathbb{C})$-semisimple representation

$$\rho : \pi_1(C \setminus \{t_1, \ldots, t_n\}, *) \longrightarrow SL_r(\mathbb{C})$$

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of type $(g, \boldsymbol{\mu})$ which satisfies the condition $\rho(\gamma_i) \in \mathcal{C}_i$ for each i where γ_i is a anticlockwise loop around the point t_i. We can define the $SL_r(\mathbb{C})$-*character variety* $\mathcal{R}_{g,\boldsymbol{\mu}}$ of the n-punctured compact Riemann surface of genus g by the following categorical quotient

$$\begin{aligned}\mathcal{R}_{g,\boldsymbol{\mu}} := \{(A_1, B_1, \dots, A_g, B_g; M_1, \dots, M_n) \in SL_r(\mathbb{C})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \\ \mid (A_1, B_1) \cdots (A_g, B_g) M_1 \cdots M_n = I_r\} /\!/ SL_r(\mathbb{C}).\end{aligned}$$

Here, we set $(A, B) = ABA^{-1}B^{-1}$ and I_r is the identity matrix. The variety depends on the actual choice of eigenvalues, but for simplicity we drop this choice from the notation. The categorical quotient $\mathcal{R}_{g,\boldsymbol{\mu}}$ can be considered as a moduli space of monodoromy $SL_r(\mathbb{C})$-semisimple representations of type $(g, \boldsymbol{\mu})$. The variety $\mathcal{R}_{g,\boldsymbol{\mu}}$, if nonempty, is a nonsingular affine variety of dimension

$$d_{g,\boldsymbol{\mu}} := r^2(2g - 2 + n) - \sum_{i,j} (\mu_j^i)^2 + 2 - 2g.$$

(See [4]). In the case where $g = 0$ and $d_{g,\boldsymbol{\mu}} = 2$, $SL_r(\mathbb{C})$-character varieties can be classified into four cases, which can be listed as follows:

$$\begin{aligned}\boldsymbol{\mu} &= ((1,1),(1,1),(1,1),(1,1)) \\ \boldsymbol{\mu} &= ((1,1,1),(1,1,1),(1,1,1)) \\ \boldsymbol{\mu} &= ((2,2),(1,1,1,1),(1,1,1,1)) \\ \boldsymbol{\mu} &= ((3,3),(2,2,2),(1,1,1,1,1,1)).\end{aligned} \tag{1.1}$$

In the first and second types, the $SL_r(\mathbb{C})$-character varieties are known to be an affine cubic surface. ([3], [10], [9], [12]).

The purpose of this paper is to study the configuration of boundary divisor of compactifications of $SL_r(\mathbb{C})$-character varieties. This study is motivated by a conjecture due to Simpson [18], which is explained as follows. We choose a smooth compactification $\overline{\mathcal{R}_{g,\boldsymbol{\mu}}}$ of $\mathcal{R}_{g,\boldsymbol{\mu}}$ such that $D^B_{g,\boldsymbol{\mu}} = \overline{\mathcal{R}_{g,\boldsymbol{\mu}}} \setminus \mathcal{R}_{g,\boldsymbol{\mu}}$ is a divisor with normal crossings. We call the divisor $D^B_{g,\boldsymbol{\mu}}$ a *boundary divisor* of the compactification $\overline{\mathcal{R}_{g,\boldsymbol{\mu}}}$. Let $\overline{N}^B_{g,\boldsymbol{\mu}}$ be a small neighborhood of $D^B_{g,\boldsymbol{\mu}}$ in $\overline{\mathcal{R}_{g,\boldsymbol{\mu}}}$, and let $N^B_{g,\boldsymbol{\mu}} = \overline{N}^B_{g,\boldsymbol{\mu}} \cap \mathcal{R}_{g,\boldsymbol{\mu}} = \overline{N}^B_{g,\boldsymbol{\mu}} \setminus D^B_{g,\boldsymbol{\mu}}$. Let $\Delta(D^B_{g,\boldsymbol{\mu}})$ be a simplicial complex whose n-dimensional simplices correspond to the irreducible components of intersections of $k+1$ distinct components of $D^B_{g,\boldsymbol{\mu}}$. This is called the *boundary complex* or *Stepanov complex* of a compactification of $\mathcal{R}_{g,\boldsymbol{\mu}}$ (see [22], [23], and [16]).

Theorem 1.1 ([22], [23], and [16]). — *The homotopy type of boundary complex $\Delta(D^B_{g,\boldsymbol{\mu}})$ is independent of the choice of compactifications.*

We have a continuous map, well-defined up to homotopy,

$$N^B_{g,\boldsymbol{\mu}} \longrightarrow \Delta(D^B_{g,\boldsymbol{\mu}}). \tag{1.2}$$

On the other hand, let $\mathcal{M}_{g,\boldsymbol{\mu}}$ be the moduli space of parabolic Higgs bundles, which is diffeomorphic to the character variety $\mathcal{R}_{g,\boldsymbol{\mu}}$ via the non-abelian Hodge theory [19]. In particular, we have $\dim \mathcal{M}_{g,\boldsymbol{\mu}} = d_{g,\boldsymbol{\mu}}$. We have the Hitchin fibration $\mathcal{M}_{g,\boldsymbol{\mu}} \to \mathbb{A}^{\frac{d_{g,\boldsymbol{\mu}}}{2}}$. The moduli space $\mathcal{M}_{g,\boldsymbol{\mu}}$ has a canonical orbifold compactification, where the divisor at infinity is the quotient

$$D^{Dol}_{g,\boldsymbol{\mu}} := \mathcal{M}^*_{g,\boldsymbol{\mu}}/\mathbb{C}^*.$$

Here, $\mathcal{M}^*_{g,\boldsymbol{\mu}}$ is the complement of the nilpotent cone. Let $\overline{N}^{Dol}_{g,\boldsymbol{\mu}}$ be a small neighborhood of $D^{Dol}_{g,\boldsymbol{\mu}}$, and let $N^{Dol}_{g,\boldsymbol{\mu}} = \overline{N}^{Dol}_{g,\boldsymbol{\mu}} \cap \mathcal{R}_{g,\boldsymbol{\mu}} = \overline{N}^{Dol}_{g,\boldsymbol{\mu}} \setminus D^{Dol}_{g,\boldsymbol{\mu}}$. The Hitchin fibration gives us a continuous map to the sphere at infinity in the Hitchin base

$$N^{Dol}_{g,\boldsymbol{\mu}} \longrightarrow S^{d_{g,\boldsymbol{\mu}}-1}. \tag{1.3}$$

Conjecture 1.2 ([18]).

(1) *There exists a homotopy-commutative diagram*

$$\begin{array}{ccc} N^{Dol}_{g,\boldsymbol{\mu}} & \xrightarrow{\cong} & N^B_{g,\boldsymbol{\mu}} \\ \downarrow & & \downarrow \\ S^{d_{g,\boldsymbol{\mu}}-1} & \xrightarrow{\cong} & \Delta(D^B_{g,\boldsymbol{\mu}}). \end{array}$$

(2) *In particular, there exists a non-singular compactification of $\mathcal{R}_{g,\boldsymbol{\mu}}$ such that the boundary complex is a simplicial decomposition of sphere $S^{d_{g,\boldsymbol{\mu}}-1}$.*

Remark 1.3 (See [18]*).* — The assertion (1) of Conjecture 1.2 is true in the first case of the list (1.1).

The main theorem of this paper is the following

Theorem 1.4 (Theorem 6.2). — *The assertion (2) of Conjecture 1.2 is true in the following cases:*

(1) $g = 0, r = 3, n = 3, \boldsymbol{\mu} = ((1,1,1),(1,1,1),(1,1,1)),\ d_{g,\boldsymbol{\mu}} = 2;$
(2) $g = 0, r = 2, n = 5, \boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1),(1,1)),\ d_{g,\boldsymbol{\mu}} = 4.$

For the case (1) of Theorem 1.4, the assertion (2) of Conjecture 1.2 can be verified by the classical invariant theory. ([3], [10], [9], [12]). However, it seems that the application of the classical invariant theory is difficult for general cases. Then, we construct compactifications of $SL_r(\mathbb{C})$-character varieties as follows. Following [13], we can construct a compactification of the *representation variety* [13]

$$\begin{aligned}\mathrm{Rep}_{g,\boldsymbol{\mu}} := \{(A_1, B_1, \dots, A_g, B_g; M_1, \dots, M_n) \in SL_r(\mathbb{C})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \\ \mid (A_1, B_1) \cdots (A_g, B_g) M_1 \cdots M_n = I_r\}.\end{aligned}$$

Then, we take the GIT quotient of this compactification of $\mathrm{Rep}_{g,\boldsymbol{\mu}}$, which gives a compactification $\overline{\mathcal{R}_{g,\boldsymbol{\mu}}}$ of $\mathcal{R}_{g,\boldsymbol{\mu}}$. As special cases, we consider the case where $g = 0, r = 2, n \geqslant 4, \boldsymbol{\mu} = ((1,1), \dots, (1,1))$. For $n = 4$, we obtain the same result as the classical invariant theory [3]. For $n = 5$ (i.e., the case (2) of Theorem 1.4), $\overline{\mathcal{R}_{g,\boldsymbol{\mu}}}$ has singular points. A suitable blowing up of $\overline{\mathcal{R}_{g,\boldsymbol{\mu}}}$ shows that the assertion (2) of Conjecture 1.2 holds. It seems that the configuration of the boundary divisor $D^B_{0,\boldsymbol{\mu}}$ is rather complicated for $n \geqslant 6$.

Conjecture 1.2 is related to the P=W conjecture due to Hausel et al ([1]). First, we consider compact curve cases. The non-abelian Hodge theory for compact curves states that character varieties $\mathcal{R}$ are diffeomorphic to moduli spaces $\mathcal{M}$ of semi-stable Higgs bundles. Then, we have the induced isomorphism between the rational cohomology groups of $\mathcal{R}$ and $\mathcal{M}$. The P=W conjecture assert that the isomorphism of the rational cohomology groups exchanges the weight filtration on the cohomology groups of $\mathcal{R}$ with the perverse Leray filtration associated with the Hitchin fibration on the cohomology groups of $\mathcal{M}$. The P=W conjecture is verified in the case where $r = 2$ ([1]). We may extend the conjecture to punctured curve cases. On the other hand, there exists a natural isomorphism from the reduced homology of the boundary complex $\Delta(D^B_{g,\boldsymbol{\mu}})$ to the $2l$-th graded piece of the weight filtration on the cohomology of $\mathcal{R}_{g,\boldsymbol{\mu}}$:

$$\widetilde{H}_{i-1}(\Delta(D^B_{g,\boldsymbol{\mu}}), \mathbb{Q}) \cong Gr^W_{2l} H^{2l-i}(\mathcal{R}_{g,\boldsymbol{\mu}}, \mathbb{Q}).$$

(For example, see [16, Theorem 4.4]). By the isomorphism, the assertion (2) of Conjecture 1.2 implies that there exists only 1-dimensional weight $2d_{g,\boldsymbol{\mu}}$ part in the middle degree $d_{g,\boldsymbol{\mu}}$ cohomology of the character variety, which is also a consequence of the P=W conjecture.

Remark 1.5. — The structure groups of character varieties studied in [1] are $GL_n(\mathbb{C})$, $PGL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$. However, for $g = 0$, those character varieties are the same.

The organization of this paper is as follows. In Section 2, we give the definition of a $SL_r(\mathbb{C})$-character variety. In Section 3, we consider the case where $g = 0, r = 2, n = 4$ and $g = 0, r = 3, n = 3$. In those cases, the character varieties are describe by invariants and a relation of invariants.

We recall that the character varieties are affine cubic surfaces. In Section 4, we consider the construction of compactifiations of $SL_2(\mathbb{C})$-character varieties of $g = 0, \boldsymbol{\mu} = ((1,1), \dots, (1,1))$. In Section 5 and 6, we describe the boundary divisor of the compactifiations of the cases where $n = 4$ and $n = 5$.

2. Preliminaries

We fix integers g, r, n with $g \geqslant 0, r > 0, n > 0$, and let $(C, \boldsymbol{t}) = (C, t_1, \dots, t_n)$ be an n-pointed compact Riemann surface of genus g, which consists of a compact Riemann surface C of genus g and a set of n-distinct points $\boldsymbol{t} = \{t_i\}_{1 \leqslant i \leqslant n}$ on C. We put $D(\boldsymbol{t}) = t_1 + \cdots + t_n$ for each $(C, \boldsymbol{t}) = (C, t_1, \dots, t_n)$. We denote by

$$\Gamma_{C,\boldsymbol{t}} := \pi_1(C \setminus D(\boldsymbol{t}), *) \tag{2.1}$$

the fundamental group of $C \setminus D(\boldsymbol{t})$ with the base point $* \in C \setminus D(\boldsymbol{t})$. The group $\Gamma_{C,\boldsymbol{t}}$ is generated by $(2g+n)$-element $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_n$ with one relation

$$(\alpha_1, \beta_1) \cdots (\alpha_g, \beta_g) \gamma_1 \cdots \gamma_n = 1.$$

Here, we set $(\alpha, \beta) = \alpha\beta\alpha^{-1}\beta^{-1}$. The set of generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_n$ is called *canonical generators* of $\Gamma_{C,\boldsymbol{t}}$.

Definition 2.1. — *An $SL_r(\mathbb{C})$-representation of the fundamental group $\Gamma_{C,\boldsymbol{t}}$ is a group homomorphism*

$$\rho : \Gamma_{C,\boldsymbol{t}} \longrightarrow SL_r(\mathbb{C}). \tag{2.2}$$

Let $\mathrm{Hom}(\Gamma_{C,\boldsymbol{t}}, SL_r(\mathbb{C}))$ be the set of all $SL_r(\mathbb{C})$-representations of $\Gamma_{C,\boldsymbol{t}}$. If we fix a set of canonical generators of $\Gamma_{C,\boldsymbol{t}}$, we have the identification

$$\mathrm{Hom}(\Gamma_{C,\boldsymbol{t}}, SL_r(\mathbb{C})) \xrightarrow{\simeq} SL_r(\mathbb{C})^{2g+n-1}.$$

Definition 2.2. — *Two $SL_r(\mathbb{C})$-representations ρ_1 and ρ_2 are isomorphic to each other, if and only if there exists a matrix $P \in SL_r(\mathbb{C})$ such that*

$$\rho_2(\gamma) = P^{-1} \cdot \rho_1(\gamma) \cdot P \text{ for all } \gamma \in \Gamma_{C,\boldsymbol{t}}.$$

Let $R^r_{(g,n-1)}$ denote the affine coordinate ring of $SL_r(\mathbb{C})^{2g+n-1}$. We consider the simultaneous action of $SL_r(\mathbb{C})$ on $SL_r(\mathbb{C})^{2g+n-1}$ as

$$\begin{aligned} P \curvearrowright &(A_1, \dots, A_g, B_1, \dots, B_g; M_1, \dots, M_{n-1}) \\ &\mapsto (P^{-1}A_1P, \dots, P^{-1}A_gP, P^{-1}B_1P, \dots, P^{-1}B_gP; \\ &\qquad\qquad P^{-1}M_1P, \dots, P^{-1}M_{n-1}P). \end{aligned}$$

The invariant ring $(R^r_{(g,n-1)})^{Ad(SL_r(\mathbb{C}))}$ is finitely generated. For any $(C,\boldsymbol{t})$, there exists the universal categorical quotient map

$$\begin{aligned}\Phi^r_{(C,\boldsymbol{t})} : \mathrm{Hom}(\Gamma_{C,\boldsymbol{t}}, SL_r(\mathbb{C})) \cong SL_r(\mathbb{C})^{2g+n-1} \\ \to \mathcal{R}^r_{(C,\boldsymbol{t})} = SL_r(\mathbb{C})^{2g+n-1}//SL_r(\mathbb{C})\end{aligned}$$

where

$$\mathcal{R}^r_{(C,\boldsymbol{t})} = \mathrm{Spec}[(R^r_{(g,n-1)})^{Ad(SL_r(\mathbb{C}))}].$$

The following lemme is due to Simpson.

Lemma 2.3 ([21, Proposition 6.1]). — *The closed points of $\mathcal{R}^r_{(C,\boldsymbol{t})}$ represent the Jordan equivalence classes of $SL_r(\mathbb{C})$-representations of $\Gamma_{C,\boldsymbol{t}}$.*

Let us set

$$\mathcal{A}^{(n)}_r := \left\{\boldsymbol{a} = (a^{(i)}_j)^{1\leqslant i\leqslant n}_{1\leqslant j\leqslant r-1} \in \mathbb{C}^{nr-n}\right\}.$$

For $\boldsymbol{a} = (a^{(i)}_j) \in \mathcal{A}^{(n)}_r$, we set

$$\chi_i(s) := s^r + a^{(i)}_{r-1}s^{r-1} + \cdots + a^{(i)}_1 s + (-1)^r,\ (i = 1, \ldots, n).$$

Moreover, we define the morphism

$$\phi^r_{(C,\boldsymbol{t})} : \mathcal{R}^r_{(C,\boldsymbol{t})} \to \mathcal{A}^{(n)}_r$$

by the relation

$$\det(sI_r - \rho(\gamma_i)) = \chi_i(s)$$

where $[\rho] \in \mathcal{R}^r_{(C,\boldsymbol{t})}$ and γ_i is a anticlockwise loop around the point t_i. The fiber of $\phi^r_{(C,\boldsymbol{t})}$ at $\boldsymbol{a} \in \mathcal{A}^{(n)}_r$ is given by the affine subscheme of $\mathcal{R}^r_{(C,\boldsymbol{t})}$:

$$\begin{aligned}\mathcal{R}^r_{(C,\boldsymbol{t}),\boldsymbol{a}} &:= (\phi^r_{(C,\boldsymbol{t})})^{-1}(\boldsymbol{a}) \\ &= \{[\rho] \in \mathcal{R}^r_{(C,\boldsymbol{t})} \mid \det(sI_r - \rho(\gamma_i)) = \chi_i(s), 1 \leqslant i \leqslant n\}.\end{aligned}$$

For $\boldsymbol{a} \in \mathcal{A}^{(n)}_r$, let $\mu^i = (\mu^i_1, \mu^i_2, \ldots)$ be the partition of r which implies the multiplicity of the solutions of the equation $\chi_i(s) = 0$. Put $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^n)$, called the *multiplicity of $\boldsymbol{a} \in \mathcal{A}^{(n)}_r$*. Moreover, we define the subvariety

$$\mathcal{A}^{(n)}_{r,\boldsymbol{\mu}} := \left\{\boldsymbol{a} = (a^{(i)}_j)^{1\leqslant i\leqslant n}_{1\leqslant j\leqslant r-1} \in \mathbb{C}^{nr-n} \;\middle|\; \text{the multiplicity of } \boldsymbol{a} \text{ is } \boldsymbol{\mu}\right\} \subset \mathcal{A}^{(n)}_r.$$

Definition 2.4. — *We fix a k-tuple $\boldsymbol{\mu}$ of partitions of r. Let $\boldsymbol{a}$ be a element of $\mathcal{A}^{(n)}_{r,\boldsymbol{\mu}}$. Then, we define*

$$\begin{aligned}\mathcal{R}^{r,s}_{(C,\boldsymbol{t}),\boldsymbol{\mu},\boldsymbol{a}} &:= \{[\rho] \in \mathcal{R}^r_{(C,\boldsymbol{t})} \mid \det(sI_r - \rho(\gamma_i)) = \chi_i(s), \rho(\gamma_i) : \\ &\qquad\qquad \textit{diagonalizable}, 1 \leqslant i \leqslant n\} \\ &= \{(A_1, B_1, \dots, A_g, B_g; M_1, \dots, M_n) \in SL_r(\mathbb{C})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \\ &\qquad \mid (A_1, B_1) \cdots (A_g, B_g) M_1 \cdots M_n = I_r\} // SL_r(\mathbb{C})\end{aligned}$$

where $\mathcal{C}_i = \{M \in SL_r(\mathbb{C}) \mid \det(sI_r - M) = \chi_{\boldsymbol{a}^{(i)}}(s),\ M : \text{diagnalizable}\}$. In Section 1, we denoted by $\mathcal{R}_{g,\boldsymbol{\mu}}$ the variety instead of $\mathcal{R}^{r,s}_{(C,\boldsymbol{t}),\boldsymbol{\mu},\boldsymbol{a}}$, for simplicity. The affine subvariety $\mathcal{R}^{r,s}_{(C,\boldsymbol{t}),\boldsymbol{\mu},\boldsymbol{a}}$ is called a $SL_r(\mathbb{C})$-character variety of the n-punctured compact Riemann surface of genus g. In particular, we denote by $\mathcal{R}^r_{n,\boldsymbol{a}}$ this variety in the case where $g = 0, \boldsymbol{\mu} = ((1, \dots, 1), \dots, (1, \dots, 1))$.

If we take a generic $\boldsymbol{a} \in \mathcal{A}^{(n)}_{r,\boldsymbol{\mu}}$, the affine algebraic variety $\mathcal{R}^{r,s}_{(C,\boldsymbol{t}),\boldsymbol{\mu},\boldsymbol{a}}$ is a non-singular irreducible variety of dimension

$$d_{g,\boldsymbol{\mu}} := r^2(2g - 2 + n) - \sum_{i,j} (\mu^i_j)^2 + 2 - 2g,$$

and has a holomorphic symplectic structure, if nonempty. (See [4],[6]). In particular, for $g = 0, \boldsymbol{\mu} = ((1, \dots, 1), \dots, (1, \dots, 1))$, the dimension of $\mathcal{R}^r_{n,\boldsymbol{a}}$ is

$$d_{0,((1,1),\dots,(1,1))} = 2n - 6.$$

3. Invariant ring

We recall the explicit description of the invariant ring $(R^r_{(g,n-1)})^{Ad(SL_r(\mathbb{C}))}$ for the two cases $g = 0, r = 2, n = 4$ and $g = 0, r = 3, n = 3$. The following proposition follows from the fundamental theorem for matrix invariants. (See [2] or [17]).

Proposition 3.1.

$$(R^r_{(0,n-1)})^{Ad(SL_r(\mathbb{C}))} = \mathbb{C}[\mathrm{Tr}(M_{i_1} M_{i_2} \cdots M_{i_k}) \mid 1 \leqslant i_1, \dots, i_k \leqslant n - 1].$$

In particular, for $r = 2$, the elements $\mathrm{Tr}(M_{i_1} M_{i_2} \cdots M_{i_k})$ of degree $k \leqslant 3$ generate the invariant ring, that is,

$$\begin{aligned}&(R^2_{(0,n-1)})^{Ad(SL_2(\mathbb{C}))} \\ &\qquad = \mathbb{C}[\mathrm{Tr}(M_i), \mathrm{Tr}(M_i M_j), \mathrm{Tr}(M_i M_j M_k) \mid 1 \leqslant i, j, k \leqslant n - 1].\end{aligned}$$

First, we consider the case where $g = 0, r = 2, n = 4$. Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. Then, the invariant ring $(R^2_{(0,3)})^{Ad(SL_2(\mathrm{C}))}$ is generated by

$$\begin{aligned} x_i &:= \mathrm{Tr}(M_k M_j) \ (i = 1, 2, 3), \\ a_i &:= \mathrm{Tr}(M_i) \ (i = 1, 2, 3), \\ a_4 &:= \mathrm{Tr}(M_3 M_2 M_1). \end{aligned} \tag{3.1}$$

The following proposition is due to Frike-Klein, Jimbo, and Iwasaki, ([3], [10], [9]).

Proposition 3.2. — *The invariant ring $(R^2_{(0,3)})^{Ad(SL_2(\mathrm{C}))}$ is generated by seven elements $x_1, x_2, x_3, a_1, a_2, a_3, a_4$ and there exists a relation*

$$f_{\boldsymbol{a}}(x) := x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\boldsymbol{a})x_1 - \theta_2(\boldsymbol{a})x_2 - \theta_3(\boldsymbol{a})x_3 + \theta_4(\boldsymbol{a}) = 0$$

where

$$\begin{aligned} \theta_i(\boldsymbol{a}) &= a_i a_4 + a_j a_k \quad (i, j, k), \\ \theta_4(\boldsymbol{a}) &= a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4. \end{aligned}$$

Therefore, we have an isomorphism

$$(R^2_{(0,3)})^{Ad(SL_2(\mathbb{C}))} \cong \mathbb{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f_{\boldsymbol{a}}(x)).$$

We have the surjective morphism

$$\begin{aligned} \phi^2_{(\mathbb{P}^1,0,1,t,\infty)} : \mathcal{R}^2_{(\mathbb{P}^1,0,1,t,\infty)} = \mathrm{Spec}[(R^2_{(0,3)})^{Ad(SL_2(\mathbb{C}))}] \\ \to \mathcal{A}_2^{(4)} = \mathrm{Spec}[\mathbb{C}[a_1, a_2, a_3, a_4]] \end{aligned}$$

where t is a point of $\mathbb{P}^1$ such that $t \neq 0, 1, \infty$. The fiber at $\boldsymbol{a} \in \mathcal{A}_2^{(4)}$, such that the type of the multiplicities of eigenvalues is $((1,1),(1,1),(1,1),(1,1))$, is an affine cubic hypersurface in $\mathbb{C}^3$. Hence, the $SL_2(\mathbb{C})$-character variety of the 4-punctured projective line is an affine cubic hypersurface

$$\mathcal{R}_{4,\boldsymbol{a}} \cong \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid f_{\boldsymbol{a}}(x) = 0\}.$$

The affine cubic hypersurface is called a *Fricke-Klein cubic surface.*

We consider the natural compactification $\mathbb{C}^3 \hookrightarrow \mathbb{P}^3$ as follows. Set $x_1 = X/W, x_2 = Y/W, x_3 = Z/W$. Then, we obtain the following homogeneous polynomial

$$\begin{aligned} XYZ + X^2W + Y^2W + Z^2W - \theta_1(a)XW^2 \\ - \theta_2(a)YW^2 - \theta_3(a)ZW^2 + \theta_4(a)W^3 = 0. \end{aligned}$$

Substitute $W = 0$ to this equation. Then, we obtain the equation $XYZ = 0$. Hence, the boundary divisor of the natural compactification of $\mathcal{R}_{4,\boldsymbol{a}}$ consists

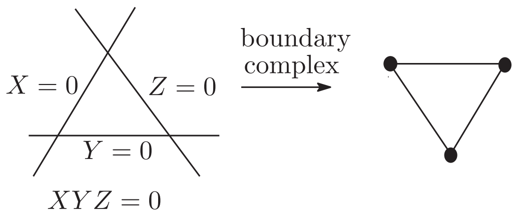

Figure 3.1.

of three lines. The boundary complex is shown in Figure 3.1. The boundary complex is a simplicial decomposition of S^1.

Next, we consider the case where $g = 0, r = 3, n = 3$. We describe generators and defining relations for the invariant ring $(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}$. The following proposition is due to Lawton [12].

Proposition 3.3. — *The invariant ring $(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}$ is generated by*

$$\begin{aligned} a_1 &:= \mathrm{Tr}(M_1) & a_2 &:= \mathrm{Tr}(M_1^{-1}) \\ b_1 &:= \mathrm{Tr}(M_2) & b_2 &:= \mathrm{Tr}(M_2^{-1}) \\ c_1 &:= \mathrm{Tr}(M_1^{-1}M_2^{-1}) = \mathrm{Tr}(M_3) & c_2 &:= \mathrm{Tr}(M_1M_2) = \mathrm{Tr}(M_3^{-1}) \\ x_1 &:= \mathrm{Tr}(M_1M_2^{-1}) & x_2 &:= \mathrm{Tr}(M_1^{-1}M_2) \\ x_3 &:= \mathrm{Tr}(M_1M_2M_1^{-1}M_2^{-1}), \end{aligned}$$

and there exists a relation

$$x_3^2 - fx_3 + g = 0$$

where f, g are polynomials of x_1, x_2 over $\mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2]$, more precisely,

$$\begin{aligned} f &= x_1x_2 - a_2b_1x_1 - a_1b_2x_2 + (\textit{constant terms in } x_1, x_2) \\ g &= x_1^3 + x_2^3 + (\textit{terms that order is at most } 2 \textit{ in } x_1, x_2). \end{aligned}$$

We consider the subring $A_3^{(3)} = \mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2]$ of $(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}$. We have a natural morphism

$$\phi^3_{(\mathbb{P}^1,0,1,\infty)} : \mathcal{R}^3_{(\mathbb{P}^1,0,1,\infty)} = \mathrm{Spec}[(R^3_{(0,2)})^{Ad(SL_3(\mathbb{C}))}] \to \mathcal{A}_3^{(3)} = \mathrm{Spec}[A_3^{(3)}].$$

The fiber at $\boldsymbol{a} \in \mathcal{A}_3^{(3)}$, such that the type of the multiplicities of eigenvalues is $((1,1,1),(1,1,1),(1,1,1))$, is an affine cubic hypersurface in $\mathbb{C}^3$. Hence,

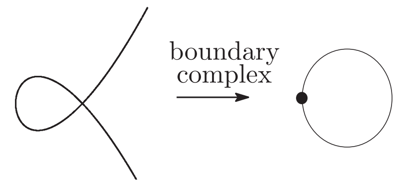

Figure 3.2.

the $SL_3(\mathbb{C})$-character variety of the 3-punctured projective line is an affine cubic hypersurface

$$\mathcal{R}^3_{3,\boldsymbol{a}} \cong \{(x_1,x_2,x_3)\in\mathbb{C}^3 \mid x_3^2 - fx_3 + g = 0\}.$$

We consider the compactification $\mathbb{C}^3 \hookrightarrow \mathbb{P}^3$ as follows. Set $x_1 = X/W, x_2 = Y/W, x_3 = Z/W$. Then, we obtain the following homogeneous polynomial

$$X^3+Y^3-XYZ+(\text{term containing } W)=0.$$

We substitute $W=0$ to this equation. Then, we obtain the equation $X^3 + Y^3 - XYZ = 0$. This equation defines a plane cubic curve having a node. The boundary complex is shown in Figure 3.2. The boundary complex is a simplicial decomposition of S^1.

4. A compactification of the character variety

We construct a compactification of the $SL_2(\mathbb{C})$-character variety $\mathcal{R}_{n,\boldsymbol{k}}$ ($\boldsymbol{k}$ of the n-punctured projective line is date of coefficient of characteristic polynomials) by means of the geometric invariant theory for a compactification of the following variety

Definition 4.1. — *We put*

(4.1)
$$\begin{aligned}\mathrm{Rep}_{n,\boldsymbol{k}} &:= \{(M_1,\dots,M_{n-1})\in\mathcal{C}_1\times\cdots\times\mathcal{C}_{n-1}| M_{n-1}^{-1}\cdots M_1^{-1}\in\mathcal{C}_n\}\\ &= \{(M_1,\dots,M_{n-1})\in\mathcal{C}_1\times\cdots\times\mathcal{C}_{n-1}|\mathrm{Tr}(M_{n-1}^{-1}\cdots M_1^{-1}) = k_n\}\end{aligned}$$

where $\mathcal{C}_i = \{M \in SL_2(\mathbb{C}) \mid \mathrm{Tr}(M) = k_i\}$ *and* $\boldsymbol{k} = (k_1,\dots,k_n)\in\mathbb{C}^n$*. The affine variety* $\mathrm{Rep}_{n,\boldsymbol{k}}$ *is said to the* $SL_2(\mathbb{C})$-representation variety *of the* n*-punctured line.*

We will introduce a compactification of the representation variety due to Benjamin [13]. First, we consider a construction of a compactification of the algebraic group $SL_2(\mathbb{C})$. We pick an embedding $\alpha\colon SL_2(\mathbb{C}) \hookrightarrow PGL_3(\mathbb{C})$. Such an embedding always exists: we consider the natural embedding $SL_2(\mathbb{C}) \to GL_2(\mathbb{C})$ and we take the composition of the embedding and the map $GL_2(\mathbb{C}) \xrightarrow{\xi} GL_3(\mathbb{C}) \to PGL_3(\mathbb{C})$ where

$$\xi(A) = \left(\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right)$$

and the second arrow is the canonical projection. We regard $PGL_3(\mathbb{C})$ as an open subvariety of $\mathbb{P}(M_3(\mathbb{C}))$, and define the compactification $\overline{SL_2(\mathbb{C})}$ of $SL_2(\mathbb{C})$ as the closure of $\alpha(SL_2(\mathbb{C}))$ in $\mathbb{P}(M_3(\mathbb{C}))$, that is,

$$\overline{SL_2(\mathbb{C})} = \left\{ \left(\begin{array}{cc|c} a & b & \\ c & d & \\ \hline & & e \end{array}\right) \in \mathbb{P}(M_3(\mathbb{C})) \;\middle|\; ad - bc = e^2 \right\}.$$

Then, we obtain a compactification of the semisimple conjugacy class $\mathcal{C}_i$, denoted by $\overline{\mathcal{C}_i}$, that is,

$$\overline{\mathcal{C}_i} = \left\{ \left(\begin{array}{cc|c} a & b & \\ c & d & \\ \hline & & e \end{array}\right) \in \mathbb{P}(M_3(\mathbb{C})) \;\middle|\; ad - bc = e^2,\ a + d = k_i e \right\}.$$

We can define a compactification of the representation variety.

Definition 4.2. — *We put*

$$\begin{aligned} \overline{\mathrm{Rep}}_{n,\boldsymbol{k}} := \{(M_1, \ldots, M_{n-1}) \in \overline{\mathcal{C}}_1 \times \cdots \times \overline{\mathcal{C}}_{n-1} \mid \\ \mathrm{Tr}(A_1 \cdots A_{n-1}) = k_n e_1 \cdots e_{n-1}\} \end{aligned} \tag{4.2}$$

where

$$M_1 = \left(\begin{array}{c|c} A_1 & \\ \hline & e_1 \end{array}\right), \ldots, M_{n-1} = \left(\begin{array}{c|c} A_{n-1} & \\ \hline & e_{n-1} \end{array}\right).$$

Remark 4.3. — In general, for $X \in \overline{SL_2(\mathbb{C})}$, there is no inverse. Since

$$\mathrm{Tr}(A_{n-1}^{-1} \cdots A_1^{-1}) = \mathrm{Tr}(A_1 \cdots A_{n-1})$$

for $\forall A_i \in SL_2(\mathbb{C})$, we use the condition $\mathrm{Tr}(A_1 \cdots A_{n-1}) = k_n$, instead of $\mathrm{Tr}(A_{n-1}^{-1} \cdots A_1^{-1}) = k_n$.

We have the following action of $SL_2(\mathbb{C})$ on $\overline{\mathrm{Rep}_{n,\boldsymbol{k}}}$, which is compatible with the simultaneous action of $SL_2(\mathbb{C})$ on $\mathrm{Rep}_{n,\boldsymbol{k}}$

$$\begin{aligned} P \curvearrowright & \left(\left(\begin{array}{c|c} A_1 & \\ \hline & e_1 \end{array} \right), \dots, \left(\begin{array}{c|c} A_{n-1} & \\ \hline & e_{n-1} \end{array} \right) \right) \\ & \longmapsto \left(\left(\begin{array}{c|c} PA_1P^{-1} & \\ \hline & e_1 \end{array} \right), \dots, \left(\begin{array}{c|c} PA_{n-1}P^{-1} & \\ \hline & e_{n-1} \end{array} \right) \right). \end{aligned} \tag{4.3}$$

We regard $\overline{\mathrm{Rep}_{n,\boldsymbol{k}}} \subset \overline{\mathcal{C}_1} \times \cdots \times \overline{\mathcal{C}_{n-1}}$ as the closed subset in $\mathbb{P}^4 \times \cdots \times \mathbb{P}^4$. Then, we obtain an embedding in the projective space by the Segre embedding. Let L be an ample line bundle associated with this embedding, that is,

$$L = \bigotimes_{i=1}^{n-1} p_i^*(\mathcal{O}_{\mathbb{P}^4}(1))$$

where $p_i \colon \overline{\mathrm{Rep}_{n,\boldsymbol{k}}} \to \mathbb{P}^4$ is the i-th projection. Then, L admits the $SL_2(\mathbb{C})$-linearization with respect to the action.

For $x = (M_1, \dots, M_{n-1}) \in \overline{\mathrm{Rep}_{n,\boldsymbol{k}}}$, we put

$$I^{nil} := \{ i \in \{1, \dots, n-1\} \mid M_i \text{ is nilpotent i.e. } e_i = 0 \}.$$

If I^{nil} is not empty, we decompose

$$I^{nil} = I_1^{nil} \cup \cdots \cup I_k^{nil} \tag{4.4}$$

where the index set $I_l^{nil} \subset I^{nil}$ $(1 \leqslant l \leqslant k)$ consists of indexes of same matrices, that is, matrices indexed by elements of I_l^{nil} are same each other and two matrices which respectively have indexes in I_l^{nil} and $I_{l'}^{nil}$ where $l \neq l'$ are not equal. Let $\sharp I_l^{nil}$ be the cardinality of I_l^{nil}, and let m_1 be a maximum value in $\sharp I_1^{nil}, \dots, \sharp I_k^{nil}$. We put

$$\begin{aligned} J_l := \{ j \in \{1, \dots, n-1\} \mid & \\ & M_j \text{ is not nilpotent}, M_j * M_i = M_i * M_j = M_i, i \in I_l^{nil} \}. \end{aligned}$$

Here, we define the product $*$ as

$$M * M' := \left(\begin{array}{c|c} AA' & \\ \hline & e \end{array} \right) \in \mathbb{P}M_3(\mathbb{C})$$
$$\text{for } M := \left(\begin{array}{c|c} A & \\ \hline & e \end{array} \right) \text{ and } M' := \left(\begin{array}{c|c} A' & \\ \hline & e' \end{array} \right).$$

Note that the product $*$ is well-defined in the case where M (resp. M') is nilpotent and M' (resp. M) is not nilpotent where $M \in \overline{\mathcal{C}}$ and $M' \in \overline{\mathcal{C}'}$. Let m_2 be a maximum value in $\{ \sharp J_l \mid l \text{ is satisfied } \sharp I_l^{nil} = m_1, 1 \leqslant l \leqslant k \}$. If I^{nil} is empty, then we put $m_1 = m_2 = 0$.

Remark 4.4. — Let $(M_1, \dots, M_{n-1}) \in \overline{\mathrm{Rep}_{n,\boldsymbol{k}}}$. Suppose that $i \in I^{nil}$. We normalize the nilpotent matrix M_i:

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right). \tag{4.5}$$

For a matrix M_j $(j \neq i)$, the condition which, by this transformation, the matrix M_j is transformed to the following form

$$\begin{pmatrix} a_j & b_j \\ 0 & d_i \end{pmatrix}$$

is equivalent to the condition $M_j * M_i = M_i * M_j = M_i$.

Proposition 4.5. — *The point $x = (M_1, \dots, M_{n-1})$ is semi-stable (resp. stable) point if and only if x is satisfied the following condition,*

$$n - 1 \geqslant 2m_1 + m_2 \quad (\textit{resp.} \ > \ ). \tag{4.6}$$

Proof. — For any integer $r > 0$, let λ_r be the 1-parameter subgroup (1-PS) of $SL_2(\mathbb{C})$ given by

$$\lambda_r \colon t \longmapsto \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}, t \in \mathbb{C}^\times. \tag{4.7}$$

The matrix $\lambda_r(t)$ acts on $\overline{\mathrm{Rep}_{n,\boldsymbol{k}}}$ as follows.

$$\begin{aligned} &\begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix} \curvearrowright \left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \dots, \left(\begin{array}{cc|c} a_{n-1} & b_{n-1} & \\ c_{n-1} & d_{n-1} & \\ \hline & & e_{n-1} \end{array}\right) \right) \\ &\longmapsto \left(\left(\begin{array}{cc|c} a_1 & t^{2r} b_1 & \\ t^{-2r} c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \dots, \left(\begin{array}{cc|c} a_{n-1} & t^{2r} b_{n-1} & \\ t^{-2r} c_{n-1} & d_{n-1} & \\ \hline & & e_{n-1} \end{array}\right) \right). \end{aligned}$$

We put $n' := 5^{n-1}$. Let $\mathbb{A}^{n'}$ be the affine cone over the projective space $\mathbb{P}^{n'-1}$ which is the target space of the Segre embedding. We take a base change of the affine cone $\mathbb{A}^{n'}$ via $\overline{\mathrm{Rep}_{n,\boldsymbol{k}}} \hookrightarrow \mathbb{P}^{n'-1}$, denoted by the same notation $\mathbb{A}^{n'}$. Let $x^* = (M_1^*, \dots, M_{n-1}^*)$ be the closed point of $\mathbb{A}^{n'}$ lying over $x \in \overline{\mathrm{Rep}_{n,\boldsymbol{k}}}$, that is, $x^* \neq 0$ and x^* projects to x. The action (4.3) and the linearization L define a linear action of $SL_2(\mathbb{C})$ on $\mathbb{A}^{n'}$. In particular, the matrix $\lambda_r(t)$ acts on $\mathbb{A}^{n'}$ as follows. For each $i = 1, \dots, n-1$, let

$e_1^{(i)},\dots,e_5^{(i)}$ be a basis of $\mathbb{A}^5$ such that the matrix

$$M_i^* = \left(\begin{array}{cc|c} a_i & b_i & \\ c_i & d_i & \\ \hline & & e_i \end{array}\right)$$

is describe by

$$M_i^* = a_i e_1^{(i)} + b_i e_2^{(i)} + c_i e_3^{(i)} + d_i e_4^{(i)} + e_i e_5^{(i)}.$$

Let $e_{i_1,\dots,i_{n-1}}$ be the base $e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_{n-1}}^{(n-1)}$ of $\mathbb{A}^{n'}$ where $i_1,\dots,i_{n-1} \in \{1,\dots,5\}$. Then, the action of $\lambda_r(t)$ on $\mathbb{A}^5$ is given by

$$\lambda_r(t)\cdot e_{i_1,\dots,i_{n-1}} = t^{2r(r^+_{i_1,\dots,i_{n-1}} - r^-_{i_1,\dots,i_{n-1}})} e_{i_1,\dots,i_{n-1}}$$

where $i_1,\dots,i_{n-1} \in \{1,\dots,5\}$ and $r^+_{i_1,\dots,i_{n-1}}$ (resp. $r^-_{i_1,\dots,i_{n-1}}$) is the number of 2 (resp. 3) in the index set $\{i_1,\dots,i_{n-1}\}$. For $x^* \in \mathbb{A}^{n'}$ lying over $x \in \overline{\mathrm{Rep}_{n,\boldsymbol{k}}}$, we write $x^* = \sum x^*_{i_1,\dots,i_{n-1}} e_{i_1,\dots,i_{n-1}}$, so that

$$\lambda_r(t)\cdot x^* = \sum t^{2r r_{i_1,\dots,i_{n-1}}} x^*_{i_1,\dots,i_{n-1}} e_{i_1,\dots,i_{n-1}}$$

where $r_{i_1,\dots,i_{n-1}} = r^+_{i_1,\dots,i_{n-1}} - r^-_{i_1,\dots,i_{n-1}}$, and we put

$$\begin{aligned} \mu^L(x,\lambda_r) &:= \max\{-r_{i_1,\dots,i_{n-1}} \,|\, i_1,\dots,i_{n-1} \text{ such that } x^*_{i_1,\dots,i_{n-1}} \neq 0\} \\ &= \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, c_i \neq 0 \right\} \\ &\qquad - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right\}. \end{aligned} \tag{4.8}$$

On the other hand, we have

$$\begin{aligned} &\sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, c_i \neq 0 \right\} \\ &\quad = (n-1) - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right\} \\ &\qquad - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, e_i \neq 0 \right\}. \end{aligned}$$

Then, we have

$$\begin{aligned} &\mu^L(x,\lambda_r) \\ &\quad = (n-1) - 2\sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right\} \\ &\qquad - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, e_i \neq 0 \right\}. \end{aligned} \tag{4.9}$$

By the Hilbert-Mumford criterion (see [14, Theorem 2.1] or [15, Proposition 4.11]), the point x is stable (resp. semi-stable) for this action if and only if $\mu^L(g\cdot x, \lambda_r) > 0$ (resp. $\geqslant 0$) for every $g \in SL_2(\mathbb{C})$ and every 1-PS λ_r of the form (4.7). If the point x satisfies the condition $2m_1 < \sharp I^{nil}$, then we have $\mu^L(g\cdot x, \lambda_r) > 0$ for any $g \in SL_2(\mathbb{C})$. On the other hand, we consider the case where the point x satisfies the condition $2m_1 \geqslant \sharp I^{nil}$. There are at most two components of the decomposition (4.4) of I^{nil} such that the cardinalities are m_1. We denote by I^{nil}_{max} the union of the components. If the index set $I^{nil} \setminus I^{nil}_{max}$ is nonempty, then we have $\mu^L(g \cdot x, \lambda_r) > 0$ for $g \in SL_2(\mathbb{C})$ such that gM_ig^{-1} is the matrix (4.5) where $i \in I^{nil} \setminus I^{nil}_{max}$. For $g \in SL_2(\mathbb{C})$ such that gM_ig^{-1} is the matrix (4.5) where $i \in I^{nil}_{max}$, we have

$$\mu^L(g\cdot x, \lambda_r) \geqslant (n-1) - (2m_1 + m_2). \tag{4.10}$$

If the index $i \in I^{nil}_{max}$ of the normalized matrix is a element of I^{nil}_l such that $\sharp I^{nil}_l = m_1$ and $\sharp J_l = m_2$, then the equality of (4.10) holds. For the other matrix $g \in SL_2(\mathbb{C})$, we have $\mu^L(g \cdot x, \lambda_r) > 0$. We have thus proved the proposition. $\square$

We obtain a compactification of the character variety $\mathcal{R}_{n,\boldsymbol{k}}$.

Definition 4.6. —

$$\overline{\mathcal{R}_{n,\boldsymbol{k}}} := \mathrm{Proj}\ H^0(\overline{\mathrm{Rep}_{n,\boldsymbol{k}}}, L^{\otimes r})^{Ad(SL_2(\mathbb{C}))}.$$

The variety $\overline{\mathcal{R}_{n,\boldsymbol{k}}}$ is a projective algebraic variety. This variety may have singular points on the boundary. Then, we should take a resolution of singular points of $\overline{\mathcal{R}_{n,\boldsymbol{k}}}$. In general, it is not easy to give a systematic resolution of singularities for any n. On the following sections, we treat the cases for $n = 4, 5$. We will show that $\overline{\mathcal{R}_{n,\boldsymbol{k}}}$ is non-singular and the boundary divisor is a triangle of $\mathbb{P}^1$. On Section 6, we will treat the case for $n = 5$.

5. $n = 4$

Let

$$\left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} a_2 & b_2 & \\ c_2 & d_2 & \\ \hline & & e_2 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right) \right) \in \overline{\mathrm{Rep}_{4,\boldsymbol{k}}}. \tag{5.1}$$

The compactification $\overline{\mathrm{Rep}_{4,\boldsymbol{k}}}$ is defined by the following equations in $\mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^4$

$$a_i + d_i = k_i e_i,\ (i = 1, 2, 3), \tag{5.2}$$

$$a_i d_i - b_i c_i = e_i^2, \ (i = 1, 2, 3), \tag{5.3}$$

$$\mathrm{Tr}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) = k_4 e_1 e_2 e_3. \tag{5.4}$$

We analyze the stability. If $e_i = 0$ and $e_j e_k \neq 0$ ($j, k \in \{1, 2, 3\}\setminus\{i\}$), then x is an unstable point if and only if x is a point of the orbit of (M_1, M_2, M_3) where

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_j = \left(\begin{array}{cc|c} a_j & b_j & \\ 0 & d_j & \\ \hline & & e_j \end{array}\right), M_k = \left(\begin{array}{cc|c} a_k & b_k & \\ 0 & d_k & \\ \hline & & e_k \end{array}\right).$$

If $e_i = 0, e_j = 0$, then x is an unstable point if and only if x is a point of the orbit of (M_1, M_2, M_3) where two matrices in M_1, M_2, M_3 are

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right).$$

Lemma 5.1. — *The point $x \in \overline{\mathrm{Rep}_{4,\boldsymbol{k}}}$ is stable if and only if x is semistable.*

Proof. — The point $x = (M_1, M_2, M_3)$ is not stable if only x is normalized as follows.

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_j = \left(\begin{array}{cc|c} a_j & b_j & \\ c_j & d_j & \\ \hline & & e_j \end{array}\right), \text{ where } c_j \neq 0,$$

$$M_k = \left(\begin{array}{cc|c} a_k & b_k & \\ 0 & d_k & \\ \hline & & e_k \end{array}\right),$$

or

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_j = \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), M_k = \left(\begin{array}{cc|c} a_k & b_k & \\ 0 & d_k & \\ \hline & & e_k \end{array}\right).$$

However, the matrices are not satisfied the equation (5.4). Then, there are no strictly semistable points. □

The following theorem shows that our compactification $\overline{\mathcal{R}_{4,\boldsymbol{k}}}$ of $\mathcal{R}_{4,\boldsymbol{k}}$ has the same configuration of the boundary divisor as the natural compactification of the Fricke-Klein cubic surface.

Theorem 5.2. — *The boundary divisor of the compactification $\overline{\mathcal{R}_{4,\boldsymbol{k}}}$ is a triangle of three projective lines.*

Proof. — We describe the boundary divisor explicitly. Let E_i be the image of the divisor $[e_i = 0]$ on $\overline{\mathrm{Rep}_{4,\boldsymbol{k}}}$ by the quotient $\overline{\mathrm{Rep}_{4,\boldsymbol{k}}} \to \overline{\mathcal{R}_{4,\boldsymbol{k}}}$ $(i = 1, 2, 3)$. First, we describe $[e_1 = 0]$. We normalize M_1 by the $SL_2(\mathbb{C})$-conjugate action as the matrix (4.5). The stabilizer subgroup of the matrix is $\left\{\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}\right\}$.

By the stability, we obtain $c_2 \neq 0$ and $c_3 \neq 0$. Since $c_2 \neq 0$, the matrices of the component $[e_1 = 0]$ are normalized by the action of this stabilizer subgroup:

$$\left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & -e_2^2 & \\ c_2^2 & k_2c_2e_2 & \\ \hline & & c_2e_2 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right)\right). \tag{5.5}$$

The stabilizer subgroup of the normalized matrices is the torus group $\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\}$.

Before we consider the quotient by the torus group, we consider the normalized matrices (5.5). The normalized matrices are defined by the following equations

$$\begin{cases} a_3 + d_3 = k_3e_3, \\ a_3d_3 - b_3c_3 = e_3^2, \\ c_2a_3 + k_2e_2c_3 = 0 \end{cases} \tag{5.6}$$

in the Zariski open set $c_2c_3 \neq 0$ of $\mathbb{P}^1 \times \mathbb{P}^4$. By the equations $a_3 + d_3 = k_3e_3$ and $a_3d_3 - b_3c_3 = e_3^2$, we obtain the equation

$$(-a_3^2 + k_3a_3e_3 - e_3^2) - b_3c_3 = 0.$$

Note that the equation define a hypersurface of degree 2 in $\mathbb{P}^3$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We put the coordinate $([S_3 : T_3], [U_3 : V_3]) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$\begin{aligned} (S_3U_3)(T_3V_3) &= -a_3^2 + k_3a_3e_3 - e_3^2 = -(a_3 - \alpha_3^+ e_3)(a_3 - \alpha_3^- e_3) \\ S_3V_3 &= b_3 \\ T_3U_3 &= c_3 \end{aligned}$$

where α_i^+, α_i^- are eigenvalues of a matrix of the semisimple conjugacy class $\mathcal{C}_i$. Then, we obtain the following transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to the hypersurface of degree 2 on $\mathbb{P}^3$:

$$\begin{aligned} a_3 &= \frac{\alpha_3^- S_3U_3 + \alpha_3^+ T_3V_3}{\alpha_3^+ - \alpha_3^-}, & b_3 &= S_3V_3, \\ c_3 &= T_3U_3, & d_3 &= \frac{\alpha_3^+ S_3U_3 + \alpha_3^- T_3V_3}{\alpha_3^+ - \alpha_3^-}, \\ e_3 &= \frac{S_3U_3 + T_3V_3}{\alpha_3^+ - \alpha_3^-}. \end{aligned} \tag{5.7}$$

Therefore, the normalized matrices are defined by

$$c_2(\alpha_3^- S_3U_3 + \alpha_3^+ T_3V_3) + k_2(\alpha_3^+ - \alpha_3^-)e_2(T_3U_3) = 0 \tag{5.8}$$

in the Zariski open set $c_2T_3U_3 \neq 0$ of $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$.

We consider the quotient by the torus group. The torus action on $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ is

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \curvearrowright & ([c_2 : e_2], [S_3 : T_3], [U_3 : V_3]) \\ & \longmapsto ([a^{-1}c_2 : ae_2], [aS_3 : a^{-1}T_3], [a^{-1}U_3 : aV_3]). \end{aligned}$$

We consider the $SL_2(\mathbb{C})$-linearization $L = \bigotimes_{i=1}^3 p_i^*(\mathcal{O}_{\mathbb{P}^4}(1))$ on $\overline{\mathrm{Rep}_{4,\boldsymbol{k}}}$. We take a pull-back of L via the embedding

$$p_{e_1} : \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow \overline{\mathrm{Rep}_{4,\boldsymbol{k}}} \tag{5.9}$$

defined by the matrices (5.5) and the transform (5.7). Let L_{e_1} be the pull-back of L on $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$. We obtain the T-linearization on L_{e_1} induced by the $SL_2(\mathbb{C})$-linearization L on $\overline{\mathrm{Rep}_{4,\boldsymbol{k}}}$. We consider the dual action on $H^0(\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1), L_{e_1})$. We have the following basis of the subspace consisting of invariant sections:

$$\begin{aligned} s_1 &= b_1 \otimes c_2^2 \otimes S_3U_3, & s_2 &= b_1 \otimes c_2^2 \otimes T_3V_3, \\ s_3 &= b_1 \otimes c_2e_2 \otimes T_3U_3 \end{aligned} \tag{5.10}$$

where $b_1 \in H^0(\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1), (p_{e_1} \circ p_1)^*(\mathcal{O}_{\mathbb{P}^4}(1)))$ corresponding to the $(1,2)$-entry of the matrix M_1. The sections have the relation

$$\alpha_3^- s_1 + \alpha_3^+ s_2 + k_2(\alpha_3^+ - \alpha_3^-)s_3 = 0$$

by the equation (5.8). Therefore, we obtain $E_1 \cong \mathbb{P}^1$. In the same way, we also obtain $E_i \cong \mathbb{P}^1$ $(i = 2, 3)$.

We show that E_1 and E_2 intersect at one point. We substitute $e_2 = 0$ for (5.6). Then, we have the following equations

$$\begin{cases} a_3 + d_3 = k_3 e_3, \\ a_3 d_3 - b_3 c_3 = e_3^2, \\ a_3 = 0. \end{cases}$$

The locus defined by the equations above is a quadric curve in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$. There are two unstable points in the locus, $[b_3 : c_3 : e_3] = [0 : 1 : 0]$ and $[b_3 : c_3 : e_3] = [1 : 0 : 0]$. The intersection is the quotient of $\mathbb{P}^1$ minus the two points by the torus action. Then, the intersection is a point. In the same way, the intersection of E_2 and E_3 (resp. E_3 and E_1) is a point. □

6. $n = 5$

Let

$$\left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array} \right), \left(\begin{array}{cc|c} a_2 & b_2 & \\ c_2 & d_2 & \\ \hline & & e_2 \end{array} \right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array} \right), \left(\begin{array}{cc|c} a_4 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array} \right) \right) \in \overline{\mathrm{Rep}_{5,\boldsymbol{k}}}.$$

The compactification $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ is defined by the following equations in $(\mathbb{P}^4)^4$

$$a_i + d_i = k_i e_i, \ (i = 1, 2, 3, 4), \tag{6.1}$$

$$a_i d_i - b_i c_i = e_i^2, \ (i = 1, 2, 3, 4), \tag{6.2}$$

$$\mathrm{Tr}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix} \right) = k_5 e_1 e_2 e_3 e_4. \tag{6.3}$$

We consider the stability condition.

Lemma 6.1. — *The closures of orbits of properly semistable points contain the point*

$$s_1 = \left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array} \right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array} \right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array} \right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array} \right) \right) \tag{6.4}$$

or

$$(6.5)\quad s_2 = \left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right) \right).$$

Expect for the points of the orbits of s_1 and s_2, the stabilizer groups of every points are finite. Each stabilizer group of the orbits of s_1 and s_2 is conjugate to the torus group $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$.

Proof. — Let $x = (M_1, \dots, M_4)$ be a property semistable point. By Proposition 4.5, we have $2m_1 + m_2 = 4$. First, we consider the case where $m_1 = 1, m_2 = 2$. We put

$$(6.6)\quad \begin{aligned} M_{i_1} &= \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), & M_{i_2} &= \left(\begin{array}{cc|c} * & * & \\ 0 & * & \\ \hline & & * \end{array}\right), \\ M_{i_3} &= \left(\begin{array}{cc|c} * & * & \\ 0 & * & \\ \hline & & * \end{array}\right), & M_{i_4} &= \left(\begin{array}{cc|c} * & * & \\ c_{i_4} & * & \\ \hline & & * \end{array}\right) \end{aligned}$$

where $\{i_1, \dots, i_4\} = \{1, \dots, 4\}$ and $c_{i_4} \neq 0$. However, by the condition $c_{i_4} \neq 0$, the matrices do not satisfy the equation (6.3).

Second, we consider the case where $m_1 = 2, m_2 = 0$. We put

$$(6.7)\quad \begin{aligned} M_{i_1} &= \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), & M_{i_2} &= \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \\ M_{i_3} &= \left(\begin{array}{cc|c} * & * & \\ c_{i_3} & * & \\ \hline & & * \end{array}\right), & M_{i_4} &= \left(\begin{array}{cc|c} * & * & \\ c_{i_4} & * & \\ \hline & & * \end{array}\right) \end{aligned}$$

where $\{i_1, \dots, i_4\} = \{1, \dots, 4\}$, $c_{i_3} \neq 0$, and $c_{i_3} \neq 0$. If $(i_1, i_2) = (1, 3)$ or $(2, 4)$, then the matrices do not satisfy the equation (6.3). Therefore, we consider the case where $(i_1, i_2) = (1, 2)$, $(2, 3)$, or $(3, 4)$. The 1-parameter subgroup (4.7) acts on the matrices (6.7). For the matrices M_{i_1} and M_{i_2}, the action is trivial. The actions of the 1-parameter subgroup $\lambda_r(t)$ on M_{i_3}

and M_{i_4} are

(6.8)

$$\lambda_r(t)\cdot M_{i_3} = \left(\begin{array}{cc|c} * & t^{2r}* & \\ t^{-2r}c_{i_3} & * & \\ \hline & & * \end{array}\right) \qquad \lambda_r(t)\cdot M_{i_4} = \left(\begin{array}{cc|c} * & t^{2r}* & \\ t^{-2r}c_{i_4} & * & \\ \hline & & * \end{array}\right)$$
$$= \left(\begin{array}{cc|c} t^{2r}* & t^{4r}* & \\ c_{i_3} & t^{2r}* & \\ \hline & & t^{2r}* \end{array}\right), \qquad = \left(\begin{array}{cc|c} t^{2r}* & t^{4r}* & \\ c_{i_4} & t^{2r}* & \\ \hline & & t^{2r}* \end{array}\right).$$

Then, the limit $\lim_{t\to 0}\lambda_r\cdot M$ is the matrices (6.4) or (6.5).

Since the orbits of the points s_1 and s_2 are closed, the orbits have the maximum dimension of the stabilizer group, which is one dimension. □

We consider a resolution of properly semistable points. We take the blowing up along the orbits of s_1 and s_2:

$$\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}} \longrightarrow \overline{\mathrm{Rep}_{5,\boldsymbol{k}}}. \tag{6.9}$$

The simultaneous action of $SL_2(\mathbb{C})$ on $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ induces an action on $\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}$. By taking the blowing up (6.9), the condition for stability and unstability is unchanging. On the other hand, the points of the exceptional divisors are stable points. The points of orbits which are not closed are unstable points. Hence, there is no properly semistable point in $\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}$. (See [11, Section 6]). We will show that the quotient of the blowing up is non-singular. First, we describe the blowing up of $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ along the orbit of s_1. Let U_1 and U_2 be the Zariski open sets $U_1 = [b_1 \neq 0, b_2 \neq 0, c_3 \neq 0, c_4 \neq 0]$ and $U_2 = [c_1 \neq 0, c_2 \neq 0, b_3 \neq 0, b_4 \neq 0]$ of $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}} \subset \overline{\mathcal{C}}_1 \times \cdots \times \overline{\mathcal{C}}_4$. Note that the orbit of s_1 is contained in $U_1 \cup U_2$. Since $\overline{\mathcal{C}}_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \dots, 4$ by the transformation (5.7), we have

$$U_i \subset \overline{\mathrm{Rep}_{5,\boldsymbol{k}}} \subset (\mathbb{P}^1 \times \mathbb{P}^1)^4 \text{ for } i = 1, 2. \tag{6.10}$$

In the open sets U_1 and U_2, we put the following affine coordinates

$$([1:x_1],[y_1:1]),([1:x_2],[y_2:1]),([x_3:1],[1:y_3]),([x_4:1],[1:y_4]),$$

and

$$([z_1:1],[1:w_1]),([z_2:1],[1:w_2]),([1:z_3],[w_3:1]),([1:z_4],[w_4:1]),$$

respectively. In the open set U_1, the ideal of the orbit of s_1 is $(X_1, X_2, X_3, X_4, X_5)$ where

$$\begin{aligned} X_0 &:= e_1 = \frac{y_1 + x_1}{\alpha_1^+ - \alpha_1^-}, & X_1 &:= e_2 = \frac{y_2 + x_2}{\alpha_2^+ - \alpha_2^-}, \\ X_2 &:= e_3 = \frac{y_3 + x_3}{\alpha_3^+ - \alpha_3^-}, & X_3 &:= e_4 = \frac{y_4 + x_4}{\alpha_4^+ - \alpha_4^-}, \\ X_4 &:= x_1 - x_2, & X_5 &:= x_3 - x_4. \end{aligned}$$

We can extend the torus action on $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ to the torus action on $\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}$ by

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \curvearrowright & [X_0 : X_1 : X_2 : X_3 : X_4 : X_5] \\ & \longmapsto [a^{-2}X_0 : a^{-2}X_1 : a^2X_2 : a^2X_3 : a^{-2}X_4 : a^2X_5]. \end{aligned}$$

On the other hand, in the open set U_2, the ideal of the orbit of s_1 is $(Y_1, Y_2, Y_3, Y_4, Y_5)$ where

$$\begin{aligned} Y_0 &:= e_1 = \frac{z_1 + w_1}{\alpha_1^+ - \alpha_1^-}, & Y_1 &:= e_2 = \frac{z_2 + w_2}{\alpha_2^+ - \alpha_2^-}, \\ Y_2 &:= e_3 = \frac{z_3 + w_3}{\alpha_3^+ - \alpha_3^-}, & Y_3 &:= e_4 = \frac{z_4 + w_4}{\alpha_4^+ - \alpha_4^-}, \\ Y_4 &:= z_1 - z_2, & Y_5 &:= z_3 - z_4. \end{aligned}$$

We can extend the torus action on $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ to the torus action on $\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}$ by

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \curvearrowright & [Y_0 : Y_1 : Y_2 : Y_3 : Y_4 : Y_5] \\ & \longmapsto [a^2Y_0 : a^2Y_1 : a^{-2}Y_2 : a^{-2}Y_3 : a^2Y_4 : a^{-2}Y_5]. \end{aligned}$$

Hence, we have

$$\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}_{s_1} \hookrightarrow (\overline{\mathrm{Rep}_{5,\boldsymbol{k}}} \setminus U_1 \cup U_2) \cup (U_1 \times \mathbb{P}^5) \cup (U_2 \times \mathbb{P}^5)$$

where $\widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}_{s_1}$ is the blowing up along the orbit of s_1. The stabilizer group of any point in the exceptional divisor is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

This action is trivial. In the same way, we can describe the blowing up along the orbit of s_2.

Theorem 6.2. — *In the case of $n = 5$, there exists a non-singular compactification of $\mathcal{R}_{5,\boldsymbol{k}}$ such that the boundary complex is a simplicial decomposition of sphere S^3.*

Proof. — The outline of the proof is as follows. We put

$$\widetilde{\overline{\mathcal{R}_{5,\boldsymbol{k}}}} := \widetilde{\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}}//SL_2(\mathbb{C}).$$

We have the six components of the boundary divisor of $\widetilde{\overline{\mathcal{R}_{5,\boldsymbol{k}}}}$: the quotients of the proper transformations of the divisors $[e_1=0],[e_2=0],[e_3=0],[e_4=0]$ of $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ and the quotients of the exceptional divisors associated with blowing up along s_1 and s_2. We denote by E_1,E_2,E_3,E_4 and ex_1,ex_2 each component. In Step 1, we describe the components E_1,E_2,E_3 and E_4 explicitly. In Step 2, we describe the intersections $E_i\cap E_j,\ i\neq j$. In particular, the intersections $E_i\cap E_{i+1}, i=1,2,3,4$ (where E_5 implies E_1) are nonempty and irreducible. On the other hand, the intersections $E_i\cap E_{i+2}, i=1,2$ are not irreducible. The intersection $E_i\cap E_{i+2}$ consists of two components, denoted by $E^+_{i,i+2},E^-_{i,i+2}$. Then, we take the blowing up along the components $E^+_{1,3},E^-_{1,3},E^+_{2,4},E^-_{2,4}$:

$$\widetilde{X}\longrightarrow X := \widetilde{\overline{\mathcal{R}_{5,\boldsymbol{k}}}}. \tag{6.11}$$

We use the same notation E_i which is the proper transform of E_i. We denote by $ex^+_{1,3},ex^-_{1,3},ex^+_{2,4},ex^-_{2,4}$ the exceptional divisors associated with the blowing up (6.11). Consequently, the components of the boundary divisor of the compactification $\widetilde{X}$ of $\mathcal{R}_{5,\boldsymbol{k}}$ are

$$E_1,E_2,E_3,E_4,ex_1,ex_2,ex^+_{1,3},ex^-_{1,3},ex^+_{2,4},ex^-_{2,4}.$$

Next, we see how ex_i and the other components intersect. In Step 3, we describe the 2-dimensional simplices and the 3-dimensional simplices. Finally, we can describe the boundary complex of the boundary divisor of the compactification of the character variety.

Step 1. — We describe the component E_i (i.e. $[e_i=0]//SL_2(\mathbb{C})$) explicitly. We consider the case where $e_1=0$. Let D_i be the divisor $[e_i=0]$ on $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$ for $i=1,\dots,4$. Let $(M_1,\dots,M_4)$ be a point on D_1. We normalize the matrix M_1 by the $SL_2(\mathbb{C})$-conjugate action as the matrix (4.5). The stabilizer subgroup of the matrix is the group of upper triangular matrices. From the stability, we obtain $c_2\neq 0$, $c_3\neq 0$ or $c_4\neq 0$. In the case of $c_2\neq 0$, the matrices of the divisor D_1 are normalized by the action of this stabilizer subgroup:

$$\left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0\end{array}\right),\left(\begin{array}{cc|c} 0 & -e_2^2 & \\ c_2^2 & k_2c_2e_2 & \\ \hline & & c_2e_2\end{array}\right),\left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3\end{array}\right),\left(\begin{array}{cc|c} a_4 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4\end{array}\right)\right). \tag{6.12}$$

Then, we have the locus defined by the following equations

$$\begin{cases} a_3 + d_3 = k_3 e_3 \\ a_3 d_3 - b_3 c_3 = e_3^2 \\ a_4 + d_4 = k_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ c_2 a_3 a_4 + k_2 e_2 c_3 a_4 + c_2 b_3 c_4 + k_2 e_2 d_3 c_4 = 0 \end{cases} \tag{6.13}$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4 \times \mathbb{P}^4)) \cap [c_2 \neq 0]$. The locus defined by $a_i + d_i = k_i e_i$ and $a_i d_i - b_i c_i = e_i^2$ in $\mathbb{P}^4$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We put the coordinates S_3, T_3, U_3, V_3 and S_4, T_4, U_4, V_4 of $(\mathbb{P}^1 \times \mathbb{P}^1)^2$ in the same way as in Section 5. Then, the locus of the normalized matrices is defined by the following equation

$$\begin{aligned} &c_2(\alpha_3^- S_3 U_3 + \alpha_3^+ T_3 V_3)(\alpha_4^- S_4 U_4 + \alpha_4^+ T_4 V_4) \\ &\quad + k_2 e_2 (\alpha_3^+ - \alpha_3^-)(T_3 U_3)(\alpha_4^- S_4 U_4 + \alpha_4^+ T_4 V_4) \\ &\qquad + c_2 (\alpha_3^+ - \alpha_3^-)(\alpha_4^+ - \alpha_4^-)(S_3 V_3)(T_4 U_4) \\ &\qquad\quad + k_2 e_2 (\alpha_4^+ - \alpha_4^-)(\alpha_3^+ S_3 U_3 + \alpha_3^- T_3 V_3)(T_4 U_4) = 0 \end{aligned}$$

in $(\mathbb{P}^1)^5 \cap [c_2 \neq 0]$. Let $D_1^{c_2 \neq 0}$ be the Zariski open set of the hypersrface in $(\mathbb{P}^1)^5$. The torus action on $D_1^{c_2 \neq 0}$ is the following action:

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} &\curvearrowright ([c_2 : e_2], [S_3 : T_3], [U_3 : V_3], [S_4 : T_4], [U_4 : V_4]) \\ &\mapsto ([a^{-1} c_2 : a e_2], [a S_3 : a^{-1} T_3], [a^{-1} U_3 : a V_3], \\ &\qquad\qquad\qquad [a S_3 : a^{-1} T_3], [a^{-1} U_3 : a V_3]). \end{aligned}$$

In the same way as in the case $c_2 \neq 0$, we have the Zariski open sets of the hypersurfaces in $(\mathbb{P}^1)^5$ corresponding to $c_3 \neq 0$ and $c_4 \neq 0$, denoted by $D_1^{c_3 \neq 0}$ and $D_1^{c_4 \neq 0}$. We glue $D_1^{c_2 \neq 0}$, $D_1^{c_3 \neq 0}$ and $D_1^{c_4 \neq 0}$, denoted by D_1'. We take the blowing up (6.9). Let $\widetilde{D}_1'$ be the proper transform of D_1'. Then, the component of the boundary divisor E_1 is the quotient of $\widetilde{D}_1'$ by the torus action. Similarly, we may describe the components E_j $(j = 2, 3, 4)$.

Step 2. — We denote by $D_{i,j}$ the intersection of the divisors $[e_i = 0]$ and $[e_j = 0]$ on $\overline{\mathrm{Rep}_{5,\boldsymbol{k}}}$. First, we consider the intersection of E_1 and E_2. We substitute $e_2 = 0$ for (6.13). Then, we have the locus defined by the following equations

$$\begin{cases} a_3 + d_3 = k_3 e_3 \\ a_3 d_3 - b_3 c_3 = e_3^2 \\ a_4 + d_4 = k_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ a_3 a_4 + b_3 c_4 = 0 \end{cases}$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_2 \neq 0]$. By the transform (5.7), we have the Zariski open set of the hypersurface in $(\mathbb{P}^1)^5$, denoted by $D_{12}^{c_2 \neq 0}$. Next, we consider the case where $c_3 \neq 0$. In the same way as in the case where $c_2 \neq 0$, we have the locus defined by the following equations

$$\begin{cases} a_2 + d_2 = 0 \\ a_2 d_2 - b_2 c_2 = 0 \\ a_4 + d_4 = k_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ d_2 c_3^2 a_4 - c_2 e_3^2 c_4 + k_3 d_2 c_3 e_3 c_4 = 0 \end{cases}$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_3 \neq 0]$. Since we may put $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} st & s^2 \\ -t^2 & -st \end{pmatrix}$ where $a + d = 0, ad - be = 0$, we have

$$\begin{cases} a_4 + d_4 = k_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ t(s c_3^2 a_4 - t e_3^2 c_4 + k_3 s c_3 e_3 c_4) = 0. \end{cases}$$

By the transform (5.7), we have the Zariski open set of the hypersurface in $(\mathbb{P}^1)^5$, denoted by $D_{1,2}^{c_3 \neq 0}$. The locus $D_{1,2}^{c_3 \neq 0}$ is not irreducible. Now, we take the blowing up along the orbits of s_1 and s_2. Let $\widetilde{D}_{1,2}^{c_3 \neq 0}$ be the proper transform of $D_{1,2}^{c_3 \neq 0}$. Since an orbit of a point of

$$[t = 0] \setminus ([t = 0] \cap [s c_3^2 a_4 - t e_3^2 c_4 + k_3 s c_3 e_3 c_4]) \subset D_{1,2}^{c_3 \neq 0} \tag{6.14}$$

are not closed, the points of the inverse image of (6.14) on $\widetilde{D}_{1,2}^{c_3 \neq 0}$ are unstable (see [11, Lemma 6.6]). Then, the quotient of $\widetilde{D}_{1,2}^{c_3 \neq 0}$ by the torus action is irreducible. Next, we consider the case where $c_4 \neq 0$. In the same way as in the case where $c_3 \neq 0$, we have the Zariski open set of the hypersurface in $(\mathbb{P}^1)^5$, denoted by $D_{1,2}^{c_4 \neq 0}$. We glue $D_{1,2}^{c_2 \neq 0}$, $D_{1,2}^{c_3 \neq 0}$ and $D_{1,2}^{c_4 \neq 0}$, denoted by $D'_{1,2}$. We take the proper transform of $D'_{1,2}$ of the blowing up along the orbits of s_1 and s_2, denoted by $\widetilde{D}'_{1,2}$. Then, the intersection of E_1 and E_2 is the quotient of $\widetilde{D}'_{1,2}$ by the torus action, denoted by $E_{1,2}$. The intersection $E_{1,2}$ is irreducible.

Second, we consider the intersection of E_1 and E_3. We substitute $e_3 = 0$ for (6.13). Then, we have the locus defined by the following equations

$$\begin{cases} a_3 + d_3 = 0 \\ a_3 d_3 - b_3 c_3 = 0 \\ a_4 + d_4 = k_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ c_2 a_3 a_4 + k_2 e_2 c_3 a_4 + c_2 b_3 c_4 + k_2 e_2 d_3 c_4 = 0 \end{cases}$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_2 \neq 0]$. We put $a_3 = st, b_3 = s^2, c_3 = -t^2, d_3 = -st$. Then, we have the equations

$$\begin{cases} a_4 + d_4 = k_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ (ta_4 + sc_4)(c_2 s - k_2 e_2 t) = 0. \end{cases} \tag{6.15}$$

We denote the two components $[ta_4 + sc_4 = 0]$ and $[c_2 s - k_2 e_2 t = 0]$ by $D_{1,3}^{c_2 \neq 0,+}$ and $D_{1,3}^{c_2 \neq 0,-}$.

Remark 6.3. — Any point (M_1, M_2, M_3, M_4) on $D_{1,3}^{c_2 \neq 0,+}$ is conjugate to the following matrices

(6.16)
$$\left(\begin{pmatrix} \begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c} a_2 & b_2 & \\ c_2 & d_2 & \\ \hline & & e_2 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c} 0 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array} \end{pmatrix} \right).$$

In fact, we normalize the third matrix M_3 instead of M_2. Then, we have

$$M_3 = \begin{pmatrix} \begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array} \end{pmatrix} \text{ or } \begin{pmatrix} \begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array} \end{pmatrix}.$$

In the former case, by the stability, we have $c_4 \neq 0$. However, the matrices do not satisfy the condition (6.3). In the latter case, the equation $ta_4 + sc_4 = 0$ implies that $a_4 = 0$. On the other hand, any point on $D_{1,3}^{c_2 \neq 0,-}$ is conjugate to the following matrices

(6.17)
$$\left(\begin{pmatrix} \begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c} a_2 & b_2 & \\ c_2 & 0 & \\ \hline & & e_2 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc|c} a_4 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array} \end{pmatrix} \right).$$

We consider the cases where $c_3 \neq 0$ and $c_4 \neq 0$. In the same way as in the case where $c_2 \neq 0$, we have the Zariski open sets

$$D_{1,3}^{c_3\neq 0,+}, D_{1,3}^{c_3\neq 0,-}, D_{1,3}^{c_4\neq 0,+}, D_{1,3}^{c_4\neq 0,-}$$

of the hypersurfaces in $(\mathbb{P}^1)^5$. We glue $D_{1,3}^{c_2\neq 0,+}$, $D_{1,3}^{c_3\neq 0,+}$ and $D_{1,3}^{c_4\neq 0,+}$ (resp. $D_{1,3}^{c_2\neq 0,-}$, $D_{1,3}^{c_3\neq 0,-}$ and $D_{1,3}^{c_4\neq 0,-}$), denoted by ${}'D_{1,3}^{+}$ (resp. ${}'D_{1,3}^{-}$). We take the blowing up (6.9). Let ${}'\widetilde{D}_{1,3}^{+}$ and ${}'\widetilde{D}_{1,3}^{-}$ be the proper transforms of ${}'D_{1,3}^{+}$ and ${}'D_{1,3}^{-}$, respectively. Then, the intersections of E_1 and E_3 are the quotients of ${}'\widetilde{D}_{1,3}^{+}$ and ${}'\widetilde{D}_{1,3}^{-}$ by the torus action, denoted by $E_{1,3}^{+}$ and $E_{1,3}^{-}$.

We consider the intersections $E_2 \cap E_3$, $E_3 \cap E_4$ and $E_1 \cap E_4$. In the same way as in the case $E_1 \cap E_2$, the intersections are irreducible, denoted by $E_{2,3}$, $E_{3,4}$ and $E_{1,4}$.

We consider the intersection of E_2 and E_4. In the same way as in the case $E_1 \cap E_3$, the intersection $E_2 \cap E_4$ is not irreducible. The intersection has two components, denoted by $E_{2,4}^{+}$ and $E_{2,4}^{-}$. Here, the components $E_{2,4}^{+}$ and $E_{2,4}^{-}$ correspond respectively to the following matrices

$$\left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & 0 & \\ \hline & & e_3 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right)\right)$$

and

$$\left(\left(\begin{array}{cc|c} 0 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right)\right).$$

Now, we take the blowing up along the components $E_{1,3}^{+}, E_{1,3}^{-}, E_{2,4}^{+}, E_{2,4}^{-}$:

$$\widetilde{X} \longrightarrow X := \widetilde{\overline{\mathcal{R}_{5,\boldsymbol{k}}}}.$$

We use the same notation E_i which is the proper transforms of E_i. We denote by $ex_{1,3}^{+}, ex_{1,3}^{-}, ex_{2,4}^{+}, ex_{2,4}^{-}$ the quotients of the exceptional divisors associated with this blowing up. Consequently, we have the ten components of the boundary divisor of the compactification $\widetilde{X}$ of $\mathcal{R}_{5,\boldsymbol{k}}$

$$E_1, E_2, E_3, E_4, ex_1, ex_2, ex_{1,3}^{+}, ex_{1,3}^{-}, ex_{2,4}^{+}, ex_{2,4}^{-},$$

and we obtain that the intersections

$$E_1 \cap E_2, \quad E_2 \cap E_3, \quad E_3 \cap E_4, \quad E_4 \cap E_1$$

and

$$E_1 \cap ex_{1,3}^{\pm},\ E_3 \cap ex_{1,3}^{\pm},\ E_2 \cap ex_{2,4}^{\pm},\ E_4 \cap ex_{2,4}^{\pm}$$

are nonempty and irreducible.

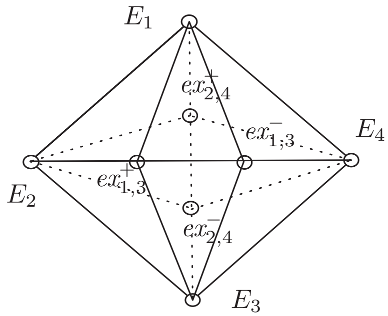

Figure 6.1.

We describe the intersections of the other pairs. We consider the intersection of $ex^+_{1,3}$ and E_4. If we substitute $e_4 = 0$ for the matrix (6.16), then we have $d_4 = 0$. Moreover, we have $b_4 = 0$ or $c_4 = 0$. Then, we obtain that

$$'D^+_{1,3} \cap [e_4 = 0] = \{s_1, s_2\} \cup [\text{points whose orbits are not closed}].$$

By the blowing up along s_1 and s_2, we obtain that the intersection of $E^+_{1,3}$ and E_4 is empty (see [11, Lemma 6.6]). Then, the intersection of $ex^+_{1,3}$ and E_4 is empty. In the same way as above, the intersections

$$ex^-_{1,3} \cap E_2, \quad ex^+_{2,4} \cap E_3, \quad ex^-_{2,4} \cap E_1$$

are empty. On the other hand, the intersections

$$ex^+_{1,3} \cap E_2,\ ex^-_{1,3} \cap E_4,\ ex^+_{2,4} \cap E_1,\ ex^-_{2,4} \cap E_3,\ ex^+_{1,3} \cap ex^-_{1,3},\ ex^+_{2,4} \cap ex^-_{2,4}$$

are nonempty and irreducible. Next, we consider the intersections of the pairs containing ex_1 or ex_2. The orbit of the point s_1 (resp. s_2) is contained in the components $D_1, \ldots, D_4$ and $D^{\pm}_{1,3}, D^{\pm}_{2,4}$, respectively. Here, $D^{\pm}_{1,3}$ and $D^{\pm}_{2,4}$ are the irreducible components of $D_{1,3}$ and $D_{2,4}$. Then, the intersections $ex_i \cap E_j$ and $ex_i \cap ex^{\pm}_{k,k+2}$ are nonempty and irreducible for $i = 1, 2$, $j = 1, \ldots, 4$ and $k = 1, 2$. On the other hand, the orbits of the point s_1 and s_2 are not intersect. Then, the intersection of ex_1 and ex_2 is empty.

Step 3. — We draw the vertexes and the 1-dimensional simplices except ex_1 and ex_2. Then, we obtain the graph of Figure 6.1. We consider the following sphere

$$\mathbb{R}^4 \supset S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}.$$

We arrange the vertexes except ex_1 and ex_2 on $S^2 = S^3 \cap [w = 0]$ and arrange the vertexes ex_1 and ex_2 at $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$ respectively.

We glue together the vertex ex_i ($i = 1, 2$) and each vertex on $S^2 = S^3 \cap [w = 0]$.

Next, we describe the 2-dimensional simplices. First, we consider the intersections $E_1 \cap E_2 \cap ex^+_{1,3}$ and $E_2 \cap E_3 \cap ex^+_{1,3}$. The intersection $E_1 \cap E_2 \cap E_3 = E^+_{1,3} \cap E_2$ is nonempty and irreducible in $\widetilde{\overline{\mathcal{R}_{5,\boldsymbol{k}}}}$. We take the blowing up along $E^+_{1,3}$. Then, the intersections $E_1 \cap E_2 \cap ex^+_{1,3}$ and $E_2 \cap E_3 \cap ex^+_{1,3}$ are irreducible. Second, we consider the intersections $E_1 \cap ex^+_{1,3} \cap ex^-_{1,3}$ and $E_3 \cap ex^+_{1,3} \cap ex^-_{1,3}$. We substitute $d_2 = 0$ for the matrices (6.16). Then, we have that $D^{c_2 \neq 0,+}_{1,3} \cap [d_2 = 0]$ is irreducible. Therefore, the intersection $E^+_{1,3} \cap E^-_{1,3}$ is irreducible. We take the blowing up along $E^+_{1,3}$. Then, the intersections $E_1 \cap ex^+_{1,3} \cap ex^-_{1,3}$ and $E_3 \cap ex^+_{1,3} \cap ex^-_{1,3}$ are irreducible. Then, we glue together the triangles

$$(E_1, E_2, ex^+_{1,3}), (E_2, E_3, ex^+_{1,3}), (E_1, ex^+_{1,3}, ex^-_{1,3}) \text{ and } (E_3, ex^+_{1,3}, ex^-_{1,3})$$

in the graph of Figure 6.1. In the same way as above, we glue together each triangle. Then, we obtain that the complex of Figure 3 is a simplicial decomposition of S^2. Third, we consider the intersection of 3-tuple of components of the boundary divisor containing ex_1 or ex_2. The divisors ex_1 and ex_2 are the exceptional divisors of the blowing up along the orbits of s_1 and s_2. The orbits of s_1 and s_2 are contained in $D_i \cap D_{i+1}$ ($i = 1, \dots, 4$), $D^+_{1,3}$, $D^-_{1,3}$, $D^+_{2,4}$ and $D^-_{2,4}$, respectively. Then, the intersections $E_i \cap E_{i+1} \cap ex_j$, $E^+_{k,k+2} \cap ex_j$ and $E^-_{k,k+2} \cap ex_j$ are nonempty and irreducible for $i = 1, \dots, 4$, $j = 1, 2$, and $k = 1, 2$. We take the blowing up along $E^+_{1,3}$ and $E^-_{1,3}$. Then, we can glue together the 3-tuples which have either ex_i or ex_i in the graph.

Lastly, we describe the 3-dimensional simplices. We can glue together the 4-tuples of components of the boundary divisor such that the 4-tuples have either ex_i or ex_i and 3-tuples expect ex_i or ex_i are glued together. On the other hand, the intersections of the 4-tuples which have the vertexes expect ex_i or ex_i are empty. Then, we obtain that the boundary complex of the compactification $\widetilde{X}$ of $\mathcal{R}_{5,\boldsymbol{k}}$ is simplicial decomposition of S^3. □

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