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INVARIANT SUBSPACES WITH NO GENERATOR AND A PROBLEM OF H. HELSON

by Jun-ichi TANAKA (*)

Dedicated to the memory of Henry Helson

Abstract. — In the almost-periodic context, the H_0^2−space cannot be generated by one of its elements. Together with a cocycle argument, this implies that there exist all kinds of invariant subspaces without a single generator, from which we answer some questions on invariant subspace theory.

Résumé. — Dans le contexte presque périodique, aucun espace H_0^2 ne peut être engendré par un de ses éléments. En tenant compte d'un argument faisant intervenir les cocycles, on peut en déduire qu'il existe de nombreux types de sous-espaces invariants qui ne peuvent pas être engendrés par un seul de leurs éléments; ceci permet de répondre à quelques questions de la théorie des sous-espaces invariants.

1. Introduction

The theory of invariant subspaces has been developed in the context of compact abelian groups with ordered duals, which is a natural generalization of such a theory on the unit circle $\mathbb{T}$. Many classical results extend to these cases, nevertheless, one also meets new difficulties. The purpose of this paper is to resolve a longstanding problem formulated by H. Helson in the 1950s.

Let Γ be a countable dense subgroup of the real line $\mathbb{R}$, endowed with the discrete topology. Then the dual group K of Γ is a compact abelian group that is metrizable. For λ in Γ, it is customary to denote by χ_λ the character on K defined by $\chi_\lambda(x) = x(\lambda)$. Let σ be the normalized Haar

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measure on K. A function ϕ in $L^1(\sigma)$ is *analytic* if its Fourier coefficients

$$a_\lambda(\phi) = \int_K \phi\,\overline{\chi_\lambda}\,d\sigma \tag{1.1}$$

vanish for all negative λ in Γ. The *Hardy space* $H^p(\sigma), 1 \leqslant p \leqslant \infty$, is defined to be the space of all analytic functions in $L^p(\sigma)$. For technical reasons, it is useful to define $H^p_0(\sigma)$ as the subspace of all ϕ in $H^p(\sigma)$ with $a_0(\phi) = 0$. A (weak*-, if $p = \infty$) closed subspace $\mathfrak{M}$ of $L^p(\sigma)$ is *invariant* if $\mathfrak{M}$ contains $\chi_\lambda\mathfrak{M}$ for all positive λ in Γ. When the inclusion is strict, $\mathfrak{M}$ is said to be *simply* invariant. Of course, both $H^p(\sigma)$ and $H^p_0(\sigma)$ are simply invariant subspaces of $L^p(\sigma)$. If ϕ is in $L^p(\sigma)$, and if $\mathfrak{M}[\phi]$ denotes the smallest invariant subspace of $L^p(\sigma)$ containing ϕ, then ϕ is called a *single generator* of $\mathfrak{M}[\phi]$. Recall that a function of modulus one is said to be *unitary* and an analytic unitary function is called an *inner* function. We say a function ϕ in $H^p(\sigma)$ is *outer* if it satisfies that

$$\log|\,a_0(\phi)\,| = \int_K \log|\,\phi\,|\,d\sigma > -\infty\,.$$

Let $1 \leqslant q \leqslant p \leqslant \infty$, and let $\mathfrak{M}$ be a simply invariant subspace of $L^p(\sigma)$. It follows from the properties of outer functions that $[\mathfrak{M} \cap L^\infty(\sigma)]_q \cap L^p(\sigma) = \mathfrak{M}$, where $[\mathfrak{M} \cap L^\infty(\sigma)]_q$ is the closure of $\mathfrak{M} \cap L^\infty(\sigma)$ in $L^q(\sigma)$ (see [3, Chapter V, Section 6] for details). This fact assures that there is a one-to-one correspondence between the invariant subspaces in $L^p(\sigma)$ and those in $L^q(\sigma)$. Therefore, in dealing with invariant subspaces, we may restrict our attention to the case of $p = 2$, in which Hilbert space theory works well. It follows from Szegö's theorem that ϕ is a single generator of $H^2(\sigma)$ if and only if ϕ is outer in $H^2(\sigma)$. However, it has been unknown for a long time whether every simply invariant subspace is singly generated or not. In the literature this has come to be known as the *single generator problem* (refer to [4, §5.4], [2, Remark, p. 158] and [3, p. 138 and p. 177]). The difficulty seems to center on the case of invariant subspace $H^2_0(\sigma)$. In [6, p. 183], it is raised in an equivalent form in connection with stochastic processes.

Our objective in this note is to show a negative answer to this problem in the almost periodic settings:

Theorem. — *The invariant subspace $H^2_0(\sigma)$ cannot be generated by one of its elements.*

To the best of author's knowledge, $H^2_0(\sigma)$ is the first known example of invariant subspace which cannot be singly generated. On the other hand, by [4, §5.3, Theorem 33], it was shown that every invariant subspace is generated by two of its elements. In more general setting, we can artificially make H^2_0-spaces to have a single generator.

For each t in $\mathbb{R}$, let us denote by e_t the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for λ in Γ. The map sending t to e_t embeds $\mathbb{R}$ continuously onto a dense subgroup of K. Define a one-parameter group $\{T_t\}_{t\in\mathbb{R}}$ of homeomorphisms on K by

$$T_t x \;=\; x + e_t\,, \qquad x \in K. \tag{1.2}$$

Then the pair $(K, \{T_t\}_{t\in\mathbb{R}})$ is a strictly ergodic flow, for which σ is the unique invariant probability measure. The flow $(K, \{T_t\}_{t\in\mathbb{R}})$ is called an *almost periodic flow*, because if ϕ is continuous on K, then $t \to \phi(x + e_t)$ is a uniformly almost periodic function with exponents in Γ. Let $H^\infty(dt/\pi(1+t^2))$ be the space of all boundary functions of bounded analytic functions in the upper half-plane $\mathcal{H}$, and let $H^p(dt/\pi(1+t^2)), 1 \leqslant p < \infty$, be the closure of $H^\infty(dt/\pi(1+t^2))$ in $L^p(dt/\pi(1+t^2))$. For a function $u(x,t)$ on $K \times \mathbb{R}$, the assertion " $t \to u(x,t)$ for $\sigma - a.e.\, x$ in K " is sometimes abbreviated to "*almost every* $t \to u(x,t)$ ". Then ϕ in $L^p(\sigma)$ lies in $H^p(\sigma)$ if and only if almost every $t \to \phi(x + e_t)$ lies in $H^p(dt/\pi(1+t^2))$. This fact enables us to define Hardy spaces on every ergodic flow (see the end of the next section).

Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$. Set $\mathfrak{M}_\lambda = \chi_\lambda \mathfrak{M}$ for each λ in Γ. Define

$$\mathfrak{M}_+ \;=\; \bigwedge_{\lambda<0} \mathfrak{M}_\lambda \qquad \text{and} \qquad \mathfrak{M}_- \;=\; \bigvee_{\lambda>0} \mathfrak{M}_\lambda.$$

Since these spaces are at most one dimension apart, $\mathfrak{M}$ coincides with either or both its versions $\mathfrak{M}_+$ and $\mathfrak{M}_-$. When $\mathfrak{M} = \mathfrak{M}_+$, $\mathfrak{M}$ is said to be *normalized*. For ϕ in $L^2(\sigma)$, the subspace $\mathfrak{M}[\phi]$ is simply invariant if and only if

$$\int_{-\infty}^{\infty} \log|\phi(x + e_t)|\, \frac{dt}{1+t^2} \;>\; -\infty\,, \qquad \sigma - a.e.\; x \in K, \tag{1.3}$$

(see [4, §3.3, Theorem 22]). It is well-known that there is a function ϕ in $L^2(\sigma)$ satisfying the inequality (1.3), while $\log|\phi|$ does not belong to $L^1(\sigma)$. Our Theorem asserts that any such function ϕ must satisfy $\mathfrak{M}[\phi]_+ = \mathfrak{M}[\phi]_-$.

A unitary Borel function $A(x,t)$ on $K \times \mathbb{R}$ is said to be a *cocycle* on K if $A(x,t)$ satisfies the *cocycle identity*

$$A(x, t+s) \;=\; A(x,t) \cdot A(x + e_t, s)\,, \qquad (x,s,t) \in K \times \mathbb{R} \times \mathbb{R}\,.$$

We identify two cocycles which differ only on a set of $d\sigma \times dt-$measure zero in $K \times \mathbb{R}$. A one-to-one correspondence is established between normalized invariant subspaces and cocycles (as discussed in [4, §2.3]). More precisely, let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. Then a function ϕ in $L^2(\sigma)$ lies in $\mathfrak{M}_+$ if and only if almost every $t \to A(x,t)\phi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$ (see [4, §3.2]). It is easy to see that $\mathfrak{M}_+ \neq \mathfrak{M}_-$ if and only if $\mathfrak{M}_+ = qH^2(\sigma)$ for some unitary function q on K. Then the cocycle of $\mathfrak{M}$ has the form $q(x)\cdot\overline{q(x+e_t)}$, which is called a *coboundary*. If a cocycle is a coboundary multiplied by $\exp(i\alpha t)$ for some α in $\mathbb{R}$, then such a cocycle is said to be *trivial*. A trivial cocycle $\exp(i\alpha t)$ is not a coboundary only if α lies in $\mathbb{R}\setminus\Gamma$.

We already know from [5] and [10] that some singly generated subspaces have nontrivial cocycles, but we can strengthen this fact by noting the following:

Corollary 1.1. — *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$. If the cocycle of $\mathfrak{M}$ is trivial, then $\mathfrak{M}_-$ has no single generator. In other words, if $\mathfrak{M}_-$ is singly generated, then the cocycle of $\mathfrak{M}$ is always nontrivial, so that $\mathfrak{M}_+ = \mathfrak{M}_-$.*

A cocycle with values in $\{-1,1\}$ is called a *real* cocycle. It follows from [7] that there exist real cocycles which are nontrivial.

Corollary 1.2. — *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ with real cocycle. Then $\mathfrak{M}_-$ has no single generator.*

A cocycle $A(x,t)$ is said to be *analytic* if almost every $t \to A(x,t)$ lies in $H^\infty(dt/\pi(1+t^2))$. Then a normalized invariant subspace with analytic cocycle contains always $H^2(\sigma)$. We say that an analytic cocycle $A(x,t)$ is a *Blaschke* or a *singular* cocycle, if almost every $t \to A(x,t)$ is an inner function of that type in $H^\infty(dt/\pi(1+t^2))$. Two cocycles are called *cohomologous* if one is a coboundary times the other. It is known that every cocycle is cohomologous to a Blaschke cocycle in some restricted class (see [4, §4.6, Theorem 26] and [15]). This fact makes Blaschke cocycles so important for the subject. Using our Theorem, we may answer some questions on analytic cocycles:

Corollary 1.3. — *In the class of analytic cocycles, the following properties hold:*

(a) *There is a Blaschke cocycle not being cohomologous to any singular cocycle.*
(b) *There is a Blaschke cocycle not having exactly the same zeros as any function in $H^2(\sigma)$.*

It would be helpful to understand the basic idea behind the proof of our Theorem. On the one hand, we claim that if ϕ is a single generator of $H_0^2(\sigma)$, then ϕ must have a very special form. Assume that Γ is the smallest group determined by the nonzero Fourier coefficients of ϕ (see below for details). Similarly, let Λ be the smallest group determined by the nonzero coefficients of $|\phi|$. Since Λ is a subgroup of Γ, the dual group of Λ is represented as K/H, where H is the annihilator of Λ in K. Let τ be the normalized Haar measure on K/H, and fix an element α in Γ with $a_\alpha(\phi) \neq 0$. Then it can be shown that $\overline{\chi}_\alpha \phi$ lies in $L^2(\tau)$ and generates the simply invariant subspace of $L^2(\tau)$ with trivial cocycle $\exp(i\alpha t)$. We also see that α is independent of Λ, meaning that $n\alpha$ lies in Λ only for $n = 0$ in the integer group $\mathbb{Z}$. This implies that K and $d\sigma$ are respectively identified with $K/H \times \mathbb{T}$ and $d\tau \times d\theta/2\pi$, since H is regarded as $\mathbb{T}$. Thus, for each single generator ϕ of $H_0^2(\sigma)$, we derive that $\Gamma \neq \Lambda$. On the other hand, if $H_0^2(\sigma)$ is singly generated, we may construct a generator ϕ of $H_0^2(\sigma)$ with the property that $\Gamma = \Lambda$, which contradicts the existence of single generator of $H_0^2(\sigma)$.

In the next section, we establish some notation and elementary facts about invariant subspaces in the almost periodic setting. Using group characters, we develop certain properties of single generators of H_0^2-spaces in Section 3. In Section 4, the proof of our Theorem is provided and then Corollaries are proved by using a lemma on cocycles. We conclude the paper with some remarks in Section 5.

We refer the reader to [9], [3, Chapter VII], [4] and [14, Chapter VIII] for further details on analyticity on compact abelian groups. Basic results concerning the Hardy space theory based on uniform algebras can be found in [3, Chapter IV] and [11].

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2. Extension of almost periodic functions

It is easy to show that a function ϕ in $H^2(\sigma)$ is outer if and only if $a_0(\phi) \neq 0$ and almost every $t \to \phi(x + e_t)$ is outer in $H^2(dt/\pi(1 + t^2))$. A weak version of this fact stated below is often used in what follows:

Lemma 2.1. — *Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. A function ϕ in $L^2(\sigma)$ generates $\mathfrak{M}_-$ if and only if* $\log|\phi|$ *does not lie in $L^1(\sigma)$ and almost every $t \to A(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. In particular, $H^2_0(\sigma)$ is singly generated by ϕ if and only if $a_0(\phi) = 0$ and almost every $t \to \phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$.*

Proof. — Suppose that $\mathfrak{M}[\phi] = \mathfrak{M}_-$ for ϕ in $L^2(\sigma)$. If $\log|\phi|$ lies in $L^1(\sigma)$, then there is a unitary function q on K such that $\mathfrak{M}[\phi] = qH^2(\sigma)$ by Szegö's theorem. This implies that $\mathfrak{M}[\phi] \neq \mathfrak{M}_-$, so $\log|\phi|$ cannot lie in $L^1(\sigma)$. Let $B(x,t)$ be the analytic cocycle defined by the inner part of $t \to A(x,t)\phi(x+e_t)$. Let $\mathfrak{N}$ be the invariant subspace with cocycle $A\overline{B}(x,t)$. By [4, §3.2, Theorem 21], we see that $\mathfrak{N}_-$ is contained in $\mathfrak{M}_-$. On the other hand, since almost every $t \to A\overline{B}(x,t)\psi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$ for each ψ in $\mathfrak{M}[\phi]$, $\mathfrak{N}_+$ includes $\mathfrak{M}[\phi]$. This shows that $\mathfrak{N}_+ = \mathfrak{M}_+$, so $B(x,t) \equiv 1$. Then almost every $t \to A(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$.

Conversely, suppose that $\mathfrak{M}[\phi]$ is contained strictly in $\mathfrak{M}_-$. Then there is a nonzero function q in $\mathfrak{M}_-$ such that

$$\int_K \psi\phi\overline{q}\,d\sigma = 0, \qquad \psi \in H^\infty(\sigma).$$

This shows that $\phi\overline{q}$ lies in $H^1(\sigma)$, so almost every $t \to \phi\overline{q}(x+e_t)$ lies in $H^1(dt/\pi(1+t^2))$. Notice that $t \to A(x,t)q(x+e_t)$ is in $H^2(dt/\pi(1+t^2))$. Since

$$\phi(x+e_t)\overline{q(x+e_t)} = A(x,t)\phi(x+e_t)\overline{A(x,t)q(x+e_t)},$$

and since $t \to A(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$, we see that almost every $t \to \overline{A(x,t)q(x+e_t)}$ is also in $H^2(dt/\pi(1+t^2))$. This shows that $t \to A(x,t)q(x+e_t)$ is constant for $\sigma - a.e.\, x$ in K, and so is $t \to |q(x+e_t)|$. It follows from the ergodic theorem that $|q(x)|$ is constant. We then assume q is a unitary function on K. Therefore, $A(x,t)$ is the coboundary $q(x)\overline{q(x+e_t)}$ and $\mathfrak{M}_- = qH^2_0(\sigma)$. Thus q does not lie in $\mathfrak{M}_-$, which is a contradiction.

The last part of assertion follows from the fact that the cocycle of $H^2(\sigma)$ equals 1. Under the assumption that almost every $t \to \phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$, we see easily $a_0(\phi) = 0$ if and only if $\log|\phi|$ does not lie in $L^1(\sigma)$. Then $\mathfrak{M}[\phi] = H^2_0(\sigma)$, so the proof is complete. □

Let $L^1(dt)$ be the usual Lebesgue space on $\mathbb{R}$. Using $\{T_t\}_{t\in\mathbb{R}}$, one may convolve a function ϕ in $L^p(\sigma), 1 \leqslant p < \infty$, with a function f in $L^1(dt)$ by setting

$$(\phi * f)(x) = \int_{-\infty}^{\infty} \phi(x+e_t)f(-t)\,dt = \int_{-\infty}^{\infty} \phi(x-e_t)f(t)\,dt\,,$$

where the integral is a Bochner integral. When $p=\infty$, the convolution $\phi * f$ is defined in the same way as the weak*-convergent integral. Under the operation of convolution, $L^p(\sigma)$ becomes an $L^1(dt)$-module such that

$$\|\phi * f\|_p \leqslant \|\phi\|_p\|f\|_1\,, \qquad \phi \in L^p(\sigma)\,,$$

for f in $L^1(dt)$. The Fourier transform $\hat{f}$ of f is defined by the formula

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\,dt, \qquad \lambda \in \mathbb{R}\,, \tag{2.1}$$

as usual. We see easily $a_\lambda(\phi * f) = a_\lambda(\phi)\hat{f}(\lambda)$, if λ is in Γ. The Poisson kernel $P_{ir}(t)$ for $\mathcal{H}$ is given by $P_{ir}(t) = r/\pi(t^2+r^2)$ for an $r>0$. If ϕ is in $L^1(\sigma)$, then the convolution $\phi * P_{ir}$ is considered as the Poisson integral of $t \to \phi(x+e_t)$, that is,

$$(\phi * P_{ir})(x+e_s) = \int_{-\infty}^{\infty} \phi(x+e_t)P_{ir}(s-t)\,dt\,.$$

Lemma 2.2. — *Suppose that $H_0^2(\sigma)$ is singly generated. Then we obtain the following properties:*

(a) *There is a single generator of $H_0^2(\sigma)$ that is bounded.*
(b) *If ϕ is a bounded generator of $H_0^2(\sigma)$, then so is each of the functions $\phi * P_{ir}$ with $r>0$ and ϕ^n for $n=1,2,\cdots$.*

Proof. — Let ψ be a single generator of $H_0^2(\sigma)$. Then there is an outer function h in $H^2(\sigma)$ such that $|h| = \min(1,|\psi|^{-1})$. From Lemma 2.1, we deduce that the bounded function ψh generates $H_0^2(\sigma)$, thus we obtain (a).

To show (b), we observe that $t \to (\phi * P_{ir})(x+e_t)$ as well as $t \to \phi^n(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$ for $\sigma-a.e.\,x$ in K. Since $a_0(\phi * P_{ir}) = a_0(\phi^n) = 0$, (b) follows from Lemma 2.1 immediately. □

We next introduce a local product decomposition of K, which is useful for studying analytic functions on K. Fix a positive γ in Γ, and let K_γ be the closed subgroup of all x in K such that $\chi_\gamma(x)=1$. Then $K_\gamma \times [0,2\pi/\gamma)$ is identified with K via the map $(y,s) \to y+e_s$. Let σ_1 be the normalized Haar measure on K_γ. Then the probability measure $(\gamma/2\pi)d\sigma_1 \times dt$ on $K_\gamma \times [0,2\pi/\gamma)$ is carried by the map to $d\sigma$ on K. The one-parameter group $\{T_t\}_{t\in\mathbb{R}}$ given by (1.2) is represented as

$$T_t(y,s) = (y+[(t+s)\gamma/2\pi]e_{2\pi/\gamma}, t+s-[(t+s)\gamma/2\pi]2\pi/\gamma)$$

on $K_\gamma \times [0, 2\pi/\gamma)$, where $[t]$ is the largest integer not exceeding t. Define the homeomorphism T on K_γ by $Ty = y + e_{2\pi/\gamma}$. We denote by $\mathcal{O}(\omega, T)$ the orbit of a point ω in (K_γ, T), that is, the set of all $T^n\omega$ for n in $\mathbb{Z}$. Since $\mathcal{O}(\omega, T)$ is dense in K_γ, the discrete flow (K_γ, T) is also a strictly ergodic flow, on which σ_1 is the unique invariant probability measure. Since Γ is countable, K_γ is metrizable (see [14, 2.2.6]).

A function ϕ on K has the *automorphic extension* $\phi^\sharp$ to $K_\gamma \times \mathbb{R}$ defined by

$$\phi^\sharp(y, t) \;=\; \phi(y + [t\gamma/2\pi]e_{2\pi/\gamma}\,, t - [t\gamma/2\pi]2\pi/\gamma).$$

Since a function f in $H^1(dt/\pi(1+t^2))$ extends analytically to $\mathcal{H}$ by $f(s + ir) = (f * P_{ir})(s)$, we write

$$\phi^\sharp(y, z) \;=\; (\phi^\sharp * P_{ir})(y, s), \qquad z = s + ir \in \mathcal{H}\,,$$

for each ϕ in $H^1(\sigma)$. It is clear that $(\phi^\sharp * P_{ir})(y, s) = (\phi * P_{ir})^\sharp(y, s)$ on $K_\gamma \times \mathbb{R}$.

The following is due to a property of Lebesgue sets.

Lemma 2.3. — *If E_1 is a compact subset of K_γ with $\sigma_1(E_1) > 0$, then there is a closed subset E of E_1 with $\sigma_1(E_1) = \sigma_1(E)$ such that $\mathcal{O}(\omega, T) \cap E$ is dense in E, for $\sigma_1 - a.e.\,\omega$ in K_γ.*

Proof. — Recall that the *metric density* of E_1 is 1 at $\sigma_1 - a.e.\,\omega$ in E_1, meaning that

$$\lim_{\delta \to 0} \frac{\sigma_1(E_1 \cap B(\omega, \delta))}{\sigma_1(B(\omega, \delta))} \;=\; 1\,,$$

where $B(\omega, \delta)$ is the open ball with center ω and radius $\delta > 0$. Define E to be the closure of the set of points of E_1 at which the metric density of E_1 is 1. Clearly, we have $\sigma_1(E_1) = \sigma_1(E)$, since E_1 is closed. If $\sigma_1(E) = 1$, then $E = K_\gamma$. Since (K_γ, T) is strictly ergodic every orbit $\mathcal{O}(\omega, T)$ is dense in E. Assume that $0 < \sigma_1(E) < 1$. It follows from the ergodic theorem that there is a σ_1-null set N in K_γ outside which

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} I_E(T^j\omega) \;=\; \sigma_1(E)\,,$$

where I_E denotes the characteristic function of E. Let H_ω be the closure of $\mathcal{O}(\omega, T) \cap E$ in K_γ. We claim that if $E \neq H_\omega$, then ω lies in N. Indeed, we see that $\sigma_1(E \setminus H_\omega) > 0$, since the metric density of E does not vanish identically on $E \setminus H_\omega$. Let p be a continuous function on K_γ such that $0 \leqslant p \leqslant 1$, $p \equiv 1$ on H_ω, and $\int_{K_\gamma} p\, d\sigma < \sigma_1(E)$. Since $I_E(T^j\omega) = I_{H_\omega}(T^j\omega)$

for j in $\mathbb{Z}$ and since (K_γ, T) is strictly ergodic, we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} I_E(T^j\omega) \leqslant \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} p(T^j\omega) = \int_{K_\gamma} p\, d\sigma_1 < \sigma_1(E)$$

by [13, §4.2, Proposition 2.8]. Thus ω has to lie in the null set N. □

For each ϕ in $H^\infty(\sigma)$, there is a σ_1-null set of K_γ outside which $z \to \phi^\sharp(y,z)$ is analytic and uniformly bounded on the upper half plane $\mathcal{H}$. Recall that if a family of analytic functions is uniformly bounded, then it forms a normal family. The next proposition may be regarded as a strengthened form of Lusin's theorem for analytic functions on K, so that it has some interest of its own. Here we denote by $cl(\mathcal{H})$ the closure of $\mathcal{H}$ in $\mathbb{R}^2$.

Proposition 2.4. — *Let ϕ be a function in $H^\infty(\sigma)$, and let $\epsilon > 0$. Then there is a closed subset E of K_γ with $\sigma_1(E) > 1-\epsilon$ having the following properties:*

(a) *The convolution $(\phi^\sharp * P_{ir})(y,t)$ is continuous on $E\times\mathbb{R}$, for a given $r > 0$.*
(b) *For $\sigma_1 - a.e.\, \omega$ in K_γ, the function $(\phi^\sharp * P_{ir})(T^j\omega, z)$ on $(\mathcal{O}(\omega,T)\cap E)\times cl(\mathcal{H})$ extends to $(\phi^\sharp * P_{ir})(y,z)$ on $E\times cl(\mathcal{H})$.*

Proof. — Since $\phi * P_{ir}$ lies in $H^\infty(\sigma)$, Lusin's theorem asserts that there is a compact subset F of K with $\sigma(F) > 1-\epsilon^2$ on which $\phi * P_{ir}$ is continuous. Regarding F as a subset of $K_\gamma\times[0,2\pi/\gamma)$, we choose a compact subset E of K_γ with $\sigma_1(E) > 1-\epsilon$ such that E satisfies the property of Lemma 2.3 and

$$\frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} I_F(y,s)\, ds > 1-\epsilon, \qquad y\in E. \tag{2.2}$$

In addition, we assume that $z\to(\phi^\sharp * P_{ir/2})(y,z)$ is analytic on $\mathcal{H}$ and

$$|(\phi^\sharp * P_{ir/2})(y,z)| \leqslant \|\phi\|_\infty, \qquad y\in E.$$

Then the family

$$\mathcal{F} = \left\{ (\phi^\sharp * P_{ir/2})(y,z)\,;\, y\in E\right\}$$

forms a normal family on $\mathcal{H}$. Let $\{y_n\}$ be a sequence in E tending to y. Since $\mathcal{F}$ is normal, there is a subsequence $\{y_j\}$ of $\{y_n\}$ such that $(\phi^\sharp * P_{ir/2})(y_j,z)$ converges uniformly on compact subsets of $\mathcal{H}$ to a bounded analytic function $f(z)$ on $\mathcal{H}$. Let us show that $f(z) = (\phi^\sharp * P_{ir/2})(y,z)$. Indeed, we observe by (2.2) that $F\cap(\{y\}\times[0,2\pi/\gamma))$ contains an infinite compact set of the form $\{y\}\times J$. Since

$$(\phi^\sharp * P_{ir})(y,t) = (\phi^\sharp * P_{ir/2})(y,t+ir/2) = f(t+ir/2), \qquad t\in J,$$

it follows from the uniqueness principle that $f(z) = (\phi^\sharp * P_{ir/2})(y,z)$. This shows that if (y_n, t_n) tends to (y,t), then $(\phi^\sharp * P_{ir})(y_n,t_n)$ tends to $(\phi^\sharp * P_{ir})(y,t)$. Thus (a) holds. We notice that $(\phi^\sharp * P_{ir/2})(y,z)$ is also continuous on $E \times \mathcal{H}$.

On the other hand, by Lemma 2.3, $\mathcal{O}(\omega,T) \cap E$ is dense in E for $\sigma_1 - a.e.\,\omega$ in K_γ. Since $(\mathcal{O}(\omega,T) \cap E) \times cl(\mathcal{H})$ is dense in $E \times cl(\mathcal{H})$ and since $(\phi^\sharp * P_{ir})(y,z)$ is continuous on $E \times cl(\mathcal{H})$, the function $(\phi^\sharp * P_{ir})(T^j\omega,t)$ on $(\mathcal{O}(\omega,T) \cap E) \times cl(\mathcal{H})$ extends to $(\phi^\sharp * P_{ir})(y,t)$ on $E \times \mathcal{H}$. Thus (b) follows immediately. □

We make some remarks on Proposition 2.4. Since $t \to \phi^\sharp(y,t)$ lies in $H^\infty(dt/\pi(1+t^2))$ for each y in E, we see that $(\phi^\sharp * P_{ir})(y, t + 2\pi/\gamma) = (\phi^\sharp * P_{ir})(Ty,t)$. Then $E \cup TE \cup \cdots \cup T^nE$ also satisfies the properties (a) and (b) and $\sigma_1(E \cup TE \cup \cdots \cup T^nE)$ converges to 1, as $n \to \infty$, by the recurrence theorem (see [13, §2.3, Theorem 3.2]). However, to obtain ϕ itself, we need a version of Fatou's theorem as discussed in [12, Theorem II]. Denote by $\mathcal{O}(x, \{T_t\}_{t\in\mathbb{R}})$ the *orbit* of x in $(K, \{T_t\}_{t\in\mathbb{R}})$. With the notation above, when $x = (y,s)$ in $K_\gamma \times [0, 2\pi/\gamma)$, we see that $\mathcal{O}(x, \{T_t\}_{t\in\mathbb{R}}) = \mathcal{O}(y,T) \times [0,2\pi/\gamma)$. For x in K, we say that $t \to (\phi * P_{ir})(x + e_t)$ *extends to* $\phi * P_{ir}$ if, for each $\epsilon > 0$, there is a compact subset $F = F(\epsilon, \phi)$ of K with $\sigma(F) > 1 - \epsilon$ such that $\phi * P_{ir}$ is continuous on F and $\mathcal{O}(x, \{T_t\}_{t\in\mathbb{R}}) \cap F$ is dense in F. The above proof may be modified so as to apply to functions in $H^1(\sigma)$ as well.

The next lemma is an immediate consequence of Proposition 2.4.

Lemma 2.5. — *Let ϕ be a function in $H^\infty(\sigma)$, and let $r > 0$. Then there is an invariant $\sigma-$null set $N = N(\phi)$ in K outside which $t \to (\phi * P_{ir})(x + e_t)$ extends to $\phi * P_{ir}$.*

Proof. — For a given $\epsilon > 0$, let E be a closed subset of K_γ with $\sigma_1(E) > 1 - \epsilon$ which has the property (a) and (b) of Proposition 2.4. Putting $F = E \times [0, 2\pi/\gamma]$, we regard F as a compact subset of K. By (b) of Proposition 2.4, we choose an invariant null set $N' = N'(\phi)$ in (K_γ, T) outside which $\mathcal{O}(\omega,T) \cap E$ is dense in E. If we set $N = N' \times [0, 2\pi/\gamma)$, then the $\sigma-$null set N satisfies the desired property. □

Let Ω be a compact metric space on which $\mathbb{R}$ acts as a Borel transformation group. This means that there is a one-parameter group $\{U_t\}_{t\in\mathbb{R}}$ of Borel isomorphisms on Ω such that the map $(\omega,t) \to U_t\omega$ of $\Omega \times \mathbb{R}$ to Ω is a Borel map. The pair $(\Omega, \{U_t\}_{t\in\mathbb{R}})$ is referred to a *Borel flow*. Especially, $(\Omega, \{U_t\}_{t\in\mathbb{R}})$ is called a *continuous flow*, if U_t is a homeomorphism on Ω and the map $(\omega,t) \to U_t\omega$ is continuous on $\Omega \times \mathbb{R}$. We often write $\omega + t$

for the translate $U_t\omega$ of ω by t. Let μ be an invariant probability measure on $(\Omega, \{U_t\}_{t\in\mathbb{R}})$ which is *ergodic*, meaning that $\mu(E) = 1$ or 0 for each invariant subset E of Ω. A function ϕ in $L^1(\mu)$ is *analytic* if $t \to \phi(\omega + t)$ lies in $H^1(dt/\pi(1+t^2))$ for $\mu - a.e.\,\omega$ in Ω. Then the *ergodic Hardy space* $H^p(\mu), 1 \leqslant p \leqslant \infty$, is defined to be the space of all analytic functions in $L^p(\mu)$. It follows from [11, Theorem I] that μ is a representing measure for $H^\infty(\mu)$, for which $H^\infty(\mu)$ is a weak*-Dirichlet algebra in $L^\infty(\mu)$. This fundamental result enables us to apply the Hardy space theory based on uniform algebras, and most of the machinery of invariant subspaces on an almost periodic flow $(K, \{T_t\}_{t\in\mathbb{R}})$ can be reconstructed (see [1], [11] and [12] for related topics). As we mentioned earlier, the H_0^2-spaces may be singly generated in the situation of ergodic flows other than almost periodic flows (see [16] and §5 (b)).

Let $A(x,t)$ be a cocycle on an almost periodic flow $(K, \{T_t\}_{t\in\mathbb{R}})$ and define the Borel flow $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ by

$$S_t(x, e^{i\theta}) = (T_t x, A(x,t)e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T}, \tag{2.3}$$

which is called the *skew product* of K and $\mathbb{T}$ induced by $A(x,t)$. Then $d\sigma \times d\theta/2\pi$ is an invariant probability measure on $K\times\mathbb{T}$. Observe that each function f in $L^2(d\sigma\times d\theta/2\pi)$ is represented as

$$f(x, e^{i\theta}) = \sum_{n=-\infty}^{\infty} \phi_n(x)e^{in\theta},$$

where the coefficients ϕ_n are in $L^2(\sigma)$. From this fact, it follows easily that $d\sigma\times d\theta/2\pi$ is ergodic on $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ if and only if $A(x,t)^n$ is a coboundary only for $n=0$ (see [4, §6.2] for details).

3. Approximation to generators

We now turn to the structure of compact group K, under the assumption that $H_0^2(\sigma)$ is singly generated by ϕ in $H_0^2(\sigma)$. By multiplying by a suitable outer function, if necessary, we can assume that ϕ is a function in $L^\infty(\sigma)$ with $1 \leqslant \|\phi\|_\infty < \infty$. Furthermore, we also assume that Γ is the smallest group containing all λ such that $a_\lambda(\phi) \neq 0$, that is, the smallest group over which Fourier series,

$$\phi(x) \sim \sum_{\Gamma\ni\lambda>0} a_\lambda(\phi)\chi_\lambda(x),$$

holds. Similarly, denote by Λ the smallest group containing all λ such that $a_\lambda(|\phi|) \neq 0$. We observe that the Fourier series of

$$\left(|\phi|^2 + \epsilon\right)^{1/2} \;=\; \exp\left\{\frac{1}{2}\,\log\left(\phi\overline{\phi} + \epsilon\right)\right\}, \qquad \epsilon > 0\,,$$

is represented on Γ, by considering the Taylor series of $z \to \log z$ at a large positive. This shows that Λ is a subgroup of Γ, since

$$a_\lambda(|\phi|) \;=\; \lim_{\epsilon\to+0}\, a_\lambda\left(\left(|\phi|^2 + \epsilon\right)^{1/2}\right)$$

by (1.1). Since $\log|\phi|$ does not lie in $L^1(\sigma)$, the generator ϕ cannot be periodic in $(K, \{T_t\}_{t\in\mathbb{R}})$. Then Γ as well as Λ is a countable dense subgroup of $\mathbb{R}$, endowed with discrete topology. Let H be the annihilator of Λ, meaning that H is the closed subgroup of all x in K such that $\chi_\lambda(x) = 1$ for all λ in Λ. Then the dual group of Λ is identified with the quotient group K/H (see [14, 2.1]). We denote by τ the normalized Haar measure on K/H. Let π be the canonical homomorphism of K onto K/H. For each x in K, we write $\bar{x}$ for $\pi(x) = x + H$. When a function ψ on K is represented as $\psi = \tilde{\psi}\circ\pi$ for a function $\tilde{\psi}$ on K/H, we usually identify ψ with $\tilde{\psi}$, so that $\psi(x) = \psi(\bar{x})$. Then we say descriptively that ψ *is generated by a function on* K/H. If $1 \leqslant p \leqslant \infty$, then $L^p(\tau)$ and $H^p(\tau)$ are subspaces of $L^p(\sigma)$ and $H^p(\sigma)$, respectively.

Since almost every $t \to \phi(x + e_t)$ is outer in $H^\infty(dt/\pi(1 + t^2))$ by Lemma 2.1, we see that

$$-\infty \;<\; \log|(\phi * P_{ir})(x)| \;=\; (\log|\phi| * P_{ir})(x)$$

for a given $r > 0$. Since $\log|\phi|$ is not in $L^1(\sigma)$ and $\log|\phi| \leqslant \|\phi\|_\infty$, Fubini's theorem shows that

$$\int_K \log|\phi * P_{ir}|\, d\sigma \;=\; \int_K (\log|\phi| * P_{ir})\, d\sigma \;=\; \int_K \log|\phi|\, d\sigma \;=\; -\infty\,.$$

Let $g = \phi * P_{ir}$. Then Lemma 2.1 shows that g is also a bounded generator of $H_0^2(\sigma)$. Since $\hat{P}_{ir}(\lambda) = e^{-r|\lambda|}$ by (2.1), we obtain $a_\lambda(g) = a_\lambda(\phi * P_{ir}) = a_\lambda(\phi)e^{-r|\lambda|}$, hence $a_\lambda(\phi) \neq 0$ if and only if $a_\lambda(g) \neq 0$. Thus the generator g plays the same role as ϕ. For $n = 1, 2, \ldots$, we then denote by ϕ_n the outer function in $H^\infty(\tau)$ with $|\phi_n| = \max(1/n, |\phi|)$. Since $-\log n \leqslant \log|\phi_n| \leqslant \|\phi\|_\infty$, each ϕ_n^{-1} is also an outer function in $H^\infty(\tau)$. Putting $g_n = \phi_n * P_{ir}$, we obtain a sequence $\{g_n\}$ of outer functions in $H^\infty(\tau)$ with $\|g_n\|_\infty \leqslant \|\phi\|_\infty$. Notice that $t \to g(x + e_t)$ and $t \to g_n(x + e_t)$ extend analytically up to $\{Re\, z > -r\}$. Let us look into the relation between g and g_n. Since

$$|g_n(x)| \;=\; \exp\{(\log|\phi_n| * P_{ir})(x)\}\,,$$

we obtain

$$(3.1)\qquad |g_1(x)| \geqslant |g_2(x)| \geqslant \cdots \geqslant |g_n(x)| \longrightarrow |g(x)|, \qquad n \to \infty,$$

for $\sigma - a.e.\, x$ in K. Although g may not be in $L^\infty(\tau)$, we observe that $|g_n(x)| = |g_n(\bar{x})|$ and $|g(x)| = |g|(\bar{x})$. By (3.1), it is easy to see that almost every $t \to |(g/g_n)(x + e_t)|$ converges pointwise to 1 on $\mathbb{R}$. Put $G^x_n(t) = g_n(x + e_t)$ and $G^x(t) = g(x + e_t)$. Let N_0 be an invariant null set in K outside which the property of Lemma 2.5 holds simultaneously for ϕ and all ϕ_n. Moreover, for x in $K \setminus N_0$, we may assume $G^x_n(t)$ and $G^x(t)$ are outer functions in $H^\infty(dt/\pi(1 + t^2))$. Then the family of all analytic extensions $G^x_n(z)$ of $G^x_n(t)$ to $\{Re\, z > -r\}$ forms a normal family, since $|G^x_n(z)| \leqslant \|\phi\|_\infty$.

The following lemma is crucial in our proof of the Theorem.

Lemma 3.1. — *For a bounded generator ϕ of $H^2_0(\sigma)$, let Λ, H and τ be as above. Choose an α in Γ with $a_\alpha(\phi) \neq 0$. Then $\overline{\chi_\alpha}\phi$ is generated by a function on K/H, so lies in $L^\infty(\tau)$. Consequently, Γ is generated by Λ and α.*

Proof. — Let $\{\delta_k\}$ be a decreasing sequence tending to 0. Then there is a sequence $\{f_k\}$ in $L^1(dt)$ such that $\hat{f}_k(\alpha) = 1$, $\|f_k\|_1 = 1$ and $\hat{f}_k = 0$ outside $(\alpha - \delta_k, \alpha + \delta_k)$, by modifying the function $t \to (1/\pi) \sin^2 t/t^2$ in $L^1(dt)$. Since $a_\lambda(g) = a_\lambda(\phi)e^{-r|\lambda|}$, we see that $\overline{\chi_\alpha}\phi$ lies in $L^2(\tau)$ if and only if so does $\overline{\chi_\alpha}g$. Thus we may replace ϕ with g in our argument. Since $a_\lambda(g * f_k) = a_\lambda(g)\hat{f}_k(\lambda)$, we observe that

$$\|g * f_k - a_\alpha(g)\chi_\alpha\|_2^2 = \sum_{0<|\lambda|<\delta_k} |a_{\alpha+\lambda}(g)\hat{f}_k(\alpha + \lambda)|^2 \to 0, \qquad k \to \infty,$$

by the Parseval theorem and that

$$\|\overline{(g * f_k)}g - \overline{a_\alpha(g)}(\overline{\chi_\alpha}g)\|_2 \leqslant \|g * f_k - a_\alpha(g)\chi_\alpha\|_2 \, \|g\|_\infty .$$

From these facts, we conclude that if each $\overline{(g * f_k)}g$ lies in $L^\infty(\tau)$, then so does $\overline{\chi_\alpha}g$. Since the outer function ϕ_n lies in $L^\infty(\tau)$, so do g_n and $g_n * f_k$. Then each $\overline{(g_n * f_k)}g_n$ lies in $L^\infty(\tau)$. Let us show that the sequence $\{\overline{(g_n * f_k)}g_n\}$ converges to $\{\overline{(g * f_k)}g\}$ in $L^2(\sigma)$, from which we obtain that $\overline{(g * f_k)}g$ lies in $L^\infty(\tau)$. Indeed, in the notation above, if we fix an x in $K \setminus N_0$, there is a subsequence $\{g_m\}$ of $\{g_n\}$ such that $\{G^x_m(t)\}$ converges pointwise to $e^{i\gamma}G^x(t)$ in $H^\infty(dt/\pi(1 + t^2))$ with $0 \leqslant \gamma < 2\pi$, where γ depends on x and $\{g_m\}$. This implies that

$$\overline{(g_m * f_k)}(x + e_t) \to e^{-i\gamma}\overline{(g * f_k)}(x + e_t), \qquad m \to \infty,$$

pointwise in $L^\infty(dt/\pi(1+t^2))$. Note that every subsequence of $\{g_n\}$ contains such a subsequence $\{g_m\}$. Since $e^{-i\gamma}e^{i\gamma} = 1$, the sequence $\{g_n\}$ itself satisfies

$$\overline{(g_n * f_k)}\, g_n(x+e_t) \to \overline{(g * f_k)}\, g(x+e_t)\,, \qquad n\to\infty,$$

pointwise in $L^\infty(dt/\pi(1+t^2))$. Since

$$\|\overline{(g_n * f_k)}\, g_n\|_\infty \;\leqslant\; \|g_n\|_\infty^2 \|f_k\|_1 \;\leqslant\; \|\phi\|_\infty^2 \|f_k\|_1,$$

it follows from the bounded convergence theorem that

$$\|\overline{(g_n * f_k)}\, g_n - \overline{(g * f_k)}\, g\|_2 \to 0\,, \qquad n\to\infty,$$

so that $\overline{(g * f_k)}\, g$ lies in $L^\infty(\tau)$. Therefore, $\overline{\chi_\alpha}\, g$ as well as $\overline{\chi_\alpha}\, \phi$ is generated by a function on K/H. On the other hand, by the property of Γ, each element in Γ has the form $\lambda + n\alpha$ for λ in Λ and n in $\mathbb{Z}$, thus the proof is complete. □

Recall that K/H coincides with the dual group of Λ. Let α be as in Lemma 3.1 and let $C(\bar{x},t)$ be the trivial cocycle on K/H defined by $C(\bar{x},t) = \exp(i\alpha t)$. Since α is positive, $C(\bar{x},t)$ is an analytic cocycle. We denote by $(K/H\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ the skew product of K/H and $\mathbb{T}$ induced by $C(\bar{x},t)$, which is the continuous flow obtained by

$$S_t(\bar{x}, e^{i\theta}) \;=\; (T_t\,\bar{x}, C(\bar{x},t)e^{i\theta}), \qquad (\bar{x}, e^{i\theta}) \in K/H\times\mathbb{T}\,.$$

Then $d\tau\times d\theta/2\pi$ is the invariant probability measure on $K/H\times\mathbb{T}$ (see the end of the preceding section). Let us represent the generator g and all the limits of subsequences of $\{g_n\}$ on $K/H\times\mathbb{T}$, which is the smallest product group with such property. Each function ψ on K/H extends naturally to the one on $K/H\times\mathbb{T}$ by setting $\psi(\bar{x},e^{i\theta}) = \psi(\bar{x})$. Since $|g|$ and g_n are functions on K/H, they belong to $L^\infty(d\tau\times d\theta/2\pi)$.

With the above notation, we fix a w in $K\setminus N_0$. Since $G_n^w(t)$ and $G^w(t)$ are outer functions in $H^2(dt/\pi(1+t^2))$ which extend analytically to $\{Re\, z > -r\}$, we may assume that $G_n^w(t)$ converges pointwise to $G^w(t)$ on $\mathbb{R}$, by multiplying each g_n by a suitable constant of modulus one. By regarding Lemma 2.5, the functions $G_n^w(t)$ and $G^w(t)$ extend to g_n and g, respectively. However, we obtain the following:

Lemma 3.2. — *For $\sigma - a.e.\ x$ in K, $G_n^x(t)$ never converges pointwise on $\mathbb{R}$. Consequently, we find two subsequences $\{g_m\}$ and $\{g_k\}$ of $\{g_n\}$ such that $G_m^x(t)$ and $G_k^x(t)$ converge to $e^{i\beta}G^x(t)$ and $e^{i\gamma}G^x(t)$ with $0\leqslant\beta<\gamma<2\pi$, respectively.*

Proof. — Since $1/n \leqslant |g_n(x)| \leqslant \|\phi\|_\infty$, each g_n^{-1} is also an outer function in $H^\infty(\sigma)$. This implies that almost every $t \to (g/g_n)(x+e_t)$ is an outer function in $H^\infty(dt/\pi(1+t^2))$. Furthermore, since

$$a_0(g/g_n) \;=\; \int_K g/g_n\,d\sigma \;=\; \int_K g\,d\sigma \int_K g_n^{-1}\,d\sigma \;=\; 0\,,$$

Lemma 2.1 assures that each g/g_n is also a single generator of $H_0^2(\sigma)$.

Denote by F the invariant set of all x in K for which $\{G_n^x(t)\}$ itself converges. Suppose that F has positive measure. By (3.1) and the ergodic theorem, $(g/g_n)(x)$ converges to an invariant function on F, so to a constant of modulus one on K. Then the bounded convergence theorem shows that $a_0(g/g_n) \neq 0$ for large n. Such g/g_n cannot be a single generator of $H_0^2(\sigma)$, which contradicts the above observation. □

Let us mention a few remarks derived from Lemma 3.2. When $0 \leqslant \beta < 2\pi$, $\mathcal{Z}(\beta)$ denotes the subgroup of $\mathbb{T}$ generated by $e^{i\beta}$, that is,

$$\mathcal{Z}(\beta) \;=\; \left\{e^{ij\beta}\;;\; j \in \mathbb{Z}\right\}\,.$$

If $\beta/2\pi$ is rational, then the order of $\mathcal{Z}(\beta)$ is finite. Fix two points w and x in $K \setminus N_0$. We assume by Lemma 3.2 that a subsequence $\{g_k\}$ of $\{g_n\}$ satisfies that $G_k^w(t)$ and $G_k^x(t)$ converge respectively to $e^{ij\beta}G^w(t)$ and $e^{i(j+1)\beta}G^x(t)$ for j in $\mathbb{Z}$, by multiplying each g_k by a suitable constant of modulus one. Denote by $\mathcal{O}(\bar{w})$ the orbit $\mathcal{O}(\bar{w},\{T_t\}_{t\in\mathbb{R}})$ of $\bar{w}$ in $(K/H,\{T_t\}_{t\in\mathbb{R}})$. Then g is determined naturally on $\mathcal{O}(\bar{w}) \times \mathcal{Z}(\beta)$ and $\mathcal{O}(\bar{x}) \times \mathcal{Z}(\beta)$ to represent the limits of the subsequence $\{g_k\}$ of $\{g_n\}$ on them. For each m in $\mathbb{Z}$, we see also that every limit of $\{g_k^m\}$ is represented on these product subsets.

If ℓ is a positive integer, then g^ℓ as well as ϕ^ℓ is also a bounded generator of $H_0^2(\sigma)$ by Lemma 2.2. We choose an invariant null set $N(\ell)$ including N_0 outside which a subsequence $\{G_j^x(t)^\ell\}$ of $\{G_n^x(t)^\ell\}$ converges to $e^{i\gamma}G^x(t)^\ell$ with $0 < \gamma < 2\pi$. Define the invariant null set N_1 by $N_1 \;=\; \cup_{\ell=1}^\infty N(\ell)\,.$ When $\ell \;=\; m\,!$, we take again a subsequence $\{G_k^x(t)\}$ of $\{G_j^x(t)\}$ converging to $e^{i\beta(m)}G^x(t)$ with $e^{i\beta(m)\ell} \;=\; e^{i\gamma}$. Then the order of $\mathcal{Z}(\beta(m))$ is larger than m, so $\cup_{m=1}^\infty \mathcal{Z}(\beta(m))$ is dense in $\mathbb{T}$. Therefore, to represent g and all the limits of subsequences of $\{g_n\}$ on each orbit, the product group $K/H \times \mathbb{T}$ is the smallest one. Let us explain the meaning more precisely. Under the assumption of Lemma 3.1, we put $h_\alpha = \overline{\chi_\alpha}g$. Then h_α lies in $L^2(\tau)$. Define the group character $\mathcal{P}_\alpha$ of $K/H \times \mathbb{T}$ by the projection $\mathcal{P}_\alpha(\bar{x},e^{i\theta}) \;=\; e^{i\theta}$. Since

$$(h_\alpha\mathcal{P}_\alpha)(S_t(\bar{x},e^{i\theta})) \;=\; h_\alpha(\bar{x}+e_t)C(\bar{x},t)e^{i\theta} \;=\; h_\alpha(\bar{x}+e_t)e^{i\alpha t}e^{i\theta},$$

the function $t \to (h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta}))$ is an outer function in $H^\infty(dt/\pi(1+t^2))$ for $d\tau \times d\theta/2\pi - a.e.\,(\bar{x}, e^{i\theta})$ in $K/H \times \mathbb{T}$. Then the outer function $G^x(t)$ equals $t \to (h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta}))$ for some θ with $0 \leqslant \theta < 2\pi$. In order to represent consistently all kinds of limits of subsequences $\{G^x_k(t)\}$, we require the family of all outer functions $t \to (h_\alpha \mathcal{P}_\alpha)(S_t(\bar{x}, e^{i\theta}))$ with $0 \leqslant \theta < 2\pi$.

Lemma 3.3. — *Let Γ and Λ be as above. Then Λ cannot be equal to Γ.*

Proof. — Let α be as in Lemma 3.1. Then α lies in Λ if and only if $\Lambda = \Gamma$. We suppose, on the contrary, that α lies in Λ. Since $K/H = K$, let us consider the skew product $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ of K and $\mathbb{T}$ induced by the cocycle $C(x,t) = e^{i\alpha t}$. We use freely the notation above. Since

$$\mathcal{F}(x, e^{i\theta}) = (\overline{\chi_\alpha}\,\mathcal{P}_\alpha)(x, e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T},$$

is an invariant function that is not constant, $d\sigma \times d\theta/2\pi$ is not an ergodic measure on $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Now K is represented as the local product decomposition $K_\alpha \times [\,0, 2\pi/\alpha)$, in which K_α is the closed subgroup of all x in K such that $\chi_\alpha(x) = 1$. If we put

$$\mathcal{G}(x, e^{i\theta}) = h_\alpha(x)\,\mathcal{P}_\alpha(x, e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T},$$

then, for each $x = (y, s)$ in $K_\alpha \times [0, 2\pi/\alpha)$, the equation

$$\mathcal{G}(S_t(x, e^{i\theta})) = e^{i(\theta+\alpha t)} h_\alpha(x + e_t) = e^{i(\theta-\alpha s)} g(x + e_t) \tag{3.2}$$

holds, since $e^{i(\theta+\alpha t)}\overline{\chi_\alpha}(y+e_s+e_t) = e^{i(\theta-\alpha s)}$ and $h_\alpha = \overline{\chi_\alpha} g$. By regarding $\mathbb{T}$ as the interval $[\,0, 2\pi/\alpha)$, $K\times\mathbb{T}$ is identified with $K_\alpha\times[\,0, 2\pi/\alpha)\times[\,0, 2\pi/\alpha)$. Let E be the subset of $K \times \mathbb{T}$ defined by

$$E = K_\alpha \times \{(s,s)\,; 0 \leqslant s < 2\pi/\alpha\}.$$

Then E is a closed invariant set in $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$, for which $(K, \{T_t\}_{t\in\mathbb{R}})$ is isomorphic to $(E, \{S_t\}_{t\in\mathbb{R}})$ via the map $(y, s) \to (y, s, s)$. We see also that the ergodic measure $d\sigma$ is carried to $(\alpha/2\pi)d\sigma_1 \times ds$ on E by this map, where σ_1 is the normalized Haar measure on K_α. We regard g_n, g and h_α as the functions on $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Recall that almost every $G^x_n(t)$ and $G^x(t)$ are outer functions in $H^\infty(dt/\pi(1+t^2))$.

Let x be in $K \setminus N_1$ and let $\{g_k\}$ be a subsequence of $\{g_n\}$ such that $G^x_k(t)$ converges pointwise to $t \to e^{i\alpha\beta}e^{i\alpha t}h_\alpha(x + e_t)$ with $0 \leqslant \beta < 2\pi/\alpha$. Notice that $t \to e^{i\alpha\beta}e^{i\alpha t}h_\alpha(x + e_t)$ is an outer function in $H^\infty(dt/\pi(1+t^2))$ and that $|h_\alpha(x + e_t)| = |g(x + e_t)|$. Let $x = (y, s)$ in $K_\alpha \times [\,0, 2\pi/\alpha)$ as above.

Since x may be replaced by any point in the orbit $\mathcal{O}(x)$ of x, we consider x as a function of s on $[0, 2\pi/\alpha)$. It follows from (3.2) that

$$e^{i\alpha\beta} e^{i\alpha t} h_\alpha(y + e_s + e_t) \;=\; e^{i\alpha(\beta - s)} \mathcal{G}(S_t(y + e_s, e^{i\alpha s}))\,, \qquad (s, t) \in [0, 2\pi/\alpha) \times \mathbb{R}.$$

Putting $t = 0$ and replacing y with $y + e_{[s\alpha/2\pi]}$, if necessary, we observe that

$$e^{i\alpha(\beta - s)} \mathcal{G}(y + e_s, e^{i\alpha s}) \;=\; e^{i\alpha\beta} e^{-i\alpha s} G^y(s)\,, \qquad s \in \mathbb{R}.$$

This shows that $G^y_k(s)$ converges pointwise to $s \to e^{i\alpha\beta}(\overline{\chi_\alpha}\, g)(y + e_s)$, which cannot be an outer function in $H^\infty(dt/\pi(1 + t^2))$. Hence any subsequence of $\{G^x_n(t)\}$ cannot converge to an outer function in $H^\infty(dt/\pi(1 + t^2))$ for $\sigma - a.e.\, x$ in K. Thus we have a contradiction. $\square$

In view of Lemma 3.3, we know that there are two possibilities in relation to α and Λ. Either $n\alpha$ lies in Λ only for $n = 0$ or $\ell\alpha$ lies in Λ for an integer $\ell \geqslant 2$. We claim that the latter case cannot occur, meaning that α is independent to Λ.

Lemma 3.4. — *Let Λ, H and α be as above. Then $n\alpha$ lies in Λ if and only if $n = 0$ in $\mathbb{Z}$. Consequently, H is isomorphic to $\mathbb{T}$, so that K and $d\sigma$ are identified with $K/H \times \mathbb{T}$ and $d\tau \times d\theta/2\pi$, respectively.*

Proof. — Suppose that $\ell\alpha$ lies in Λ for some $\ell \geqslant 2$. By Lemma 2.2, ϕ^ℓ is also a bounded generator of $H^2_0(\sigma)$. It follows from Lemma 3.2 that $\chi_{\ell\alpha}$ and $(\overline{\chi_\alpha}\phi)^\ell$ lie in $L^2(\tau)$, so does ϕ^ℓ itself. Let Γ_ℓ and Λ_ℓ be the smallest groups determined by the nonzero Fourier coefficients of ϕ^ℓ and $|\phi^\ell|$ as above. Then they both are subgroups of Λ. On the other hand, since

$$a_\lambda(|\phi|) \;=\; \lim_{\epsilon \to +0} \, a_\lambda\left((|\phi|^\ell + \epsilon)^{1/\ell} \right)\,,$$

each λ in Λ with $a_\lambda(|\phi|) \neq 0$ lies in Λ_ℓ. This implies that $\Lambda = \Lambda_\ell = \Gamma_\ell$. By replacing ϕ with ϕ^ℓ in Lemma 3.3, this gives a contradiction. Thus $n\alpha$ lies in Λ if and only if $n = 0$.

Since $C(\bar{x}, t)^n$ is a coboundary only for $n = 0$, the measure $d\tau \times d\theta/2\pi$ is ergodic on $(K/H \times \mathbb{T}, \{S_t\}_{t \in \mathbb{R}})$. Define the isomorphism of $\Lambda \times \mathbb{Z}$ onto Γ by

$$\varrho(\lambda, n) \;=\; \lambda + n\alpha\,, \qquad (\lambda, n) \in \Lambda \times \mathbb{Z}.$$

Then the conjugate map ϱ^* of ϱ is given by $\varrho^*(x) = (\bar{x}, e^{i\theta})$ on K, where $\chi_\alpha(x) = e^{i\theta}$. Indeed, we observe that

$$\chi_\lambda(\bar{x}) e^{in\theta} \;=\; \langle (\lambda, n), (\bar{x}, e^{i\theta}) \rangle \;=\; \chi_{\lambda + n\alpha}(x) \;=\; \chi_\lambda(\bar{x}) \chi_\alpha(x)^n\,,$$

for each (λ, n) in $\Lambda \times \mathbb{Z}$. Via the map ϱ^*, K is identified with $K/H \times \mathbb{T}$, and $d\tau \times d\theta/2\pi$ is carried by the map to $d\sigma$ on K. □

We notice that the annihilator H of Λ is isomorphic to $\mathbb{T}$, and $|g(x)|$ as well as $|\phi(x)|$ is constant on almost every coset $\bar{x} = x + H$ in K/H.

4. Contradiction to existence

We may now offer our proof of the main result stated in Section 1.

Proof of the Theorem. — Suppose, on the contrary, that a bounded function ϕ generates $H_0^2(\sigma)$. Let Γ and Λ be the dense subgroups of $\mathbb{R}$ defined as in Section 3 with respect to ϕ and $|\phi|$, respectively. Choose an α in Γ with $a_\alpha(\phi) \neq 0$. It follows from Lemma 3.4 that α is independent of Λ and Γ is generated by α and Λ. Let $0 < \beta < 1$. Since the function

$$(1 + \beta\chi_\alpha)^{-1} = \sum_{k=0}^{\infty} (-\beta)^k \chi_{k\alpha}$$

lies in $H^\infty(\sigma)$, $(1 + \beta\chi_\alpha)^2$ is an outer function in $H^\infty(\sigma)$. Define $\phi_1 = (1 + \beta\chi_\alpha)^2 \phi$. In view of Lemma 2.1, ϕ_1 is also a bounded generator of $H_0^2(\sigma)$. As above, let Γ_1 and Λ_1 be the smallest groups determined by the nonzero Fourier coefficients of ϕ_1 and $|\phi_1|$, respectively. Notice that Γ_1 is a subgroup of Γ. We claim that the generator ϕ_1 cannot satisfy the property of Lemma 3.3. Indeed, since $|\phi_1| = (1 + \beta^2 + \beta\overline{\chi_\alpha} + \beta\chi_\alpha)\,|\phi|$, we obtain by (1.1) that

$$a_\lambda(|\phi_1|) = (1 + \beta^2) a_\lambda(|\phi|) + \beta a_{\lambda+\alpha}(|\phi|) + \beta a_{\lambda-\alpha}(|\phi|).$$

Since α does not lie in Λ, if λ is in Λ, then $a_{\lambda+\alpha}(|\phi|) = a_{\lambda-\alpha}(|\phi|) = 0$. Then we have

$$a_\lambda(|\phi_1|) = (1 + \beta^2) a_\lambda(|\phi|) \quad \text{and} \quad a_{\lambda+\alpha}(|\phi_1|) = \beta a_\lambda(|\phi|),$$

for each λ in Λ. These facts imply that Λ_1 contains Λ and α, so that $\Gamma = \Lambda_1 = \Gamma_1$, which contradicts Lemma 3.3. □

The next proof is of independent interest, because it suggests that our Theorem is regarded essentially as the converse to Corollary 1.1.

Proof of Corollary 1.1. — We consider the case where the cocycle $C(x, t)$ of $\mathfrak{M}$ has the form $C(x, t) = e^{i\alpha t}$. Then $\mathfrak{M}_-$ is the space of all ψ in $L^2(\sigma)$ satisfying that

$$\psi(x) \sim \sum_{\Gamma \ni \lambda > -\alpha} a_\lambda(\psi) \chi_\lambda(x).$$

Suppose that $\mathfrak{M}_-$ has a generator ϕ. Then $\log|\phi|$ does not lie in $L^1(\sigma)$ and we may assume that ϕ is bounded. If $\ell\alpha$ is in Γ for a positive integer ℓ, then the bounded function $(\chi_\alpha\phi)^\ell$ is a single generator of $H_0^2(\sigma)$ by Lemma 2.1, which is contrary to Theorem. We next consider the case that

$$\alpha \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (1/n)\Gamma\,.$$

Since $C(x,t)^n$ is a coboundary only for $n=0$, the measure $d\sigma \times d\theta/2\pi$ is ergodic on the skew product $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ induced by $C(y,t)$, that is,

$$S_t(x, e^{i\theta}) \;=\; (x + e_t\,,\, e^{i\alpha t} e^{i\theta}), \qquad (x, e^{i\theta}) \in K \times \mathbb{T}.$$

Let Γ_1 be the discrete group generated by Γ and α, and let K_1 be the dual group of Γ_1. Since $\varrho(\lambda, n) \;=\; \lambda + \alpha n$ is an isomorphism of $\Gamma \times \mathbb{Z}$ onto Γ_1, the almost periodic flow on K_1 is identified with $(K \times \mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. Then, via the dual map ϱ^* of ϱ, the normalized Haar measure $d\mu$ on K_1 is identified with $d\sigma \times d\theta/2\pi$. Define the function ϕ_1 in $L^2(\mu)$ by $\phi_1(x, e^{i\theta}) \;=\; \phi(x)e^{i\theta}$. Since $\log|\phi|$ does not lie in $L^1(\sigma)$, neither does $\log|\phi_1|$ in $L^1(\mu)$. Since $t \to \phi_1 \circ S_t(x, e^{i\theta})$ is outer in $H^2(dt/\pi(1+t^2))$ for $\mu - a.e.\,(x, e^{i\theta})$ in $K \times \mathbb{T}$, Lemma 2.1 implies that ϕ_1 is a single generator of $H_0^2(\mu)$, which contradicts our Theorem. □

Proof of Corollary 1.2. — Denote by $C(x,t)$ the real cocycle of $\mathfrak{M}$. Suppose that $\mathfrak{M}_-$ has a generator ϕ, for which $\log|\phi|$ does not lie in $L^1(\sigma)$. It follows from Lemma 2.1 that almost every $t \to C(x,t)\phi(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. We may assume that ϕ is bounded. Since $C(x,t)^2 \;\equiv\; 1$, ϕ^2 is a single generator of $H_0^2(\sigma)$ by Lemma 2.1, which contradicts our Theorem. □

By the same way as above, we may show that if $C(x,t)$ takes only finite values, then $\mathfrak{M}_-$ cannot be singly generated. Indeed, by the cocycle identity, the set of values of $C(x,t)$ forms a group of order k,

$$\mathcal{Z}(2\pi/k) \;=\; \left\{ e^{i2\pi j/k}\;;\; j = 0, \ldots, k-1 \right\}.$$

Then if ϕ generates $\mathfrak{M}_-$, then ϕ^k is a generator of $H_0^2(\sigma)$.

Let $\mathfrak{M}$ be the normalized simply invariant subspace of $L^2(\sigma)$ with cocycle $A(x,t)$. Recall that ψ lies in $\mathfrak{M}$ if and only if almost every $t \to A(x,t)\psi(x+e_t)$ lies in $H^2(dt/\pi(1+t^2))$. Denote by $\widetilde{\mathfrak{M}}$ the invariant subspace with cocycle $\overline{A(x,t)}$ (as discussed in [4, §3.2]). To prove Corollary 1.3. we need the following:

Lemma 4.1. — *Let $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ be as above. If $\mathfrak{M}$ is singly generated, then $(\widetilde{\mathfrak{M}})_-$ cannot be singly generated.*

Proof. — Since $A(x,t) \cdot \overline{A(x,t)} \equiv 1$, $H_0^2(\sigma)$ is the smallest subspace of $L^2(\sigma)$ containing all $\psi_1\psi_2$ with ψ_1 in $\mathfrak{M} \cap L^\infty(\sigma)$ and ψ_2 in $(\widetilde{\mathfrak{M}})_- \cap L^\infty(\sigma)$ (see [4, §3.2, Theorem 20]). Suppose that $(\widetilde{\mathfrak{M}})_-$ is singly generated. Then Lemma 2.1 shows that there are bounded single generators ϕ_1 and ϕ_2 of $\mathfrak{M}$ and $(\widetilde{\mathfrak{M}})_-$, respectively. Thus $\phi_1\phi_2$ is a single generator of $H_0^2(\sigma)$, which contradicts our Theorem. □

Proof of Corollary 1.3.

(a) Let $\mathfrak{M}$ be a simply invariant subspace with nontrivial cocycle $A(x,t)$. It follows from [8] that $\mathfrak{M}$ is singly generated if and only if $A(x,t)$ is cohomologous to a singular cocycle. On the other hand, by [4, §4.6, Theorem 26], every cocycle is cohomologous to a Blaschke cocycle. By virtue of Lemma 4.1, we obtain easily a desired Blaschke cocycle.

(b) From Lemma 4.1, we choose a Blaschke cocycle $B(x,t)$ such that the invariant subspace $\mathfrak{N}$ having the cocycle $\overline{B(x,t)}$ is not singly generated. We claim that $B(x,t)$ satisfies the desired property. Suppose, on the contrary, that some function ψ in $H^2(\sigma)$ has exactly the same zeros as $B(x,t)$. By multiplying by a suitable outer function, we assume that ψ is bounded. Then ψ generates the invariant subspace with cocycle $\overline{B(x,t)S(x,t)}$, where $S(x,t)$ is the singular cocycle determined by the inner part of $t \to \overline{B(x,t)}\psi(x+e_t)$ in $H^2(dt/\pi(1+t^2))$. On the other hand, it follows from [8] and Lemma 2.1 that there is a function h in $L^2(\sigma)$ such that almost every $t \to S(x,t)h(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. Observe that

$$(h\psi)(x+e_t) \;=\; B(x,t) \cdot S(x,t)h(x+e_t) \cdot \overline{B(x,t)S(x,t)}\psi(x+e_t)\,.$$

Since the inner part of $t \to (h\psi)(x+e_t)$ is $t \to B(x,t)$, the subspace $\mathfrak{N}$ is singly generated by $h\psi$, thus we have a contradiction.

□

In the proof of (b) above, if the singular cocycle $S(x,t)$ is a coboundary, then h is taken as a unitary function, otherwise $\log|h|$ does not lie in $L^1(\sigma)$.

5. Remarks

Remark A. It is sometimes useful to study the spectral measures associated with invariant subspaces. Let $\mathfrak{M}$ be a simply invariant subspace of $L^2(\sigma)$ and put

$$\mathfrak{M}_\lambda \;=\; \bigwedge_{\lambda \geqslant \nu} \chi_\nu \mathfrak{M}.$$

for each λ in $\mathbb{R}$. Denote by P_λ the orthogonal projection of $L^2(\sigma)$ onto $\mathfrak{M}_\lambda$. By the property that

$$\bigwedge_{-\infty<\lambda<\infty} \mathfrak{M}_\lambda \;=\; \{0\} \qquad \text{and} \qquad \bigvee_{-\infty<\lambda<\infty} \mathfrak{M}_\lambda \;=\; L^2(\sigma)\,,$$

we obtain the continuity of the spectral resolution of identity $\{I - P_\lambda\}_{\lambda\in\mathbb{R}}$ on $L^2(\sigma)$, where I is the identity map on $L^2(\sigma)$. Let $A(x,t)$ be the cocycle of $\mathfrak{M}$. By Stone's theorem, a unitary group $\{V_t\}_{t\in\mathbb{R}}$ on $L^2(\sigma)$ is defined as

$$V_t\,\phi(x) \;=\; A(x,t)T_t\,\phi(x) \;=\; -\int_{-\infty}^{\infty} e^{i\lambda t}\,dP_\lambda\phi(x)\,, \qquad \phi\in L^2(\sigma)\,,$$

where $T_t\,\phi(x) = \phi(x+e_t)$. For a nonzero function ϕ in $L^2(\sigma)$, $-d(P_\lambda\phi,\phi)$ is a finite positive measure on $\mathbb{R}$. On almost periodic flows, by comparing with Lebesgue measure $d\lambda$, the type of such measures is uniquely determined. We then say that each of $\mathfrak{M}$, $A(x,t)$ and $\{V_t\}_{t\in\mathbb{R}}$ is of *absolutely continuous*, or *singular continuous*, or *discrete* type (as discussed in [4, §2.4]). This fact plays an important role to classify invariant subspaces in this special context. It is easy to observe that $A(x,t)$ and $\overline{A(x,t)}$ have the same spectral type, so the following is an immediate consequence of Lemma 4.1.

PROPOSITION 5.1. — *There is a simply invariant subspace of $L^2(\sigma)$ of either absolutely continuous or singular continuous type which has no single generator.*

Let w be a nonnegative function in $L^2(\sigma)$ satisfying (1.3), while $\log w$ does not lie in $L^1(\sigma)$. We know that a cocycle is trivial if and only if it is of discrete type (see [4, §2.4, Theorem 15]). It follows from Corollary 1.1 that the type of $\mathfrak{M}[w]$ has to be continuous. However, we have no idea to decide what kind of continuous spectrum $\mathfrak{M}[w]$ may have.

Remark B. Using a suitable cocycle, we may construct a skew product on which the H^2_0−space is singly generated. Indeed, let w be a bounded function as above and let $A(x,t)$ be the cocycle of $\mathfrak{M}[w]$. By Lemma 2.1 we see that almost every $t \to A(x,t)w(x+e_t)$ is outer in $H^2(dt/\pi(1+t^2))$. Denote by $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$ the skew product induced by $A(x,t)$. If $A(x,t)^n, n \geqslant 1$, is a coboundary $\overline{q(x)}q(x+e_t)$ with unitary function q on K, then qw^n is a single generator of $H^2_0(\sigma)$. It then follows from Theorem that $A(x,t)^n$ is a coboundary only for $n=0$. Hence $d\mu = d\sigma\times d\theta/2\pi$ is an ergodic measure on $(K\times\mathbb{T}, \{S_t\}_{t\in\mathbb{R}})$. If we set

$$\phi(x,e^{i\theta}) \;=\; w(x)e^{i\theta}\,, \qquad (x,e^{i\theta})\in K\times\mathbb{T}\,,$$

then ϕ is a single generator of $H^2_0(\mu)$, since $\log|\phi|$ does not lie in $L^1(\mu)$ and almost every $t \to \phi(S_t(x, e^{i\theta}))$ is outer in $H^2(dt/\pi(1+t^2))$ (see [16] for another construction).

Remark C. We have a bit of information on the distribution of zeros of functions in $H^2(\sigma)$ which are connected with Dirichlet series (refer to [17] for related topics). Let $\{\lambda_n\}$ be a sequence in Γ such that

$$0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n \longrightarrow \lambda, \qquad n \to \infty,$$

for some λ in Γ. Define a function ψ in $H^2(\sigma)$ by

$$\psi = \sum_{n=1}^{\infty} a_n \chi_{\lambda_n}$$

with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Observe that almost every $t \to \psi(x + e_t)$ extends to an entire function.

Proposition 5.2. — *Let ψ be as above and let $\delta > 0$. Then there is a decreasing sequence $\{m_n\}$ with $m_n \to -\infty$ such that the number of zeros of $z \to \psi(x + e_z)$ in the strip*

$$S_n = \{ z = t + iu \,;\, m_n > u > m_n - \delta\}$$

is infinite, for $\sigma - a.e.\, x$ in K.

Proof. — Putting $\nu_n = \lambda - \lambda_n$, we let $\phi = \sum_{n=1}^{\infty} \overline{a_n}\, \chi_{\nu_n}$. Since $z \to e^{i\lambda z}$ has no zero, $z \to \psi(x+e_z)$ has zero at z if and only if so does $z \to \phi(x+e_z)$ at $\bar{z}$. For each $r > 0$, $t \to \phi * P_{ir}(x + e_t)$ cannot be an outer function in $H^2(dt/\pi(1 + t^2))$, even if $\log|\phi|$ does not lie in $L^1(\sigma)$. Since ϕ has no weight at infinity, the inner part of $t \to \phi * P_{ir}(x + e_t)$ derives a Blaschke cocycle being not constant. From this fact, we may choose easily a desired decreasing sequence $\{m_n\}$. □

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