ANNALES

DE

L'INSTITUT FOURIER

Osamu FUJINO & Yoshinori GONGYO

On the moduli b-divisors of lc-trivial fibrations

Tome 64, nº 4 (2014), p. 1721-1735.

<http://aif.cedram.org/item?id=AIF_2014__64_4_1721_0>

Ann. Inst. Fourier, Grenoble
64, 4 (2014) 1721-1735

ON THE MODULI B-DIVISORS OF LC-TRIVIAL FIBRATIONS

by Osamu FUJINO & Yoshinori GONGYO

Abstract. — Roughly speaking, by using the semi-stable minimal model program, we prove that the moduli part of an lc-trivial fibration coincides with that of a klt-trivial fibration induced by adjunction after taking a suitable generically finite cover. As an application, we obtain that the moduli part of an lc-trivial fibration is b-nef and abundant by Ambro's result on klt-trivial fibrations.

Résumé. — Grosso modo, en utilisant le programme des modèles minimaux semi-stables, nous montrons que la partie modulaire d'une fibration lc-triviale coïncide avec celle d'une fibration klt-triviale induite par adjonction aprés changement de base par un morphisme génériquement fini. Comme application, eu utilisant le résultat de Ambro sur fibrations klt-triviales, on obtient que la partie modulaire d'une fibration lc-triviale est b-nef et abondante.

1. Introduction

In this paper, we prove the following theorem. More precisely, we reduce Theorem 1.1 to Ambro's result (see [2, Theorem 3.3]) by using the semi-stable minimal model program (see, for example, [14]). For a related result, see [6, Theorem 1.4].

Theorem 1.1 (cf. [2, Theorem 3.3]). — *Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that (X, B) is log canonical and $K_X + B \sim_{\mathbb{Q},Y} 0$. Then the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-nef and abundant.*

Let us recall the definition of *b-nef and abundant* $\mathbb{Q}$-b-divisors.

Keywords: semi-stable minimal model program, canonical bundle formulae, lc-trivial fibrations, klt-trivial fibrations.
Math. classification: 14N30, 14E30, 14J10.

Definition 1.2 ([2, Definition 3.2]). — *A* $\mathbb{Q}$*-b-divisor* $\mathbf{M}$ *of a normal complete algebraic variety* Y *is called* b-nef and abundant *if there exists a proper birational morphism* $Y' \to Y$ *from a normal variety* Y'*, endowed with a proper surjective morphism* $h : Y' \to Z$ *onto a normal variety* Z *with connected fibers, such that:*

(1) $\mathbf{M}_{Y'} \sim_{\mathbb{Q}} h^*H$*, for some nef and big* $\mathbb{Q}$*-divisor* H *of* Z*;*
(2) $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$*.*

Let us quickly explain the idea of the proof of Theorem 1.1. We assume that the pair (X, B) in Theorem 1.1 is dlt for simplicity. Let W be a log canonical center of (X, B) which is dominant onto Y and is minimal over the generic point of Y. We set $K_W + B_W = (K_X + B)|_W$ by adjunction. Then we have $K_W + B_W \sim_{\mathbb{Q},Y} 0$. Let $h : W \to Y'$ be the Stein factorization of $f|_W : W \to Y$. Note that (W, B_W) is klt over the generic point of Y'. We prove that the moduli part $\mathbf{M}$ of $f : (X, B) \to Y$ coincides with the moduli part $\mathbf{M}^{\min}$ of $h : (W, B_W) \to Y'$ after taking a suitable generically finite base change by using the semi-stable minimal model program. We do not need the *mixed* period map nor the infinitesimal *mixed* Torelli theorem (see [2, Section 2] and [23]) for the proof of Theorem 1.1. We just reduce the problem on lc-trivial fibrations to Ambro's result on klt-trivial fibrations, which follows from the theory of period maps. Our proof of Theorem 1.1 partially answers the questions in [21, 8.3.8 (Open problems)].

It is conjectured that $\mathbf{M}$ is b-semi-ample (see, for example, [1, 0. Introduction], [22, Conjecture 7.13.3], [6], [3], and [16, Section 3]). The b-semi-ampleness of the moduli part has been proved only for some special cases (see, for example, [19], [9], and [22, Section 8]). See also Remark 4.1 below.

Acknowledgments. — *The first author was partially supported by the Grant-in-Aid for Young Scientists (A) ♯24684002 from JSPS. The second author was partially supported by the Grant-in-Aid for Research Activity Start-up ♯24840009 from JSPS. The authors would like to thank Enrica Floris for giving them some comments and checking the French title and abstract. They also would like to thank the referee for some comments, suggestions, and drawing a big diagram in the proof of Theorem 1.1.*

We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will make use of the standard notation as in [12].

2. Preliminaries

Throughout this paper, we do not use $\mathbb{R}$-divisors. We only use $\mathbb{Q}$-divisors.

2.1 (Pairs). — *A pair (X, B) consists of a normal variety X over $\mathbb{C}$ and a $\mathbb{Q}$-divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. A pair (X, B) is called* subklt *(resp.* sublc*) if for any projective birational morphism $g : Z \to X$ from a normal variety Z, every coefficient of B_Z is < 1 (resp. $\leqslant 1$) where $K_Z + B_Z := g^*(K_X + B)$. A pair (X, B) is called* klt *(resp.* lc*) if (X, B) is subklt (resp. sublc) and B is effective. Let (X, B) be an lc pair. If there is a log resolution $g : Z \to X$ of (X, B) such that $\mathrm{Exc}(g)$ is a divisor and that the coefficients of the g-exceptional part of B_Z are < 1, then the pair (X, B) is called* divisorial log terminal *(*dlt*, for short). Let (X, B) be a sublc pair and let W be a closed subset of X. Then W is called a* log canonical center *of (X, B) if there are a projective birational morphism $g : Z \to X$ from a normal variety Z and a prime divisor E on Z such that $\mathrm{mult}_E B_Z = 1$ and that $g(E) = W$. Moreover we say that W is* minimal *if it is minimal with respect to inclusion.*

In this paper, we use the notion of *b-divisors* introduced by Shokurov. For details, we refer to [4, 2.3.2] and [15, Section 3].

2.2 (Canonical b-divisors). — *Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the* canonical b-divisor *of X.*

2.3 ($\mathbf{A}(X, B)$ and $\mathbf{A}^*(X, B)$). — *The* discrepancy b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ *of a pair (X, B) is the $\mathbb{Q}$-b-divisor of X with the trace $\mathbf{A}_Y$ defined by the formula*

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $\mathbf{A}^* = \mathbf{A}^*(X, B)$ by

$$\mathbf{A}^*_Y = \sum_{a_i > -1} a_i A_i$$

for

$$K_Y = f^*(K_X + B) + \sum a_i A_i,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Note that $\mathbf{A}(X, B) = \mathbf{A}^*(X, B)$ when (X, B) is subklt.

By the definition, we have $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) \simeq \mathcal{O}_X$ if (X, B) is lc (see [15, Lemma 3.19]*). We also have $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) \simeq \mathcal{O}_X$ when (X, B) is klt.*

2.4 (b-nef and b-semi-ample $\mathbb{Q}$-b-divisors). — *Let X be a normal variety and let $X \to S$ be a proper surjective morphism onto a variety S. A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is* b-nef over S *(resp.* b-semi-ample over S*) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is nef (resp. semi-ample) relative to the induced morphism $X' \to S$.*

2.5. — *Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety, where D_i is a prime divisor for every i, $D_i \neq D_j$ for $i \neq j$, and $d_i \in \mathbb{Q}$ for every i. Then we set*

$$D^{\geqslant 0} = \sum_{d_i \geqslant 0} d_i D_i \quad \text{and} \quad D^{\leqslant 0} = \sum_{d_i \leqslant 0} d_i D_i.$$

3. A quick review of lc-trivial fibrations

In this section, we quickly recall some basic definitions and results on *klt-trivial fibrations* and *lc-trivial fibrations* (see also [16, Section 3]).

Definition 3.1 (Klt-trivial fibrations). — *A* klt-trivial fibration $f : (X, B) \to Y$ *consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a pair (X, B) satisfying the following properties:*

(1) *(X, B) is subklt over the generic point of Y;*
(2) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = 1$;
(3) *There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that*

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

Note that Definition 3.1 is nothing but [1, Definition 2.1], where a klt-trivial fibration is called an lc-trivial fibration. So, our definition of lc-trivial fibrations in Definition 3.2 is different from the original one in [1, Definition 2.1].

Definition 3.2 (Lc-trivial fibrations). — *An* lc-trivial fibration $f : (X, B) \to Y$ *consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a pair (X, B) satisfying the following properties:*

(1) *(X, B) is sublc over the generic point of Y;*
(2) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) = 1$;
(3) *There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that*

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

In Section 4, we sometimes take various base changes and construct the induced lc-trivial fibrations and klt-trivial fibrations. For the details, see [1, Section 2].

3.3 (Induced lc-trivial fibrations by base changes). — *Let $f : (X, B) \to Y$ be a klt-trivial (resp. an lc-tirivial) fibration and let $\sigma : Y' \to Y$ be a generically finite morphism. Then we have an induced klt-trivial (resp. lc-trivial) fibration $f' : (X', B_{X'}) \to Y'$, where $B_{X'}$ is defined by $\mu^*(K_X + B) = K_{X'} + B_{X'}$:*

$$\begin{array}{ccc} (X', B_{X'}) & \xrightarrow{\mu} & (X, B) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\sigma]{} & Y, \end{array}$$

Note that X' is the normalization of the main component of $X \times_Y Y'$. We sometimes replace X' with X'' where X'' is a normal variety such that there is a proper birational morphism $\varphi : X'' \to X'$. In this case, we set $K_{X''} + B_{X''} = \varphi^*(K_{X'} + B_{X'})$.

Let us explain the definitions of the *discriminant* and *moduli* $\mathbb{Q}$-b-divisors.

3.4 (Discriminant $\mathbb{Q}$-b-divisors and moduli $\mathbb{Q}$-b-divisors). — *Let $f : (X, B) \to Y$ be an lc-trivial fibration as in Definition 3.2. Let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. We set*

$$b_P = \max \left\{ t \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X, B + tf^*P) \text{ is sublc over} \\ \text{the generic point of } P \end{array} \right\}$$

and set

$$B_Y = \sum_P (1 - b_P)P,$$

where P runs over prime divisors on Y. Then it is easy to see that B_Y is a well-defined $\mathbb{Q}$-divisor on Y and is called the discriminant $\mathbb{Q}$-divisor *of $f : (X, B) \to Y$. We set*

$$M_Y = D - K_Y - B_Y$$

and call M_Y the moduli $\mathbb{Q}$-divisor *of $f : (X, B) \to Y$. Let $\sigma : Y' \to Y$ be a proper birational morphism from a normal variety Y' and let $f' : (X', B_{X'}) \to Y'$ be the induced lc-trivial fibration by $\sigma : Y' \to Y$ (see 3.3). We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that $\sigma^*D = K_{Y'} + B_{Y'} + M_{Y'}$, $\sigma_*B_{Y'} = B_Y$, $\sigma_*K_{Y'} = K_Y$ and $\sigma_*M_{Y'} = M_Y$. Hence there exist a unique $\mathbb{Q}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'} = B_{Y'}$ for every $\sigma : Y' \to Y$ and a unique*

$\mathbb{Q}$-b-divisor $\mathbf{M}$ *such that* $\mathbf{M}_{Y'} = M_{Y'}$ *for every* $\sigma : Y' \to Y$*. Note that* $\mathbf{B}$ *is called the* discriminant $\mathbb{Q}$-b-divisor *and that* $\mathbf{M}$ *is called the* moduli $\mathbb{Q}$-b-divisor *associated to* $f : (X, B) \to Y$*. We sometimes simply say that* $\mathbf{M}$ *is the* moduli part *of* $f : (X, B) \to Y$*.*

For the basic properties of the discriminant and moduli $\mathbb{Q}$-b-divisors, see [1, Section 2].

Let us recall the main theorem of [1]. Note that a klt-trivial fibration in the sense of Definition 3.1 is called an lc-trivial fibration in [1].

Theorem 3.5 (see [1, Theorem 2.7]). — *Let* $f : (X, B) \to Y$ *be a klt-trivial fibration and let* $\pi : Y \to S$ *be a proper morphism. Let* $\mathbf{B}$ *and* $\mathbf{M}$ *be the induced discriminant and moduli* $\mathbb{Q}$*-b-divisors of* f*. Then,*

(1) $\mathbf{K} + \mathbf{B}$ *is* $\mathbb{Q}$*-b-Cartier, that is, there exists a proper birational morphism* $Y' \to Y$ *from a normal variety* Y' *such that* $\mathbf{K} + \mathbf{B} = \overline{K_{Y'} + \mathbf{B}_{Y'}}$,
(2) $\mathbf{M}$ *is b-nef over* S*.*

Theorem 3.5 has some important applications, see, for example, [13, Proof of Theorem 1.1] and [15, The proof of Theorem 1.1].

By modifying the arguments in [1, Section 5] suitably with the aid of [10, Theorems 3.1, 3.4, and 3.9] (see also [17]), we can generalize Theorem 3.5 as follows.

Theorem 3.6. — *Let* $f : (X, B) \to Y$ *be an lc-trivial fibration and let* $\pi : Y \to S$ *be a proper morphism. Let* $\mathbf{B}$ *and* $\mathbf{M}$ *be the induced discriminant and moduli* $\mathbb{Q}$*-b-divisors of* f*. Then,*

(1) $\mathbf{K} + \mathbf{B}$ *is* $\mathbb{Q}$*-b-Cartier,*
(2) $\mathbf{M}$ *is b-nef over* S*.*

Theorem 3.5 is proved by using the theory of variations of Hodge structure. On the other hand, Theorem 3.6 follows from the theory of variations of *mixed* Hodge structure. We do not adopt the formulation in [7, Section 4] (see also [21, 8.5]) because the argument in [1] suits our purposes better. For the reader's convenience, we include the main ingredient of the proof of Theorem 3.6, which easily follows from [10, Theorems 3.1, 3.4, and 3.9] (see also [17]).

Theorem 3.7 (cf. [1, Theorem 4.4]). — *Let* $f : X \to Y$ *be a projective morphism between algebraic varieties. Let* Σ_X *(resp.* Σ_Y*) be a simple normal crossing divisor on* X *(resp.* Y*) such that* f *is smooth over* $Y \setminus \Sigma_Y$*,* Σ_X *is relatively normal crossing over* $Y \setminus \Sigma_Y$*, and* $f^{-1}(\Sigma_Y) \subset \Sigma_X$*. Assume that*

f is semi-stable in codimension one. Let D be a simple normal crossing divisor on X such that $\operatorname{Supp} D \subset \Sigma_X$ and that every irreducible component of D is dominant onto Y. Then the following properties hold.

(1) *$R^p f_* \omega_{X/Y}(D)$ is a locally free sheaf on Y for every p.*
(2) *$R^p f_* \omega_{X/Y}(D)$ is semi-positive for every p.*
(3) *Let $\rho : Y' \to Y$ be a projective morphism from a smooth variety Y' such that $\Sigma_{Y'} = \rho^{-1}(\Sigma_Y)$ is a simple normal crossing divisor on Y'. Let $\pi : X' \to X \times_Y Y'$ be a resolution of the main component of $X \times_Y Y'$ such that π is an isomorphism over $Y' \setminus \Sigma_{Y'}$. Then we obtain the following commutative diagram:*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \underset{\rho}{\longrightarrow} & Y. \end{array}$$

Assume that f' is projective, D' is a simple normal crossing divisor on X' such that D' coincides with $D \times_Y Y'$ over $Y' \setminus \Sigma_{Y'}$, and every stratum of D' is dominant onto Y'. Then there exists a natural isomorphism

$$\rho^*(R^p f_* \omega_{X/Y}(D)) \simeq R^p f'_* \omega_{X'/Y'}(D')$$

which extends the base change isomorphism over $Y \setminus \Sigma_Y$ for every p.

Remark 3.8. — For the proof of Theorem 3.6, Theorem 3.7 for $p = 0$ is sufficient. Note that all the local monodromies on $R^q(f_0)_* \mathbb{C}_{X_0 \setminus D_0}$ around Σ_Y are unipotent for every q because f is semi-stable in codimension one, where $X_0 = f^{-1}(Y \setminus \Sigma_Y)$, $D_0 = D|_{X_0}$, and $f_0 = f|_{X_0 \setminus D_0}$. More precisely, let $C_0^{[d]}$ be the disjoint union of all the codimension d log canonical centers of (X_0, D_0). If $d = 0$, then we put $C_0^{[0]} = X_0$. In this case, we have the following weight spectral sequence

$${}_W E_1^{-d, q+d} = R^{q-d}(f|_{C_0^{[d]}})_* \mathbb{C}_{C_0^{[d]}} \Longrightarrow R^q(f_0)_* \mathbb{C}_{X_0 \setminus D_0}$$

which degenerates at E_2 (see, for example, [5, Corollaire (3.2.13)]). Since f is semi-stable in codimension one, all the local monodromies on $R^{q-d}(f|_{C_0^{[d]}})_* \mathbb{C}_{C_0^{[d]}}$ around Σ_Y are unipotent for every q and d (see, for example, [18, VII. The Monodromy theorem]). By the above spectral sequence, we obtain that all the local monodromies on $R^q(f_0)_* \mathbb{C}_{X_0 \setminus D_0}$ around Σ_Y are unipotent.

We add a remark on the proof of Theorem 3.6. In Remark 3.9, we explain how to modify the arguments in the proof of [1, Lemma 5.2] in order to treat lc-trivial fibrations. It will help the reader to understand the main difference between klt-trivial fibrations and lc-trivial fibrations and the reason why we need Theorem 3.7.

Remark 3.9. — We use the notation in [1, Lemma 5.2]. We only assume that (X, B) is sublc over the generic point of Y in [1, Lemma 5.2]. We write

$$B = \sum_{i \in I} d_i B_i$$

where B_i is a prime divisor for every i and $B_i \neq B_j$ for $i \neq j$. We set

$$J = \{ i \in I \mid B_i \text{ is dominant onto } Y \text{ and } d_i = 1 \}$$

and set

$$D = \sum_{i \in J} B_i.$$

In Ambro's original setting in [1, Lemma 5.2], we have $D = 0$ because (X, B) is subklt over the generic point of Y. In the proof of [1, Lemma 5.2 (4)], we have to replace

$$\widetilde{f}_* \omega_{\widetilde{X}/Y} = \bigoplus_{i=0}^{b-1} f_* \mathcal{O}_X(\lceil (1-i) K_{X/Y} - iB + if^* B_Y + if^* M_Y \rceil) \cdot \psi^i.$$

with

$$\widetilde{f}_* \omega_{\widetilde{X}/Y}(\pi^* D) = \bigoplus_{i=0}^{b-1} f_* \mathcal{O}_X(\lceil (1-i) K_{X/Y} - iB + D + if^* B_Y + if^* M_Y \rceil) \cdot \psi^i$$

in order to treat lc-trivial fibrations. We leave the details as exercises for the reader.

The following theorem is a part of [2, Theorem 3.3].

Theorem 3.10 (see [2, Theorem 3.3]). — *Let $f : (X, B) \to Y$ be a klt-trivial fibration such that Y is complete, the geometric generic fiber $X_{\overline{\eta}} = X \times \operatorname{Spec} \overline{\mathbb{C}(\eta)}$ is a projective variety, and $B_{\overline{\eta}} = B|_{X_{\overline{\eta}}}$ is effective, where η is the generic point of Y. Then the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-nef and abundant.*

4. Proof of Theorem 1.1

Let us give a proof of Theorem 1.1.

Proof of Theorem 1.1. — By taking a dlt blow-up, we may assume that the pair (X, B) is $\mathbb{Q}$-factorial and dlt (see, for example, [14, Section 4]). If (X, B) is klt over the generic point of Y, then Theorem 1.1 follows from [2, Theorem 3.3] (see Theorem 3.10). Therefore, we may also assume that (X, B) is not klt over the generic point of Y. Let $\sigma_1 : Y_1 \to Y$ be a suitable projective birational morphism such that $\mathbf{M} = \overline{\mathbf{M}_{Y_1}}$ and $\mathbf{M}_{Y_1}$ is nef by Theorem 3.6. Let W be an arbitrary log canonical center of (X, B) which is dominant onto Y and is minimal over the generic point of Y. We set

$$K_W + B_W = (K_X + B)|_W$$

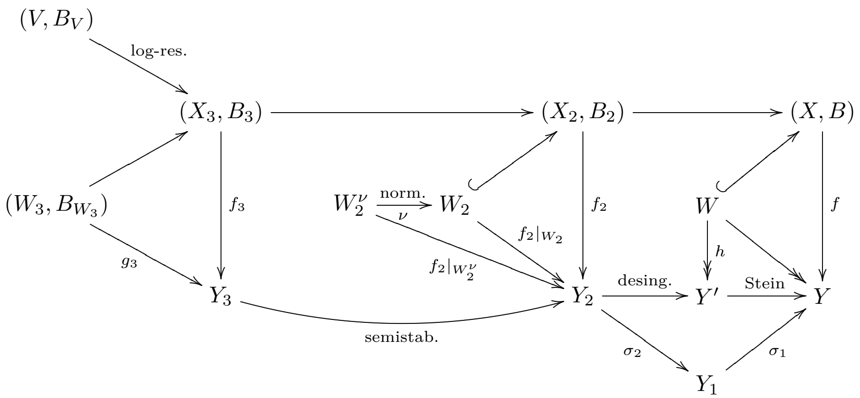
by adjunction (see, for example, [11, 3.9]). By the construction, we have $K_W + B_W \sim_{\mathbb{Q},Y} 0$. We consider the Stein factorization of $f|_W : W \to Y$ and denote it by $h : W \to Y'$. Then $K_W + B_W \sim_{\mathbb{Q},Y'} 0$. We see that $h : (W, B_W) \to Y'$ is a klt-trivial fibration since the general fibers of $f|_W$ are klt pairs. Let Y_2 be a suitable resolution of Y' which factors through $\sigma_1 : Y_1 \to Y$. By taking the base change by $\sigma_2 : Y_2 \to Y_1$, we obtain $\mathbf{M}_{Y_2} = \sigma_2^* \mathbf{M}_{Y_1}$ (see [1, Proposition 5.5]). Note that the proof of [1, Proposition 5.5] works for lc-trivial fibrations by some suitable modifications. By the construction, on the induced lc-trivial fibration $f_2 : (X_2, B_{X_2}) \to Y_2$, where X_2 is the normalization of the main component of $X \times_Y Y_2$, there is a log canonical center W_2 of (X_2, B_{X_2}) such that $f_2|_{W_2^\nu} : (W_2^\nu, B_{W_2^\nu}) \to Y_2$ is a klt-trivial fibration, which is birationally equivalent to $h : (W, B_W) \to Y'$. Note that $\nu : W_2^\nu \to W_2$ is the normalization, $K_{W_2^\nu} + B_{W_2^\nu} = \nu^*(K_{X_2} + B_{X_2})|_{W_2}$, and $f_2|_{W_2^\nu} = f_2|_{W_2} \circ \nu$. It is easy to see that

$$K_{Y_2} + \mathbf{M}_{Y_2} + \mathbf{B}_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + \mathbf{M}^{\min}_{Y_2} + \mathbf{B}^{\min}_{Y_2}$$

where $\mathbf{M}^{\min}$ and $\mathbf{B}^{\min}$ are the induced moduli and discriminant $\mathbb{Q}$-b-divisors of $f_2|_{W_2^\nu} : (W_2^\nu, B_{W_2^\nu}) \to Y_2$ such that

$$K_{W_2^\nu} + B_{W_2^\nu} \sim_{\mathbb{Q}} (f_2|_{W_2^\nu})^*(K_{Y_2} + \mathbf{M}^{\min}_{Y_2} + \mathbf{B}^{\min}_{Y_2}).$$

By replacing Y_2 birationally, we may further assume that $\mathbf{M}^{\min} = \overline{\mathbf{M}^{\min}_{Y_2}}$ by Theorem 3.5. By Theorem 3.10, we see that $\mathbf{M}^{\min}_{Y_2}$ is nef and abundant. Let $\sigma_3 : Y_3 \to Y_2$ be a suitable generically finite morphism such that the induced lc-trivial fibration $f_3 : (X_3, B_{X_3}) \to Y_3$ has a semi-stable resolution in codimension one (see, for example, [20], [23, (9.1) Theorem], and [1, Theorem 4.3]). Note that X_3 is the normalization of the main component of $X \times_Y Y_3$. Here we draw the following big diagram for the reader's convenience.



Note that $g_3 : (W_3, B_{W_3}) \to Y_3$ is the induced klt-trivial fibration from $f_2|_{W_2^\nu} : W_2^\nu \to Y_2$ by $\sigma_3 : Y_3 \to Y_2$. On Y_3, we will see the following claim by using the semi-stable minimal model program.

Claim. — *The following equality*

$$\mathbf{B}_{Y_3} = \mathbf{B}_{Y_3}^{\min}$$

holds.

Proof of Claim. — By taking general hyperplane cuts, we may assume that Y_3 is a curve. We write

$$\mathbf{B}_{Y_3} = \sum_P (1 - b_P)P \quad \text{and} \quad \mathbf{B}_{Y_3}^{\min} = \sum_P (1 - b_P^{\min})P.$$

Let $\varphi : (V, B_V) \to (X_3, B_{X_3})$ be a resolution of (X_3, B_{X_3}) with the following properties:

- $K_V + B_V = \varphi^*(K_{X_3} + B_{X_3})$;
- π^*Q is a reduced simple normal crossing divisor on V for every $Q \in Y_3$, where $\pi : V \to X_3 \to Y_3$;
- $\operatorname{Supp} \pi^*Q \cup \operatorname{Supp} B_V$ is a simple normal crossing divisor on V for every $Q \in Y_3$;
- π is projective.

Let Σ be a reduced divisor on Y_3 such that π is smooth over $Y_3 \setminus \Sigma$ and that $\operatorname{Supp} B_V$ is relatively normal crossing over $Y_3 \setminus \Sigma$. We consider the set of prime divisors $\{E_i\}$ where E_i is a prime divisor on V such that $\pi(E_i) \in \Sigma$ and

$$\operatorname{mult}_{E_i}(B_V + \sum_{P \in \Sigma} b_P \pi^* P)^{\geqslant 0} < 1.$$

We run the minimal model programs with ample scaling with respect to

$$K_V + (B_V + \sum_{P\in\Sigma} b_P \pi^* P)^{\geqslant 0} + \varepsilon \sum_i E_i$$

over X_3 and Y_3 for some small positive rational number ε. Note that

$$(V, (B_V + \sum_P b_P \pi^* P)^{\geqslant 0} + \varepsilon \sum_i E_i)$$

is a $\mathbb{Q}$-factorial dlt pair because $0 < \varepsilon \ll 1$. We set

$$E = -(B_V + \sum_P b_P \pi^* P)^{\leqslant 0} + \varepsilon \sum_i E_i.$$

Then it holds that

$$K_V + (B_V + \sum_P b_P \pi^* P)^{\geqslant 0} + \varepsilon \sum_i E_i \sim_{\mathbb{Q}, Y_3} E \geqslant 0.$$

First we run a minimal model program with ample scaling with respect to

$$K_V + (B_V + \sum_P b_P \pi^* P)^{\geqslant 0} + \varepsilon \sum_i E_i \sim_{\mathbb{Q}, X_3} E \geqslant 0$$

over X_3. Note that every irreducible component of E which is dominant onto Y_3 is exceptional over X_3 by the construction. Thus, if E is dominant onto Y_3, then it is not contained in the relative movable cone over X_3. Therefore, after finitely many steps, we may assume that every irreducible component of E is contained in a fiber over Y_3 (see, for example, [14, Theorem 2.2]). Next we run a minimal model program with ample scaling with respect to

$$K_V + (B_V + \sum_P b_P \pi^* P)^{\geqslant 0} + \varepsilon \sum_i E_i \sim_{\mathbb{Q}, Y_3} E \geqslant 0$$

over Y_3. Then the minimal model program terminates at V' (see, for example, [14, Theorem 2.2]). Note that all the components of $E + \sum_i E_i$ are contracted by the above minimal model programs. Thus, we have

$$K_{V'} + (B_{V'} + \sum_P b_P \pi'^* P)^{\geqslant 0} \sim_{\mathbb{Q}, Y_3} 0,$$

where $\pi' : V' \to Y_3$ and $B_{V'}$ is the pushforward of B_V on V'. Note that $B_{V'} + \sum_P b_P \pi'^* P$ is effective since $\operatorname{Supp}(E + \sum_i E_i)$ is contracted by the above minimal model programs. Of course, we see that

$$(V', (B_{V'} + \sum_P b_P \pi'^* P)^{\geqslant 0}) = (V', B_{V'} + \sum_P b_P \pi'^* P)$$

is a $\mathbb{Q}$-factorial dlt pair. By the construction, the induced proper birational map

$$(V, B_V + \sum_P b_P \pi^* P) \dashrightarrow (V', B_{V'} + \sum_P b_P \pi'^* P)$$

over Y_3 is B-birational (see [8, Definition 1.5]), that is, we have a common resolution

$$\begin{array}{ccc} & Z & \\ {\scriptstyle a} \swarrow & & \searrow {\scriptstyle b} \\ V & \dashrightarrow & V' \end{array}$$

over Y_3 such that

$$a^*(K_V + B_V + \sum_{P \in \Sigma} b_P \pi^* P) = b^*(K_{V'} + B_{V'} + \sum_{P \in \Sigma} b_P \pi'^* P).$$

Let S be any log canonical center of $(V', B_{V'} + \sum_P b_P \pi'^* P)$ which is dominant onto Y_3 and is minimal over the generic point of Y_3. Then (S, B_S), where

$$K_S + B_S = (K_{V'} + B_{V'} + \sum_P b_P \pi'^* P)|_S,$$

is not klt but lc over every $P \in \Sigma$ since it holds that

$$B_{V'} + \sum_{P \in \Sigma} b_P \pi'^* P \geqslant \sum_{P \in \Sigma} \pi'^* P. \qquad (\spadesuit)$$

Note that ($\spadesuit$) follows from the fact that all the components of $\sum_i E_i$ are contracted in the minimal model programs. Let $g_3 : (W_3, B_{W_3}) \to Y_3$ be the induced klt-trivial fibration from $(W_2^\nu, B_{W_2^\nu}) \to Y_2$ by $\sigma_2 : Y_3 \to Y_2$. By [8, Claims (A_n) and (B_n) in the proof of Lemma 4.9], there is a log canonical center S_0 of $(V', B_{V'} + \sum_P b_P \pi'^* P)$ which is dominant onto Y_3 and is minimal over the generic point of Y_3 such that there is a B-birational map

$$(W_3, B_{W_3} + \sum_{P \in \Sigma} b_P g_3^* P) \dashrightarrow (S_0, B_{S_0})$$

over Y_3, where

$$K_{S_0} + B_{S_0} = (K_{V'} + B_{V'} + \sum_{P \in \Sigma} b_P \pi'^* P)|_{S_0}.$$

This means that there is a common resolution

$$\begin{array}{ccc} & T & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ W_3 & \dashrightarrow & S_0 \end{array}$$

over Y_3 such that

$$\alpha^*(K_{W_3} + B_{W_3} + \sum_P b_P g_3^* P) = \beta^*(K_{S_0} + B_{S_0}).$$

This implies that $b_P = b_P^{\min}$ for every $P \in \Sigma$. Therefore, we have $\mathbf{B}_{Y_3} = \mathbf{B}_{Y_3}^{\min}$. □

Then we obtain

$$\mathbf{M}_{Y_3} \sim_{\mathbb{Q}} \mathbf{M}_{Y_3}^{\min} = \sigma_3^* \mathbf{M}_{Y_2}^{\min}$$

because

$$K_{Y_3} + \mathbf{M}_{Y_3} + \mathbf{B}_{Y_3} \sim_{\mathbb{Q}} K_{Y_3} + \mathbf{M}_{Y_3}^{\min} + \mathbf{B}_{Y_3}^{\min}.$$

Thus, $\mathbf{M}_{Y_3}$ is nef and abundant. Since

$$\mathbf{M}_{Y_3} = \sigma_3^* \mathbf{M}_{Y_2} = \sigma_3^* \sigma_2^* \mathbf{M}_{Y_1},$$

$\mathbf{M}$ is b-nef and abundant. Moreover, by replacing Y_3 with a suitable generically finite cover, we have that $\mathbf{M}_{Y_3}$ and $\mathbf{M}_{Y_3}^{\min}$ are both Cartier (see [1, Lemma 5.2 (5), Proposition 5.4, and Proposition 5.5]) and $\mathbf{M}_{Y_3} \sim \mathbf{M}_{Y_3}^{\min}$. □

We close this paper with a remark on the b-semi-ampleness of $\mathbf{M}$. For some related topics, see [16, Section 3].

Remark 4.1 (b-semi-ampleness). — Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that (X, B) is log canonical and $K_X + B \sim_{\mathbb{Q},Y} 0$. Without loss of generality, we may assume that (X, B) is dlt by taking a dlt blow-up. We set

$$d_f(X, B) = \left\{ \dim W - \dim Y \;\middle|\; \begin{array}{l} W \text{ is a log canonical center of} \\ (X, B) \text{ which is dominant onto } Y \end{array} \right\}.$$

If $d_f(X, B) \in \{0, 1\}$, then the b-semi-ampleness of the moduli part $\mathbf{M}$ follows from [19] and [22] by the proof of Theorem 1.1. Moreover, it is obvious that $\mathbf{M} \sim_{\mathbb{Q}} 0$ when $d_f(X, B) = 0$.

BIBLIOGRAPHY

[1] F. Ambro, " Shokurov's boundary property", *J. Differential Geom.* **67** (2004), no. 2, p. 229-255.

[2] ———, "The moduli b-divisor of an lc-trivial fibration", *Compos. Math.* **141** (2005), no. 2, p. 385-403.

[3] C. Birkar & Y. Chen, "Images of manifolds with semi-ample anti-canonical divisor", to appear in J. Algebraic Geom.

[4] A. Corti, "3-fold flips after Shokurov", in *Flips for 3-folds and 4-folds*, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, p. 18-48.

[5] P. Deligne, "Théorie de Hodge. II", (French) *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), p. 5-57.

[6] E. Floris, "Inductive approach to effective b-semiampleness", *Int. Math. Res. Not. IMRN* **2014** (2014), no. 6, p. 1645-1492.

[7] O. Fujino, "Higher direct images of log canonical divisors and positivity theorems", preprint (2003). arXiv:math/0302073v1.

[8] ———, "Abundance theorem for semi log canonical threefolds", *Duke Math. J.* **102** (2000), no. 3, p. 513-532.

[9] ———, "A canonical bundle formula for certain algebraic fiber spaces and its applications", *Nagoya Math. J.* **172** (2003), p. 129-171.

[10] ———, "Higher direct images of log canonical divisors", *J. Differential Geom.* **66** (2004), no. 3, p. 453-479.

[11] ———, "What is log terminal?", in *Flips for 3-folds and 4-folds*, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, p. 49-62.

[12] ———, "Fundamental theorems for the log minimal model program", *Publ. Res. Inst. Math. Sci.* **47** (2011), no. 3, p. 727-789.

[13] ———, "On Kawamata's theorem", in *Classification of algebraic varieties*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, p. 305-315.

[14] ———, "Semi-stable minimal model program for varieties with trivial canonical divisor", *Proc. Japan Acad. Ser. A Math. Sci.* **87** (2011), no. 3, p. 25-30.

[15] ———, "Basepoint-free theorems: saturation, b-divisors, and canonical bundle formula", *Algebra Number Theory* **6** (2012), no. 4, p. 797-823.

[16] ———, "Some remarks on the minimal model program for log canonical pairs", to appear in Kodaira Centennial issue of Journal of Mathematical Sciences, the University of Tokyo.

[17] O. Fujino & T. Fujisawa, "Variations of mixed Hodge structure and semipositivity theorems", to appear in Publ. Res. Inst. Math. Sci.

[18] N. M. Katz, "The regularity theorem in algebraic geometry", in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, Gauthier-Villars, Paris, 1971, p. 437-443.

[19] Y. Kawamata, " Subadjunction of log canonical divisors for a subvariety of codimension 2", in *Birational algebraic geometry (Baltimore, MD, 1996)*, Contemp. Math., Amer. Math. Soc., Providence, RI, 1997, p. 79-88.

[20] G. Kempf, F. Knudsen, D. Mumford & S. B., *Toroidal embeddings. I*, Lecture Notes in Mathematics, vol. 339, Springer-Verlag, Berlin-New York, 1973.

[21] J. Kollár, "Kodaira's canonical bundle formula and adjunction", in *Flips for 3-folds and 4-folds*, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, p. 134-162.

[22] Y. G. Prokhorov & V. V. Shokurov, "Towards the second main theorem on complements", *J. Algebraic Geom.* **18** (2009), no. 1, p. 151-199.

[23] M.-H. Saito, Y. Shimizu & S. Usui, “Variation of mixed Hodge structure and the Torelli problem”, in *Algebraic geometry, Sendai, 1985*, Adv. Stud. Pure Math., North-Holland, Amsterdam, 1987, p. 649-693.

Manuscrit reçu le 19 octobre 2012,
révisé le 13 novembre 2013,
accepté le 3 avril 2014.

Osamu FUJINO
Department of Mathematics,
Faculty of Science,
Kyoto University,
Kyoto 606-8502, Japan
fujino@math.kyoto-u.ac.jp

Yoshinori GONGYO
Graduate School of Mathematical Sciences,
The University of Tokyo, 3-8-1 Komaba,
Meguro,
Tokyo, 153-8914 Japan.
Department of Mathematics,
Imperial College London,
180 Queen’s Gate,
London SW7 2AZ, UK
gongyo@ms.u-tokyo.ac.jp
y.gongyo@imperial.ac.uk