ANNALES

DE

L'INSTITUT FOURIER

Jan-Li LIN & Elizabeth WULCAN

Stabilization of monomial maps in higher codimension

Tome 64, nº 5 (2014), p. 2127-2146.

<http://aif.cedram.org/item?id=AIF_2014__64_5_2127_0>

Ann. Inst. Fourier, Grenoble
64, 5 (2014) 2127-2146

STABILIZATION OF MONOMIAL MAPS IN HIGHER CODIMENSION

by Jan-Li LIN & Elizabeth WULCAN

ABSTRACT. — A monomial self-map f on a complex toric variety is said to be k-stable if the action induced on the $2k$-cohomology is compatible with iteration. We show that under suitable conditions on the eigenvalues of the matrix of exponents of f, we can find a toric model with at worst quotient singularities where f is k-stable. If f is replaced by an iterate one can find a k-stable model as soon as the dynamical degrees λ_k of f satisfy $\lambda_k^2 > \lambda_{k-1}\lambda_{k+1}$. On the other hand, we give examples of monomial maps f, where this condition is not satisfied and where the degree sequences $\deg_k(f^n)$ do not satisfy any linear recurrence. It follows that such an f is not k-stable on any toric model with at worst quotient singularities.

RÉSUMÉ. — Une application monomiale f d'une variété torique complexe dans elle-même est dite k-stable si l'action induite sur le $2k$-ème groupe de cohomologie est compatible avec l'itération. Nous démontrons que sous des conditions appropriées sur les valeurs propres de la matrice des exposants associés de f, il existe un modèle torique à singularités quotients pour laquelle f est k-stable. De plus, si l'on remplace f par une de ses itérés, l'existence d'un modèle torique k-stable pour f est garantie dès lors que les degrés dynamiques de f satisfont la condition $\lambda_k^2 > \lambda_{k-1}\lambda_{k+1}$. Par ailleurs, nous donnons des exemples d'applications monomiales f pour lesquelles cette condition n'est pas satisfaite, et dont la suite de degrés $\deg_k(f^n)$ ne satisfait aucune condition de récurrence linéaire. Il en résulte qu'une telle application f ne peut être k-stable pour aucune modèle torique à singularités quotients.

Introduction

When studying the dynamics of a dominant meromorphic self-map $f\colon X \dashrightarrow X$ on a compact complex manifold X it is often desirable that the action of f on the cohomology of X be compatible with iterations. Following Sibony [25] and Dinh-Sibony [7] (see also [11]) we then say that f is *(algebraically) stable*. More precisely, if f^* denotes the induced action on

Keywords: Algebraic stability, monomial maps, degree growth.
Math. classification: 14M25, 37F10.

$H^{2k}(X)$ we say that f is *k-stable* if $(f^n)^* = (f^*)^n$ for all n. For examples of classes of k-stable maps see, *e.g.* [5, 7].

If f is not stable, one can look for a model X', birational to X, such that the induced self-map on X' is stable. As shown by Favre [8] this is not always possible to achieve. However, for large classes of surface maps and monomial maps, one can find models $X' \to X$ (with at worst quotient singularities), so that f lifts to a 1-stable map, see [4, 8, 9, 17, 18, 19].

In this paper we address the question of finding a k-stable model for the special class of monomial maps, but for arbitrary k. Monomial maps on complex projective space $\mathbf{P}^m$, or more generally, on toric varieties, correspond to integer-valued $m \times m$-matrices, $M_m(\mathbf{Z})$. For $A \in M_m(\mathbf{Z})$ with entries a_{ij} we write f_A for the corresponding monomial map

$$f_A(z_1, \ldots, z_m) = (z_1^{a_{11}} \cdots z_m^{a_{m1}}, \ldots, z_1^{a_{1m}} \cdots z_m^{a_{mm}})$$

with $(z_1, \ldots, z_m) \in (\mathbf{C}^*)^m$. This mapping is holomorphic on the torus $(\mathbf{C}^*)^m$ and extends as a rational map to $\mathbf{P}^m$ or to any toric variety. It is dominant precisely if $\det A \neq 0$. Note that $f_A^\ell = f_{A^\ell}$.

Theorem A. — *Assume that the eigenvalues of $A \in M_m(\mathbf{Z})$ are real and satisfy $\mu_1 > \ldots > \mu_m > 0$ or $\mu_1 < \ldots < \mu_m < 0$. Then there is a projective toric variety X, with at worst quotient singularities, such that $f_A \colon X \dashrightarrow X$ is k-stable for $k = 1, \ldots, m-1$.*

The definition of k-stable extends verbatim to toric varities with at worst quotient singularities, *cf.* Section 2.

If the eigenvalues of A only satisfy $|\mu_1| > \ldots > |\mu_m| > 0$, it is not always possible to find a stable model, see, *e.g.* [17, Example 6.3]. Still, since A^2 has positive and distinct eigenvalues, by Theorem A we can find a model so that f_A^2 becomes stable. In fact, there is an ℓ_0 such that f^ℓ is k-stable for $\ell \geqslant \ell_0$ and each k.

Theorem B. — *Assume that the eigenvalues of $A \in M_m(\mathbf{Z})$, ordered so that $|\mu_1| \geqslant \ldots \geqslant |\mu_m|$, satisfy $|\mu_{k_j}| > |\mu_{k_j+1}|$ for $j = 1, \ldots, s$ and $|\mu_m| > 0$. Then there is a projective toric variety X, with at worst quotient singularities, and $\ell_0 \in \mathbf{N}$, such that $f_A^\ell \colon X \dashrightarrow X$ is k_j-stable for $\ell \geqslant \ell_0$ and $j = 1, \ldots, s$.*

Recall that the *kth degree* $\deg_k(f)$ of the rational self-map $f \colon \mathbf{P}^m \dashrightarrow \mathbf{P}^m$ is defined as $\deg f^{-1}(L_k)$ where L_k is a generic linear subspace of $\mathbf{P}^m$ of codimension k. In [10, 20] it was proved that the *kth dynamical degree*

$$\lambda_k = \lambda_k(f_A) := \lim_n (\deg_k(f_A^n))^{1/n}$$

of f_A, introduced by Russakovskii-Shiffman [24], is equal to $|\mu_1|\cdots|\mu_k|$, if the eigenvalues of A are ordered so that $|\mu_1| \geqslant \ldots \geqslant |\mu_m|$. It follows that the condition $|\mu_k| > |\mu_{k+1}|$ is equivalent to $\lambda_k^2 > \lambda_{k-1}\lambda_{k+1}$. In general the dynamical degrees satisfy $\lambda_k^2 \geqslant \lambda_{k-1}\lambda_{k+1}$; for this and other basic properties of dynamical degrees, see, *e.g.* [6, 14, 24]. Thus, in particular, Theorem B says that if we are only interested in the action of f_A^* on $H^{2k}(X)$, we can find good models as soon as $\lambda_k^2 > \lambda_{k-1}\lambda_{k+1}$. One could ask if this is true for a general meromorphic map $f\colon X \dashrightarrow X$. Is it always possible to find a model X' birational to X so that $f^\ell\colon X' \dashrightarrow X'$ is k-stable for ℓ large enough when $\lambda_k^2 > \lambda_{k-1}\lambda_{k+1}$?

The problem of finding stable models for f is related to the question whether the degree sequence $\deg_k(f^n)$ satisfies a linear recurrence.

THEOREM C. — *Assume that $1 \leqslant k \leqslant m-1$ and that the eigenvalues of $A \in M_m(\mathbf{Z})$ satisfy*

$$|\mu_{k-1}| > |\mu_k| = |\mu_{k+1}| > |\mu_{k+2}| \tag{0.1}$$

and moreover that μ_k/μ_{k+1} is not a root of unity. Then the degree sequence $\deg_k(f_A^n)$ does not satisfy any linear recurrence.

If $k = 1$ or $k = m-1$ the condition (0.1) on the moduli of the μ_j should be interpreted as $|\mu_1| = |\mu_2| > |\mu_3|$ and $|\mu_{m-2}| > |\mu_{m-1}| = |\mu_m|$, respectively.

COROLLARY D. — *Assume that $1 \leqslant k \leqslant m-1$ and that $A \in M_m(\mathbf{Z})$ satisfies the assumption of Theorem C. Then for each toric projective variety X with at worst quotient singularities, $f_A\colon X \dashrightarrow X$ is not k-stable.*

In fact, Corollary D follows from a slight generalization of Theorem C, Theorem C', which asserts that $\deg_{D,k}(f_A^n)$ does not satisfy any linear recurrence, where $\deg_{D,k}(f_A)$ is the kth degree of f_A on a projective toric variety X with respect to the ample divisor D on X, see Section 6.

Note that if A satisfies the assumption of Theorem C, then all powers A^ℓ of A satisfy the assumption as well. Thus we get that for each X as in Corollary D and each $\ell \in \mathbf{N}$, $f_A^\ell\colon X \dashrightarrow X$ is not k-stable.

It would be interesting to investigate whether one can remove the conditions $|\mu_{k-1}| > |\mu_k|$ and $|\mu_{k+1}| > |\mu_{k+2}|$. Is it true that f_A cannot be made k-stable as soon as $|\mu_k| = |\mu_{k+1}|$ and μ_k/μ_{k+1} is not a root of unity?

For $m = 2$, Theorems A and B follow from [8] and for $m = 3$ they follow from [18, Theorem 1.1]. Moreover, for $k = 1$ Theorem A follows from [17, Theorem A]. In fact, if f_A is a monomial map on a toric variety X, under the assumption $\mu_1 > \ldots > \mu_m > 0$ one can find a birational modification $\pi\colon X' \to X$, with at worst quotient singularities, such that the lifted mapping $\pi^{-1} \circ f_A \circ \pi\colon X' \dashrightarrow X'$ is 1-stable.

Geometrically, $f\colon X \dashrightarrow X$ is 1-stable if no iterate of f sends a hypersurface into the indeterminacy set of f, see [11, 25]. If $f = f_A$ is monomial and X is toric this translates into a certain condition in terms of the action of A on the fan of X, see [19, Section 4] and [17, Section 2.4]. The construction of a 1-stable model $X' \to X$ amounts to carefully refining the fan corresponding to X. A model X' that is only birationally equivalent to X can be obtained in a much less technical way and also for a larger class of monomial mappings; for $s = 1$ and $k_1 = 1$, Theorem B appeared in [19, Theorem 4.7] and [17, Theorem B'], *cf.* Remark 5.2.

For $k \geqslant 2$ we do not in general understand what it means geometrically to be k-stable, nor if there is a translation into the language of fans for monomial maps. For sufficient conditions to be k-stable, see *e.g.* [7]. In this paper we consider a certain class of toric varieties, where the action of f_A^* is particularly simple. Given a basis $\epsilon_1, \ldots, \epsilon_m$ of $\mathbf{Q}^m$ we construct a toric variety X, see Section 3, for which the entries of the matrix of $f_A^*\colon H^{2k}(X) \to H^{2k}(X)$ are the absolute values of the $k \times k$-minors of A in the basis ϵ_j (modulo multiplication by a positive constant). It turns out that a sufficient condition for f_A to be k-stable is that all $k \times k$-minors have the same sign, see Lemma 3.2 and Remark 3.3.

The basic idea of the proofs of Theorems A and B is to find bases ϵ_j so that this condition is satisfied. The construction will be based on (strictly) totally positive matrices, *i.e.* matrices whose minors are all (strictly) positive. Typical examples of totally positive matrices are certain Vandermonde matrices.

Corollary D is due to Favre [8] for $m = 2$; he showed that if $|\mu_1| = |\mu_2|$ and μ_1/μ_2 is not a root of unity there is no model such that (any power of) f_A is stable. Bedford-Kim [2] proved Theorem C for $k = 1$ and some cases when $k > 1$, see also [19, Theorem 4.7]. Following ideas due to Hasselblatt-Propp [15] and Bedford-Kim [2], we prove Theorem C by comparing the degree sequence $\deg_k(f_A^n)$ to a certain other sequence β_n, which satisfies a linear recurrence. If $\deg_k(f_A^n)$ satisfied a linear recurrence the set of n for which $\deg_k(f_A^n) = \beta_n$ would be eventually periodic, which we show cannot be the case. To do this we express $\deg_k(f_A^n)$ in terms of minors of A^n using a result from [10], which expresses $\deg_k(f_A)$ as a mixed volume of certain polytopes, and a method due to Huber-Sturmfels [16] of computing mixed volumes of polytopes.

Acknowledgment. — We thank Eric Bedford, Mattias Jonsson, and Pavlo Pylyavskyy for fruitful discussions. We also thank the referee for many helpful suggestions. Part of this work was done while the authors were visiting the Institute for Computational and Experimental Research in Mathematics (ICERM); we would like to thank the ICERM for their hospitality. The second author was supported by the Swedish Research Council.

1. Toric varieties

A complex toric variety is a (partial) compactification of the torus $T \cong (\mathbf{C}^*)^m$, which contains T as a dense subset and which admits an action of T that extends the natural action of T on itself. We briefly recall some of the basic definitions, referring to [12] and [22] for details.

1.1. Fans and toric varieties

Let N be a lattice isomorphic to $\mathbf{Z}^m$ and let $M = \mathrm{Hom}(N, \mathbf{Z})$ denote the dual lattice. Set $N_{\mathbf{Q}} := N \otimes_{\mathbf{Z}} \mathbf{Q}$, $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$, and define $M_{\mathbf{Q}}$ and $M_{\mathbf{R}}$ analogously. Let $\mathbf{R}_+$ and $\mathbf{R}_-$ denote the sets of non-negative and non-positive numbers, respectively.

A *cone* σ in $N_{\mathbf{R}}$ is a set that is closed under positive scaling. If σ is convex and does not contain any line in $N_{\mathbf{R}}$, it is said to be *strictly convex*. If σ is of the form $\sigma = \sum \mathbf{R}_+ v_i$ for some $v_i \in N$, we say that σ is a *convex rational cone generated* by the vectors v_i. A *face* of σ is the intersection of σ and a *supporting hyperplane*, *i.e.* a hyperplane through the origin such that the whole cone σ is contained in one of the closed half-spaces determined by the hyperplane. The *dimension* of σ is the dimension of the linear space $\mathbf{R}\sigma$ spanned by σ. One-dimensional faces of σ are called *edges* and one-dimensional cones are called *rays*. Given a ray σ, the associated *primitive vector* is the first non-zero lattice point met along σ. The *multiplicity* $\mathrm{mult}(\sigma)$ of σ is the index of the lattice generated by the primitive elements of the edges of σ in the lattice generated by σ. A k-dimensional cone is *simplicial* if it can be generated by k vectors. A cone is *regular* if it is simplicial and of multiplicity one.

A *fan* Δ in N is a finite collection of rational strongly convex cones in $N_{\mathbf{R}}$ such that each face of a cone in Δ is also a cone in Δ and, moreover, the intersection of two cones in Δ is a face of both of them. Let Δ_k denote the set of cones in Δ of dimension k. The fan Δ is said to be *complete* if the union of all cones in Δ equals $N_{\mathbf{R}}$. If all cones in Δ are simplicial then Δ is said to be *simplicial*, and if all cones are regular, Δ is said to be *regular*. A fan $\widetilde{\Delta}$ is a *refinement* of Δ if each cone in Δ is a union of cones in $\widetilde{\Delta}$.

A fan Δ determines a toric variety $X(\Delta)$ obtained by patching together affine toric varieties U_σ corresponding to the cones $\sigma \in \Delta$. It is compact if and only if Δ is complete. Toric varieties are normal and Cohen-Macaulay. The variety $X(\Delta)$ is nonsingular if and only if Δ is regular. Moreover, $X(\Delta)$ has at worst quotient singularities, *i.e.* it is locally the quotient of a smooth variety by the action of a finite group, if and only if Δ is simplicial, see *e.g.* [12, Section 2.2]. In this case, we will also say that the variety $X(\Delta)$ is *simplicial*. For any fan Δ in N there is a fan $\widetilde{\Delta}$ that refines Δ and such that $X(\widetilde{\Delta}) \to X(\Delta)$ is a resolution of singularities.

1.2. Cohomology of toric varieties and piecewise linear functions

Let Δ be a simplicial complete fan. Then the odd cohomology groups of $X := X(\Delta)$ vanish and the even cohomology groups are generated by varieties invariant under the action of T. More precisely $H^{2k}(X) := H^{2k}(X;\mathbf{R})$ is generated by T-invariant varieties of codimension k. There is a Hodge decomposition $H^k(X) \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{p+q=k} H^{p,q}(X)$ of the cohomology groups of X and, moreover, $H^{p,q}(X) = 0$ if $p \neq q$, see, *e.g.* [3, Proposition 12.11] and [23, Chapter 2.5]. In particular,

$$H^{2k}(X) = H^{k,k}(X;\mathbf{R}) := H^{k,k}(X) \cap H^{2k}(X;\mathbf{R}).$$

Each cone $\sigma \in \Delta_k$ determines an irreducible subvariety $V(\sigma)$ of X of codimension k that is invariant under the action of T. If we use $[V(\sigma)]$ to denote the class of $V(\sigma)$ in $H^{2k}(X)$, then the classes $[V(\sigma)]$, as σ runs through all cones of codimension k, generate $H^{2k}(X)$. In particular, each ray ρ in Δ determines a T-invariant prime Weil divisor $D(\rho)$ and these divisors generate $H^2(X)$. Since Δ is simplicial,

$$\frac{1}{\mathrm{mult}(\sigma)}[V(\sigma)] = \prod [D(\rho_i)]$$

in $H^*(X)$, where ρ_i are the edges of σ, *i.e.* the $[D(\rho_i)]$ genererate $H^*(X)$ as an $\mathbf{R}$-algebra.

Let $\mathrm{PL}(\Delta)$ be the set of all continuous functions $h\colon \bigcup_{\sigma\in\Delta} \sigma \to \mathbf{R}$ that are *piecewise linear with respect to* Δ, *i.e.* for each cone $\sigma \in \Delta$ there exists $m = m(\sigma) \in M$ with $h|_\sigma = m$. A function in $\mathrm{PL}(\Delta)$ is said to be *strictly convex* if it is convex and defined by different elements $m(\sigma)$ for different cones $\sigma \in \Delta_m$. A compact toric variety $X(\Delta)$ is projective if and only if there is a strictly convex $h \in \mathrm{PL}(\Delta)$. We then say that Δ is *projective*.

Functions in $\mathrm{PL}(\Delta)$ are in one-to-one correspondence with T-invariant Cartier divisors. If D is a T-invariant Cartier divisor of the form $D = \sum a_i D(\rho_i)$, then the corresponding function $h_D \in \mathrm{PL}(\Delta)$ is determined by $h_D(v_i) = a_i$ if v_i is a primitive vector for ρ_i. Conversely $h \in \mathrm{PL}(\Delta)$ determines the Cartier divisor $D(h) := \sum h(v_i)D(\rho_i)$. Given $h_1, h_2 \in \mathrm{PL}(\Delta)$, the corresponding divisors are linearly equivalent if and only if $h_1 - h_2$ is linear. The function h_D is strictly convex if and only if D is ample.

A function $h \in \mathrm{PL}(\Delta)$ determines a polyhedron

$$P(h) := \{m \in M_{\mathbf{R}},\, m \leqslant h\} \subset M_{\mathbf{R}};$$

in particular,

$$P_D := P(h_D) = \{m \in M_{\mathbf{R}},\, m(v_i) \leqslant a_i\}.$$

If h is convex, then $P(h)$ is a compact *lattice polytope* in $M_{\mathbf{R}}$, *i.e.* it is the convex hull of finitely many points in the lattice M. Conversely, if $P \subset M_{\mathbf{R}}$ is a lattice polytope, then the function

$$h_P(u) := \sup\{m(u),\, m \in P\} \tag{1.1}$$

is a piecewise linear convex function on $N_{\mathbf{R}}$. If $h_P \in \mathrm{PL}(\Delta)$ then Δ is said to be *compatible* with P. We write D_P for the corresponding divisor on $X(\Delta)$.

1.3. Mixed volume and intersection of divisors

Given any finite collection of convex compact sets $K_1, \dots, K_s \subset M_{\mathbf{R}}$, we let $K_1 + \cdots + K_s$ denote the *Minkowski sum*

$$K_1 + \cdots + K_s := \{x_1 + \cdots + x_s \mid x_j \in K_j\},$$

and for $r \in \mathbf{R}_+$, we write $rK_j := \{rx \mid x \in K_j\}$. Let Vol be the Lebesque measure on $M_{\mathbf{R}} \cong \mathbf{R}^m$ normalized so that the parallelepiped

$$Q_e := \Big\{\textstyle\sum_{j=1}^m a_j e_j \mid 0 \leqslant a_j \leqslant 1\Big\},$$

spanned by a basis $e_1, \dots, e_m$ of M, has volume 1.

A theorem by Minkowski and Steiner asserts that $\mathrm{Vol}(r_1K_1 + \cdots + r_sK_s)$ is a homogeneous polynomial of degree m in the variables $r_1, \dots, r_s \in \mathbf{R}$. In particular, there is a unique expansion:

$$\begin{aligned} &\mathrm{Vol}\,(r_1K_1 + \cdots + r_sK_s) \\ &\quad = \sum_{k_1+\cdots+k_s=m} \binom{m}{k_1, \dots, k_s} \mathrm{Vol}\,(K_1[k_1], \dots, K_s[k_s])\; r_1^{k_1} \cdots r_s^{k_s}; \end{aligned} \tag{1.2}$$

the coefficients $\mathrm{Vol}(K_1[k_1], \dots, K_s[k_s]) \in \mathbf{R}$ are called *mixed volumes*. Here the notation $K_j[k_j]$ refers to the repetition of K_j k_j times.

Example 1.1. — Pick $u_1, \dots, u_m \in M_{\mathbf{R}}$ and let P_j be the line segments $[0, u_j] \subset M_{\mathbf{R}}$. Then $r_1 P_1 + \dots + r_m P_m$ is the parallelepiped Q_{ru}, where ru denotes the tuple $r_1 u_1, \dots, r_m u_m$, and so

$$\mathrm{Vol}(r_1 P_1 + \dots + r_m P_m) = r_1 \cdots r_m \, \mathrm{Vol}(Q_u).$$

Hence $\mathrm{Vol}(P_1, \dots, P_m) = \mathrm{Vol}(Q_u)/m!$. Note that $\mathrm{Vol}(Q_u)$ is strictly positive if and only if the u_j are linearly independent.

If Δ is compatible with $P_1, \dots, P_s$, then the intersection product (*i.e.* the cup product for cohomology classes) of the corresponding divisor classes equals

$$[D_{P_1}]^{k_1} \cdots [D_{P_s}]^{k_s} = m! \, \mathrm{Vol}(P_1[k_1], \dots, P_s[k_s]) \tag{1.3}$$

if $k_1 + \dots + k_s = m$, see [22, p. 79].

2. Monomial maps

Given a group homomorphism $A \colon M \to M$, we will write A also for the induced linear maps $M_{\mathbf{Q}} \to M_{\mathbf{Q}}$ and $M_{\mathbf{R}} \to M_{\mathbf{R}}$. Moreover, we let $\check{A}$ denote the dual map $N \to N$, as well as the dual linear maps $N_{\mathbf{Q}} \to N_{\mathbf{Q}}$ and $N_{\mathbf{R}} \to N_{\mathbf{R}}$. It turns out to be convenient to use this notation rather than writing A for the map on N and $\check{A}$ for the map on M.

Let Δ be a fan in $N \cong \mathbf{Z}^m$. Then any group homomorphism $\check{A} \colon N \to N$ gives rise to a rational map $f_A \colon X(\Delta) \dashrightarrow X(\Delta)$, which is equivariant with respect to the action of T. Let $e_1, \dots, e_m$ be a basis of M and let $e_1^*, \dots, e_m^*$ be the dual basis of N. Then the dual map $A \colon M \to M$ is of the form $A = \sum a_{ij} e_i \otimes e_j^*$ for some $a_{ij} \in \mathbf{Z}$. If $z_1, \dots, z_m$ are the induced coordinates on T, then f_A is the monomial map

$$f_A(z_1, \dots, z_m) = (z_1^{a_{11}} \cdots z_m^{a_{m1}}, \dots, z_1^{a_{1m}} \cdots z_m^{a_{mm}})$$

restricted to T. Conversely, any rational, equivariant map $f \colon X(\Delta) \dashrightarrow X(\Delta)$ comes from a group homomorphism $N \to N$, see [22, p.19].

The map $f_A \colon X(\Delta) \dashrightarrow X(\Delta)$ is holomorphic precisely if $\check{A} \colon N_{\mathbf{R}} \to N_{\mathbf{R}}$ satisfies that for each $\sigma \in \Delta$ there is a $\sigma' \in \Delta$, such that $\check{A}(\sigma) \subseteq \sigma'$. Then $f_A^*[D(h)] = [D(h \circ \check{A})]$, see, *e.g.* [21, Chapter 6, Exercise 8], and, moreover, $P(h \circ \check{A}) = AP(h)$. Given a fan Δ and a group homomorphism $\check{A} \colon N \to N$ one can find a regular refinement $\widetilde{\Delta}$ of Δ such that the induced equivariant map $\tilde{f}_A \colon X(\widetilde{\Delta}) \to X(\Delta)$ is holomorphic. We denote by π the modification

$X(\widetilde{\Delta}) \to X(\Delta)$ induced by the identity map id: $N \to N$. Furthermore, we have the relation $\tilde{f}_A = f_A \circ \pi$, *i.e.* the following diagram commutes.

$$\begin{array}{ccc} & X(\widetilde{\Delta}) & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \tilde{f}_A} \\ X(\Delta) & \overset{}{\underset{f_A}{\dashrightarrow}} & X(\Delta) \end{array}$$

Now the pullback of a T-invariant Cartier divisor D under $f_A\colon X(\Delta) \dashrightarrow X(\Delta)$ is defined as $f_A^* D = \pi_* \tilde{f}_A^* D$; in fact, this definition does not depend on the particular choice of $\widetilde{\Delta}$. The divisor $f_A^* D$ is in general only $\mathbf{Q}$-Cartier, *cf.* [12, Chapter 3.3]. Note that, since $H^*(X(\Delta))$ is generated (as an algebra) by Cartier divisors, f_A induces an action f_A^* on $H^*(X(\Delta))$.

3. An important example

We will prove Theorems A and B by constructing toric varieties of a certain type. Throughout this paper we let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_\ell\}$ be strictly increasing multi-indices in $\{1, \dots, m\}$. If $|I| = |J| = k$, we let B_{IJ} denote the minor corresponding to the sub-matrix of B with rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$. Moreover, we write $[\ell]$ for the multi-index $\{1, \dots, \ell\}$ and I^C for the complement $[m] \smallsetminus I$ of I.

Pick linearly independent vectors $v_1, \dots, v_m \in N_{\mathbf{Q}}$ and let Δ be the fan

$$\Delta = \left\{ \textstyle\sum_{j=1}^m \mathbf{R}_+ \varepsilon_j v_j \right\}_{\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{0, -1, +1\}^m}.$$

In particular, the rays of Δ are of the form $\mathbf{R}_+ v_j$ and $\mathbf{R}_- v_j$. For simplicity we will assume that v_j is the primitive vector of the ray $\mathbf{R}_+ v_j$ for each j. Note that Δ is complete and simplicial and that there are strictly convex functions in $\mathrm{PL}(\Delta)$; hence the resulting toric variety $X(\Delta)$ is projective and has at worst quotient singularities. If the v_j form a basis of N, then $X(\Delta)$ is isomorphic to $(\mathbf{P}^1)^m$.

Note that the rays of Δ determine divisors $D_j := D(\mathbf{R}_- v_j)$ and $E_j := D(\mathbf{R}_+ v_j)$, such that E_j is linearly equivalent to D_j for each j. The polytope $P_j := P_{D_j}$ associated to the divisor D_j is the line segment in $M_{\mathbf{R}}$ with the origin and $u_j \in M_{\mathbf{R}}$ as endpoints, where u_j is the point in $M_{\mathbf{R}}$ such that $\langle v_i, u_j \rangle = \delta_{ij}$ (Kronecker's delta). Notice that the u_j, as vectors, are linearly independent. By [12, Section 5.2] $H^{2k}(X)$ will be generated by the intersection (cup) product of divisor classes:

$$[D_I] := [D_{i_1}] \cdots [D_{i_k}]$$

for $I=\{i_1,\dots,i_k\}\subseteq[m]$. In particular

$$f_A^*[D_I]=\sum_{|J|=k}\alpha_{IJ}[D_J]$$

for some coefficients α_{IJ}. From (1.3) we get

$$[D_I]\cdot[D_{J^C}]=\begin{cases} m!\operatorname{Vol}(P_1,\dots,P_m)>0 & \text{if } J=I\\ 0 & \text{otherwise}\end{cases}, \tag{3.1}$$

cf. Example 1.1. It follows that

$$f_A^*[D_I]\cdot[D_{J^C}]=\alpha_{IJ}\cdot m!\operatorname{Vol}(P_1,\dots,P_m)$$

On the other hand, for $I=\{i_1,\dots,i_k\}$ and $J^C=\{j_1,\dots,j_{m-k}\}$, by the projection formula [13, p. 325], we have

$$\begin{aligned} f_A^*[D_I]\cdot[D_{J^C}] &= \pi_*\tilde f_A^*[D_I]\cdot[D_{J^C}]\\ &= \tilde f_A^*[D_I]\cdot\pi^*[D_{J^C}]\\ &= \tilde f_A^*\big([D_{i_1}]\cdots[D_{i_k}]\big)\cdot\pi^*\big([D_{j_1}]\cdots[D_{j_{m-k}}]\big)\\ &= \tilde f_A^*[D_{i_1}]\cdots\tilde f_A^*[D_{i_k}]\cdot\pi^*[D_{j_1}]\cdots\pi^*[D_{j_{m-k}}], \end{aligned}$$

where the last step follows since $\tilde f_A$ and π are holomorphic. Recall from Section 2 that the polytopes associated to $\tilde f_A^*[D_i]$ and $\pi^*[D_j]$ are AP_i and $\operatorname{id}P_j=P_j$, respectively. Thus in light of (1.3),

$$\begin{aligned} &\tilde f_A^*[D_{i_1}]\cdots\tilde f_A^*[D_{i_k}]\cdot\pi^*[D_{j_1}]\cdots\pi^*[D_{j_{m-k}}]\\ &\qquad\qquad = m!\operatorname{Vol}(AP_{i_1},\dots,AP_{i_k},P_{j_1},\dots,P_{j_{m-k}}). \end{aligned}$$

Let $A_{IJ}=A_{IJ}(u_j)$ denote the minors of $A\colon M_{\mathbf R}\to M_{\mathbf R}$ with respect to the basis $u_1,\dots,u_m$. Then, in light of Example 1.1,

$$\operatorname{Vol}(AP_{i_1},\dots,AP_{i_k},P_{j_1},\dots,P_{j_{m-k}})=|A_{IJ}|\operatorname{Vol}(P_1,\dots,P_m).$$

To conclude, $\alpha_{IJ}=|A_{IJ}|$, and thus we have proved the following result.

Lemma 3.1. — *Let Δ be a fan of the form*

$$\Delta=\Big\{\sum_{j=1}^m \mathbf R_+\varepsilon_j v_j\Big\}_{\varepsilon\in\{0,-1,+1\}^m}.$$

Using the notation above,

$$f_A^*[D_I]=\sum_{|J|=k}|A_{IJ}|[D_J].$$

Hence

$$(f_A^*)^\ell[D_I] = \sum_{|J_1|=\ldots=|J_{\ell-1}|=|J|=k} |A_{IJ_1}||A_{J_1J_2}|\cdots|A_{J_{\ell-2}J_{\ell-1}}||A_{J_{\ell-1}J}|[D_J]$$

and

$$(f_A^\ell)^*[D_I] = (f_{A^\ell})^*[D_I] = \sum_{|J|=k} |A^\ell_{IJ}|[D_J],$$

where A^ℓ_{IJ} denotes the IJ-minor of A^ℓ. Recall the Cauchy-Binet formula:

$$(AB)_{IJ} = \sum_{|K|=k} A_{IK}B_{KJ}. \tag{3.2}$$

It follows that a sufficient condition for $(f_A^*)^\ell = (f_A^\ell)^*$ is that $A_{IJ} \geqslant 0$ for all I, J or $A_{IJ} \leqslant 0$ for all I, J. Let us summarize this:

Lemma 3.2. — *Let Δ be a fan of the form*

$$\Delta = \Big\{\sum_{j=1}^m \mathbf{R}_+\varepsilon_j v_j\Big\}_{\varepsilon\in\{0,-1,+1\}^m}.$$

Using the notation above, if $A_{IJ} \geqslant 0$ for all $I = \{i_1,\ldots,i_k\}, J = \{j_1,\ldots,j_k\} \subseteq [m]$ or if $A_{IJ} \leqslant 0$ for all I, J, then $f_A\colon X(\Delta) \dashrightarrow X(\Delta)$ is k-stable.

Remark 3.3. — One can also construct the fan Δ above starting from a basis $\epsilon_1,\ldots,\epsilon_m$ of $M_{\mathbf{Q}}$. For $j = 1,\ldots,m$, let V_j be the one-dimensional subspace of $N_{\mathbf{Q}}$ defined by

$$V_j = \big\{v \in N_{\mathbf{Q}} \mid \epsilon_\ell(v) = 0 \text{ for } \ell \neq j\big\}.$$

Then each V_j determines two rational rays in $N_{\mathbf{R}}$, which will be the rays of Δ; more precisely, pick v_j to be a primitive vector of one of the rays in V_j and construct Δ as above. Now the polytopes P_{D_j} and P_{E_j} will lie in the one-dimensional vector space spanned by ϵ_j. By the choice of v_j we can arrange so that u_j is a positive multiple of ϵ_j. Then the minor A_{IJ} of A in the basis u_j is just a positive constant times the IJ-minor $A_{IJ}(\epsilon_j)$ of A in the basis ϵ_j. More precisely, if $\epsilon_j = \alpha_j u_j$, then

$$A_{IJ} = \frac{\alpha_{i_1}\cdots\alpha_{i_k}}{\alpha_{j_1}\cdots\alpha_{j_k}} A_{IJ}(\epsilon_j).$$

Example 3.4. — Consider the monomial map

$$f_A(z_1,\ldots,z_m) = (z_1^{a_{11}}\cdots z_m^{a_{m1}},\ldots,z_1^{a_{1m}}\cdots z_m^{a_{mm}}).$$

If all $k\times k$-minors of the matrix (a_{ij}) are either nonnegative (or nonpositive), then $f_A\colon (\mathbf{P}^1)^n \dashrightarrow (\mathbf{P}^1)^n$ is k-stable. In particular, if (a_{ij}) is totally positive (or totally negative) f_A is stable on $(\mathbf{P}^1)^n$ for all k. Indeed, since (a_{ij}) is the matrix of the group homomorphism $A\colon M \to M$ associated with f_A with respect to the basis e_j of M, *cf.* Section 2, Lemma 3.2 implies that f_A is stable on

$$X\Big(\Big\{\sum_{j=1}^{m}\mathbf{R}_{+}\varepsilon_j e_j^*\Big\}_{\varepsilon\in\{0,-1,+1\}^m}\Big)=(\mathbf{P}^1)^n.$$

4. Proof of Theorem A

Given a basis $\xi_1,\dots,\xi_m$ of $M_{\mathbf{R}}$, and a linear transformation A, let $A(\xi_j)$ denote the matrix of A with respect to this basis.

Assume that A has distinct positive eigenvalues $\mu_1>\dots>\mu_m>0$. Then so has the matrix $A(\xi_j)$, for any basis $\xi_1,\dots,\xi_m$ of $M_{\mathbf{R}}$. By [1], one can find a strictly totally positive matrix A^+ with eigenvalues $\mu_1,\dots,\mu_m$. Since both matrices $A(\xi_j)$ and A^+ are diagonalizable over $\mathbf{R}$ and they have the same set of eigenvalues, it follows that they are conjugate to each other over $\mathbf{R}$. Thus, without loss of generality, we can perform a change of basis and assume that $A(\xi_j)=A^+$.

The coefficients and the minors of the matrix $A(\xi_j)$ change continuously as one perturbs the basis ξ_j. Moreover, being strictly totally positive is an open condition on the space of matrices. Hence, by perturbing ξ_j, we can find a basis $\epsilon_1,\dots,\epsilon_m$ of $M_{\mathbf{Q}}$ such that $A(\epsilon_j)$ is strictly totally positive. Given this basis, following Remark 3.3, we construct a fan of the form

$$\Delta=\Big\{\textstyle\sum_{j=1}^{m}\mathbf{R}_{+}\varepsilon_j v_j\Big\}_{\varepsilon\in\{0,-1,+1\}^m}.$$

In view of Remark 3.3, using the notation of Section 3, all $k\times k$-minors A_{IJ} in the basis u_j are then positive for $k=1,\dots,m-1$, and thus Lemma 3.2 asserts that $f_A\colon X(\Delta)\dashrightarrow X(\Delta)$ is k-stable for $k=1,\dots,m-1$.

If A has negative and distinct eigenvalues, by arguments as above, we can find a basis of $M_{\mathbf{Q}}$ so that the matrix of A is of the form $-B$, where B is strictly totally positive. Constructing Δ as above, the $k\times k$-minors of A in the basis u_j will then all have sign $(-1)^k$ and so, by Lemma 3.2, $f_A\colon X(\Delta)\dashrightarrow X(\Delta)$ is k-stable for $k=1,\dots,m-1$.

5. Proof of Theorem B

Given vectors $w_1,\dots,w_m\in M_{\mathbf{R}}$ we will write $w_I=w_{i_1}\wedge\dots\wedge w_{i_k}$ for $I=\{i_1,\dots,i_k\}\subseteq[m]$. Note that if $A\colon M_{\mathbf{R}}\to M_{\mathbf{R}}$ is a linear map with eigenvalues $\mu_1,\dots,\mu_m$, then the induced linear map $\Lambda^kA\colon\Lambda^kM_{\mathbf{R}}\to\Lambda^kM_{\mathbf{R}}$ has eigenvalues $\mu_I:=\mu_{i_1}\cdots\mu_{i_k}$ for $I=\{i_1,\dots,i_k\}\subseteq[m]$. Throughout we will assume that the eigenvalues of A are ordered so that $|\mu_1|\geqslant\dots\geqslant|\mu_m|$.

LEMMA 5.1. — *Given a basis $\rho_1, \dots, \rho_m$ of $M_{\mathbf{R}}$, there is a basis $\epsilon_1, \dots, \epsilon_m$ of $M_{\mathbf{Q}}$, such that for $k = 1, \dots, m$, $\rho_{[k]}$ lies in the interior of the first orthant $\sigma_k := \sum_{|I|=k} \mathbf{R}_+ \epsilon_I \subset \Lambda^k M_{\mathbf{R}}$, and, moreover, the hyperplane $H_k \subset \Lambda^k M_{\mathbf{R}}$, spanned by ρ_I, $I \neq [k]$, intersects σ_k only at the origin.*

Proof. — Pick real numbers $\mu_1 > \dots > \mu_m > 0$ and let $A\colon M_{\mathbf{R}} \to M_{\mathbf{R}}$ be a linear map given by a diagonal matrix in the basis ρ_j with diagonal entries $\mu_1, \dots, \mu_m$. As in the proof of Theorem A we can then choose a basis $\epsilon_1, \dots, \epsilon_m$ of $M_{\mathbf{Q}}$ such that $A(\epsilon_j)$ is strictly totally positive. In particular, for a given k, $A_{IJ}(\epsilon_j) > 0$ for all I, J such that $|I| = |J| = k$, which means that $\Lambda^k A\colon \Lambda^k M_{\mathbf{R}} \to \Lambda^k M_{\mathbf{R}}$ maps the first orthant σ_k into itself. Since the ρ_j are the eigenvectors of A, it follows by the Perron-Frobenius theorem that the eigenvector $\rho_{[k]}$ (or $-\rho_{[k]}$) corresponding to the largest eigenvalue $\mu_{[k]}$ of $\Lambda^k A$ is contained in the interior of σ_k and, moreover, that $H_k \cap \sigma_k$ is the origin. □

To prove Theorem B, we choose a basis ρ_j such that the linear map $A\colon M_{\mathbf{R}} \to M_{\mathbf{R}}$ is in real Jordan form, *i.e.* with blocks

$$\begin{bmatrix} \mu_j & 1 & & \\ & \mu_j & \ddots & \\ & & \ddots & 1 \\ & & & \mu_j \end{bmatrix} \text{ and } \begin{bmatrix} C_j & I & & \\ & C_j & \ddots & \\ & & \ddots & I \\ & & & C_j \end{bmatrix}, \text{ where } C_j = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

and I is the 2×2 identity matrix; we have the first block type for real eigenvalues μ_j and the second type for complex eigenvalues $a_j \pm ib_j$. We order the blocks so that moduli of the eigenvalues are in decreasing order along the diagonal and the vectors ρ_j so that ρ_1 is an eigenvector corresponding to the largest eigenvalue etc. Next, we let $\epsilon_1, \dots, \epsilon_m$ be a basis of $M_{\mathbf{Q}}$ constructed as in Lemma 5.1, and from ϵ_j, following Remark 3.3, we construct a fan Δ of the form

$$\Delta = \left\{ \textstyle\sum_{j=1}^m \mathbf{R}_+ \varepsilon_j v_j \right\}_{\varepsilon \in \{0,-1,+1\}^m}.$$

Assume that $|\mu_k| > |\mu_{k+1}|$. Then $\mu_{[k]}$ is the unique eigenvalue of $\Lambda^k A\colon \Lambda^k M_{\mathbf{R}} \to \Lambda^k M_{\mathbf{R}}$ of largest modulus. Since A, and thus $\Lambda^k A$, is real, $\mu_{[k]}$ is real. Moreover, $\Lambda^k A \rho_{[k]} = \mu_{[k]} \rho_{[k]}$, so that $\rho_{[k]}$ spans the one-dimensional eigenspace of $\mu_{[k]}$. By Lemma 5.1 $\rho_{[k]}$ is the unique (up to scaling) eigenvector of $\Lambda^k A$ that is contained in σ_k and the hyperplane in $\Lambda^k M_{\mathbf{R}}$ spanned by the other eigenvectors intersects σ_k only at the origin, and thus, since $\mu_{[k]}$ is the unique eigenvalue of largest modulus, σ_k will get attracted to the eigenspace $\mathbf{R}\rho_{[k]} \subseteq \Lambda^k M_{\mathbf{R}}$. Hence there is an $\ell_k \in \mathbf{N}$, such that

$(\Lambda^k A)^\ell \sigma_k \subset \sigma_k$ or

$$(\Lambda^k A)^\ell \sigma_k \subset -\sigma_k := \{x \in M_{\mathbf{R}} \mid -x \in \sigma_k\}$$

for all $\ell \geqslant \ell_k$. In particular, for such an ℓ, $(\Lambda^k A)^\ell \epsilon_I \in \sigma_k$ for all $I = \{i_1, \dots, i_k\} \subseteq [m]$ or $(\Lambda^k A)^\ell \epsilon_I \in -\sigma_k$ for $(\Lambda^k A)^\ell$ all I. This means that the entries of $(\Lambda^k A)^\ell$, *i.e.* the $k \times k$-minors of A^ℓ, in the basis ϵ_j are either all positive or all negative. In view of Remark 3.3, using the notation of Section 3, this implies that $A^\ell_{IJ}(u_j) \geqslant 0$ for all $I = \{i_1, \dots, i_k\}, J = \{j_1, \dots, j_k\} \subseteq [m]$ or $A^\ell_{IJ}(u_j) \leqslant 0$ for all I, J. Now Lemma 3.2 asserts that $f^\ell_A \colon X(\Delta) \dashrightarrow X(\Delta)$ is k-stable.

Finally let $\ell_0 = \max_j \ell_{k_j}$. Then $f^\ell_A \colon X(\Delta) \dashrightarrow X(\Delta)$ is k_j-stable for $\ell \geqslant \ell_0$ and $j = 1, \dots, s$.

Remark 5.2. — For $s = 1$ and $k_1 = 1$ Theorem B appeared as Theorem 4.7 in [19] and Theorem B' in [17]. Also in these papers the idea of the proof is to find a basis (of $N_{\mathbf{R}}$) such that the first orthant is mapped into itself and then construct a toric variety as in Section 3.

6. Degrees of monomial maps

Let Δ be a complete simplicial projective fan and let D be an ample divisor on $X(\Delta)$. Then the *kth degree of* $f_A \colon X(\Delta) \dashrightarrow X(\Delta)$ *with respect to* D is defined as

$$\deg_{D,k} := f_A^* D^k \cdot D^{m-k}.$$

If $X(\Delta) = \mathbf{P}^m$ and $\mathcal{O}(D) = \mathcal{O}_{\mathbf{P}^m}(1)$, then $\deg_{D,k}$ coincides with the kth degree $\deg_k$ as defined in the introduction.

We have the following more general version of Theorem C. Indeed, Theorem C corresponds to the case when $X(\Delta) = \mathbf{P}^m$ and $\mathcal{O}(D) = \mathcal{O}_{\mathbf{P}^m}(1)$.

Theorem C'. — *Let Δ be a complete simplicial fan and let D be an ample divisor on $X(\Delta)$. Assume that $1 \leqslant k \leqslant m-1$ and that the eigenvalues of $A \in M_m(\mathbf{Z})$ satisfy*

$$|\mu_{k-1}| > |\mu_k| = |\mu_{k+1}| > |\mu_{k+2}| \tag{6.1}$$

and moreover that μ_k/μ_{k+1} is not a root of unity. Then the degree sequence $\deg_{D,k}(f^n_A)$ does not satisfy any linear recurrence.

If $k = 1$ or $k = m - 1$ the condition (6.1) on the moduli of the μ_j should be interpreted as $|\mu_1| = |\mu_2| > |\mu_3|$ and $|\mu_{m-2}| > |\mu_{m-1}| = |\mu_m|$, respectively.

Remark 6.1. — Note that there exist maps that satisfy the assumption of Theorem C'. For example, choose integers $a_1, \dots, a_{k-1}, b_1, b_2, a_{k+2}, \dots, a_m$ such that

$$|a_1| \geqslant \dots \geqslant |a_{k-1}| > \sqrt{b_1^2 + b_2^2} > |a_{k+2}| \geqslant \dots \geqslant |a_m|$$

and $b_1 + ib_2 = \sqrt{b_1^2 + b_2^2} \cdot e^{2\pi i\theta}$, where $\theta \notin \mathbf{Q}$. Then (the matrix of) the monomial map

$$(z_1, \dots, z_m) \mapsto \left(z_1^{a_1}, \dots, z_{k-1}^{a_{k-1}}, z_k^{b_1} z_{k+1}^{b_2}, z_k^{-b_2} z_{k+1}^{b_1}, z_{k+2}^{a_{k+2}}, \dots, z_m^{a_m}\right)$$

satisfies the assumption of Theorem C'.

Corollary D follows immediately from Theorem C' and the following result. This is probably well-known, but we include a proof for completeness.

Proposition 6.2. — *Assume that Δ is a simplicial projective fan and let D be an ample divisor on $X(\Delta)$. Assume that the monomial map $f_A\colon X(\Delta) \dashrightarrow X(\Delta)$ is k-stable. Then the degree sequence $\deg_{D,k}(f_A^n)$ satisfies a linear recurrence.*

For the proof we will need the *Caley-Hamilton theorem*: Let $B \in M_L(\mathbf{Z})$ and assume that

$$\chi(r) = r^L + \varphi_{L-1} r^{L-1} + \dots + \varphi_1 r + \varphi_0$$

is the characteristic polynomial of B. Then the Caley-Hamilton theorem asserts that

$$B^L + \varphi_{L-1} B^{L-1} + \dots + \varphi_1 B + \varphi_0 I = 0$$

where I is the identity matrix. In particular, for each $1 \leqslant i, j \leqslant L$, the entry $b_{ij}^n =: b_n$ of B^n satisfies the linear recurrence

$$\chi(b_n)\colon b_{n+L} + \varphi_{L-1} b_{n+L-1} + \dots + \varphi_1 b_{n+1} + \varphi_0 b_n = 0. \tag{6.2}$$

Proof of Proposition 6.2. — Since D is ample, the class $[D]^k$ in $H^{2k}(X)$ is non-zero, where $X = X(\Delta)$, and thus we can extend it to a basis $[D]^k, \theta_1, \dots, \theta_r$ for $H^{2k}(X)$ such that $\theta_j \cdot [D]^{m-k} = 0$ for $j = 1, \dots, r$. Note that then $\deg_{D,k}(f_A)$ is equal to $[D]^k \cdot [D]^{m-k} = [D]^m$ times the $(1,1)$-entry of the matrix B of $f_A^*\colon H^{2k}(X) \to H^{2k}(X)$ with respect to the basis $[D]^k, \theta_1, \dots, \theta_r$. Since by assumption f_A is k-stable, *i.e.* $(f_A^n)^* = (f_A^*)^n$ for all $n \in \mathbf{N}$, it follows that $\deg_{D,k}(f_A^n)$ is equal to $[D]^m$ times the $(1,1)$-entry of B^n. Therefore by the Caley-Hamilton theorem $\deg_{D,k}(f_A^n) =: b_n$ satisfies the linear recurrence (6.2), where χ is the characteristic equation of B. □

Note that Proposition 6.2 implies that if A satisfies the assumption of Theorem A, $X(\Delta)$ is the toric variety constructed in the proof of Theorem A, and D is an ample divisor on $X(\Delta)$, then the degree sequence $\deg_{D,k}(f_A^n)$ of $f_A\colon X(\Delta) \dashrightarrow X(\Delta)$ satisfies a linear recurrence for $k = 1,\dots,n$. Similarly if A and $X(\Delta)$ are as in the (proof of) Theorem B, then $\deg_{D,k}(f_A^{\ell n})$ satisfies a linear recurrence for $\ell \geqslant \ell_0$ and for $j = 1,\dots,s$.

Moreover, even if f_A is not k-stable, as long as we can lift it to a k-stable map, we still have a linear recurrence for its degree sequence.

Proposition 6.3. — *Suppose that X is a simplicial projective toric variety, and that $\pi\colon \widetilde{X} \to X$ is a nonsingular projective modification of X such that $f_A\colon X \dashrightarrow X$ lifts to a k-stable map $f_A\colon \widetilde{X} \dashrightarrow \widetilde{X}$. Then, for any ample divisor D on X, the degree sequence $\deg_{D,k}(f_A^n)$ satisfies a linear recurrence.*

Proof. — Since D is ample, $\pi^*([D]^k)$ is nonzero. As in the proof of Proposition 6.2, we can extend it to a basis of $H^{2k}(\widetilde{X})$ in such a way that $\deg_{D,k}(f_A)$ is equal to $\pi^*([D]^m)$ times the $(1,1)$-entry of the matrix B of f_A^*. Thus, again as in the proof of Proposition 6.2, $\deg_{D,k}(f_A^n)$ satisfies the linear recurrence given by the characteristic equation of B. ☐

It follows from Theorem C' and Proposition 6.3 that if A satisfies the assumption of Theorem C, one cannot k-stabilize f_A by blowing up $\mathbf{P}^m$ or any other simplicial projective toric variety.

6.1. Computing $\deg_{D,k}(f_A)$

To prove Theorem C' we will express $\deg_{D,k}(f_A)$ in terms of the $k \times k$-minors of A. First, for a T-invariant divisor D, Proposition 4.1 in [10] says that $\deg_{D,k}(f_A)$ can be computed as a mixed volume:

$$\deg_{D,k}(f_A) = m!\,\mathrm{Vol}\big(AP_D[k], P_D[m-k]\big). \tag{6.3}$$

In the case of a general ample divisor D', notice that the degrees only depend on the cohomology class of D', and there is always a T-invariant divisor D such that $[D] = [D']$, *cf.* Section 1.2.

Next, we will describe a method of computing the mixed volume of polytopes that we learned from Huber-Sturmfels [16]. A more detail exposition can be found in their paper.

Let $\mathcal{P} = (P_1,\dots,P_s)$ be a tuple of polytopes in $\mathbf{R}^m$ such that $P := P_1 + \cdots + P_s$ has dimension m. A *cell* of $\mathcal{P}$ is a tuple $\mathcal{C} = (C_1,\dots,C_s)$ of polytopes $C_i \subset P_i$. Let $\#C_i$ be the number of vertices of C_i and let

$C := C_1 + \cdots + C_s$. A *fine mixed subdivision* of $\mathcal{P}$ is a collection of cells $\mathcal{S} = \{\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(r)}\}$ such that $C^{(j)}$ has dimension m,

$$\dim C_1^{(j)} + \cdots + \dim C_s^{(j)} = m \quad \text{and} \quad \#(C_1^{(j)}) + \cdots + \#(C_s^{(j)}) - s = m$$

for $j = 1, \ldots, r$, and moreover, $C^{(j)} \cap C^{(j')}$ is a face of both $C^{(j)}$ and $C^{(j')}$ for all j, j', and $\bigcup_j C^{(j)} = P$. If $\mathcal{S}$ is a fine mixed subdivsion of $\mathcal{P}$, then Theorem 2.4 in [16] asserts that

$$\text{(6.4)} \quad \operatorname{Vol}(P_1[k_1], \ldots, P_s[k_s]) = k_1! \cdots k_s! \cdot \sum_{\mathcal{C}^{(j)} \in \mathcal{S}, \dim C_i^{(j)} = k_i, i=1,\ldots,s} \operatorname{Vol}(C^{(j)}).$$

Moreover, Algorithm 2.9 in [16] gives a method of finding fine mixed subdivisions; in particular, each tuple of polytopes $\mathcal{P}$ admits a fine mixed subdivision, where, for each i, j, $C_i^{(j)}$ is the convex hull of a subset of the vertices of P_i.

We want to apply this method to the right hand side of (6.3). Assume that P_D has vertices $v_1, \ldots, v_N$. Then AP_D has vertices $Av_1, \ldots, Av_N$ and thus we can find a fine mixed subdivision $\mathcal{S}$ of $\big(AP_D, P_D\big)$ with cells of the form

$$\mathcal{C}_{IJ} := \big(\operatorname{conv}(Av_{i_0}, \ldots, Av_{i_k}), \operatorname{conv}(v_{j_0}, \ldots, v_{j_{m-k}})\big),$$

where $\operatorname{conv}(v_{i_0}, \ldots, v_{i_k})$ denotes the convex hull of $v_{i_0}, \ldots, v_{i_k}$, for some $I = \{i_0, \ldots, i_k\}$ and $J = \{j_0, \ldots, j_{m-k}\} \subset [N]$. Let $\mathcal{S}_k \subset \mathcal{S}$ be the set of cells $\mathcal{C}_{IJ}$, where $|I| = k+1$. Then (6.4) gives

$$\operatorname{Vol}(AP_D[k], P_D[m-k]) = k!(m-k)! \sum_{\mathcal{C}_{IJ} \in \mathcal{S}_k} \operatorname{Vol}(C_{IJ}).$$

Note that C_{IJ} is the Minkowski sum of the k-simplex $\operatorname{conv}(Av_{i_0}, \ldots, Av_{i_k})$ with edges $A(v_{i_1} - v_{i_0}), \ldots, A(v_{i_k} - v_{i_0})$ and the $(m-k)$-simplex $\operatorname{conv}(v_{j_0}, \ldots, v_{j_{m-k}})$ with edges $v_{j_1} - v_{j_0}, \ldots, v_{j_{m-k}} - v_{j_0}$. From now on, let us fix a basis of M. It follows that $k!(m-k)! \operatorname{Vol}(C_{IJ})$ equals the modulus of the determinant of the matrix B_{IJ} with the vectors $A(v_{i_1} - v_{i_0}), \ldots, A(v_{i_k} - v_{i_0})$ and $v_{j_1} - v_{j_0}, \ldots, v_{j_{m-k}} - v_{j_0}$ as columns. Since P_D is a lattice polytope, the determinant of B_{IJ} is an integer linear combination of $k \times k$-minors of A. Hence $\deg_{D,k}(f_A)$ is of the form $\sum \sigma_{IJ} A_{IJ}$ where $\sigma_{IJ} \in \mathbf{Z}$ and A_{IJ} is the IJ-minor of A. Observe that the matrix σ with entries σ_{IJ} only depends on the set of multi-indices I, J such that C_{IJ} is in $\mathcal{S}_k$ and the sign of the determinant of B_{IJ}. Since there are only finitely many choices of I, J and signs, we conclude the following.

LEMMA 6.4. — *There is a finite set* Σ *of matrices* $\sigma = (\sigma_{IJ}) \in \mathbf{Z}^{\binom{m}{k}^2}$, *such that for each* $A\colon M \to M$ *there is a* $\sigma = \sigma(A) \in \Sigma$ *such that*

$$\deg_{D,k}(f_A) = \sum_{IJ} \sigma_{IJ} A_{IJ}.$$

6.2. Proof of Theorem C'

Our proof is much inspired by the proof of Proposition 3.1 in [2] and the proof of Proposition 7.3 in [15]. We will argue by contradiction using a result from combinatorics, which says that if α_n and β_n are two sequences that each satisfies a linear recurrence, then the set of $n \in \mathbf{N}$, for which $\alpha_n = \beta_n$, is either finite or eventually periodic, see [26, Chapter 4, Exercise 3]. In particular if $\alpha_n = \beta_n$ for infinitely many n, then for some $a, b \in \mathbf{N}$,

$$\alpha_{a+b\ell} = \beta_{a+b\ell} \text{ for } \ell \gg 0. \tag{6.5}$$

Now, let $\alpha_n = \deg_{D,k}(f_A^n)$. By Lemma 6.4, $\alpha_n = \sum_{IJ} \sigma_{IJ}(n) A^n_{IJ}$, where A^n_{IJ} is the IJ-minor of A^n, for some matrix $\sigma(n) \in \Sigma$. Since Σ is a finite set, there is at least one $\sigma \in \Sigma$ such that $\sigma(n) = \sigma$ for infinitely many n. Pick such a $\sigma = (\sigma_{IJ}) \in \Sigma$ and let $\beta_n = \sum_{IJ} \sigma_{IJ} A^n_{IJ}$. Let $\chi(r)$ be the characteristic polynomial of $\Lambda^k A$. By the Caley-Hamilton theorem the entries A^n_{IJ} of $(\Lambda^k A)^n$, *cf.* (3.2), satisfy the linear recurrence $\chi(A^n_{IJ})$, see (6.2). It follows that β_n satisfies the linear recurrence $\chi(\beta_n)$, and $\alpha_n = \beta_n$ for infinitely many n.

Next, we claim that if A is as in the assumption of Theorem C', then for each choice of $a, b \in \mathbf{N}$, $\beta_{a+b\ell} < 0$ for infinitely many ℓ. Since the eigenvalues of A satisfy

$$|\mu_{k-1}| < |\mu_k| = |\mu_{k+1}| < |\mu_{k+2}|$$

it follows that $\mu_{[k]} =: \mu$ and $\mu_{\{1,\dots,k-1,k+1\}}$ are the two eigenvalues of $\Lambda^k A$ of largest modulus, and that the other eigenvalues $\mu_{I_3}, \dots, \mu_{I_{\binom{m}{k}}}$ are of strictly smaller modulus. Moreover, since $\mu_{k+1} = \bar{\mu}_k$ and $\mu_k/\bar{\mu}_k$ is not a root of unity, it follows that $\mu_{\{1,\dots,k-1,k+1\}} = \bar{\mu}$ and that $\mu/\bar{\mu}$ is not a root of unity. Hence we can write

$$(\Lambda^k A)^n = P \begin{bmatrix} \mu^n & 0 & 0 & \cdots \\ 0 & \bar{\mu}^n & 0 & \cdots \\ 0 & 0 & \mu^n_{I_3} & \\ \vdots & \vdots & & \ddots \end{bmatrix} P^{-1}$$

for some invertible matrix P. It follows that

$$\beta_n = \sum \sigma_{IJ} A_{IJ}^n = C\mu^n + D\bar{\mu}^n + \mathcal{O}(|\mu_{I_3}|^n)$$

where C and D are independent of n. Since β_n is real it follows that $D = \bar{C}$, so that

$$\beta_n = 2\,\mathrm{Re}(C) \cdot \mathrm{Re}(\mu^n) + \mathcal{O}(|\mu_{I_3}|^n).$$

Since $\mu = |\mu| \cdot e^{2\pi i\theta}$ with $\theta \notin \mathbf{Q}$, it follows that $\arg(\mu^{a+b\ell})$ is dense in $[0, 2\pi)$. In particular, $\mathrm{Re}(\mu^{a+b\ell}) < 0$ for infinitely many ℓ and $\mathrm{Re}(\mu^{a+b\ell}) > 0$ for infinitely many ℓ, and thus, since $|\mu_{I_3}| < |\mu|$, $\beta_{a+b\ell} < 0$ for infinitely many ℓ.

Assume that α_n satisfies a linear recurrence. Then, since $\alpha_n = \beta_n$ for infinitely many n, (6.5) holds for some a, b, but this contradicts that $\alpha_n > 0$. This proves Theorem C'.

BIBLIOGRAPHY

[1] W. Barrett & C. R. Johnson, "Possible spectra of totally positive matrices", *Linear Algebra Appl.* **62** (1984), p. 231-233.

[2] E. Bedford & K. Kim, "Linear recurrences in the degree sequences of monomial mappings", *Ergodic Theory Dynam. Systems* **28** (2008), no. 5, p. 1369-1375.

[3] V. I. Danilov, "The geometry of toric varieties", *Uspekhi Mat. Nauk* **33** (1978), no. 2(200), p. 85-134, 247.

[4] J. Diller & C. Favre, "Dynamics of bimeromorphic maps of surfaces", *Amer. J. Math.* **123** (2001), no. 6, p. 1135-1169.

[5] T.-C. Dinh & N. Sibony, "Dynamics of regular birational maps in $\mathbb{P}^k$", *J. Funct. Anal.* **222** (2005), no. 1, p. 202-216.

[6] ———, "Une borne supérieure pour l'entropie topologique d'une application rationnelle", *Ann. of Math. (2)* **161** (2005), no. 3, p. 1637-1644.

[7] ———, "Super-potentials of positive closed currents, intersection theory and dynamics", *Acta Math.* **203** (2009), no. 1, p. 1-82.

[8] C. Favre, "Les applications monomiales en deux dimensions", *Michigan Math. J.* **51** (2003), no. 3, p. 467-475.

[9] C. Favre & M. Jonsson, "Dynamical compactifications of $\mathbf{C}^2$", *Ann. of Math. (2)* **173** (2011), no. 1, p. 211-248.

[10] C. Favre & E. Wulcan, "Degree growth of monomial maps and McMullen's polytope algebra", *Indiana Univ. Math. J.* **61** (2012), no. 2, p. 493-524.

[11] J. E. Fornaess & N. Sibony, "Complex dynamics in higher dimension. II", in *Modern methods in complex analysis (Princeton, NJ, 1992)*, Ann. of Math. Stud., vol. 137, Princeton Univ. Press, Princeton, NJ, 1995, p. 135-182.

[12] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry, xii+157 pages.

[13] ———, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998, xiv+470 pages.

[14] V. Guedj, "Ergodic properties of rational mappings with large topological degree", *Ann. of Math. (2)* **161** (2005), no. 3, p. 1589-1607.

[15] B. Hasselblatt & J. Propp, "Degree-growth of monomial maps", *Ergodic Theory Dynam. Systems* **27** (2007), no. 5, p. 1375-1397.

[16] B. Huber & B. Sturmfels, "A polyhedral method for solving sparse polynomial systems", *Math. Comp.* **64** (1995), no. 212, p. 1541-1555.

[17] M. Jonsson & E. Wulcan, "Stabilization of monomial maps", *Michigan Math. J.* **60** (2011), no. 3, p. 629-660.

[18] J.-L. Lin, "On Degree Growth and Stabilization of Three Dimensional Monomial Maps Jan-Li Lin", Michigan Math. J., to appear.

[19] ———, "Algebraic stability and degree growth of monomial maps", *Math. Z.* **271** (2012), no. 1-2, p. 293-311.

[20] ———, "Pulling back cohomology classes and dynamical degrees of monomial maps", *Bull. Soc. Math. France* **140** (2012), no. 4, p. 533-549 (2013).

[21] M. Mustaţă, "Lecture notes on toric varieties", Available on the author's webpage: `www.math.lsa.umich.edu/~mmustata`.

[22] T. Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese, viii+212 pages.

[23] C. A. M. Peters & J. H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008, xiv+470 pages.

[24] A. Russakovskii & B. Shiffman, "Value distribution for sequences of rational mappings and complex dynamics", *Indiana Univ. Math. J.* **46** (1997), no. 3, p. 897-932.

[25] N. Sibony, "Dynamique des applications rationnelles de $\mathbf{P}^k$", in *Dynamique et géométrie complexes (Lyon, 1997)*, Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, p. ix-x, xi-xii, 97-185.

[26] R. P. Stanley, *Enumerative combinatorics. Vol. I*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986, With a foreword by Gian-Carlo Rota, xiv+306 pages.

Manuscrit reçu le 3 décembre 2012,
révisé le 4 février 2013,
accepté le 13 mars 2013.

Jan-Li LIN
University of Notre Dame
Department of Mathematics
Notre Dame, IN 46556 (USA)
jlin4@nd.edu

Elizabeth WULCAN
Chalmers University of Technology
and the University of Gothenburg
SE-412 96 Göteborg (Sweden)
wulcan@chalmers.se