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ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF FOURIER SERIES (*)

by NIRANJAN SINGH

1.1. DEFINITIONS. — Let Σa_n be a given infinite series with S_n as its n -th partial sum. The series Σa_n is said to be absolutely summable (C, α) , or summable $|C, \alpha|$, if the sequence $\{\sigma_n^\alpha\}$ is of bounded variation, that is

$$\sum_n |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

where $\{\sigma_n^\alpha\}$ is the n -th Cesàro mean of order α , $\alpha > -1$, of the sequence $\{S_n\}$.

If $\{t_n^\alpha\}$ be the n -th Cesàro mean of order α of the sequence $\{na_n\}$, then we have the following identity [6].

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

For any sequence $\{u_n\}$, we write

$$\Delta u_n = u_n - u_{n+1}$$

and

$$\Delta^r u_n = \sum_{p=0}^{\infty} A_p^{-r-1} u_{n+p},$$

provided the series on the right converges.

If S is a +ve integer, then

$$\Delta^S (u_n \nu_n) = \sum_{r=0}^S \binom{S}{r} \Delta^r u_n \Delta^{S-r} \nu_{n+r}$$

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By repeated partial summation, we observe that, for $k = 0, 1, \dots$

$$\sum_{p=0}^q A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^q S_p^k \Delta^{k+1} (A_{n-p}^{r-1} u_p) + \sum_{j=0}^k \Delta^j (A_{n-q-1}^{r-1} u_{q+1}) S_q^j$$

where S_n^k denotes the n -th Cesàro sum of order k of the sequence $\{S_n\}$. Hence, putting $q = n$, we get

$$(1.1.1) \quad \sum_{p=0}^n A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^n S_p^k \Delta^{k+1} (A_{n-p}^{r-1} u_p).$$

A sequence $\{\lambda_n\}$ is said to be convex, if $\Delta^2 \lambda_n \geq 0$, and it is said to be hyper-convex of order h , if

$$\Delta^{h+2} \lambda_n \geq 0, \quad (h = 0, 1, 2, \dots).$$

By definition hyper-convexity of order zero is the same as convexity.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is

$$\int_{-\pi}^{\pi} f(t) dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations :

$$\begin{aligned} \Phi(t) &= \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\}, \\ \Phi_{\alpha}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \Phi(u) du, \quad \alpha > 0, \\ \Phi_0(t) &= \Phi(t), \\ \varphi_{\alpha}(t) &= \Gamma(\alpha+1) t^{-\alpha} \Phi_{\alpha}(t), \quad \alpha \geq 0, \\ (F(t))_h &= \frac{\partial^h F(t)}{\partial t^h}, \\ \varepsilon(n) &= (\log n)^{-\beta}, \quad \beta \geq 0, \\ \varepsilon^{-1}(n) &= \frac{1}{\varepsilon(n)}. \end{aligned}$$

1.2. Various results on summability factors of Fourier series due to Prasad [12], Izumi and Kwata [5], Cheng [3], Pati [8] and Dikshit [4] were generalized by Pati and Sinha [11] in the form of the following theorem.

THEOREM A. — *Let h be an integer ≥ 0 , and let $\{\lambda_n\}$ be a monotonic non-increasing sequence when $h = 0$, and a hyper-convex sequence of order $(h - 1)$ when $h \geq 1$, such that*

$$(i) \quad \sum \frac{\lambda_n}{n} < \infty, \quad (ii) \quad \sum n^h \Delta^{h+1} \lambda_n < \infty.$$

If

$$\int_0^t |\varphi_h(u)| du = 0(t),$$

as $t \rightarrow 0$, then $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ is summable $|C, h + 1 + \delta|$ for every $\delta > 0$.

Later on Ahmad [1] obtained the following theorem which includes as a special case for $\beta = 0$ the above theorem of Pati and Sinha.

THEOREM B. — *Let $\{\lambda_n\}$ be a sequence such that for all non-negative integral values of h , $\Delta^{h+1} \lambda_n \geq 0$, and $\sum \frac{\lambda_n}{n} < \infty$. If*

$$\int_0^t |\varphi_h(u)| du = 0 \left\{ t \varepsilon^{-1} \left(\frac{1}{t} \right) \right\},$$

as $t \rightarrow 0$, then $\sum_{n=1}^{\infty} \varepsilon(n + 1) \lambda_n A_n(x)$ is summable $|C, h + 1 + \delta|$ for every $\delta > 0$.

In this paper we prove the following theorem for summability $|C, 1 + h|$ by imposing suitable conditions on the sequence $\{\lambda_n\}$.

We prove the following theorem.

THEOREM. — *Let $\{\lambda_n\}$ be a sequence such that for non-negative integral values of h , $\Delta^{h+1} \lambda_n \geq 0$, and*

$$(1.2.1) \quad \sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty.$$

If

$$(1.2.2) \quad \int_0^t |\varphi_h(u)| du = O\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\}, \quad t \rightarrow 0,$$

then, $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ is summable $|C, h+1|$.

It may be remarked that this theorem generalizes the following theorem of the author [14] which in turn, includes a theorem of Pati [10].

THEOREM C. — Let $\{\lambda_n\}$ be a convex sequence such that $\sum \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty$.

If

$$\int_0^t |\Phi(u)| du = O\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \rightarrow 0$, then $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ is summable $|C, 1|$.

1.3. For the proof of our theorem we require the following lemmas :

LEMMA 1 [9]. — Let $C_{n,\rho}^k$ and $S_n^k(t)$ denote the n -th Cesàro-sums of order k corresponding to the series $\sum_1^\infty (-1)^{n\rho}$ and $\sum_1^\infty (\sin nt)_{h+1} (h \geq 0)$, respectively, then

$$(i) \quad C_{n,\rho}^k = O(n^k) \quad k \geq \rho$$

$$(ii) \quad S_n^k(t) = O(n^{k+h+2}) \quad \left(0 < t \leq \frac{1}{n}\right), \quad k \geq 0 \\ = O(n^{h+1}t^{-k-1}) + O(n^k t^{-h-2}), \quad (n^{-1} < t \leq \pi) k \geq 0.$$

LEMMA 2 [2]. — If $k \geq -1, r \geq 0$, necessary and sufficient conditions for $\sum a_n \varepsilon_n$ to be summable $|C, r|$ whenever

$$S_n = a_0 + a_1 + \dots + a_n = O(1)(C, k)$$

are

$$(i) \quad \sum n^{k-r} |\varepsilon_n| < \infty, \\ (ii) \quad \sum n^{-1} |\varepsilon_n| < \infty, \\ (iii) \quad \sum n^k \left| \Delta^{k+1} \varepsilon_n \right| < \infty.$$

LEMMA 3 [1]. — Let $R_n^k(t)$ denote the n -th Cesàro sum of order k ($0 \leq k < h + 1$) of the series $\sum_1^\infty \varepsilon(n + 1) (\sin nt)_{h+1}$ ($h \geq 0$), then

- (i) $R_n^k(t) = O\{\varepsilon(n + 1)n^{k+h+2}\} \quad \left(0 < t \leq \frac{1}{n}\right),$
- (ii) $R_n^k(t) = O\{\varepsilon(n + 1)n^{h+1}t^{-k-1}\} \quad (n^{-1} < t \leq \pi).$

LEMMA 4 [1]. — If (1.2.2) holds, then

$$\int_{\frac{1}{n}}^\pi t^{-1} |\varphi_h(t)| dt = O\{\varepsilon^{-1}(n + 1) \log n\}.$$

LEMMA 5 [1]. — Let h be a positive integer, and $\{\lambda_n\}$ be a sequence such that $\Delta^h \lambda_n \geq 0$, and $\sum \frac{\lambda_n}{n} < \infty$, then

- (a) $\Delta^r \lambda_n \downarrow_0 \quad (r = 0, 1, \dots, h - 1).$
- (b) $\lambda_n = \begin{cases} \sum_{m=n}^\infty \Delta \lambda_m & \text{for } h = 1 \\ \left(\frac{h-1}{h-1}\right)^{-1} \sum_{m=n}^\infty (m-n+1)(m-n+2) \dots \\ \qquad \qquad \qquad (m-n+h-1) \Delta^h \lambda_m & (h > 1) \end{cases}$
- (c) $\sum m^{r-1} \Delta^r \lambda_m < \infty \quad (r = 1, 2, \dots, h - 1).$

LEMMA 6 [11]. — Let $\{\lambda_n\}$ be a hyper-convex sequence of order $(h - 1)$ when $h \geq 1$, or monotonic non-increasing when $h = 0$, such that

$$\sum \frac{\lambda_n}{n} < \infty.$$

If

$$\sum n^h \Delta^{h+1} \lambda_n < \infty,$$

then

$$\sum \log(n + 1) n^h \Delta^{h+1} \lambda_n < \infty.$$

LEMMA 7 [13]. — If

$$\int_0^t |\varphi_\alpha(u)| du = O\left\{t \left(\log \frac{1}{t}\right)^\beta\right\},$$

then

$$\sum_{m=0}^n |\sigma_m^{\alpha}|^2 = O\{n (\log n)^{2\beta+1}\} \quad \text{for } \beta > -\frac{1}{2}$$

and $\alpha \geq 0$ where σ_m^{α} is the m -th (C, α) mean of the series $\Sigma A_n(x)$.

LEMMA 8. — We have for $r = 0, 1, \dots, h$

$$\Delta^{h+1-r} \{(\mu + r)\varepsilon_{\mu+r+1}\} = O\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu + 1)}\right\}.$$

Proof. — Since $\Delta^p(\mu + r) = 0$ for $p \geq 2$ we have

$$\begin{aligned} \Delta^{h+1-r} \{(\mu + r)\varepsilon_{\mu+r+1}\} &= \sum_{p=0}^{h+1-r} \binom{h+1-r}{p} \Delta^p(\mu + r) \Delta^{h+1-r-p} \varepsilon_{\mu+r+1} \\ &= (\mu + r) \Delta^{h+1-r} \varepsilon_{\mu+r+1} - (h+1-r) \Delta^{h-r} \varepsilon_{\mu+r+2} \\ &= O\left\{\frac{(\mu + 1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu + 1)}\right\}. \end{aligned}$$

1.4. *Proof of the Theorem.* — Since

$$\begin{aligned} A_n(x) &= \frac{2}{\pi} \int_0^{\pi} \Phi(t) \cos nt \, dt \\ &= \frac{2}{\pi} \left[\sum_{\rho=1}^h (-1)^{\rho-1} \Phi_{\rho}(t) (\cos nt)_{\rho-1} \right]_0^{\pi} \\ &\quad + (-1)^h \frac{2}{\pi} \int_0^{\pi} \Phi_h(t) (\cos nt)_h \, dt \\ &= A_{n,1}(x) + A_{n,2}(x), \quad \text{say.} \end{aligned}$$

Thus by virtue of the consistency theorem for absolute Cesàro-summability, it is sufficient for our purpose, to prove that each of the series

$$(1.4.1) \quad \sum_{n=1}^{\infty} \varepsilon(n+1) \lambda_n A_{n,1}(x),$$

and

$$(1.4.2) \quad \sum_{n=1}^{\infty} \varepsilon(n+1) \lambda_n A_{n,2}(x),$$

is summable $|C, h+1|$.

Now since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, for proving

the summability $|C, h + 1|$ of (1.4.1), it is enough to show that if ρ is an odd integer, $1 \leq \rho \leq h$,

$$\sum_{n=1}^{\infty} \varepsilon(n + 1)\lambda_n(-1)^n n^{\rho-1} \quad \text{is summable} \quad |C, h + 1|.$$

Taking the series $\sum a_n$ in lemma 2 to be $\sum(-1)^n n^{\rho-1}$, $r = h$, $k = h - 1$, we have from lemma 1.

$$C_{n,\rho-1}^{h-1} = O(n^{h-1}).$$

Also by taking ε_n to be $\lambda_n \varepsilon_{n+1}$ we find that conditions (i) and (ii) of lemma 2 are satisfied. Also

$$\begin{aligned} \sum n^{h-1} \left| \Delta^h \left(\frac{\lambda_n}{(\log n + 1)} \beta \right) \right| &= O \left\{ \sum_{n=1}^{\infty} \sum_{r=0}^h n^{r-1} \Delta^r \lambda_n \right\} \\ &= O(1), \end{aligned}$$

by virtue of part (c) of lemma 5. Finally applying lemma 2 we find that $\sum \lambda_n \varepsilon_{n+1} (-1)^n n^{\rho-1}$ is summable $|C, h|$ and consequently summable $|C, h + 1|$.

Also the summability $|C, h + 1|$ of the series (1.4.2) is equivalent to the assertion that

$$(1.4.3) \quad \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \phi_h(t) L_n^{h+1}(t) dt \right| < \infty,$$

where

$$L_n^{h+1}(t) = \frac{t^h}{A_n^{h+1}} \sum_{\nu=0}^n A_{n-\nu}^h \varepsilon(\nu + 1) \lambda_{\nu} (\sin \nu t)_{h+1}.$$

Proof of (1.4.3). — We have

$$\Sigma \equiv \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu + 1) \lambda_{\nu} (\sin \nu t)_{h+1}.$$

Applying the process of repeated summation we have in the notation of Lemma 3,

$$\begin{aligned} \Sigma &= \sum_{\nu=1}^n R_{\nu}^h(t) \Delta^{h+1} (A_{n-\nu}^h \lambda_{\nu}) \\ &= \sum_{r=0}^h \binom{h+1}{r} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} R_{\nu}^h(t) \\ &\quad + \sum_{\nu=1}^n A_{n-\nu}^{-1} \lambda_{\nu+h+1} R_{\nu}^h(t) \\ &= \Sigma_1 + \Sigma_2, \quad \text{say.} \end{aligned}$$

Hence we need to prove that

$$\sum_{n=1}^{\infty} n^{-1} \left| \int_0^{\pi} \phi_h(t) \frac{t^h}{A_n^{h+1}} (\Sigma_1 + \Sigma_2) dt \right| < \infty,$$

for which it is sufficient to show that

$$(1.4.4) \quad \sum_{n=1}^{\infty} n^{-h-2} \int_0^{\pi} |\phi_h(t)| t^h |\Sigma_1| dt < \infty,$$

and

$$(1.4.5) \quad \sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} \phi_h(t) t^h \Sigma_2 dt \right| < \infty.$$

Proof of (1.4.4). — It suffices, for our purpose, to show that for $0 \leq r \leq h$,

$$\sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \int_0^{\pi} t^h |\phi_h(t)| |R_{\nu}^h(t)| dt < \infty.$$

The above expression is

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \left(\int_0^{\frac{1}{\nu}} + \int_{\frac{1}{\nu}}^{\infty} \right) |\phi_h(t)| t^h |R_{\nu}^h(t)| dt \\ = \Sigma_{11} + \Sigma_{12}, \quad \text{say.} \end{aligned}$$

Now by lemma 3 and the hypothesis we have

$$\begin{aligned} \Sigma_{11} &\leq K^{(1)} \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} (\nu^{2h+2} \varepsilon(\nu+1)) \left(\frac{\nu^{-h-1}}{\varepsilon_{\nu+1}} \right), \\ &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n \nu^{h+1} (n+1-\nu)^{h+1-r} \Delta^{h+1-r} \lambda_{\nu+r} \\ &\leq K \sum_{\nu=1}^{\infty} \nu^{h+1} \Delta^{h+1-r} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2} \\ &\leq K \sum_{\nu=1}^{\infty} \nu^{h+1} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{-r-1} \\ &\leq K \sum_{\nu=1}^{\infty} \nu^{h-r} \Delta^{h-r+1} \lambda_{\nu+r} \leq K. \end{aligned}$$

By lemma 5 and the fact that

$$\begin{aligned} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2} &= 0 \left(\int_{\nu}^{\infty} x^{-h-2} (x-\nu)^{h-r} dx \right) \\ &= 0(\nu^{-r-1}) \end{aligned}$$

(¹) K is a constant not necessarily the same at each occurrence.

Also by lemmas 3 and 4 we get

$$\begin{aligned}
 \Sigma_{12} &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \\
 &\qquad \int_{\frac{1}{\nu}}^{\pi} t^h |\varphi_h(t)| t^{-h-1} dt \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^{-1} |\varphi_h(t)| dt, \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n A_{n-\nu}^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1), \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^n (n+1-\nu)^{h-r} \Delta^{h+1-r} \lambda_{\nu+r} \nu^{h+1} \log(\nu+1), \\
 &\leq K \sum_{\nu=1}^{\infty} \nu^{h+1} \log(\nu+1) \Delta^{h+1-r} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2}, \\
 &\leq K \sum_{\nu=1}^{\infty} \log(\nu+1) \nu^{h-r} \Delta^{h-r+1} \lambda_{\nu+r}, \\
 &\leq K,
 \end{aligned}$$

by lemmas 5 and 6.

This completes the proof of (1.4.4).

Proof of (1.4.5). — Now we have to show that

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \Sigma_2 dt \right| < \infty.$$

Since

$$\Sigma_2 = \lambda_{n+h+1} R_n^h(t),$$

substituting the value of Σ_2 , we find that the above expression is

$$\begin{aligned}
 &\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \lambda_{n+h+1} R_n^h(t) dt \right| \\
 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \int_0^{\pi} \Phi_h(t) R_n^h(t) dt \right| \\
 &= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot \int_0^{\pi} \Phi_h(t) (\cos \nu t)_n dt \right| \\
 &= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot (-1)^h \right|
 \end{aligned}$$

$$\begin{aligned}
& \left\{ (-1)^h \frac{2}{\pi} \int_0^\pi \Phi_h(t) (\cos \nu t)_h dt \right. \\
& + \frac{2}{\pi} \left[\sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1} \right]_0^\pi \\
& \left. - \frac{2}{\pi} \left[\sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1} \right]_0^\pi \right\} \\
& \leq K \sum_{n=1}^\infty n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot A_\nu(x) \right| \\
& + K \sum_{n=1}^\infty n^{-h-2} \lambda_{n+h+1} \\
& \left| \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot \left[\sum_{\rho=1}^h (-1)^{\rho-1} \Phi_\rho(t) (\cos \nu t)_{\rho-1} \right]_0^\pi \right| \\
& = I_1 + I_2, \text{ say.}
\end{aligned}$$

By repeated partial summation we have

$$\sum_{\nu=0}^n A_{n-\nu}^h \varepsilon(\nu+1) \cdot \nu \cdot A_\nu(x) = \sum_{\nu=0}^n \check{S}_\nu^h \Delta^{h+1} (A_{n-\nu}^h \cdot \varepsilon_{\nu+1}),$$

where \check{S}_n^h denotes the n -th Cesàro-sum of order h of the series $\sum A_n(x)$.

Now since

$$\begin{aligned}
\Delta^{h+1} (A_{n-\nu}^h \cdot \varepsilon(\nu+1)) &= \sum_{r=0}^{h+1} \binom{h+1}{r} \Delta^r (A_{n-\nu}^h) \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&= \sum_{r=0}^h \binom{h+1}{r} A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&\quad + A_{n-\nu}^{-1} (\nu+h+1) \varepsilon_{\nu+h+2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{\nu=1}^n A_{n-\nu}^h \varepsilon(\nu+1) \nu A_\nu(x) \\
&= \sum_{r=0}^h \binom{h+1}{r} \sum_{\nu=0}^n \check{S}_\nu^h A_{n-\nu}^{h-r} \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
&\quad + \check{S}_n^h (n+h+1) \varepsilon_{n+h+2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
 I_1 &\leq K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \sum_{r=0}^h \binom{h+1}{r} \\
 &\quad \sum_{v=0}^n |\check{S}_v^h| A_{n-v}^{h-r} \left| \Delta^{n+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \right| \\
 &\quad + K \sum n^{-h-2} \lambda_{n+h+1} (n+h+1) \varepsilon_{n+h+2} |\check{S}_n^h| \\
 &= I_{11} + I_{12}, \quad \text{say.}
 \end{aligned}$$

Now

$$\begin{aligned}
 I_{12} &= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} (n+h+1) A_n^h |\check{\sigma}_n^*| \varepsilon_{n+h+2} \\
 &= 0 \left[\sum_{n=1}^{\infty} |\check{\sigma}_n^*| \lambda_n \frac{\varepsilon_{n+1}}{n} \right].
 \end{aligned}$$

Applying Abel's transformation we have by Lemma 7.

$$\begin{aligned}
 \sum_{n=1}^m |\check{\sigma}_n^*| \frac{\lambda_n \varepsilon_{n+1}}{n} &= \sum_{n=1}^{m-1} \Delta \left(\frac{\lambda_n \varepsilon_{n+1}}{n} \right) \sum_{v=0}^n |\check{\sigma}_v^*| \\
 &\quad + \frac{\lambda_m \varepsilon_{m+1}}{m} \sum_{n=0}^m |\check{\sigma}_n^*| \\
 &= 0 \left[\sum_{n=1}^{m-1} \Delta \left(\frac{\lambda_n \varepsilon_{n+1}}{n} \right) n \varepsilon^{-1} (n+1) (\log(n+1))^{\frac{1}{2}} \right] \\
 &\quad + 0 \left[\frac{\lambda_m \varepsilon_{m+1}}{m} \cdot m \varepsilon^{-1} (m+1) (\log(m+1))^{\frac{1}{2}} \right] \\
 &= 0 \left[\sum_1^{m-1} \Delta \lambda_n (\log n + 1)^{\frac{1}{2}} \right] \\
 &\quad + 0 \left[\sum_1^{m-1} \frac{\lambda_{n+1}}{n+1} (\log n + 1)^{\frac{1}{2}} \right] \\
 &= 0(1) + 0(1) = 0(1).
 \end{aligned}$$

Since $\Delta \varepsilon_n = 0 \left(\frac{\varepsilon_n}{n} \right)$ and $\lambda_m \log(m+1) = 0(1)$.

Now in order to show that $I_{11} = 0(1)$ it is sufficient to prove that

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \sum_{v=0}^n |\check{S}_v^h| A_{n-v}^{h-r} \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \\
 = 0(1) \quad \text{for } r = 0, 1, \dots, h.
 \end{aligned}$$

The above expression is by lemma 8

$$\begin{aligned}
 & \left| \sum_{v=1}^{\infty} |\dot{S}_v^h| \left| \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=v}^{\infty} (n-\nu+1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right| \right. \\
 & \leq K \sum_{v=1}^{\infty} |\dot{S}_v^h| \lambda_{\nu+h+1} \left| \Delta^{h+1-r} \{(\nu+r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=v}^{\infty} (n-\nu+1)^{h-r} \cdot n^{-h-2} \right| \\
 & = 0 \left(\sum_{v=0}^{\infty} |\dot{S}_v^h| \lambda_{\nu+h+1} \frac{(\nu+1)^{r-h} \varepsilon_{\nu+1}}{\log(\nu+1)} \nu^{-r-1} \right), \\
 & = 0 \left(\sum_{v=0}^{\infty} |\dot{\sigma}_v^h| (\nu+1)^{r-h} \frac{\varepsilon_{\nu+1}}{\log(\nu+1)} \nu^{h-r-1} \lambda_{\nu+h+1} \right) \\
 & = 0 \left(\sum_{v=0}^{\infty} |\dot{\sigma}_v^h| \lambda_{\nu+h+1} \frac{\varepsilon(\nu+1)}{\nu \log \nu + 1} \right) \\
 & = 0(1),
 \end{aligned}$$

as shown in the proof of $I_{12} = 0(1)$.

Hence

$$I_1 = 0(1).$$

Now we proceed to show that $I_2 = 0(1)$.

If ρ is an odd integer, then it is sufficient to show that

$$(1.4.6) \quad k \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{v=1}^n A_{n-v}^h \varepsilon_{v+1} (-1)^v \nu^\rho \right| < \infty,$$

for $1 \leq \rho \leq h$.

By repeated partial summation we have

$$\sum_{v=1}^n A_{n-v}^h \varepsilon_{v+1} (-1)^v \cdot \nu^\rho = \sum_{v=0}^n C_{v,\rho}^h \Delta^{h+1} (A_{n-v}^h \varepsilon(\nu+1)),$$

where $C_{n,\rho}^h$ is the n -th Cesàro sum of order h of the series $\sum (-1)^v \cdot \nu^\rho$.

Also

$$\begin{aligned}
 \Delta^{h+1} (A_{n-v}^h \varepsilon_{v+1}) &= \sum_{r=0}^{h+1} \binom{h+1}{r} \Delta^r (A_{n-v}^h) \Delta^{h+1-r} \varepsilon_{v+r+1} \\
 &= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-v}^{h-r} \Delta^{h+1-r} \varepsilon_{v+r+1} \\
 &= \sum_{r=0}^h \binom{h+1}{r} A_{n-v}^{h-r} \Delta^{h+1-r} \varepsilon_{v+r+1} + A_{n-v}^{-1} \varepsilon_{v+h+2}.
 \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\nu=1}^n A_{n-\nu}^h \varepsilon_{\nu+1} (-1)^{\nu} \rho^{\nu} &= \sum_{r=0}^h \binom{h+1}{r} \sum_{\nu=0}^n C_{\nu, \rho}^h A_{n-\nu}^{h-r} \Delta^{h+1-r} \varepsilon_{\nu+r+1} \\ &+ \sum_{\nu=0}^n C_{\nu, \rho}^h A_{n-\nu}^{-1} \varepsilon_{\nu+h+2} \\ &= 0 \left(\sum_{\nu=0}^n \nu^h (n - \nu + 1)^{h-r} \left| \Delta^{h+1-r} \varepsilon_{\nu+r+1} \right| \right) \\ &+ 0(n^h \varepsilon_{n+h+2}), \end{aligned}$$

by lemma 1.

Therefore the expression in (1.4.6) is

$$\begin{aligned} &= 0 \left(\sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \sum_{\nu=0}^n \nu^h (n - \nu + 1)^{h-r} \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \right) \\ &+ 0 \left(\sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} n^h \varepsilon_{n+h+2} \right) \\ &= 0 \left(\sum_{\nu=1}^{\infty} \nu^h \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right) \\ &+ 0 \left(\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \right) \\ &= 0 \left(\sum_{\nu=1}^{\infty} \nu^h \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \lambda_{\nu+h+1} \sum_{n=\nu}^{\infty} (n - \nu + 1)^{h-r} n^{-h-2} \right) + 0(1) \\ &= 0 \left(\sum_{\nu=1}^{\infty} \nu^h \left| \Delta^{h-r+1} \varepsilon_{\nu+r+1} \right| \lambda_{\nu+h+1} \cdot \nu^{-r-1} \right) + 0(1) \\ &= 0 \left(\sum_{\nu=1}^{\infty} \nu^{h-r-1} \frac{\varepsilon_{\nu+1} \lambda_{\nu}}{(\nu + 1)^{h-r+1}} \right) + 0(1) \\ &= 0 \left(\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu}}{\nu} \right) + 0(1) \\ &= 0(1). \end{aligned}$$

This completes the proof of the theorem.

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