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A NOTE ON DEGENERATIONS OF DEL PEZZO SURFACES

by Yuri PROKHOROV (*)

ABSTRACT. — We prove that for a \mathbb{Q} -Gorenstein degeneration X of del Pezzo surfaces, the number of non-Du Val singularities is at most $\rho(X)+2$. Degenerations with $\rho(X)+2$ and $\rho(X)+1$ non-Du Val points are investigated

RÉSUMÉ. — Nous montrons que pour une dégénérescence \mathbb{Q} -Gorenstein X de surfaces de del Pezzo, le nombre de singularités non-Du Val est au plus $\rho(X)+2$. Les dégénérescences avec $\rho(X)+2$ et $\rho(X)+1$ points non-Du Val sont étudiées.

1. Introduction

This paper continues the classification of \mathbb{Q} -Gorenstein degenerations of del Pezzo surfaces started in [11], [6]. Let $\mathcal{X} \rightarrow \mathcal{Z}$ be a family of surfaces over a smooth curve \mathcal{Z} such that a general fiber is a smooth del Pezzo surface and the special fiber $X := \mathcal{X}_o$ is reduced, normal and has only quotient singularities. Assume further that \mathcal{X} is \mathbb{Q} -Gorenstein and $-K_{\mathcal{X}}$ is ample over \mathcal{Z} . Such kind of families appear naturally in the three-dimensional minimal model program [8], [10] and in the study of certain moduli spaces [3], [4]. It is expected that the combinatorial structure of singularities of X is related to exceptional vector bundles on smooth del Pezzo surfaces but this relation still looks mysterious (cf. [5]).

In this paper we study the special fiber X of the above family under the condition that the Picard number $\rho(X)$ is large. The case $\rho(X) = 1$ was investigated completely in [6]. Our main result is the following.

Keywords: del Pezzo surface, T-singularity, deformation.

Math. classification: 14J10, 14E30.

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THEOREM 1.1. — *Let X be a del Pezzo surface with only quotient singularities and let $s(X)$ be the number of its non-Du Val points⁽¹⁾. Assume that X admits a \mathbb{Q} -Gorenstein smoothing. Then*

- (i) $s(X) \leq \rho(X) + 2$,
- (ii) if $s(X) = \rho(X) + 2$, then X is toric,
- (iii) if $s(X) = \rho(X) + 1$, then X admits an effective \mathbb{C}^* -action.

Similar to [6, Th. 1.3], as a consequence of our techniques we verify a particular case of Reid's general elephant conjecture (cf. [17, 3.4B]):

THEOREM 1.2. — *Let $f: \mathcal{X} \rightarrow \mathcal{Z}$ be a del Pezzo fibration over a smooth curve. That is, \mathcal{X} is a 3-fold with terminal singularities, f has connected fibers, and $-K_{\mathcal{X}}$ is ample over \mathcal{Z} . Fix a point $o \in \mathcal{Z}$ and assume that the fiber $f^{-1}(o)$ is reduced, irreducible, normal, and has only quotient singularities. Then, for some ample divisor A on \mathcal{Z} , a general member $S \in |-K_{\mathcal{X}} + f^*A|$ is normal and has only Du Val singularities in a neighborhood of $f^{-1}(o)$.*

Furthermore, we give a characterization of log surfaces that admit a \mathbb{C}^* -actions (Theorem 5.1) and establish the existence of 1-complements on arbitrary del Pezzo surfaces with T-singularities (Theorem 4.1).

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2. Preliminaries

Throughout this paper, we work over the field \mathbb{C} of complex numbers. $\rho(X)$ denotes the Picard group of a variety X . We use standard definitions, notation, and facts of the Minimal Model Program [9], [10].

PROPOSITION-DEFINITION [17, 3.6], [10, 5.19] 2.1. — *Let $P \in X$ be the germ of a normal variety at a point P and let D be a Weil divisor on X . Assume that D is \mathbb{Q} -Cartier at P , that is, rD is a Cartier divisor near P for some positive integer r . Suppose that r is the smallest such r . Then r is called the index of D .*

*There exists a covering $\pi: X^\sharp \rightarrow X$ which is Galois with group μ_r , such that X^\sharp is normal, π is etale over the locus $X_0 \subset X$ where K_X is Cartier, and $P^\sharp := \pi^{-1}(P)$ is a single point. The divisor $K_{X^\sharp} = \pi^*K_X$ is Cartier. Such a covering is called canonical index-one covering of $P \in X$.*

⁽¹⁾For various definitions of Du Val (rational double) singularities we refer to [2].

2.1. T-singularities [8], [19]

DEFINITION 2.2. — Let X be a normal surface such that K_X is \mathbb{Q} -Cartier. We say that a deformation $\mathcal{X}/(0 \in S)$ over a germ $(0 \in S)$ with 0-fiber $\mathcal{X}_0 = X$ is \mathbb{Q} -Gorenstein if, locally analytically at every point $P \in X$, \mathcal{X}/S is induced by an equivariant deformation of the canonical index-one covering $(X^\sharp \ni P^\sharp) \rightarrow (X \ni P)$ (see 2.1).

DEFINITION [8, Def. 3.7] 2.3. — Let $P \in X$ be a quotient singularity of dimension 2. We say $P \in X$ is a T-singularity if it admits a \mathbb{Q} -Gorenstein smoothing. That is, there exists a \mathbb{Q} -Gorenstein deformation of $P \in X$ over a smooth curve germ such that the general fiber is smooth.

PROPOSITION [8, Prop. 3.10] 2.4. — A T-singularity is either a Du Val singularity or a cyclic quotient singularity of the form $\frac{1}{dn^2}(1, dna - 1)$ for some positive integers d, n, a with $\gcd(a, n) = 1$.

2.2. Noether’s formula

For a T-singularity $P \in X$, define

$$\mu_P = \begin{cases} r & \text{if } P \in X \text{ is a Du Val singularity of type } A_r, D_r, \text{ or } E_r, \\ d - 1 & \text{if } P \in X \text{ is of type } \frac{1}{dn^2}(1, dna - 1). \end{cases}$$

This number coincides with the Milnor number of $P \in X$ [11, Sec. 3].

PROPOSITION [6]. 2.5. — Let X be a projective rational surface with T-singularities. Then

$$K_X^2 + \rho(X) + \sum_{P \in \text{Sing } X} \mu_P = 10.$$

COROLLARY 2.6. — Let X be a del Pezzo surface with T-singularities. Then $K_X^2 + \rho(X) \leq 10$.

2.3. Del Pezzo surfaces with T-singularities

PROPOSITION [6]. 2.7. — Let X be a projective surface with T-singularities such that $-K_X$ is nef and big. Then

$$\dim H^0(X, \mathcal{O}_X(-nK_X)) = \frac{1}{2}n(n + 1)K_X^2 + 1.$$

COROLLARY 2.8. — Let X be a projective surface with T-singularities such that $-K_X$ is nef and big. Then

$$\dim | -K_X | = K_X^2 > 0.$$

2.4. Divisorial adjunction [9, ch. 16]

Let X be a normal variety and $S \subset X$ a reduced subscheme of pure codimension one. Assume that the pair (X, S) is lc (log canonical [9, 2.10]) in codimension two. Then there exists a naturally defined effective \mathbb{Q} -Weil divisor $\text{Diff}_S(0)$, called the *different*, such that

$$(K_X + S)|_S = K_S + \text{Diff}_S(0).$$

Now let B be a \mathbb{Q} -divisor, which is \mathbb{Q} -Cartier in codimension two. Then the different for $K_X + S + B$ is defined by the formula

$$(K_X + S + B)|_S = K_S + \text{Diff}_S(B).$$

In particular, if B is a boundary and $(X, S + B)$ is lc in codimension two, then B is \mathbb{Q} -Cartier in codimension two. Moreover, none of the components of $\text{Diff}_S(B)$ are contained in the singular locus of S .

2.5. Classification of two-dimensional log canonical pairs with reduced boundary [9, ch. 3 & Prop. 16.6], [10, Th. 4.15]

Let $P \in (X, C)$ be the germ of a two-dimensional log pair where X is normal and C is a (possibly reducible) reduced curve. Assume that (X, C) is lc. Then one of the following possibilities holds where all isomorphisms are isomorphisms of analytic germs:

2.9.1. — (X, C) is plt (purely log terminal, [9, 2.13]). Then

$$(X, C) \simeq (\mathbb{C}^2, \{x_1 = 0\})/\mu_m(1, a), \quad \text{with } \gcd(a, m) = 1, \quad m \geq 1,$$

$$\text{Index}(K_X + C) = \text{Index}(C) = m, \quad \text{Diff}_C(0) = (1 - 1/m)P.$$

2.9.2. — (X, C) is not plt and C analytically reducible. Then

$$(X, C) \simeq (\mathbb{C}^2, \{x_1x_2 = 0\})/\mu_m(1, a), \quad \text{with } \gcd(a, m) = 1, \quad m \geq 1,$$

$$\text{Index}(K_X + C) = 1, \quad \text{Diff}_C(0) = 0.$$

2.9.3. — (X, C) is not plt and C analytically irreducible. Then

$$(X, C) \simeq (\mathbb{C}^2, \{x_1x_2 = 0\})/G,$$

$$\text{Index}(K_X + C) = 2, \quad \text{Diff}_C(0) = 1,$$

where $G \subset \text{GL}_2(\mathbb{C})$ is a finite subgroup of dihedral type without reflections (see [1] for the precise description of G).

2.6. 1-complements.

Let X be a normal variety and let D be a boundary on X (an effective \mathbb{Q} -divisor with coefficients ≤ 1). Write $D = S + B$, where $S := \lfloor D \rfloor$ (resp. $B := \{D\}$) is the integral (resp. fractional) part of D . A 1-complement of $K_X + D$ is a divisor $D^+ \in |-K_X|$ such that (X, D^+) is log canonical and $D^+ \geq S + \lfloor 2B \rfloor$. In particular, if $D = 0$, then a 1-complement of K_X is a divisor $D^+ \in |-K_X|$ such that (X, D^+) is log canonical. We say that the log divisor $K_X + D$ is 1-complementary if there exists a 1-complement of $K_X + D$.

PROPOSITION [16, Prop. 4.3.2] 2.10. — *Let $f: X \rightarrow Y$ be a birational contraction and let D be a boundary on X such that*

- (i) $K_X + D$ is nef over Y ;
- (ii) the coefficients of f_*D satisfy the inequality $d_i \geq 1/2$.

Assume that $K_Y + f_*D$ is 1-complementary. Then so is $K_X + D$.

PROPOSITION [16, Prop. 4.4.1] 2.11. — *Let $(X, D = S + B)$ be a log variety, where $S := \lfloor D \rfloor$ and $B := \{D\}$. Assume that*

- (i) $K_X + D$ is plt;
- (ii) $-(K_X + D)$ is nef and big;
- (iii) $S \neq 0$;
- (iv) the coefficients of $D = \sum d_i D_i$ satisfy the inequality $d_i \geq 1/2$.

Further, assume that there exists a 1-complement $K_S + \text{Diff}_S(B)^+$ of $K_S + \text{Diff}_S(B)$. Then there exists a 1-complement $K_X + S + B^+$ of $K_X + S + B$ such that $\text{Diff}_S(B)^+ = \text{Diff}_S(B^+)$.

2.7. Contractions of surfaces with Du Val singularities

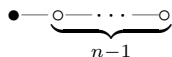
For convenience of the reader we state facts about MMP for Du Val surfaces.

DEFINITION [13]⁽²⁾ 2.12. — *Let $y \in Y$ be a smooth point on a surface and let (u, v) be a local coordinates near y . A weighted blowup with weights $(1, n)$ of a $y \in Y$ is the blowup $X \rightarrow Y$ of the ideal (u, v^n) .*

Clearly, a weighted blowup depends on n and on the choice of coordinates. For $n = 1$ the above defined map is the usual blowup of $y \in Y$. From easy local computations (see [13, §1]) we obtain the following.

(2) For general definition see e.g. [10, 4.56] or [16, 3.2].

LEMMA 2.13. — Let $y \in Y$ be a smooth surface germ and let $f : X \rightarrow Y$ be a weighted blowup with weights $(1, n)$, $n \geq 2$. Let $E = f^{-1}(y)$ be the exceptional divisor and let $\pi : \tilde{X} \rightarrow X$ be the minimal resolution. Then the exceptional locus of the composition $\tilde{X} \rightarrow Y$ is a simple normal divisor whose dual graph looks as follows:



where the vertex \bullet corresponds to a (-1) -curve (the proper transform of E) and the vertices \circ correspond to π -exceptional (-2) -curves. In particular, X has exactly one singular point and this point is Du Val of type A_{n-1} .

COROLLARY 2.14. — In the above notation we have $K_X \cdot E = -1$.

THEOREM ([13, Theorem 1.4]) 2.15. — Let X be a surface with Du Val singularities and let $f : X \rightarrow Y$ be an extremal Mori contraction. Let $E \subset X$ be the exceptional divisor and let $y := f(E)$. Then Y is smooth at y and f is a weighted blowup of $y \in Y$ with weights $(1, n)$ for some $n \geq 1$.

2.8. Contractions of surfaces with T-singularities

The following is the local variant of Theorem 4.1 below.

PROPOSITION [14, Prop. 4.7] 2.16. — Let X be a surface with T-singularities and let $f : X \rightarrow Y$ be a contraction such that $-K_X$ is f -ample. Then, near each fiber $f^{-1}(y)$, $y \in Y$, there exists a 1-complement of K_X .

COROLLARY 2.17. — Let X be a surface with T-singularities and let $f : X \rightarrow Y$ be a birational contraction such that $-K_X$ is f -ample. If the fiber $f^{-1}(y)$ is not a point, then $y \in Y$ is a cyclic quotient singularity (or smooth).

Proof. — Let $D \in |-K_X|$ be a 1-complement of K_X near the fiber. If X has only Du Val singularities, then Y is smooth at y by Theorem 2.15. So we assume that K_X is not Cartier near $f^{-1}(y)$. Then $D \neq 0$ and $D \cap f^{-1}(y) \neq \emptyset$. Denote $D_Y := f_*D$. Since $D \sim -K_X$ is f -ample, we have $\text{Supp}(D) \not\subset f^{-1}(y)$. Hence $D_Y \neq 0$. The pair (Y, D_Y) is lc because so (X, D) is. Moreover, $K_Y + D_Y \sim f_*(K_X + D) \sim 0$. By the classification 2.5 the point $y \in Y$ is a cyclic quotient singularity. \square

Warning 2.18. — In general it is not true that the singularities of Y are of type T.

3. E- and D-singularities

PROPOSITION 3.1. — *Let X be a del Pezzo surface with at worst quotient singularities and such that $\dim | -K_X | > 0$. Then X has no Du Val points of type D_n or E_n contained in $\text{Bs} | -K_X |$.*

COROLLARY 3.2. — *Let X be a del Pezzo surface with T-singularities. Then X has no Du Val points of type D_n or E_n contained in $\text{Bs} | -K_X |$.*

Let $P \in X$ be a Du Val point of type D_n or E_n such that $P \in \text{Bs} | -K_X |$. Let $D \in | -K_X |$ be a general member. Write $D = \sum d_i D_i$, where the D_i are prime divisors and $d_i > 0$.

LEMMA 3.3. — *Notation as above.*

- (i) *For any component D_i of D we have $D_i^2 \geq 0$.*
- (ii) *If $D_i^2 = 0$ for some $D_i \subset D$, then P is the only singular point of X lying on D_i , the pair (X, D_i) is lc, and $d_i = 1$.*
- (iii) *All the components D_i pass through P and do not meet each other elsewhere.*

Proof. — Let D_i be a component passing through P . Assume that $D_i^2 < 0$. Then D_i generates a birational extremal ray. If K_X is not Cartier along D_i , then by [14, Cor. 4.3] X has no Du Val points of type D_n or E_n on D_i , a contradiction. If K_X is Cartier along D_i , then X has only Du Val singularities in a neighborhood of D_i and we have a contradiction by Theorem 2.15 (because our ray is K_X -negative). This proves (i) modulo (iii).

Now assume that $D_i^2 = 0$ and $D_i \ni P$. Then D_i generates a contractible K_X -negative extremal face. Thus there is a contraction $f : X \rightarrow Z$, where Z is a smooth curve, such that $D_i = f^{-1}(z)_{\text{red}}$ for some $z \in Z$. Since $D \in | -K_X |$ is a general member, the scheme fiber f^*z is not contained in D . So, $f^*z \neq D_i$, i.e. f^*z is not reduced. By [14, Cor. 4.3] K_X is Cartier along D_i . Since $-K_X \cdot f^*z = 2$, we have $-K_X \cdot D_i = 1$ and D_i is a fiber of multiplicity 2. Since $2D_i = f^*z$ is not contained in D , we have $d_i = 1$. Finally, by [16, Prop. 7.1.3, Th. 7.1.12] the pair (X, D_i) is lc. This proves (ii) modulo (iii).

Assume that (iii) does not hold. Then there is a component D_i such that $P \in D_i$ and $D_i \cap D_j \ni Q \neq P$ for some $j \neq i$ (because $\text{Supp } D$ is connected). Put $D' := D - d_i D_i$. By the above, $D_i^2 \geq 0$. Hence the divisor $-(K_X + D_i + D') \sim (d_i - 1)D_i$ is nef. Then by the adjunction we have

$$\deg \text{Diff}_{D_i}(D') \leq -\deg K_{D_i} \leq 2.$$

On the other hand, $\text{Diff}_{D_i}(D') \geq P + Q$. Hence, $\text{Diff}_{D_i}(D') = P + Q$ and $(X, D_i + D')$ is lc near D_i [9, 17.6]. Since $P \in X$ is not a cyclic quotient singularity, the pair (X, D_i) is strictly lc (i.e. lc but not plt) at P (see 2.5). In particular, no component of D' pass through P . Hence, $K_X + D_i + D'$ is not Cartier at P (see 2.9.2), so $D \neq D_i + D'$ and $d_i > 1$. In this case, $D_i^2 > 0$ by (ii). Then

$$(K_X + D_i + D') \cdot D_i = -(d_i - 1)D_i^2 < 0$$

and $\text{deg Diff}_{D_i}(D') < 2$. The contradiction proves (iii). □

Proof of Proposition 3.1. — Let now $Q \in X$ be a non-Du Val point and let D_i be a component of D passing through Q . Write $D = d_i D_i + D'$.

If $D = D_i$, then by 2.9.1 the pair (X, D) is not plt at Q . Hence, as above, $\text{Diff}_D(0) = P + Q$. By [9, 17.6] (X, D) is lc near D . But then $K_X + D$ is not Cartier near Q , a contradiction.

Therefore, $D \neq D_i$. Since $\dim |D| > 0$, $D' \neq 0$ (and D' has a reduced movable component). Note that the divisor $-(K_X + D_i + D') \sim (d_i - 1)D_i$ is nef. Then by the adjunction we have

$$\text{deg Diff}_{D_i}(D') \leq -\text{deg } K_{D_i} \leq 2.$$

Since the coefficients of $\text{Diff}_{D_i}(0)$ are $\geq 1/2$ and (X, D_i) is not plt at P , $\text{Supp Diff}_{D_i}(D') = \{P, Q\}$. Write $\text{Diff}_{D_i}(0) = a_0P + bQ$ and $\text{Diff}_{D_i}(D') = aP + bQ$. Since (X, D_i) is not plt at P , $a > a_0 \geq 1$. Hence $b < 1$ and (X, D_i) is plt at Q . Thus, $b = 1 - 1/m$ for some $m \geq 3$ (because $Q \in X$ is not Du Val of type A_1) and $a \leq 4/3$. If (X, D_i) is not lc at P , then $a_0 \geq 1 + 1/l$, where l is the minimal positive integer such that $l(K_X + D_i)$ is Cartier at P . Recall that for the Weil divisor class group of a Du Val singularity (X, P) we have

$$\begin{array}{c|ccccc} (X, P) & \parallel & D_{2n+1} & D_{2n} & E_6 & E_7 & E_8 \\ \hline \text{Cl}(X, P) & \parallel & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/3\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 \end{array}$$

(see, e.g., [1]). So in our case we have $a_0 \geq 5/4$ and $a \geq 5/4 + 1/4 = 3/2$, a contradiction.

Thus we may assume that (X, D_i) is lc at P . In particular, (X, P) is of type D_n . Then $a_0 = 1$. Since $a \leq 4/3$, we have only one possibility: $b = 2/3$, $a = 5/4$, and $2D'$ is not Cartier at P . Moreover, D' is irreducible (and reduced), (X, D') is not lc at P (see 2.9.2), and so $\text{Diff}_{D'}(0) \geq \frac{5}{4}P$. Again by 2.9.2 $2D_i$ is Cartier at P . Hence,

$$\text{Diff}_{D'}(D_i) \geq \left(\frac{5}{4} + \frac{d_i}{2}\right)P > 2P,$$

a contradiction. □

4. Existence of 1-complements

In this section we prove the following important fact (cf. [6, Th. 7.1]).

THEOREM 4.1. — *Let X be a del Pezzo surface with \mathbb{T} -singularities. Then there exists a 1-complement of K_X .*

We need a few preliminary facts.

DEFINITION [15, §2] 4.2. — *Let X be a normal projective variety. We say that X is FT (Fano type) if there is a \mathbb{Q} -boundary Δ such that (X, Δ) is a klt (Kawamata log terminal) log Fano.*

PROPOSITION [15, §2] 4.3. — *Let X be an FT variety.*

- (i) *The Mori cone $\overline{NE}(X)$ is polyhedral and has contractible faces.*
- (ii) *If $f : X \rightarrow Z$ be any contraction of normal varieties. Then Z is FT. In particular, the FT property is preserved under MMP.*
- (iii) *Let Ξ be a boundary on X such that (X, Ξ) is lc and $-(K_X + \Xi)$ is nef. Let $f : Y \rightarrow X$ be a birational extraction such that $a(E, X, \Xi) < 0$ for every f -exceptional divisor E . Then Y is also FT.*
- (iv) *Assume the LMMP in dimension $\dim X$. Then the D-MMP works on X with respect to any divisor D .*

PROPOSITION 4.4. — *Let (Y, C) be a log pair where Y is an FT surface and C is an irreducible curve. Assume that (Y, C) is plt, $-(K_Y + C)$ is nef and big, and $|- (K_Y + C)| \neq \emptyset$. Then one of the following holds:*

- (i) *$K_Y + C$ has a 1-complement,*
- (ii) *Y has three or four singular points on C and either*
 - (a) *$C^2 < 0, K_Y \cdot C \geq 0$, or*
 - (b) *$\dim |-K_Y| = 0$ and $-K_Y \sim bC, b \geq 2$.*

Proof. — First of all note that the curve C is smooth (see 2.9.1). By Proposition 2.11 we can extend complements from $(C, \text{Diff}_C(0))$ to Y . Thus, for (i), it is sufficient to show existence of a 1-complement of $K_C + \text{Diff}_C(0)$. Assume the converse and write

$$\text{Diff}_C(0) = \sum \left(1 - \frac{1}{m_i}\right) P_i, \quad \deg K_C + \deg \text{Diff}_C(0) = (K_X + C) \cdot C \leq 0$$

(see 2.9.1). Thus, $\sum(1 - 1/m_i) \leq 2$. Since, by our assumption, the log divisor $K_C + \text{Diff}_C(0)$ is not 1-complementary, easy computations [9, 19.5]

show that $\text{Diff}_C(0)$ is supported in three or four points P_i . In particular, $\deg K_C < 0$ and so $C \simeq \mathbb{P}^1$.

Assume that $C^2 \leq 0$. Then C generates an extremal face. Since Y is FT, this extremal face is contractible: there is a contraction $\varphi : Y \rightarrow Y'$ such that $y := \varphi(C)$ is a point. By Lemma 4.6 below $K_Y \cdot C \geq 0$. If $C^2 = 0$, then Y' is a curve, φ is a rational curve fibration, and $C = \varphi^{-1}(y)_{\text{red}}$. In this case, $K_Y \cdot C < 0$, a contradiction. Thus $C^2 < 0$ and we are in the case (iia).

Assume that $C^2 > 0$. Let $D \in |- (K_Y + C)|$ be a general member. Write $D = aC + D'$, where $a \geq 0$ and C is not a component of D' . If $D' = 0$, we get case (iib). (Here $b = a + 1 \geq 2$ because $K_X + C$ is not Cartier near singular points on C , see 2.9.1.) Thus we may assume that $D' \neq 0$. Since the support of

$$(a + 1)C + D' \in |- K_Y|$$

is connected, D' meets C . Further,

$$(4.5) \quad \deg \text{Diff}_C(D') = -\deg K_C + (K_Y + C + D') \cdot C = 2 - (a - 1)C^2 \leq 2.$$

By the above, $\text{Diff}_C(D')$ has at least one point of multiplicity ≥ 1 (and multiplicities of all points are $\geq 1/2$). Since $\text{Diff}_C(D') \geq \text{Diff}_C(0)$, the only possibility is

$$\text{Diff}_C(D') = P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3,$$

where $P_1 \in C \cap \text{Supp}(D')$ and $P_2, P_3 \notin C \cap \text{Supp}(D')$. By 2.9.1 $K_Y + C + D'$ is not Cartier at P_2 and P_3 . Thus $K_Y + C + D' \not\sim 0$ and $a > 1$. On the other hand, $\deg \text{Diff}_C(D') = 2$, so by (4.5) $a = 1$, a contradiction. \square

LEMMA [16, Prop. 7.1.12] 4.6. — *Let $S \rightarrow Z$ be a K -negative extremal contraction from a surface S with log terminal singularities, where Z is not a point. Then S has at most two singular points on each fiber.*

Proof of Theorem 4.1. — Let X be a del Pezzo surface with at worst quotient singularities and such that $\dim |- K_X| > 0$ and let $D \in |- K_X|$ be a general member. Take $t \in \mathbb{Q}$ so that (X, tD) is maximally lc. If $t = 1$, then $K_X + D$ is a 1-complement. So from now on we assume that $t < 1$.

Consider the case where (X, tD) is plt. Write $tD = C + B$, where $C := \lfloor tD \rfloor \neq \emptyset$ and B is an effective fractional divisor. Since X is an FT variety, we can run $-(K + C)$ -MMP and obtain

$$\varphi : X \longrightarrow \bar{X}.$$

Since

$$-(K_X + C) \equiv B - (1 - t)K_X,$$

all the contractions are B -negative. Hence they are birational and we end up with a model (\bar{X}, \bar{C}) such that $-(K_{\bar{X}} + \bar{C})$ is nef. We have

$$-(K_{\bar{X}} + \bar{C}) \equiv \bar{B} - (1 - t)K_{\bar{X}},$$

where $-K_{\bar{X}}$ is ample and $\bar{B} := \varphi_*B$ is effective. Hence the divisor $-(K_{\bar{X}} + \bar{C})$ is big. Further,

$$K_X + C + B \equiv -(1 - t)D \equiv (1 - t)K_X.$$

Hence all the contractions in φ are $(K + C + B)$ -negative. Therefore, $(\bar{X}, \bar{C} + \bar{B})$ is plt and so is (\bar{X}, \bar{C}) . So, $\bar{B} \neq 0$ and $\bar{D} := \varphi_*D \neq \bar{C}$. Apply Proposition 4.4 to (\bar{X}, \bar{C}) . The case (iia) does not occur because $-K_{\bar{X}}$ is ample and the case (iib) does not occur because

$$\dim | -K_{\bar{X}} | \geq \dim | -K_X | > 0.$$

Hence, there exists a 1-complement of $K_{\bar{X}} + \bar{C}$. By Proposition 2.10 we can pull back 1-complements from \bar{X} to X .

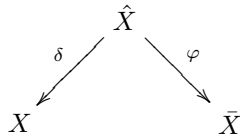
Now consider the case where (X, tD) is not plt. Put $B := tD$. Consider an inductive plt blowup [16, Prop. 3.1.4] $\delta : \hat{X} \rightarrow X$, that is, a birational extraction such that $\rho(\hat{X}/X) = 1$ and

$$K_{\hat{X}} + \hat{B} + C = \delta^*(K_X + B),$$

where C is the (irreducible) exceptional divisor and \hat{B} is the strict transform of B . Moreover, the pair $(\hat{X}, C + (1 - \epsilon)\hat{B})$ is plt for any $\epsilon > 0$. Write

$$K_{\hat{X}} + \hat{D} + aC = \delta^*(K_X + D),$$

where \hat{D} is the strict transform of D and $a > 1$. Then $\hat{D} + aC \in | -K_{\hat{X}} |$, so $\dim | -K_{\hat{X}} | > 0$. By Proposition 4.3 the variety \hat{X} is FT. Run the $-(K + C)$ -MMP. As above all the contractions are \hat{B} -negative. So we end up with a model (\bar{X}, \bar{C}) where $-(K_{\bar{X}} + \bar{C})$ is nef and big (and $\bar{C} \neq 0$):



(the case $\hat{X} = \bar{X}$ is not excluded). Since $\overline{\text{NE}}(\hat{X})$ is polyhedral, $-(K_{\hat{X}} + C + (1 - \epsilon)\hat{B})$ is ample for some $0 < \epsilon \ll 1$. Hence the plt property of the pair $(\hat{X}, C + (1 - \epsilon)\hat{B})$ is preserved. In particular, (\bar{X}, \bar{C}) is plt. Apply Proposition 4.4 to (\bar{X}, \bar{C}) . The case (iib) does not occur because

$$\dim | -K_{\bar{X}} | \geq \dim | -K_X | > 0.$$

Assume that we are in the case (iia). Then $\bar{C}^2 < 0$ and $K_{\bar{X}} \cdot \bar{C} \geq 0$. In particular, \bar{C} is contractible: there is a contraction $\psi : \bar{X} \rightarrow \check{X}$ of \bar{C} , where \check{X} is an FT surface. As in [6, Proof of Th. 7.1] we see that $\check{P} := \psi(\bar{C}) \in \check{X}$ is a singular point and it is not a cyclic quotient singularity. According to Zariski’s main theorem the composition $v = \psi \circ \varphi \circ \delta^{-1} : X \dashrightarrow \check{X}$ is a morphism. By Corollary 3.2 v is not an isomorphism (because $\delta(C) \in \text{Bs} | -K_X |$). Since X is a del Pezzo, v is a K -negative contraction. On the other hand, by Corollary 2.17 \check{P} is a cyclic quotient singularity, a contradiction. Therefore, the case (iia) does not occur and so there exists a 1-complement of $K_{\bar{X}} + \bar{C}$. Now as above by Proposition 2.10 we can pull back 1-complements from \bar{X} to X . \square

5. Tori actions

For a normal projective surface X we denote by $\varrho(X)$ the *numerical Picard number*, that is, the rank of the group of Weil divisors modulo numerical equivalence. Clearly, $\varrho(X) \geq \rho(X)$ and the equality holds if X is \mathbb{Q} -factorial. For a \mathbb{Q} -divisor $D = \sum d_i D_i$ on X we denote

$$\begin{aligned} \|D\| &:= \sum d_i, \\ \varsigma(X, D) &:= \varrho(X) + 2 - \|D\|. \end{aligned}$$

We say that a log pair (X, D) is *toric* if X is a toric variety and D is the (reduced) invariant boundary. We say that a log pair (X, D) admits an *effective \mathbb{C}^* -action* if the variety X admits such an action so that the divisor D is \mathbb{C}^* -invariant.

The statements (i) and (ii) of the following theorem were proved by Shokurov in much more general form [18]. For the convenience of the reader we provide simplified complete proofs.

THEOREM 5.1. — *Let (X, D) be a projective normal log surface such that D is an integral (effective) divisor, the pair (X, D) is lc, and $K_X + D \sim 0$. Then*

- (i) $\varsigma(X, D) \geq 0$;
- (ii) if the equality holds, then (X, D) is toric;
- (iii) if $\varsigma(X, D) = 1$, then (X, D) admits an effective \mathbb{C}^* -action.

Remark 5.2. — Let X be a projective normal surface and let $D \in |-K_X|$ be a divisor such that the pair (X, D) is lc. Then the property $\varsigma(X, D) = 0$ characterizes toric pairs. On the other hand, the condition $\varsigma(X, D) \leq 1$

is sufficient but not necessary for (X, D) to admit an effective \mathbb{C}^* -action. For example, the product $C \times \mathbb{P}^1$, where C is an elliptic curve, admits an effective \mathbb{C}^* -action but for any $D \in |-K_X|$ we have $\varsigma(X, D) \geq 2$.

The rest of this section is devoted to the proof of Theorem 5.1. We will use the following fact which is an easy consequence of the definition.

LEMMA 5.3. — *Let $\varphi : Y' \rightarrow Y$ be a birational morphism of normal surfaces, and let D' be a reduced boundary on Y' . Denote by $N(\varphi, D')$ the number of φ -exceptional curves that are not contained in the support of D' . Then*

$$\varsigma(Y', D') = \varsigma(Y, g_*D') + N(\varphi, D').$$

Let (X, D) be a projective log surface such that (X, D) is lc, $K_X + D \sim 0$, and

$$(5.4) \quad \varsigma := \varsigma(X, D) \leq 1.$$

Let $f : (X', D') \rightarrow (X, D)$ be a minimal dlt modification, that is, a birational map such that the log pair (X', D') is dlt (divisorial log terminal [9, 2.13]),

$$K_{X'} + D' \sim f^*(K_X + D), \quad f_*D' = D,$$

and any f -exceptional divisor has multiplicity 1 in D' (see e.g. [9, Prop. 21.6.1], [16, Prop. 3.1.2]). By Lemma 5.3

$$\varsigma(X', D') = \varsigma(X, D) = \varsigma \leq 1.$$

Hence, (5.4) holds for (X', D') . Since (X', D') is dlt, X' is non-singular near D' [9, Prop. 16.6]. Moreover, X' has at worst Du Val singularities outside of D' .

Run the K -MMP:

$$(X', D') = (X^{(1)}, D^{(1)}) \xrightarrow{\varphi_1} (X^{(2)}, D^{(2)}) \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{l-1}} (X^{(l)}, D^{(l)}) = (Y, D_Y).$$

Let $E^{(i)} \subset X^{(i)}$ be the φ_i -exceptional divisor.

CLAIM 5.5. — *For each $i = 1, \dots, l$ we have*

- (i) $X^{(i)}$ has at worst Du Val singularities;
- (ii) $X^{(i)}$ is non-singular near $D^{(i)}$;
- (iii) φ_i is the weighted blowup with weights $(1, n)$, $n \geq 1$ of a smooth point $\varphi_i(E^{(i)}) \in X^{(i+1)}$;
- (iv) $D^{(i)}$ is a simple normal crossing divisor;
- (v) $E^{(i)} \cdot D^{(i)} = 1$.

Proof. — One can prove (i)-(iii) by induction on i using the following scheme:

$$(i)_i, (ii)_i \implies (iii)_i, (i)_{i+1}, (ii)_{i+1}.$$

Indeed, if $(i)_i$ holds, then φ_i is a weighted blowup by Theorem 2.15 and so $X^{(i+1)}$ is smooth at $\varphi_i(E^{(i)})$.

Since the pair $(X^{(i)}, D^{(i)})$ is lc, (ii) implies (iv) and (v) follows from Corollary 2.14 because $D^{(i)} \sim -K_{X^{(i)}}$. □

CLAIM 5.6. — For each $i = 1, \dots, l$ we have $\zeta(X^{(i)}, D^{(i)}) \leq \zeta \leq 1$, in particular, $D^{(i)} \neq 0$.

Proof. — It follows from Lemma 5.3. □

By Lemma 5.3 and because $X^{(i)}$ is non-singular near $D^{(i)}$, on each step we have one of the following possibilities:

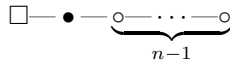
5.7.1. — $\zeta(X^{(i+1)}, D^{(i+1)}) = \zeta(X^{(i)}, D^{(i)})$, $E^{(i)} \subset D^{(i)}$, and φ_i is the usual blowup of a singular point of $D^{(i)}$;

5.7.2. — $\zeta(X^{(i+1)}, D^{(i+1)}) = \zeta(X^{(i)}, D^{(i)}) - 1$, and $E^{(i)} \not\subset D^{(i)}$.

COROLLARY 5.8. — Suppose that we are in the case 5.7.1 above. Furthermore suppose that $X^{(i+1)}$ admit an action of a connected algebraic group G so that the boundary $D^{(i+1)}$ is G -invariant. Then the action lifts to $X^{(i)}$ so that $D^{(i)}$ is G -invariant.

COROLLARY 5.9. — Suppose that we are in the case 5.7.2 above. Furthermore suppose that $(X^{(i+1)}, D^{(i+1)})$ is a toric surface. Then the action of some one-dimensional subtorus \mathcal{T} lifts to $X^{(i)}$ so that $D^{(i)}$ is \mathcal{T} -invariant.

Proof. — Since $E^{(i)} \cdot D^{(i)} = 1$ and $E^{(i)} \not\subset D^{(i)}$, the curve $E^{(i)}$ meets only one component $D_0^{(i)} \subset D^{(i)}$ so that $E^{(i)} \cdot D_0^{(i)} = 1$. Let $\pi : \bar{X}^{(i)} \rightarrow X^{(i)}$ be the minimal resolution near $E^{(i)}$. By Lemma 2.13 the dual graph of $\bar{X}^{(i)}$ has the following form:



where ● corresponds to $E^{(i)}$, □ corresponds to $D_0^{(i)}$, and the vertices ○ correspond to π -exceptional (-2) -curves. Thus $\bar{X}^{(i)}$ is obtained from $X^{(i+1)}$ by making successive blowups of a fixed point on the proper transform of $D_0^{(i)}$. The stabilizer of this point is a one-dimensional subtorus in the big torus acting on $X^{(i+1)}$. □

At the end of our MMP we get a log surface (Y, D_Y) admitting a fiber type extremal D_Y -positive contraction $h : Y \rightarrow Z$. Moreover,

$$(5.10) \quad \varsigma(Y, D_Y) \leq \varsigma \leq 1.$$

Recall that Y is non-singular near D_Y . In particular, all the component of D_Y are Cartier divisors. Moreover, $K_Y + D_Y \sim 0$ and Y has at worst Du Val singularities outside of D_Y .

First we consider the case where Z is a point. Then Y is a del Pezzo surface with at worst Du Val singularities and $\text{Pic}(Y) \simeq \mathbb{Z}$. In particular, $\varrho(Y, D_Y) = 1$. Since $\|D_Y\| \geq 2$, the divisor $-K_Y$ is not a primitive element of $\text{Pic}(Y)$. In this case, Y is either a projective plane \mathbb{P}^2 or a singular quadric $\mathbb{P}(1, 1, 2)$ (see e.g. [12, Lemma 6]). Moreover we have one of the following:

5.11.1. — $Y \simeq \mathbb{P}^2, D_Y = D_1 + D_2 + D_3$, where D_i are lines in general position, $\varsigma(Y, D_Y) = 0$;

5.11.2. — $Y \simeq \mathbb{P}^2, D_Y = D_1 + D_2$, where D_1 is a line and D_2 is a conic meeting D_1 transversely at two distinct points, $\varsigma(Y, D_Y) = 1$;

5.11.3. — $Y \simeq \mathbb{P}(1, 1, 2), D_Y = D_1 + D_2$, where the class of D_i generates $\text{Pic}(Y)$ and again D_1 and D_2 meet each other transversely at two distinct points, $\varsigma(Y, D_Y) = 1$.

CLAIM. — *The pair (Y, D_Y) is toric in the case 5.11.1 and admits an effective \mathbb{C}^* -action in cases 5.11.2 and 5.11.3.*

Proof. — Modulo change of coordinates x, y, z in \mathbb{P}^2 or $\mathbb{P}(1, 1, 2)$ we have

$$5.11.1 \implies D_Y = \{xyz = 0\},$$

$$5.11.2 \implies D_Y = \{(xy - z^2)z = 0\},$$

$$5.11.3 \implies D_Y = \{(xy - z)z = 0\}.$$

Then the statement of the claim is an easy exercise. □

In all cases 5.11.1–5.11.3 we have $\varsigma(Y, D_Y) \geq 0$. This proves (i) of Theorem 5.1. Moreover, if $\varsigma(X, D) = 0$, then we are in the case 5.11.1. In particular, (Y, D_Y) is a toric surface. Since $\varsigma(Y, D_Y) = \varsigma(X', D') = 0$, on each step of our MMP we have the possibility 5.7.1. By Corollary 5.8 both (X', D') and (X, D) are toric.

Now assume that $\varsigma(X, D) = 1$. If moreover $\varsigma(Y, D_Y) = 1$, then (Y, D_Y) is of type 5.11.2 or 5.11.3 and each step of our MMP is of type 5.7.1. By Corollary 5.8 the action of the corresponding one-dimensional torus lifts to X' . Hence (X', D') and (X, D) admit effective \mathbb{C}^* -actions. Finally assume that $\varsigma(X, D) = 1$ and $\varsigma(Y, D_Y) = 0$. Then (Y, D_Y) is toric and all but one

steps of our MMP are of type 5.7.1. As above we can apply Corollaries 5.8 and 5.9 to conclude that (X, D) admits an effective \mathbb{C}^* -action.

Now consider the case where Z is a curve. Then Z is smooth and $h : Y \rightarrow Z$ is a rational curve fibration with $\text{Pic}(Y/Z) \simeq \mathbb{Z}$. For a general fiber F we have

$$D_Y \cdot F = -K_Y \cdot F = 2.$$

Let D_0 be a h -horizontal component of D_Y . We claim that D_0 is a section. Indeed, assume that D_0 is a double section. Then by the adjunction formula

$$D_0 \cdot (D_Y - D_0) = -D_0 \cdot (K_Y + D_0) = -\text{deg } K_{D_0} \leq 2.$$

Since $\|D_Y\| \geq 3$ and D_Y is a simple normal crossing divisor, it has at least two vertical components D_i with $D_0 \cdot D_i = 2$. Thus $D_0 \cdot (D_Y - D_0) \geq 4$, a contradiction.

Hence, D_0 is a section. Then D_Y has another h -vertical component D_1 which is also a section of h . Since D_0 is a Cartier divisor, $h : Y \rightarrow Z$ is a smooth \mathbb{P}^1 -fibration. If D_0 is not a rational curve, then as above by adjunction $\text{deg } K_{D_1} = 0$ and D_0 is disjoint from $D_Y - D_0$. On the other hand, $D_Y - D_0$ has at least one h -vertical component, a contradiction. Hence, D_0 is a smooth rational curve and Y is a Hirzebruch surface \mathbb{F}_e , $e \neq 1$. Let Σ be the minimal section of \mathbb{F}_e and let F be a fiber. Since $D_i \cdot (D_Y - D_i) = -\text{deg } K_{D_i} = 2$ for each component $D_i \subset D_Y$, we have one of the following possibilities (up to permutation of D_0 and D_1):

5.12.1. — $D_0 \cdot D_1 = 0$, $\|D_Y\| = 4$, $D_0 = \Sigma$, $D_1 \sim \Sigma + eF$, D_2 and D_3 are distinct fibers, $\zeta(Y, D_Y) = 0$;

5.12.2. — $D_0 \cdot D_1 = 1$, $\|D_Y\| = 3$, $D_0 = \Sigma$, $D_1 \sim \Sigma + (e + 1)F$, D_2 is a fiber, $\zeta(Y, D_Y) = 1$.

Then we can complete the proof similar to what we did page 383 (with 5.11.1–5.11.3) by using the following.

CLAIM. — *The pair (Y, D_Y) is toric in the case 5.12.1 and admits an effective \mathbb{C}^* -action in the case 5.12.2.*

Proof. — The statement is obvious in the case $e = 0$, so we assume that $e \geq 2$. Let $\pi : (Y, D_Y) \rightarrow (Y', D'_Y)$ be the contraction of the negative section. Then Y' is the weighted projective plane $\mathbb{P}(1, 1, e)$. We may assume that in suitable orbifold coordinates x, y, z the boundary D'_Y is given by the equation $xyf_e(x, y, z) = 0$ (resp. $xf_{e+1}(x, y, z) = 0$) in the case 5.12.1 (resp. 5.12.2), where $f_d(x, y, z)$ denotes some polynomial of weighted degree d .

In the case 5.12.1, since D'_Y is a simple normal crossing divisor outside of the origin $(0 : 0 : 1)$ and D'_1 does not pass through $(0 : 0 : 1)$, the polynomial $f_e(x, y, z)$ contains z . Then by a coordinate change we get $f_e(x, y, z) = z$. Hence (Y', D'_Y) is toric. Since π is the minimal resolution, the torus action lifts to Y .

Similarly, in the case 5.12.2 $f_{e+1}(x, y, z)$ contains zy (because (Y', D'_Y) is lc). By a coordinate change we get $f_{e+1}(x, y, z) = zy + x^{e+1}$. Then (Y', D'_Y) admits an \mathbb{C}^* -action $(x, y, z) \mapsto (x, \lambda y, \lambda^{-1}z)$. □

6. Proof of main theorems

Now Theorem 1.1 is a consequence of the following.

PROPOSITION 6.1. — *Let X be a projective normal surface and let $s(X)$ be the number of its points where K_X is not Cartier. Assume that X has a 1-complement $D \in |-K_X|$. Then*

- (i) $s(X) \leq \varrho(X) + 2$,
- (ii) if $s(X) = \varrho(X) + 2$, then X is toric,
- (iii) if $s(X) = \varrho(X) + 1$, then X admits an effective \mathbb{C}^* -action.

Proof. — By the classification of log canonical singularities of pairs [10, Thm. 4.15], D is a nodal curve, and, at each singularity $P \in X$, either $D = 0$ and $P \in X$ is a Gorenstein log canonical singularity, or the pair $P \in (X, D)$ is locally analytically isomorphic to the pair $(\frac{1}{n}(1, a), (uv = 0))$ for some n and a . Moreover D has arithmetic genus 1 because $2p_a(D) - 2 = (K_X + D) \cdot D = 0$ (note that the adjunction formula holds because $K_X + D$ is Cartier [9, 16.4.3]). Thus D is either a smooth elliptic curve, or a rational curve with a node, or a cycle of smooth rational curves.

Let s' be the number of singular points of X lying on D . Then

$$\# \text{Sing}(D) \geq s' \geq s(X).$$

By the above $\# \text{Sing}(D) = \|D\|$. Then the assertion follows from Theorem 5.1. □

The proof of Theorem 1.2 is essentially the same as the proof of [6, Theorem 1.3].

7. Examples

A natural way to produce examples of del Pezzo surfaces as in (iii) of Theorem 1.1 is to use deformations:

THEOREM [6, Prop. 3.1] 7.1. — *Let X be a projective surface such that X has only T-singularities and $-K_X$ is nef and big. Then there are no local-to-global obstructions to deformations of X .*

Thus we can start with some known examples and construct new ones by deforming their singularities. The behavior of the Picard number is described by Noether’s formula 2.5 and by the following

PROPOSITION [6, Prop. 2.3] 7.2. — *Let $(P \in \mathcal{X})/(0 \in S)$ be a \mathbb{Q} -Gorenstein deformation of a T-singularity $P \in X$ of type $\frac{1}{dn^2}(1, dna - 1)$ and let P_1, \dots, P_l be all the singular points of a fiber \mathcal{X}_s , $s \in S$. Then the possible types of $P_1, \dots, P_l \in \mathcal{X}_s$ are as follows:*

- a) $A_{d_1-1}, \dots, A_{d_l-1}$ or
- b) $\frac{1}{d_1 n^2}(1, d_1 n a - 1), A_{d_2-1}, \dots, A_{d_l-1}$,

where d_1, \dots, d_l is a partition of d .

Remark 7.3. — In the above situation, the case $\text{Sing}(\mathcal{X}_s) = \emptyset$ is not excluded. This is possible only if $d = 1$ and in this case we put $l = 1$.

COROLLARY 7.4. — *Let X be a projective surface with T-singularities and let $\mathcal{X}/(0 \in S)$ be a \mathbb{Q} -Gorenstein deformation induced by a local deformation of one point $P \in X$. Then, in the notation of 7.2, for a general fiber \mathcal{X}_s , $s \in S$ we have*

$$\rho(\mathcal{X}_s) - \rho(X) = l - 1,$$

Now we can take one of the toric surfaces with T-singularities and $\rho(X) = 1$ described in [6, §4] and deform it in a suitable way.

Example 7.5. — Take the weighted projective plane $X := \mathbb{P}(a^2, b^2, 5c^2)$, where a, b, c , are given by the following Markov-type equation

$$a^2 + b^2 + 5c^2 = 5abc$$

(cf. [7]). Then X has three singular points which are of type T and $K_X^2 = 5$. More precisely,

$$\text{Sing}(X) = \left\{ \frac{1}{a^2}(b^2, 5c^2), \frac{1}{b^2}(a^2, 5c^2), \frac{1}{5c^2}(a^2, b^2) \right\}$$

For the third point we have

$$\frac{1}{5c^2}(a^2, b^2) = \frac{1}{5c^2}(1, 5c\alpha - 1),$$

where $\alpha = ab\delta$ and δ is taken so that $a^2\delta \equiv 1 \pmod{5c^2}$. Thus deforming this point to one of the following collection of singularities

- $\frac{1}{c^2}(1, c\alpha - 1), A_3;$

- $\frac{1}{2c^2}(1, 2c\alpha - 1), A_2;$
- $\frac{1}{3c^2}(1, 3c\alpha - 1), A_1;$
- $\frac{1}{4c^2}(1, 4c\alpha - 1),$

we get examples of del Pezzo surfaces as in (iii) of Theorem 1.1 with $K_X^2 = 5$, $\rho(X) = 2$, $s(X) = 3$.

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