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FINITENESS OF ERGODIC UNITARILY INVARIANT MEASURES ON SPACES OF INFINITE MATRICES

by Alexander I. BUFETOV

Abstract. — The main result of this note, Theorem 1.3, is the following: a Borel measure on the space of infinite Hermitian matrices, that is invariant and ergodic under the action of the infinite unitary group and that admits well-defined projections onto the quotient space of "corners" of finite size, must be finite. A similar result, Theorem 1.1, is also established for unitarily invariant measures on the space of all infinite complex matrices. These results imply that the infinite Hua-Pickrell measures of Borodin and Olshanski have finite ergodic components.

The proof is based on the approach of Olshanski and Vershik. First, it is shown that if the sequence of orbital measures assigned to almost every point is weakly precompact, then our ergodic measure must indeed be finite. The second step, which completes the proof, shows that if a unitarily-invariant measure admits well-defined projections onto the quotient space of finite corners, then for almost every point the corresponding sequence of orbital measures is indeed weakly precompact.

Résumé. — Le résultat principal de cet article, Théorème 1.3, affirme que si une mesure borélienne sur l'espace des matrices hermitiennes infinies, invariante et ergodique par l'action du groupe unitaire infini admet, en plus, des projections sur l'espace quotient des matrices finies, alors la mesure est elle-même finie. Un résultat similaire, Théorème 1.1, est obtenu pour les mesures invariantes par l'action du groupe unitaire sur l'espace de toutes les matrices complexes infinies. Ces résultats impliquent que toutes les composantes ergodiques des mesures infinies de Hua-Pickrell introduites par Borodin et Olshanski doivent être finies.

L'argument se base sur l'approche d'Olshanski et Vershik. On démontre d'abord que la mesure ergodique doit être finie si la suite des mesures orbitales d'un point générique est précompacte. Le deuxième pas qui conclut la preuve est la vérification de la précompacité des suites des mesures orbitales.

Keywords: Infinite-dimensional Lie groups, classification of ergodic measures, Hua-Pickrell measures, orbital measures, weak compactness.
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1. Introduction

1.1. Statement of the mainresults

1.1.1. Unitarily invariant measures on spaces of infinite complex matrices

Let $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ be the space of all infinite matrices whose rows and columns are indexed by natural numbers and whose entries are complex:

$$\mathrm{Mat}(\mathbb{N},\mathbb{C}) \;=\; \{z=(z_{ij})_{i,j\in\mathbb{N}},\; z_{ij}\in\mathbb{C}\}\,.$$

Let $U(\infty)$ be the infinite unitary group: an infinite matrix $u=(u_{ij})_{i,j\in\mathbb{N}}$ belongs to $U(\infty)$ if there exists a natural number n_0 such that the matrix

$$(u_{ij})_{i,j\in[1,n_0]}$$

is unitary, while $u_{ii}=1$ if $i>n_0$ and $u_{ij}=0$ if $i\neq j$, $\max(i,j)>n_0$.

The group $U(\infty)\times U(\infty)$ acts on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ by multiplication on both sides:

$$T_{(u_1,u_2)}z \;=\; u_1 z u_2^{-1}.$$

Recall that a $U(\infty)\times U(\infty)$-invariant measure on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$, finite or infinite, is called *ergodic* if any $U(\infty)\times U(\infty)$-invariant Borel set either has measure zero or has complement of measure zero. Finite ergodic $U(\infty)\times U(\infty)$-invariant measures on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ have been classified by Pickrell [8]. The first main result of this paper is that, under natural assumptions, an ergodic $U(\infty)\times U(\infty)$-invariant measure on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ must be finite.

Precisely, let $m\in\mathbb{N}$ and let $\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ denote the space of Borel measures ν on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that for any $R>0$ we have

$$\nu\left(\left\{z\in\mathrm{Mat}(\mathbb{N},\mathbb{C}):\max_{i,j\leqslant m}|z_{ij}|<R\right\}\right)<+\infty.$$

Note that $\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))\subset\mathfrak{F}(m+1;\mathrm{Mat}(\mathbb{N},\mathbb{C})))$ for any $m\in\mathbb{N}$.

THEOREM 1.1. — *If a $U(\infty)\times U(\infty)$-invariant Borel measure from the class $\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ is ergodic then it is finite.*

A measure $\nu\in\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ is automatically sigma-finite, clearly satisfies all assumptions of the ergodic decomposition theorem of [3] and therefore admits a decomposition into ergodic components. By definition, almost all ergodic components of a measure $\nu\in\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ must themselves lie in the class $\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))$. Let $\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ stand for the set of $U(\infty)\times U(\infty)$-invariant ergodic Borel probability measures on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$; the set $\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ is a Borel subset of the space of all Borel probability measures on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ (see, e.g., [3], where the claim

is proved for all measurable Borel actions of inductively compact groups). Theorem 1.1 and the ergodic decomposition theorem of [3] now implies the following

COROLLARY 1.2. — *For any $U(\infty)\times U(\infty)$-invariant Borel measure*

$$\nu\in\mathfrak{F}(m;\mathrm{Mat}(\mathbb{N},\mathbb{C}))$$

there exists a unique sigma-finite Borel measure $\tilde{\nu}$ on $\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N},\mathbb{C}))$ such that

$$\nu=\int\limits_{\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N},\mathbb{C}))}\eta d\tilde{\nu}(\eta). \tag{1.1}$$

The integral in (1.1) is understood in the usual weak sense: for every Borel subset $A\subset\mathrm{Mat}(\mathbb{N},\mathbb{C})$ we have

$$\nu(A)=\int\limits_{\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N},\mathbb{C}))}\eta(A)d\tilde{\nu}(\eta).$$

1.1.2. Unitarily invariant measures on spaces of infinite Hermitian matrices

Now let $H\subset\mathrm{Mat}(\mathbb{N},\mathbb{C})$ be the space of infinite Hermitian matrices:

$$H=\{h=(h_{ij})_{i,j\in\mathbb{N}},h_{ij}=\overline{h}_{ji}\}.$$

The group $U(\infty)$ naturally acts on the space H by conjugation. Finite ergodic $U(\infty)$-invariant measures on H have also been classified by Pickrell [8] (see also Olshanski and Vershik [6]). An analogue of Theorem 1.1 holds in this case as well.

Precisely, a Borel measure ν on H is said to belong to the class $\mathfrak{F}(m,H)$ if for any $R>0$ we have

$$\nu(\{h\in H:\max_{i\leqslant m,j\leqslant m}|h_{ij}|\leqslant R\})<\infty.$$

THEOREM 1.3. — *If a $U(\infty)$-invariant measure from the class $\mathfrak{F}(m,H)$ is ergodic, then it is finite.*

As before, let $\mathfrak{M}_{erg}(H)$ stand for the set of $U(\infty)$-invariant ergodic Borel probability measures on H; the set $\mathfrak{M}_{erg}(H)$ is a Borel subset of the space of all Borel probability measures on H. Theorem 1.3 now implies

COROLLARY 1.4. — *For any $U(\infty)$-invariant Borel measure $\nu \in \mathfrak{F}(m, H)$ there exists a unique sigma-finite Borel measure $\tilde{\nu}$ on $\mathfrak{M}_{erg}(H)$ such that*

$$\nu = \int\limits_{\mathfrak{M}_{erg}(H)} \eta d\tilde{\nu}(\eta). \tag{1.2}$$

The integral in (1.2) is again understood in the weak sense.

One expects similar results to hold for all the 10 series of homogeneous spaces (see, e.g., [5, 4]).

1.1.3. Infinite Hua-Pickrell measures

A natural example of measures lying in the class $\mathfrak{F}(m, H)$ is given by infinite Hua-Pickrell measures introduced by Borodin and Olshanski [2], Section 8, Subsection "Infinite measures". In fact, for any $m \in \mathbb{N}$, Borodin and Olshanski give explicit examples of measures lying in the class $\mathfrak{F}(m, H)$ but not in the class $\mathfrak{F}(m-1, H)$. Starting from the Pickrell measures [7], a similar construction can be carried out to obtain infinite $U(\infty) \times U(\infty)$-invariant measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ lying in the class $\mathfrak{F}(m, \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ but not in the class $\mathfrak{F}(m-1, \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ for any $m \in \mathbb{N}$. Corollaries 1.2, 1.4 show now that ergodic components of infinite Hua-Pickrell measures are finite.

1.2. Outline of the proofs of Theorems 1.1, 1.3.

Olshanski and Vershik [6] gave a completely different proof for Pickrell's Classification Theorem of finite $U(\infty)$-invariant ergodic measures on H, and their method has been adapted to finite ergodic $U(\infty) \times U(\infty)$-invariant measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ by Rabaoui [10], [9]. The proof of Theorems 1.1, 1.3 is based on the Olshanski-Vershik approach.

First, following Vershik [11], to each infinite matrix we assign its sequence of *orbital measures* obtained by averaging over exhausting sequences of compact subgroups in our infinite-dimensional unitary groups. A simple general argument shows that precompactness of the family of orbital measures for almost all points implies finiteness of an ergodic measure. Using the work of Olshanski and Vershik [6] and Rabaoui [10], [9], we give a sufficient condition, called "radial boundedness" of a matrix, for weak precompactness of its family of orbital measures: namely, it is shown that the sequence of orbital measures is weakly precompact as soon as the norms (and, in case of H, also the traces) of $n \times n$ "corners" of our matrix do not grow too fast

as $n \to \infty$. To complete the proof of Theorem 1.3, it remains to show that with respect to any measure in the class $\mathfrak{F}(m, H)$, almost all matrices are indeed radially bounded (the same statement, with the same proof, also holds for $\mathfrak{F}(m; \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$). This is done in two steps: first, it is shown that if a measure from the class $\mathfrak{F}(m, H)$ is $U(\infty)$-invariant, then its suitably averaged conditional measures yield a *finite* $U(\infty)$-invariant measure — with respect to which almost all points must then be radially bounded; second, applying a finite permutation of columns and rows, one deduces radial boundedness for the initial matrix and completes the proof.

1.3. Projections and conditional measures

For $n \in \mathbb{N}$, let $\mathrm{Mat}(n, \mathbb{C})$ be the space of all $n \times n$ complex matrices. Introduce a map

$$\Pi_{[1,n]} : \mathrm{Mat}(\mathbb{N}, \mathbb{C}) \to \mathrm{Mat}(n, \mathbb{C})$$

by the formula

$$\Pi_{[1,n]} z \;=\; (z_{ij})_{i,j=1,\dots,n},\; z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C}).$$

If a measure ν on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is infinite, then the projection $\left(\Pi_{[1,n]}\right)_* \nu$ may fail to be well-defined. The class $\mathfrak{F}(m; \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ consists precisely of those measures ν for which the projection $\left(\Pi_{[1,m]}\right)_* \nu$ (and, consequently, all projections $\left(\Pi_{[1,n]}\right)_* \nu$ for $n \geqslant m$) are indeed well-defined. Equivalently, by Rohlin's Theorem on existence of conditional measures, a measure ν belongs to the class $\mathfrak{F}(m; \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ if and only if:

(1) there exists a measure $\overline{\nu}$ on the space $\mathrm{Mat}(m, \mathbb{C})$ assigning finite weight to every compact set;
(2) for $\overline{\nu}$-almost every $z^{(m)} \in \mathrm{Mat}(m, \mathbb{C})$ there exists a Borel probability measure $\nu_{z^{(m)}}$ on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ supported on the set $\left(\Pi_{[1,m]}\right)^{-1} z^{(m)}$ such that for every Borel subset $A \subset \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ the map

$$z^{(m)} \to \nu_{z^{(m)}}(A)$$

is $\overline{\nu}$-measurable and that we have a decomposition

$$\nu = \int\limits_{\mathrm{Mat}(m,\mathbb{C})} \nu_{z^{(m)}}\, d\overline{\nu}\left(z^{(m)}\right) \tag{1.3}$$

again understood in the weak sense.

A similar description can be given for measures in the class $\mathfrak{F}(m;H)$: a Borel measure ν on H belongs to the class $\mathfrak{F}(m;H)$ if and only if there exists a measure $\overline{\nu}$ on the space $H(m)$ of $m\times m$-Hermitian matrices which assigns finite weight to every compact set and, for $\overline{\nu}$-almost every $h^{(m)}\in H(m)$ there exists a Borel probability measure $\nu_{h^{(m)}}$ such that

$$\nu=\int_{H(m)}\nu_{h^{(m)}}\,d\overline{\nu}\left(h^{(m)}\right), \tag{1.4}$$

where the decomposition (1.4) is understood in the same way as the decomposition (1.3).

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2. Weak recurrence

The proof is based on the following simple general observation. Let X be a complete metric space, and let G be an inductively compact group, in other words,

$$G=\bigcup_{n=1}^{\infty}K(n),\quad K(n)\subset K(n+1)$$

where the groups $K(n)$, $n\in\mathbb{N}$, are compact and metrizable. Let $\mathfrak{T}$ be a continuous action of G on X (continuity is here understood with respect to the totality of the variables).

Each group $K(n)$ is endowed with the Haar measure $\mu_{K(n)}$, and to each point $x \in X$ we assign, following Vershik [11], the corresponding sequence of *orbital measures* $\mu^x_{K(n)}$ on X given by the formula

$$\int_X f(y)\, d\mu^x_{K(n)}(y) = \int_{K(n)} f\,(T_g x)\, d\mu_{K(n)}(g),$$

valid for any bounded continuous function f on X. Given a family $\mathfrak{A}$ of Borel probability measures on X, we say that the family $\mathfrak{A}$ is *weakly recurrent* if for any positive bounded continuous function f on X we have

$$\inf_{\nu\in\mathfrak{A}} \int f\, d\nu > 0.$$

PROPOSITION 2.1. — *Let ν be an ergodic $\mathfrak{T}$-invariant measure on X that assigns finite weight to every ball and admits a set B, $\nu(B) > 0$, such that for every $x \in B$ the sequence of orbital measures $\mu^x_{K(n)}$ is weakly recurrent. Then ν is finite.*

Proof. — Consider the space $L_2(X,\nu)$; for $n \in \mathbb{N}$, let $L_2(X,\nu)^{K(n)}$ be the subspace of $K(n)$-invariant functions, and let $P_n : L_2(X,\nu) \to L_2(X,\nu)^{K(n)}$ be the corresponding orthogonal projection.

If the measure ν is ergodic and infinite, then

$$\bigcap_{n=1}^{\infty} L_2(X,\nu)^{K(n)} = 0. \tag{2.1}$$

Indeed, let $L_2(X,\nu)^G$ be the subspace of G-invariant square-integrable functions. By definition, we have

$$\bigcap_{n=1}^{\infty} L_2(X,\nu)^{K(n)} = L_2(X,\nu)^G. \tag{2.2}$$

Now, if the measure ν is ergodic and assigns finite weight to every ball, then, by results of [3], it is also indecomposable in the sense that any Borel set $A \subset X$ such that for any $g \in G$ we have $\nu(T_g A \Delta A) = 0$ must satisfy either $\nu(A) = 0$ or $\nu(X \setminus A) = 0$. Since ν is infinite, we have

$$L_2(X,\nu)^G = 0,$$

and (2.1) is proved.

For any $f \in L_2(X,\nu)$ we thus have $P_n f \to 0$ in $L_2(X,\nu)$ as $n \to \infty$. Along a subsequence we then also have $P_{n_k} f \to 0$ almost surely with the respect to the measure ν.

If f is continuous and square-integrable, then the equality

$$P_n f(x) = \int\limits_X f(y)\, d\mu^x_{K(n)}(y)$$

holds for ν-almost all x.

Take, therefore, f to be a positive, continuous, square-integrable function on X (the existence of such a function follows from the fact that the measure ν assigns finite weight to balls: indeed, taking $x_0 \in X$, letting d be the distance on X, and setting $f(x) = \psi(d(x_0, x))$, where $\psi : \mathbb{R} \to \mathbb{R}$ is positive, continuous, and decaying rapidly enough at infinity, we obtain the desired function).

If ν is ergodic and infinite, then, from the above, for almost all $x \in X$ we have

$$\lim_{n\to\infty} \int f\, d\mu^x_{K(n)} = 0.$$

In particular, for ν-almost all $x \in X$, the sequence of orbital measures is not weakly recurrent, which contradicts the assumptions of the proposition. □

Remark. — The argument above, combined with the ergodic decomposition theorem of [3], yields a slightly stronger statement: if a $\mathfrak{T}$-invariant measure ν on X that assigns finite weight to every ball is such that for ν-almost every $x \in X$ the sequence of orbital measures $\mu^x_{K(n)}$ is weakly recurrent, then the ergodic components of ν are almost surely finite.

It remains to derive Theorems 1.1, 1.3 from Proposition 2.1. We start with Theorem 1.3.

3. Proof of Theorem 1.3

3.1. Radial boundedness

A matrix $h \in H$ will be called *radially bounded* in H if

$$\sup_{n\in\mathbb{N}} \frac{|\, tr\left(\Pi_{[1,n]} h\right)|}{n} < +\infty,\ \sup_{n\in\mathbb{N}} \frac{tr\left(\Pi_{[1,n]} h\right)^2}{n^2} < +\infty.$$

We shall now see that if $h \in H$ is radially bounded in H, then the family of orbital measures μ^h_n, $n \in \mathbb{N}$, is precompact in the weak topology on H, and, consequently, weakly recurrent.

Recall that if X is a complete separable metric space, $\mathfrak{M}(X)$ the space of Borel probability measures on X, then the *weak topology* on $\mathfrak{M}(X)$ is defined as follows. Let $f_1, \ldots, f_k : X \longrightarrow \mathbb{R}$ be bounded continuous functions

on X, let $\varepsilon_1, \dots, \varepsilon_k > 0$, let $\nu_0 \in \mathfrak{M}(X)$ and consider the set

$$\left\{ \nu \in \mathfrak{M}(X) : \left| \int f_i \, d\nu - \int f_i \, d\nu_0 \right| < \varepsilon_i, \ i = 1, \dots, k \right\} \tag{3.1}$$

Sets of the form (3.1) form the basis of the weak topology on $\mathfrak{M}(X)$. Our assumptions on X imply that the space $\mathfrak{M}(X)$ endowed with the weak topology is itself metrizable and separable; for instance, the Lévy-Prohorov metric or the Kantorovich-Rubinstein metric induce the weak topology on X (see, e.g., [1], Section 8.3). The symbol $\Rightarrow$ will denote weak convergence in the space $\mathfrak{M}(X)$.

It is clear that weak precompactness of a family of probability measures implies weak recurrence.

PROPOSITION 3.1. — *If a matrix $h \in H$ is radially bounded then the sequence $\left\{\mu_n^h\right\}_{n\in\mathbb{N}}$ of orbital measures corresponding to h is weakly precompact.*

This Proposition is an immediate Corollary of Theorem 4.1 in Olshanski-Vershik [6]. Indeed, let $h \in H$ be radially bounded, let

$$h(n) = \Pi_{[1,n]} h = (h_{ij})_{i,j=1,\dots,n},$$

let

$$\lambda_1^{(n)} \geqslant \dots \geqslant \lambda_{k_n}^{(n)} \geqslant 0$$

be the nonnegative eigenvalues of $h(n)$ arranged in decreasing order, and let

$$\widetilde{\lambda}_1^{(n)} \leqslant \widetilde{\lambda}_2^{(n)} \leqslant \dots \leqslant \widetilde{\lambda}_{l_n}^{(n)} < 0$$

be the negative eigenvalues of $h(n)$ arranged in increasing order. Set

$$x_i^{(n)} = \frac{\lambda_i^{(n)}}{n}, \qquad \widetilde{x}_i^{(n)} = \frac{\widetilde{\lambda}_i^{(n)}}{n};$$

$$\gamma_1^{(n)} = \frac{tr\, h(n)}{n}, \qquad \gamma_2^{(n)} = \frac{tr\, h^2(n)}{n^2}.$$

Let h is radially bounded, and let positive constants C_1, C_2 be such that for all $n \in \mathbb{N}$ we have

$$|\, tr\left(\Pi_{[1,n]} h\right)| \leqslant C_1 n, \quad tr\left(\Pi_{[1,n]} h\right)^2 \leqslant C_2 n^2.$$

We clearly have

$$|\gamma_1^{(n)}| \leqslant C_1, \quad 0 \leqslant \gamma_2^{(n)} \leqslant C_2,$$

and, for all $i = 1, \dots, n$, we have

$$|x_i^{(n)}|, \ |\tilde{x}_i^{(n)}| \leqslant C_2.$$

Therefore, any infinite set of natural numbers contains a subsequence n_r such that sequences $\gamma_1^{(n_r)}, \gamma_2^{(n_r)}$, as well as the sequences $x_i^{(n_r)}, \widetilde{x}_i^{(n_r)}$ for all $i = 1, 2, \dots$ converge to a finite limit as $r \to \infty$. By the Olshanski-Vershik Theorem (Theorem 4.1 in [6]), in this case the sequence $\mu_{n_r}^h$ of orbital measures weakly converges (in fact, to an ergodic $U(\infty)$-invariant probability measure) as $r \to \infty$. The Proposition is proved completely.

Remark. — The converse claim (which, however, we do not need for our argument) also holds: if the sequence of orbital measures for a matrix $h \in H$ is weakly precompact, then the matrix h is radially bounded. This immediately follows from claim (ii) of Theorem 4.1 of Olshanski and Vershik [6]. Note that, while claim (ii) in [6] is only formulated for the full sequence of orbital measures, the same result, with the identical proof, is valid for any infinite subsequence of orbital measures.

Observe that Theorem 4.1 in Olshanski-Vershik [6] as well as the Ergodic Decomposition Theorem of Borodin-Olshanski [2] immediately imply the following

PROPOSITION 3.2. — *If ν is a finite Borel $U(\infty)$-invariant measure on H, then ν-almost every $h \in H$ is radially bounded.*

Proof. — Indeed, if ν is an ergodic probability measure, then the claim is part of the statement of the Olshanski-Vershik Theorem: in this case, for ν-almost all $h \in H$, the sequence of orbital measures μ_n^h weakly converges to ν. For a general finite measure, the result follows from the Ergodic Decomposition Theorem of Borodin and Olshanski [2]. □

To complete the proof of Theorem 1.3, it remains to establish

PROPOSITION 3.3. — *If a $U(\infty)$-invariant measure ν belongs to the class $\mathfrak{F}(m; H)$ for some $m \in \mathbb{N}$, then ν-almost every $h \in H$ is radially bounded.*

3.2. Proof of Proposition 3.3

For a matrix $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$, $n \in \mathbb{N}$, denote

$$\Pi_{[n,\infty)} z = (z_{ij})_{i,j=n,n+1,\dots}.$$

We start by showing that, under the assumptions of the proposition, for ν-almost every $h \in H$ the matrix $\Pi_{[m,\infty)} h$ is radially bounded.

Take a measure $\nu \in \mathfrak{F}(m; H)$ and consider the corresponding canonical decomposition (1.4) into conditional measures.

PROPOSITION 3.4. — *Let $\nu \in \mathfrak{F}(m;H)$ be $U(\infty)$-invariant. Then for $\overline{\nu}$-almost every $h^{(m)} \in H(m)$ the probability measure*

$$\left(\Pi_{[m,\infty)}\right)_* \nu_{h^{(m)}}$$

on H is also $U(\infty)$-invariant.

Proof. — Let $U_m(\infty) \subset U(\infty)$ be the subgroup of matrices $u = (u_{ij})$ satisfying the conditions:

(1) if $\min(i,j) \leqslant m$, $i \neq j$, then $u_{ij} = 0$
(2) if $i \leqslant m$, then $u_{ii} = 1$

It follows from the definitions that if $u \in U_m(\infty)$, then $\Pi_{[m,\infty)} u \in U(\infty)$, and that the map

$$\Pi_{[m,\infty)} : U_m(\infty) \longrightarrow U(\infty)$$

is a group isomorphism.

For $u \in U(\infty)$ let $\mathfrak{t}_u : H \to H$ be given by the formula

$$\mathfrak{t}_u(h) \;=\; u^{-1}hu.$$

Let $U'_m(\infty) \subset U_m(\infty)$ be a countable dense subgroup. A Borel probability measure η on H satisfying $(\mathfrak{t}_u)_*\eta = \eta$ for all $u \in U'_m(\infty)$ must clearly be invariant under the whole group $U_m(\infty)$.

Uniqueness of Rohlin's system of conditional measures implies that for $\overline{\nu}$-almost every $h^{(m)} \in H(m)$ and every $u \in U'_m(\infty)$ we have

$$\nu_{h^{(m)}} \;=\; (\mathfrak{t}_u)_*\nu_{h^{(m)}}. \tag{3.2}$$

By definition of the subgroup $U'_m(\infty)$, the equality (3.4) also holds for all $u \in U_m(\infty)$. Now let A be a measurable subset of H, and let

$$\widetilde{A}_{h^{(m)}} \;=\; \left\{h \in H : \Pi_{[1,m]}h = h^{(m)},\, \Pi_{[m,\infty)}h \in A\right\}.$$

Let $u \in U(\infty)$ and let $\widetilde{u} \in U_m(\infty)$ be defined by the formula

$$\Pi_{[m,\infty)}\widetilde{u} \;=\; u.$$

From the definitions it follows:

$$\widetilde{\mathfrak{t}_u}(\widetilde{A}_{h^{(m)}}) \;=\; \left\{h \in H : \Pi_{[1,m]}h = h^{(m)},\, \Pi_{[m,\infty)}h \in \mathfrak{t}_u(A).\right\}$$

Since

$$\nu_{h^{(m)}}(\widetilde{A}_{h^{(m)}}) \;=\; \nu_{h^{(m)}}(\widetilde{\mathfrak{t}_u}(\widetilde{A}_{h^{(m)}})),$$

we have

$$\left(\Pi_{[m,\infty)}\right)_* \nu_{h^{(m)}}(A) \;=\; \left(\Pi_{[m,\infty)}\right)_* \nu_{h^{(m)}}\left(\mathfrak{t}_u(A)\right),$$

and the proposition is proved. □

Applying Proposition 3.2 to the measure $\left(\Pi_{[m,\infty)}\right)_* \nu_{h^{(m)}}$, we now obtain

Corollary 3.5. — *If $\nu \in \mathfrak{F}(m;H)$ is $U(\infty)$-invariant, then for ν-almost every $h \in H$ the matrix $\Pi_{[m,\infty)}(h)$ is radially bounded.*

We proceed with the proof of Proposition 3.3. Let $\check{u} \in U(\infty)$ be defined as follows:

$$\check{u}_{i,m+i} = \check{u}_{m+i,i} = 1 \qquad i = 1, \dots, m \tag{3.3}$$

$$\check{u}_{2m+i,2m+i} = 1 \qquad i \in \mathbb{N} \tag{3.4}$$

$$\check{u}_{ij} = 0 \qquad \text{otherwise.} \tag{3.5}$$

Proposition 3.6. — *Let $h \in H$. If $\Pi_{[m,\infty)}(h)$ and $\Pi_{[m,\infty)}(\check{u}^{-1}h\check{u})$ are radially bounded, then h is also radially bounded.*

Proof. — If $\Pi_{[m,\infty)}(h)$ is radially bounded, then

$$\sup_{n\in\mathbb{N}} \frac{|\, tr\left(\Pi_{[1,n]}(\Pi_{[m,\infty)}h)\right)|}{n} < +\infty,$$

and, since for $n > m$ we have

$$tr\left(\Pi_{[1,n]}(h)\right) = tr\left(\Pi_{[1,n]}(\Pi_{[m,\infty)}h)\right) + tr\left(\Pi_{[1,m]}(h)\right),$$

it follows that

$$\sup_{n\in\mathbb{N}} \frac{|\, tr\left(\Pi_{[1,n]}h\right)|}{n} < +\infty.$$

It remains to show that

$$\sup_{n\in\mathbb{N}} \frac{tr\left(\Pi_{[1,n]}h\right)^2}{n^2} < +\infty.$$

Let π be a permutation of $\mathbb{N}$ defined as follows:

$$\pi(i) = \begin{cases} m+i & i = 1, \dots, m; \\ i-m & i = m+1, \dots, 2m; \\ i & i > 2m. \end{cases}$$

By definition, for any $h \in H$ we have

$$(\check{u}^{-1}h\check{u})_{ij} \;=\; \check{h}_{\pi(i)\pi(j)}.$$

Consequently, for any $N \in \mathbb{N}$ we have

$$\sum_{i,j=1}^{N} |h_{ij}|^2 \;\leqslant\; \sum_{i,j=m+1}^{N} |h_{ij}|^2 + \sum_{i,j=m+1}^{N} \left|(\check{u}^{-1}h\check{u})_{ij}\right|^2 + \sum_{i,j=1}^{2m} |h_{ij}|^2 .$$

□

Proposition 3.3 is now immediate from Corollary 3.5 and Propositions 3.4, 3.6.

Theorem 1.3 is proved completely.

4. Proof of Theorem 1.1.

The proof is similar (and simpler) in this case. Again, a matrix $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ will be called *radially bounded* if

$$\sup_{n\in\mathbb{N}} \frac{tr\left(\Pi_{[1,n]}z\right)^*\left(\Pi_{[1,n]}z\right)}{n^2} < +\infty$$

(here, as usual, the symbol z^* stands for the transpose conjugate of a matrix z). As before, we assign to a matrix $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ the sequence μ_n^z of orbital measures corresponding to the sequence of compact subgroups $U(n) \times U(n)$, $n \in \mathbb{N}$, and say that a matrix $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is *weakly recurrent* if for any bounded positive continuous function f on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ we have

$$\inf_{n\in\mathbb{N}} \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} f\, d\mu_n^z > 0$$

Again we have the following

PROPOSITION 4.1. — *If a matrix $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is radially bounded then the sequence of orbital measures μ_n^z is weakly precompact. In particular, if z is radially bounded, then z is also weakly recurrent.*

Remark. — As before, the converse statement also holds: if the sequence of orbital measures is weakly precompact, then z is radially bounded.

Proof — This, again, follows from Rabaoui's work [10], [9]. Indeed, let $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$, let

$$z(n) = \Pi_{[1,n]}z,$$

let

$$\lambda_1^{(n)} \geqslant \cdots \geqslant \lambda_n^{(n)} \geqslant 0$$

be the eigenvalues of the matrix $z(n))^*z(n)$ arranged in decreasing order, and set

$$x_i^{(n)} = \frac{\lambda_i^{(n)}}{n^2}, \quad \gamma^{(n)} = \frac{tr\left(z(n)^*z(n)\right)}{n^2}.$$

If z is radially bounded, then any infinite set of natural numbers contains a subsequence n_r such that the sequence $\gamma^{(n^r)}$ as well as all the sequences $x_i^{n_r}$, $i = 1, \ldots,$ converge (to a finite limit) as $r \to \infty$. In this case, by Rabaoui's theorems (see Theorem 5 in [10], Theorem 7 in [9]), the sequence of orbital measures $\mu_{n_r}^z$ weakly converges to a probability measure as $r \to \infty$; weak precompactness is thus established.

To conclude the proof of the Theorem, it therefore remains to establish the following

PROPOSITION 4.2. — *Let $m \in \mathbb{N}$ and let $\nu \in \mathfrak{F}(m; \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$. Then ν-almost every $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is radially bounded.*

The proof follows the same pattern as that of Proposition 3.3. Again, using Pickrell's classification of ergodic probability measures as well as the ergodic decomposition theorem of [3], we have

PROPOSITION 4.3. — *Let ν be a $U(\infty) \times U(\infty)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. Then ν-almost every $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is radially bounded.*

Given $\nu \in \mathfrak{F}(m, \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$, we consider, again, the decomposition

$$\nu \ = \int\limits_{\mathrm{Mat}(m,\mathbb{C})} \nu_{z^{(m)}} \, d\overline{\nu}(z^{(m)}).$$

Here $\mathrm{Mat}(m, \mathbb{C})$ stands for the space of all $m \times m$-matrices with complex entries; the measure $\overline{\nu}$ is the projection of ν onto $\mathrm{Mat}(m, \mathbb{C})$ which is well-defined by definition of the class $\mathfrak{F}(m, \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$; and, for $\overline{\nu}$-almost every point $z^{(m)} \in \mathrm{Mat}(m, \mathbb{C})$ the measure $\nu_{z^{(m)}}$ is the canonical conditional probability measure given by Rohlin's Theorem. Again, we have the following

PROPOSITION 4.4. — *If $\nu \in \mathfrak{F}(m, \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ is $U(\infty) \times U(\infty)$-invariant, then, for $\overline{\nu}$-almost all $z^{(m)} \in \mathrm{Mat}(m, \mathbb{C})$, the probability measure*

$$\left(\Pi_{[m,\infty)}\right)_* \nu_{z^{(m)}}$$

is also $U(\infty) \times U(\infty)$-invariant.

Proof. — The proof of this Proposition is exactly the same as that of Proposition 3.4. □

Using Proposition 4.4 and applying Proposition 4.3 to the measure

$$\left(\Pi_{[m,\infty)}\right)_* \nu_{z^{(m)}},$$

we see that for ν-almost every z, the matrix $\Pi_{[m,\infty)} z$ is radially bounded. To obtain boundedness for the matrix z itself, we again apply a permutation of rows and columns.

Denote

$$\tau_n(z) \ = \ tr\left(\left(\Pi_{[1,n]} z\right)^* \Pi_{[1,n]} z\right) \ = \ \sum_{i,j=1}^{n} |z_{ij}|^2 .$$

Let the matrix $\breve{u} \in U(\infty)$ be defined by (3.3). The following inequality clearly holds for any $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ and all $n > 3m$:

$$\tau_n(z) \ \leqslant \ \tau_{2m}(z) + \tau_n\left(\Pi_{[m,\infty)} z\right) + \tau_n\left(\Pi_{[m,\infty)}(\breve{u}^{-1} z \breve{u})\right).$$

Consequently, if $\nu \in \mathfrak{F}(m, \mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ is $U(\infty) \times U(\infty)$-invariant, then ν-almost every $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is radially bounded, and Theorem 1.1 is proved completely.

BIBLIOGRAPHY

[1] V. I. Bogachev, *Measure theory*, vol. II, Springer-Verlag, Berlin, 2007.

[2] A. Borodin & G. Olshanski, "Infinite random matrices and ergodic measures", *Comm. Math. Phys.* **223** (2001), no. 1, p. 87-123.

[3] A. I. Bufetov, "Ergodic decomposition for measures quasi-invariant under Borel actions of inductively compact groups", *Sbornik Mathematics* **205** (2014), no. 2, p. 39-71.

[4] G. Olshanski, "Unitary representations of infinite-dimensional classical groups (Russian)", D.Sci Thesis, Institute of Geography of the Russian Academy of Sciences; online at `http://www.iitp.ru/upload/userpage/52/Olshanski_thesis.pdf`.

[5] ———, "Unitary representations of infinite-dimensional pairs (G,K) and the formalism of R. Howe", in *Representation of Lie Groups and Related Topics* (A. Vershik & D. P. Zhelobenko, eds.), Advanced Studies in Contemporary Mathematics, vol. 7, Gordon and Breach, 1990, online at `http://www.iitp.ru/upload/userpage/52/HoweForm.pdf`, p. 165-189.

[6] G. Olshanski & A. Vershik, "Ergodic unitarily invariant measures on the space of infinite Hermitian matrices", in *Contemporary mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 175, Amer. Math. Soc., Providence, RI, 1999, p. 137-175.

[7] D. Pickrell, "Measures on infinite-dimensional Grassmann manifolds", *J. Funct. Anal.* **70** (1987), p. 323-356.

[8] ———, "Mackey analysis of infinite classical motion groups", *Pacific J. Math.* **150** (1991), no. 1, p. 139-166.

[9] M. Rabaoui, "A Bochner type theorem for inductive limits of Gelfand pairs", *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 5, p. 1551-1573.

[10] ———, "Asymptotic harmonic analysis on the space of square complex matrices", *J. Lie Theory* **18** (2008), no. 3, p. 645-670.

[11] A. M. Vershik, "A description of invariant measures for actions of certain infinite-dimensional groups", *(Russian) Dokl. Akad. Nauk SSSR* **218** (1974), p. 749-752.

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