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$\mathcal{H}^p$ SPACES OF HARMONIC FUNCTIONS

by Linda LUMER-NAIM (1)

Introduction.

It is a well known fact that the complex-valued harmonic functions in the unit disk, whose L^p-norms ($1 \leqslant p \leqslant +\infty$) on the concentric circles of radii $r < 1$ are uniformly bounded in r, are the Poisson integrals of L^p-functions on the unit circle, for $1 < p \leqslant +\infty$, and the Poisson integrals of finite complex measures for $p = 1$; it is also known, by Fatou's theorem, that these functions have non-tangential (hence radial) limits at almost all boundary points (see for instance Hofmann [14]). In 1951, Parreau [22] gave a similar characterization for harmonic functions on a Riemann surface, using the Martin boundary and related kernel; in 1960, Stein and Weiss [23] extended the whole classical result to harmonic functions in the n-dimensional half-space, with normal approach to the boundary; various other extensions have been mentioned since (for instance Doob [9]). It is our purpose to prove here a similar extension for the axiomatic functions of Brelot [2], using essentially Gowrisankaran's results [11, 12] on axiomatic Martin boundary and fine limits, and Doob's ideas on uniform integrability [8].

More precisely, we consider, on a locally compact connected and locally connected Hausdorff space Ω, a *harmonic class* $\mathcal{H}$ of complex-valued functions, called *harmonic functions*, defined from a presheaf of real-valued functions satisfying Brelot's axioms as recalled in I, 1. We define the class $\mathcal{H}^p$, $1 \leqslant p \leqslant +\infty$,

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as the set of those harmonic functions whose L^p-norms, with respect to the harmonic measures $\rho_{x_0}^{\omega_\iota}$ relative to a fixed point $x_0 \in \Omega$ and the open relatively compact subsets ω_ι of Ω containing x_0, are uniformly bounded in ι. $\mathcal{H}^p$ is a Banach space under appropriate norm. $\mathcal{H}^1$ is the set of elements of $\mathcal{H}$ whose real and imaginary parts are differences of two positive harmonic functions in Ω; $\mathcal{H}^\infty$ is the set of bounded elements of $\mathcal{H}$. We prove that the functions in $\mathcal{H}^p$ are the solutions of Dirichlet problems with the « minimal boundary » Δ_1 of Ω, the fine filters in Ω, and boundary functions in $L^p(\Delta_1)$, for $1 < p \leqslant +\infty$, the integrals of finite complex measures on Δ_1, for $p = 1$. It follows that every function in $\mathcal{H}^p$, $1 \leqslant p \leqslant +\infty$, has a finite fine limit $\varepsilon L^p(\Delta_1)$ at almost every point of the minimal boundary Δ_1. Similar results are obtained for classes $\mathcal{H}^\Phi$, defined, like in Parreau [22], by replacing the function t^p, $p \geqslant 1$, $t \geqslant 0$, by any positive convex increasing function Φ, defined in $[0, +\infty[$. We also study related classes $\overset{*}{\mathcal{H}}{}^p$, $\overset{*}{\mathcal{H}}{}^\Phi$, of positive subharmonic functions u in Ω, defined like the $\overset{*}{\mathcal{H}}{}^p$'s, $\overset{*}{\mathcal{H}}{}^\Phi$'s, and characterized by the fact that u has a harmonic majorant in $\mathcal{H}^p$, $\mathcal{H}^\Phi$; a finite fine limit exists at almost every point of the minimal boundary Δ_1, for any $u \in \overset{*}{\mathcal{H}}{}^p, \overset{*}{\mathcal{H}}{}^\Phi$. Various applications are given at the end, mainly in connection with the so-called « strongly subharmonic » functions of Gärding-Hörmander [10], and the notions of extremal function and reproducting kernel also considered by Parreau [22].

The question of existence or non existence of non void or non constant classes $\mathcal{H}^p$, $\mathcal{H}^\Phi$, in a given harmonic class $\mathcal{H}$ pertains to the axiomatic extension of the classification of Riemann surfaces. The notion of hyperbolic or parabolic class $\mathcal{H}$ has already been introduced by Loeb [15]; the classification of hyperbolic classes $\mathcal{H}$ by means of classes $\mathcal{H}^p$, $\mathcal{H}^\Phi$, and the extension of Parreau's results [22] on the classification of hyperbolic Riemann surfaces, will be completely developed elsewhere. We only give here a partial result, deduced immediately from the general properties of the $\mathcal{H}^\Phi$ classes.

It would be of interest to extend whole or part of the present research to other axiomatics, mainly to Bauer's [1]. In the particular case of a Green space, or an euclidian domain

Ω, with the ordinary harmonic functions, it would also be of interest to know if the functions in $\mathcal{H}^p$ have limits, in a sense or another, along the Green lines, or any other trajectories to the boundary, replacing the radii in the unit disk. This particular question will be studied elsewhere.

An abstract of the present research has been published in the Notices of the AMS [19].

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CHAPTER I

PRELIMINARIES

1. Basis of Brelot's axiomatic theory of harmonic functions [2], [4].

The set-up is a locally compact, non compact, connected and locally connected Hausdorff space Ω, with, on each open subset $\omega \subset \Omega$, a real vector space of real-valued finite continuous functions defined in ω, called *harmonic functions* in ω, and satisfying axioms 1,2,3: 1. The harmonic functions form a presheaf in Ω; 2. There exists a base of the topology of Ω consisting of regular open sets; 3. The upper envelope of an increasing directed set of harmonic functions in a domain ω is either $+\infty$ everywhere, or harmonic in ω; (this last axiom, together with 1 and 2, recently proved [16] to imply the stronger axiom 3'); the existence of a > 0 potential in Ω is usually assumed in order to avoid trivialities.

Topological notions in Ω are in general relative to the topology of the one-point compactification $\overline{\Omega}$; for instance, if $E \subset \Omega$, $\overline{E}$ usually denotes its closure in $\overline{\Omega}$, and δE its boundary in this space.

Notations. — Here, the real vector space of real-valued (positive) harmonic functions *in* Ω will be denoted by $\mathcal{H}_{\mathbf{R}}(\mathcal{H}_{\mathbf{R}}^{+})$. $\mathcal{H} = \mathcal{H}_{\mathbf{R}} + i\mathcal{H}_{\mathbf{R}}$ will denote the (complex) vector space of complex-valued harmonic functions *in* Ω, i.e. of functions of the form $f = u + iv$, where u, $v \in \mathcal{H}_{\mathbf{R}}$. This $\mathcal{H}$ we call *harmonic class in* Ω. Finally $\overset{*}{\mathcal{H}}_{\mathbf{R}}$ $(\overset{*}{\mathcal{H}}_{\mathbf{R}}^{+}, \underset{*}{\mathcal{H}}_{\mathbf{R}})$ will denote the set of all superharmonic (positive superharmonic, subharmonic) functions in Ω.

2. The (modified) Dirichlet problem with the minimal boundary and the fine filters in Ω (Gowrisankaran [11] [12]).

Adding the assumption of a countable base for the open sets of Ω, Δ_1, the « *minimal boundary* » of Ω, denotes the set of extreme harmonic functions (or minimal functions, after Martin [21]) of a compact and metrizable base Λ of the cone $\overset{*}{\mathcal{H}}{}^{+}_{\mathbf{R}}$, and, for each $h \in \mathcal{H}^{+}_{\mathbf{R}}$, μ_h denotes the unique $\geqslant 0$ measure on Λ (or $\bar{\Delta}_1 \subset \Lambda$), such that $\mu_h(\Lambda - \Delta_1) = 0$, and h has the integral representation

$$h(x) = \int_{\Delta_1} k(x)\, d\mu_h(k), \qquad \forall x \in \Omega$$

(Mrs. Hervé [13]).

$\{\mathcal{F}_k\}_{k \in \Delta_1}$, denotes the family of *fine filters* in Ω, and, (assuming $1 \in \mathcal{H}^{+}_{\mathbf{R}}$ with corresponding measure μ_1 on Δ_1), for any real-valued weakly 1-resolutive (equivalently μ_1-integrable) function $\check{f}$ on Δ_1, $\mathcal{S}_{\check{f},1}$ denotes the solution of a modified Dirichlet problem where the boundary conditions along the fine filters are taken μ_1-*almost everywhere on* Δ_1, and writes

$$\mathcal{S}_{\check{f},1}(x) = \int_{\Delta_1} k(x)\check{f}(k)\, d\mu_1(k), \qquad x \in \Omega.$$

It is proved that the solution $\mathcal{S}_{\check{f},1}$ has the (finite) fine limit $\check{f}$ at μ_1-almost every point of Δ_1, and that every positive harmonic function is (uniquely) decomposed as the sum of a function of type $\mathcal{S}_{\check{f},1} \geqslant 0$, and a function $\varepsilon \mathcal{H}^{+}_{\mathbf{R}}$ which has fine limit 0 at μ_1-almost every point of Δ_1. As every potential is also proved to have fine limit 0 at μ_1-almost every point of Δ_1, one finally obtains the extension of Fatou-Doob's theorem [7] for a Green space and its Martin boundary, namely: Every function $v > 0$ superharmonic in Ω has a finite fine limit $\check{v}$ at μ_1-almost every point of the minimal boundary Δ_1, (and this $\check{v}$ is in $L^1(\mu_1)$).

3. Doob's results on uniform integrability.

The notion of uniform integrability has been successfully used a few years ago by Doob [5, 6, 8] in some important questions of potential theory, essentially in boundary value

problems. His results, which we recall below in a slightly extended form valid in the axiomatic setting, play a central role in our study of $\mathcal{H}^p$ spaces.

Let us first recall the definition, and some equivalent forms of this notion:

$\{X_\iota\}_{\iota\in I}$ is a family of measure spaces, with corresponding measures $\{\mu_\iota\}_{\iota\in I}$ positive and such that $\sup_\iota \mu_\iota(X_\iota) < +\infty$; f_ι is a real-valued μ_ι-integrable function on the space X_ι.

(1) The family $\{f_\iota\}_{\iota\in I}$ is called *uniformly integrable* if

$$\lim_{a\to+\infty} \int_{\{|f_\iota|\geqslant a\}} |f_\iota|\, d\mu_\iota = 0,$$

uniformly on the index set I.

(2) This is equivalent to the following two simultaneous conditions:

$$\begin{cases} \text{(i)} & \sup_\iota \int_{X_\iota} |f_\iota|\, d\mu_\iota < +\infty, \\ \text{(ii)} & \lim_{\varepsilon\to 0} \left(\sup_\iota \int_{e_\iota} |f_\iota|\, d\mu_\iota\right) = 0, \end{cases}$$

for any measurable subset $e_\iota \subset X_\iota$, with $\mu_\iota(e_\iota) < \varepsilon$.

(3) The family $\{f_\iota\}$ is uniformly integrable if and only if there exists a *positive* Baire function Φ on the interval $[0, +\infty]$ such that:

$$\begin{cases} \text{(i)} & \lim_{t\to+\infty} \dfrac{\Phi(t)}{t} = +\infty, \\ \text{(ii)} & \sup_\iota \int_{X_\iota} \Phi(|f_\iota|)\, d\mu_\iota < +\infty, \end{cases}$$

and even a *convex increasing* such Φ does exist, with $\Phi(0) = 0$.

We shall mainly need the sufficient condition for

$$\Phi(t) = t^p, \qquad 1 < p < +\infty.$$

Doob was concerned with the uniform integrability condition for the family of restrictions of a fixed real harmonic function u, defined in Ω, to the boundaries $\partial\omega_\iota$ of the open relatively compact subsets ω_ι of Ω which contain a fixed point $x_0 \in \Omega$, the corresponding measures being the harmonic measures $\rho_{x_0}^{\omega_\iota}$ relative to x_0 and ω_ι (whose total masses are uniformly bounded in ι if one assumes that $1 \in \overset{*}{\mathcal{H}}{}_{\mathbf{R}}^{+}$ since then $\int_{\partial\omega_\iota} d\rho_{x_0}^{\omega_\iota} \leqslant 1$ for all ω_ι).

In view of property (3) above, and the fact that if u is subharmonic $\geqslant 0$ so is $\Phi(u)$, (see chapter III, 1), it is equivalent here to replace $\{\omega_\iota\}_{\iota\in I}$ by any increasing directed subfamily with union Ω. Also, since axiom 3′ ($\Longleftrightarrow$ 3) is satisfied, this uniform, integrability condition for u is independent of x_0; it is actually so for any family $\{f_\iota\}_{\iota\in I}$ provided each $\Phi(|f_\iota|)$ be resolutive on the corresponding $\partial\omega_\iota$.

Finally, if u is harmonic, this uniform integrability implies at once that u is the difference of two positive harmonic functions in Ω, since

$$u(x) = \int_{\partial\omega_\iota} u\, d\rho_x^{\omega_\iota} = \int_{\partial\omega_\iota} u^+\, d\rho_x^{\omega_\iota} - \int_{\partial\omega_\iota} u^-\, d\rho_x^{\omega_\iota}, \qquad \forall x \in \omega_\iota,$$

and the last two integrals are positive harmonic functions (in ω_ι), uniformly bounded at the point x_0, hence increasing to finite harmonic functions in Ω.

Doob's main results can be stated as follows (see Brelot [3] where the proofs are given in detail):

THEOREM 1 (Doob). — *Assume axioms* 1, 2, 3, *the existence of* $a > 0$ *potential in* Ω, *and* $1 \in \overset{*}{\mathcal{H}}{}_{\mathbf{R}}^{+}$. *Then a real harmonic function* u *in* Ω *is uniformly integrable with respect to the harmonic measures* $\rho_{x_0}^{\omega_\iota}$ *if and only if there exists, for any* $\varepsilon > 0$, *two functions:* v_1 *subharmonic bounded above,* v_2 *superharmonic bounded below, such that*

$$v_1 \leqslant u \leqslant v_2, \qquad \text{and} \qquad v_2(x_0) - v_1(x_0) \leqslant \varepsilon.$$

As a consequence, we obtain the following useful.

COROLLARY 1.1. — *In* Ω, *the positive harmonic functions which are uniformly integrable with respect to the harmonic measures* $\rho_{x_0}^{\omega_\iota}$ *are all the (finite) limits of increasing sequences of bounded positive harmonic functions.*

Proof. — If u is > 0, and uniformly integrable, then $\inf(u, n)$ is > 0 superharmonic, and its greatest harmonic minorant u_n satisfies $v_1 \leqslant u_n \leqslant u \leqslant v_2$ for large n(v_1, v_2, functions of the theorem), so that $u_n(x_0) \to u(x_0)$. But u_n, increasing and $\leqslant u$, converges in Ω to $a > 0$ harmonic function $u_1 \leqslant u$, and equal to u at the point x_0, hence equal to u everywhere.

The converse is obvious, by taking for functions v_2, v_1, respectively u and a suitable element of an increasing sequence converging to u.

It is easy to see that any solution of a Dirichlet problem with the minimal boundary and the fine filters, as described in 2, is uniformly integrable with respect to the harmonic measures $\rho_{x_0}^{\omega_i}$, $\forall x_0 \in \Omega$. The important fact is the converse, proved by Doob in the classical case of a Green space and its Martin boundary, now extended to the axiomatic setting in which Gowrisankaran's results hold :

THEOREM 2. (Doob). — *Assume axioms* 1, 2, 3, *the existence of* $a > 0$ *potential in* Ω, *a countable base for the open sets of* Ω *and* $1 \in \mathcal{H}_{\mathbf{R}}^{+}$. *Then every real harmonic function* u *uniformly integrable with respect to the harmonic measures* $\rho_{x_0}^{\omega_i}$ *is the solution of a Dirichlet problem with the minimal boundary* Δ_1 *and the fine filters in* Ω.

As a consequence, Doob obtains :

COROLLARY 2.1. — *A positive superharmonic function in* Ω *is a potential if and only if it has fine limit* 0 μ_1*-almost everywhere on* Δ_1, *and is uniformly integrable with respect to the harmonic measures* $\rho_{x_0}^{\omega_i}$.

Extension. — The uniform integrability condition makes sense as well for complex-valued functions $f = u + iv$, and is actually equivalent to the same condition for u and v simultaneously. Theorem 2 still holds in this case, and will be consistently used in the following.

CHAPTER II

$\mathcal{H}^p$ SPACES OF HARMONIC FUNCTIONS $1 \leqslant p \leqslant +\infty$.

1. The Banach spaces $\mathcal{H}^p$.

Hypotheses :

$$\begin{cases} \text{Axioms 1, 2, 3.} \\ \text{Existence of a} > 0 \text{ potential in } \Omega. \\ 1 \in \overset{*}{\mathcal{H}}{}^{+}_{\mathbf{R}}. \end{cases}$$

From now on we shall consider complex-valued harmonic functions $f = u + iv$ (u, v, harmonic real) in Ω, with modulus $|f| = (u^2 + v^2)^{1/2}$, this being a subharmonic function, because, more generally :

LEMMA 3. — *Let $u_1, u_2, \ldots, u_n$, be harmonic (or positive subharmonic) in Ω. Then, for any real number $p \geqslant 1$, $\left(\sum_{i=1}^{n} u_i^2\right)^{p/2}$ is subharmonic in Ω.*

Proof. — It suffices to prove the lemma for $p = 1$; more precisely, that for any $x \in \Omega$ and regular open set $\omega \ni x$,

$$\left(\sum_{i=1}^{n} u_i^2(x)\right)^{1/2} \leqslant \int_{\partial\omega} \left(\sum_{i=1}^{n} u_i^2\right)^{1/2} d\rho_x^\omega.$$

Each $|u_i|$ is subharmonic, so, except when $\int_{\partial\omega} |u_i|\, d\rho_x^\omega = 0$, $\forall\, i$, in which case the above inequality is clearly true, we have

$$\left(\sum_{i=1}^{n} u_i^2(x)\right)^{1/2} \leqslant \left(\sum_{i=1}^{n} \left(\int_{\partial\omega} |u_i|\, d\rho_x^\omega\right)^2\right)^{1/2}$$

$$= \int_{\partial\omega} \frac{\sum_{i=1}^{n} \left(|u_i| \int_{\partial\omega} |u_i|\, d\rho_x^\omega\right)}{\left(\sum_{i=1}^{n} \left(\int_{\partial\omega} |u_i|\, d\rho_x^\omega\right)^2\right)^{1/2}} d\rho_x^\omega$$

$$\leqslant \int_{\partial\omega} \frac{\left(\sum_{i=1}^{n} u_i^2\right)^{1/2} \left(\sum_{i=1}^{n} \left(\int_{\partial\omega} |u_i|\, d\rho_x^\omega\right)^2\right)^{1/2}}{\left(\sum_{i=1}^{n} \left(\int_{\partial\omega} |u_i|\, d\rho_x^\omega\right)^2\right)^{1/2}} d\rho_x^\omega = \int_{\partial\omega} \left(\sum_{i=1}^{n} u_i^2\right)^{1/2} d\rho_x^\omega.$$

We shall denote by $\{\omega_\iota\}_{\iota\in I}$ the increasing directed set of all open relatively compact subsets of Ω, containing a fixed point $x_0 \in \Omega$, and by $\rho_{x_0}^{\omega_\iota}$ the harmonic measure on $\partial\omega_\iota$, relative to x_0 and ω_ι; since $1 \in \overset{*}{\mathcal{H}}{}_{\mathbf{R}}^{+}$, $\int_{\partial\omega_\iota} d\rho_{x_0}^{\omega_\iota} \leqslant 1$, $\forall \iota \in I$.

DEFINITION. — *A harmonic function $f = u + iv$ belongs to the class $\mathcal{H}^p$, $1 \leqslant p \leqslant +\infty$, if and only if the L^p-norms, with respect to the harmonic measures $\rho_{x_0}^{\omega_\iota}$, of the restrictions of f to the boundaries $\partial\omega_\iota$, are uniformly bounded in ι.*

In other words, $f \in \mathcal{H}^p$ if and only if there exists a constant M, independent of ι, such that $\|f\|_{p,\iota} \leqslant M$, $\forall \iota \in I$, where

$$\begin{cases} \|f\|_{p,\iota} = \left(\int_{\partial\omega_\iota} |f|^p \, d\rho_{x_0}^{\omega_\iota}\right)^{1/p}, & \text{for} \quad 1 \leqslant p < +\infty, \\ \|f\|_{\infty,\iota} = \operatorname{ess.\,sup}_{\partial\omega_\iota} |f|; \end{cases}$$

$f = u + iv$ is in $\mathcal{H}^p$ if and only if both u and v are.

Note that $\|f\|_{p,\iota}^p = H_{|f|^p}^{\omega_\iota}(x_0)$, $p < +\infty$, while $\|f\|_{\infty,\iota}$ is equal to $\sup |f|$ in the connected component ω_{ι_0} of ω_ι which contains x_0, this last property coming from the fact that on $\partial\omega_{\iota_0}$ the sets of harmonic measures $\rho_x^{\omega_{\iota_0}}$ (hence the ess. sup norms with respect to these measures) zero are independent of $x \in \omega_{\iota_0}$, which implies that $\sup_{\omega_{\iota_0}} |f| = \operatorname{ess.\,sup}_{\partial\omega_{\iota_0}} |f|$, while $\rho_{x_0}^{\omega_\iota}$ which has support $\partial\omega_{\iota_0}$ coïncides there with $\rho_{x_0}^{\omega_{\iota_0}}$.

Also note that, in the definition, it is equivalent to replace $\{\omega_\iota\}_{\iota\in I}$ by any increasing directed subfamily $\{\omega_{\iota_k}\}$ with union Ω. In fact, for $p < +\infty$, $|f|^p$ is subharmonic in Ω, so $|f|^p \leqslant H_{|f|^p}^{\omega_{\iota_k}}$ in any ω_{ι_k}; a given ω_ι is contained with its closure in some ω_{ι_k}, hence $H_{|f|^p}^{\omega_\iota} \leqslant H_{|f|^p}^{\omega_{\iota_k}}$ in ω_ι, $\|f\|_{p,\iota}^p = H_{|f|^p}^{\omega_\iota}(x_0) \leqslant H_{|f|^p}^{\omega_{\iota_k}}(x_0) \leqslant M^p$, and $f \in \mathcal{H}^p$. For $p = +\infty$, since the connected component ω_{ι_0} of ω_ι which contains x_0 is contained in the similar component $\omega_{\iota_{k_0}}$ of some ω_{ι_k}, we have

$$\|f\|_{\infty,\iota} = \sup_{\omega_{\iota_0}} |f| \leqslant \sup_{\omega_{\iota_{k_0}}} |f| = \|f\|_{\infty,\iota_{k_0}} \leqslant M, \qquad \text{and} \qquad f \in \mathcal{H}^\infty.$$

Finally, axiom 3′ ($\longleftrightarrow$ 3) shows that the class $\mathcal{H}^p$,

$$1 \leqslant p < +\infty,$$

does not depend on the particular choice of the point x_0, and

it is obviously the same for the class $\mathcal{H}^\infty$. This also comes from the following:

THEOREM 4. — *The harmonic function f is in the class $\mathcal{H}^p$, $1 \leqslant p < +\infty$, if and only if $|f|^p$ has a harmonic majorant in Ω.*

Proof. — If such a U does exist, then, for each ω_ι,

$$\mathrm{H}^{\omega_\iota}_{|f|^p} \leqslant \mathrm{H}^{\omega_\iota}_{\mathrm{U}} = \mathrm{U},$$

and
$$\|f\|_{p,\iota} = (\mathrm{H}^{\omega_\iota}_{|f|^p}(x_0))^{1/p} \leqslant \mathrm{U}^{1/p}(x_0) = \mathrm{M},$$

so $f \in \mathcal{H}^p$. This in turn implies that $\mathrm{H}^{\omega_\iota}_{|f|^p}$, uniformly bounded in ι at the point x_0, increases to a finite harmonic function in Ω, which dominates $|f|^p$, (and is in fact the smallest harmonic majorant of $|f|^p$ in Ω).

COROLLARY 4.1. — *$\mathcal{H}^1$ is the class of f's whose real and imaginary parts are differences of two positive harmonic functions in Ω; $\mathcal{H}^\infty$ is the class of bounded harmonic functions in Ω; and for any finite $q \geqslant p \geqslant 1$, we have the inclusions*

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \subset \mathcal{H}^1.$$

Proof. — u real $\varepsilon \mathcal{H}^1$ implies that u^+ and u^- have smallest harmonic majorants u_1, $u_2 \geqslant 0$, (because $|u|$ does);

$$u_1(x) = \lim_\iota \int_{\partial\omega_\iota} u^+ \, d\rho_x^{\omega_\iota}, \qquad u_2(x) = \lim_\iota \int_{\partial\omega_\iota} u^- \, d\rho_x^{\omega_\iota},$$

and
$$u(x) = \int_{\partial\omega_\iota} u^+ \, d\rho_x^{\omega_\iota} - \int_{\partial\omega_\iota} u^- \, d\rho_x^{\omega_\iota}, \qquad \forall x \in \omega_\iota,$$

so that $u = u_1 - u_2$ in Ω. Any difference $u = w_1 - w_2$ of two positive harmonic functions is conversely in $\mathcal{H}^1(|u| \leqslant w_1 + w_2)$, and the corresponding above decomposition is extremal in the sense that u_1, u_2, respectively, minorize any such w_1, w_2.

Next, $f \in \mathcal{H}^\infty$ is equivalent to: $\sup\limits_{\omega_{\iota_k}} |f| \leqslant \mathrm{M}$ for any relatively compact *domain* $\omega_{\iota_k} \ni x_0$, which obviously means that f is bounded in Ω.

The final inclusion comes from the elementary inequality $|t|^p \leqslant 1 + |t|^q$, for any finite $q \geqslant p \geqslant 1$.

COROLLARY 4.2. — *Any real harmonic function $u \in \mathcal{H}^p$ is the difference of two positive harmonic functions in $\mathcal{H}^p$, and conversely.*

Proof. — By the above corollary, it suffices to show that, for $1 < p < +\infty$, the smallest harmonic majorants u_1 and u_2 of u^+ and u^-, respectively, are both in $\mathcal{H}^p$.

But, by Hölder's inequality

$$\int_{\partial\omega_\iota} u^+ d\rho_x^{\omega_\iota} \leqslant \Big(\int_{\partial\omega_\iota} (u^+)^p d\rho_x^{\omega_\iota}\Big)^{1/p} \Big(\int_{\partial\omega_\iota} d\rho_x^{\omega_\iota}\Big)^{1/q} \leqslant \Big(\int_{\partial\omega_\iota} (u^+)^p d\rho_x^{\omega_\iota}\Big)^{1/p},$$

$(1/p + 1/q = 1)$, so, $\forall x \in \Omega$,

$$u_1^p(x) = \lim_\iota \Big(\int_{\partial\omega_\iota} u^+ d\rho_x^{\omega_\iota}\Big)^p \leqslant \lim_\iota \int_{\partial\omega_\iota} (u^+)^p d\rho_x^{\omega_\iota} \leqslant \mathrm{U}(x),$$

since $|u|^p$, hence $(u^+)^p$, has a harmonic majorant U in Ω; this proves $u_1 \in \mathcal{H}^p$; similarly $u_2 \in \mathcal{H}^p$, whence the corollary.

THEOREM 5. — *For $1 \leqslant p \leqslant +\infty$, $\mathcal{H}^p$ is a Banach space under the norm $\|f\|_p = \sup_\iota \|f\|_{p,\iota}$. For $p < +\infty$, this equals $[{}_pf(x_0)]^{1/p}$, where ${}_pf$ denotes the smallest harmonic majorant of $|f|^p$ in Ω; for $p = +\infty$, this is just $\sup_\Omega |f|$.*

Proof. — It is clear that all $\mathcal{H}^p$*'s* are linear spaces over the complex field K. For $1 \leqslant p < +\infty$, we have already noted that $\|f\|_{p,\iota}^p = \mathrm{H}_{|f|^p}^{\omega_\iota}(x_0)$, and that the harmonic function $\mathrm{H}_{|f|^p}^{\omega_\iota}$ increases, following the increasing directed set $\{\omega_\iota\}_{\iota \in \mathrm{I}}$, to the smallest harmonic majorant ${}_pf$ of $|f|^p$ in Ω, so that in this case $\|f\|_p = \sup_\iota \|f\|_{p,\iota} = [{}_pf(x_0)]^{1/p} < +\infty$. This $\|f\|_p$ is a norm in $\mathcal{H}^p$: relations $\|\alpha f\|_p = |\alpha|\,\|f\|_p$, $\alpha \in \mathrm{K}$, and

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p,$$

come from the similar ones for the norms $\|f\|_{p,\iota}$, and $\|f\|_p = 0$ implies ${}_pf(x_0) = 0$, hence ${}_pf = 0$ everywhere in Ω, and $f \equiv 0$. Finally, if $\{f_n\}$ is a Cauchy sequence in $\mathcal{H}^p$, i.e. if

$$\begin{aligned} &\|f_n - f_{n'}\|_p \to 0, && n, n' \to +\infty, \\ &{}_p(f_n - f_{n'})(x_0) \to 0, && n, n' \to +\infty; \end{aligned}$$

with *axiom* 3′, this positive harmonic function ${}_p(f_n - f_{n'})$ converges to 0 everywhere in Ω, and uniformly locally; hence f_n also converges, uniformly locally, to a harmonic function f in Ω, which is in $\mathcal{H}^p$, since

$$\int_{\partial\omega_\iota} |f_n|^p d\rho_{x_0}^{\omega_\iota} \to \int_{\partial\omega_\iota} |f|^p d\rho_{x_0}^{\omega_\iota}, \qquad n \to +\infty,$$

and

$$\|f_n\|_{p,t} \leqslant \|f_n\|_p,$$

uniformly bounded in n, and is limit in norm, since

$$\|f - f_n\|_p \leqslant \|f - f_{n_0}\|_p + \|f_n - f_{n_0}\|_p,$$

where n_0 may be choosen so that $\|f_n - f_{n_0}\|_p \leqslant \varepsilon$, $\forall n \geqslant n_0$, (whence $\|f_n - f_{n_0}\|_{p,t} \leqslant \varepsilon$, $\forall n \geqslant n_0$, $\forall t \in \mathrm{I}$), while

$$\|f - f_{n_0}\|_p = \sup_t \|f - f_{n_0}\|_{p,t} = \sup_t \left(\lim_{n \to +\infty} \|f_n - f_{n_0}\|_{p,t} \right) \leqslant \varepsilon.$$

For $p = +\infty$, it is clear that $\sup_t \|f\|_{\infty,t} = \sup_\Omega |f|$, so $\|f\|_\infty$ is the usual sup norm in Ω, under which $\mathcal{H}^\infty$ is complete (and even a Banach algebra, under suitable multiplication, as was announced in a preliminary report [18], and will be developed elsewhere).

2. Characterization and boundary properties of $\mathcal{H}^p$ functions.

Hypotheses :

$$\begin{cases} \text{Axioms 1, 2, 3.} \\ \text{Existence of a} > 0 \text{ potential in } \Omega. \\ \text{A countable base for the open sets of } \Omega. \\ 1 \in \mathcal{H}^+_{\mathbf{R}}. \end{cases}$$

We now proceed to prove the extended characterization mentioned in the introduction, showing the role of uniform integrability in the study of $\mathcal{H}^p$ functions. The close connection, (and in some sense equivalence) between these two concepts will appear more clearly in chapter III, when we study the $\mathcal{H}^\Phi$ classes of harmonic functions in Ω.

We recall that μ_1 is the measure on Δ_1 representing 1 in the integral representation $1 = \int_{\Delta_1} k(x)\, d\mu_1(k)$, $x \in \Omega$.

THEOREM 6. — *A harmonic function f belongs to the class $\mathcal{H}^p$, $1 < p \leqslant +\infty$, if and only if f is the solution of a Dirichlet problem with the minimal boundary Δ_1, the fine filters in Ω, and boundary function $\check{f} \in \mathrm{L}^p(\mu_1)$.*

Moreover, if the extreme harmonic functions are normalized so as to be equal to 1 at the point x_0, $\|f\|_p = \|\check{f}\|_p$, *and the correspondence* $f \to \check{f}$ *is an isometric isomorphism of the Banach space* $\mathcal{H}^p$ *onto* $L^p(\mu_1)$.

Proof. — By property I.3.(3), any $f \in \mathcal{H}^p$, $1 < p \leqslant +\infty$, is uniformly integrable with respect to the harmonic measures $\rho_{x_0}^{\omega_1}$, hence (th. 2) is the solution of a Dirichlet problem with the minimal boundary Δ_1, the fine filters in Ω, and boundary function $\check{f}$; moreover f has the fine limit $\check{f}$ at μ_1-almost every point of Δ_1.

If $p = +\infty$, this $\check{f}$ is obviously bounded, where defined, hence $\varepsilon L^\infty(\mu_1)$; if $1 < p < +\infty$, this $\check{f} \in L^p(\mu_1)$, for $|\check{f}|^p$ is dominated by the μ_1-integrable boundary function of the positive harmonic function ${}_pf$ in Ω.

Now, let $\check{f} \in L^p(\mu_1)$, $1 < p \leqslant +\infty$, and $f = \mathcal{S}_{\check{f},1}$,

$$f(x) = \int_{\Delta_1} k(x)\check{f}(k)\, d\mu_1(k), \qquad x \in \Omega.$$

If $p = +\infty$, f is bounded in Ω, hence $\varepsilon\mathcal{H}^\infty$, and it is clear that $\|f\|_\infty = \|\check{f}\|_\infty$.

If $1 < p < +\infty$, Hölder's inequality

$$\left|\int_{\Delta_1} k(x)\check{f}(k)\, d\mu_1(k)\right| \leqslant \left(\int_{\Delta_1} k(x)|\check{f}(k)|^p\, d\mu_1(k)\right)^{1/p} \left(\int_{\Delta_1} k(x)\, d\mu_1(k)\right)^{1/q}$$

$(1/p + 1/q = 1)$ shows that

$$|f(x)|^p \leqslant \mathcal{S}_{|\check{f}|^p,1}(x), \qquad \forall x \in \Omega;$$

so $|f|^p$ has a harmonic majorant in Ω, hence $f \in \mathcal{H}^p$. As

$$|f|^p \leqslant {}_pf \leqslant \mathcal{S}_{|\check{f}|^p,1},$$

we see that ${}_pf$ is also a solution, and has fine limit $|\check{f}|^p$ μ_1-almost everywhere, therefore ${}_pf = \mathcal{S}_{|\check{f}|^p,1}$ and

$$\|f\|_p = [{}_pf(x_0)]^{1/p} = \left(\int_{\Delta_1} |\check{f}|^p\, d\mu_1\right)^{\frac{1}{p}} = \|\check{f}\|_p.$$

The isometric isomorphism of the theorem follows at once. For $p = 1$, there is a fundamental difference in the characterization of $\mathcal{H}^1$ functions. By corollary 4.1, we have:

THEOREM 6′. — *A harmonic function f belongs to the class $\mathcal{H}^1$ if and only if f is the integral, with kernel $k \in \Delta_1$, of a finite complex measure ν on the minimal boundary Δ_1, and*

$$\|f\|_1 = \int_{\Delta_1} |d\nu|.$$

Any $f \in \mathcal{H}^1$ has a finite fine limit $\check{f} \in L^1(\mu_1)$ μ_1-almost everywhere on the boundary, and if $\check{f} \in L^1(\mu_1)$, then $f = \mathcal{S}_{\check{f},1} \in \mathcal{H}^1$, ${}_1f = \mathcal{S}_{|\check{f}|,1}$, and $\|f\|_1 = \|\check{f}\|_1$. But there may exist in $\mathcal{H}^1$ non-zero functions with fine limit 0 μ_1-almost everywhere on the boundary, so the characterization of theorem 6 fails to hold, and $f \to \check{f}$ is no more isomorphic.

We emphasize the following:

COROLLARY 6.1. — *Every function $f \in \mathcal{H}^p$, $1 \leqslant p \leqslant +\infty$ has a finite fine limit $\check{f} \in L^p(\mu_1)$ at μ_1-almost every point of the minimal boundary Δ_1.*

COROLLARY 6.2. — *If $\check{f} \in L^p(\mu_1)$, $1 \leqslant p \leqslant +\infty$, and $f = \mathcal{S}_{\check{f},1}$, then the smallest harmonic majorant of $|f|$ in Ω (resp. of $|f|^p$, if $p < +\infty$) is $\mathcal{S}_{|\check{f}|,1}$ (resp. $\mathcal{S}_{|\check{f}|^p,1}$), and*

$$\begin{cases} |f| = \mathcal{S}_{|\check{f}|,1} - w_1, & 1 \leqslant p \leqslant +\infty, \\ |f|^p = \mathcal{S}_{|\check{f}|^p,1} - w, & 1 \leqslant p < +\infty, \end{cases}$$

where w_1, w are positive potentials in Ω.

These are just the Riesz decompositions of $|f|$, $|f|^p$, in Ω. Note that $\mathcal{S}_{|\check{f}|,1} \in \mathcal{H}^p$, while $\mathcal{S}_{|\check{f}|^p,1} \in \mathcal{H}^1$.

3. $\mathcal{H}^p_*$ classes of positive subharmonic functions.

If we look at the definition and properties of $\mathcal{H}^p$ functions $f = u + iv$, $1 \leqslant p \leqslant +\infty$, we see that they essentially involve the positive subharmonic functions $|f|$, and their powers $|f|^p$. Therefore it seems worth introducing in general positive subharmonic functions in Ω satisfying the same condition as the above $|f|$, $f \in \mathcal{H}^p$; they will by definition belong to the class $\mathcal{H}^p_*$, whose properties are rapidly studied in the present section.

Hypotheses are first those of section 1; $\{\omega_\iota\}_{\iota\in I}$ is the same family as above.

DEFINITION. — *A positive subharmonic function u belongs to the class* $\underset{*}{\mathcal{H}}^p$, $1 \leqslant p \leqslant +\infty$, *if and only if the* L^p*-norms, with respect to the harmonic measures* $\rho_{x_0}^{\omega_\iota}$, *of the restrictions of u to the boundaries* $\partial\omega_\iota$, *are uniformly bounded in* ι.

In other words, $u \in \underset{*}{\mathcal{H}}^p$ if and only if there exists a constant M, independent of ι, such that $\|u\|_{p,\iota} \leqslant M$, $\forall \iota \in I$, where

$$\begin{cases} \|u\|_{p,\iota} = \left(\int_{\partial\omega_\iota} u^p \, d\rho_{x_0}^{\omega_\iota}\right)^{1/p}, & \text{for} \quad 1 \leqslant p < +\infty, \\ \|u\|_{\infty,\iota} = \operatorname{ess.\,sup}_{\partial\omega_\iota} u. \end{cases}$$

Here $\|u\|_{p,\iota}^p \geqslant \underline{H}_{u^p}^{\omega_\iota}(x_0)$, $p < +\infty$, (with equality if u^p is resolutive), while $\|u\|_{\infty,\iota} \geqslant \sup u$ in the connected component of ω_ι which contains x_0.

It follows that *a positive subharmonic function u is in the class* $\underset{*}{\mathcal{H}}^p$, $1 \leqslant p < +\infty$, *if and only if* u^p *has a harmonic majorant in* Ω; *u is in the class* $\underset{*}{\mathcal{H}}^\infty$ *if and only if u is bounded in* Ω.

Consequence. — It is equivalent, in the definition, to replace $\{\omega_\iota\}_{\iota\in I}$ by any increasing directed subfamily $\{\omega_{\iota_k}\}$ with union Ω; the class $\underset{*}{\mathcal{H}}^p$ is independent of the point x_0, and for any finite $q \geqslant p \geqslant 1$, we have the inclusions

$$\underset{*}{\mathcal{H}}^\infty \subset \underset{*}{\mathcal{H}}^q \subset \underset{*}{\mathcal{H}}^p \subset \underset{*}{\mathcal{H}}^1.$$

Obviously $f = u + iv \in \mathcal{H}^p$ if and only if $|f| \in \underset{*}{\mathcal{H}}^p$. Moreover:

THEOREM 7. — *A positive subharmonic function u belongs to the class* $\underset{*}{\mathcal{H}}^p$, $1 \leqslant p \leqslant +\infty$, *if and only if u has a harmonic majorant in the class* $\mathcal{H}^p$.

Proof. — Any $u \in \underset{*}{\mathcal{H}}^p$ has a smallest harmonic majorant $U = \lim_\iota \underline{H}_u^{\omega_\iota}$, obviously bounded if u is, otherwise in $\mathcal{H}^p$, $p < +\infty$, because U^p is then dominated by any harmonic majorant of u^p in Ω. The converse is obvious.

We shall denote by ${}_1u$ (resp. ${}_pu$, if $p < +\infty$) the smallest harmonic majorant of u (resp. u^p) in Ω; ${}_1u \in \mathcal{H}^p$, while ${}_pu \in \mathcal{H}^1$, and we have the Riesz decompositions

$$u = {}_1u - w_1, \qquad u^p = {}_pu - w,$$

where w_1, w are positive potentials in Ω.

$\mathcal{H}^p$ is a convex cone containing the cone $(\mathcal{H}^p)^+_{\mathbf{R}}$ of positive functions in $\mathcal{H}^p$, but obviously not a linear space. Let $\mathcal{E}^p_{\mathbf{R}}$, $1 \leqslant p \leqslant +\infty$, be the real vector space of differences of functions in $\mathcal{H}^p$, with the obvious addition and scalar multiplication; $\mathcal{E}^p_{\mathbf{R}}$ contains the real vector space $\mathcal{H}^p_{\mathbf{R}}$. For $u - u' \in \mathcal{E}^p_{\mathbf{R}}$, $(u,\ u' \in \mathcal{H}^p)$, *define* $\|u - u'\|^*_p$ as $\|{}_1u - {}_1u'\|_p$, norm of the function ${}_1u - {}_1u'$ in the Banach space $\mathcal{H}^p$. It is easy to see that this depends only on the difference $u - u'$, and is a *semi-norm* on $\mathcal{E}^p_{\mathbf{R}}$, the condition $\|u - u'\|^*_p = 0$ being equivalent to $u - u'$ equal to the difference of two potentials in Ω. On $\mathcal{H}^p$, $\| \ \|^*_p$ is equal to the $\mathcal{H}^p$-norm; and if $u \in \mathcal{H}^p$, $\|u\|^*_p = \sup_t \|u\|_{p,t}$, while $\|u\|^*_p = 0$ here implies that $u \equiv 0$. Finally, if $\{(u_n - u'_n)\}$ is a Cauchy sequence in $\mathcal{E}^p_{\mathbf{R}}$, i.e. if $\|(u_n - u'_n) - (u_m - u'_m)\|^*_p \to 0$, $m, n, \to +\infty$, then $({}_1u_n - {}_1u'_n)$ is a Cauchy sequence in the Banach space $\mathcal{H}^p_{\mathbf{R}}$, hence converges there to a function $h - h'$:

$$\|({}_1u_n - {}_1u'_n) - (h - h')\|_p \to 0, \qquad n \to +\infty.$$

It follows that $\{(u_n - u'_n)\}$ is $\| \ \|^*_p$-convergent to $h - h'$, but this $\| \ \|^*_p$-limit is only defined up to the difference of two positive potentials in Ω.

If we want uniqueness, we must therefore introduce the linear subspace $\mathcal{O}$ of $\mathcal{E}^p_{\mathbf{R}}$ consisting of the above differences, and the quotient space $\mathcal{E}^p_{\mathbf{R}}/\mathcal{O}$. On this, $\| \ \|^*_p$ induces a norm, because $\| \ \|^*_p$ is the same for two elements of $\mathcal{E}^p_{\mathbf{R}}$ equal mod $\mathcal{O}$, whence the required uniqueness, and $\mathcal{E}^p_{\mathbf{R}}/\mathcal{O}$ is a Banach space in which the equivalence classes mod $\mathcal{O}$ of the elements of $\mathcal{H}^p$ form a complete although non linear subspace. Note that the latter are also the equivalence classes of the elements of $(\mathcal{H}^p_{\mathbf{R}})^+$, while $\mathcal{E}^p_{\mathbf{R}}/\mathcal{O}$ is the set of equivalence classes of elements of $\mathcal{H}^p_{\mathbf{R}}$.

The above could as well be done in the complex setting, with $\mathcal{E}^p = \mathcal{E}^p_{\mathbf{R}} + i\mathcal{E}^p_{\mathbf{R}}$, but all this has limited interest so far, and will not be pushed further.

Under the hypotheses of section 2, we now study the boundary behavior of $\underset{*}{\mathcal{H}}^p$ functions. We have:

THEOREM 8. — *Every function $u \in \underset{*}{\mathcal{H}}^p$, $1 \leqslant p \leqslant +\infty$, has a finite fine limit $\check{u} \in L^p(\mu_1)$ at μ_1-almost every point of the minimal boundary Δ_1. If $1 < p \leqslant +\infty$, then the smallest harmonic majorant of u (resp. of u^p, if $p < +\infty$), in Ω is $\mathcal{S}_{\check{u},1}$ (resp. $\mathcal{S}_{\check{u}^p,1}$) and*

$$\begin{cases} u = \mathcal{S}_{\check{u},1} - w_1, & 1 < p \leqslant +\infty, \\ u^p = \mathcal{S}_{\check{u}^p,1} - w, & 1 < p < +\infty, \end{cases}$$

where w_1, w are positive potentials in Ω.

Proof. — We already know that $u \in \underset{*}{\mathcal{H}}^p$ has the Riesz decomposition $u = {}_1u - w_1$, where ${}_1u \in \underset{*}{\mathcal{H}}^p$, and w_1 is a positive potential in Ω. So u has a finite fine limit $\check{u} \in L^p(\mu_1)$ μ_1-almost everywhere on Δ_1, as ${}_1u$ does, and $\check{u} = {}_1\check{u}$ μ_1-almost everywhere on Δ_1.

This, for $p > 1$, implies that ${}_1u = \mathcal{S}_{\check{u},1}$ (th. 6), whence $u = \mathcal{S}_{\check{u},1} - w_1$, then, for $1 < p < +\infty$, $u^p \leqslant \mathcal{S}_{\check{u}^p,1}$, whence ${}_pu = \mathcal{S}_{\check{u}^p,1}$, and the second decomposition of the theorem.

As a *consequence*, we see that if u real $\varepsilon \mathcal{H}^p$, $p > 1$, then the positive harmonic functions u_1, u_2 of the extremal decomposition $u = u_1 - u_2$ of corollary 4.1 are $\mathcal{S}_{\check{u}^+,1}$, $\mathcal{S}_{\check{u}^-,1}$ respectively.

Application. — Instead of a single function u, one may consider an n-tuple $(u_1, u_2, \ldots, u_n)$ of real harmonic (or positive subharmonic) functions in Ω, and the condition $u \in \underset{*}{\mathcal{H}}^p$, for the positive subharmonic function $u = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}$. This is equivalent to the simultaneous conditions

$$|u_i| \in \underset{*}{\mathcal{H}}^p, \qquad i = 1, 2, \ldots, n,$$

and gives various properties of the components u_i, in particular boundary properties, which were consistently used in the already mentioned paper by Stein and Weiss [23].

4. Relative classes $\mathcal{H}_h^p$, $\overset{*}{\mathcal{H}}{}_h^p$.

The study of the preceding sections can be carried over to h-harmonic and sub-h-harmonic functions (the quotients of harmonic and subharmonic functions by a fixed > 0 finite continuous function h in Ω, which satisfy the basic axioms) and the related h-harmonic measures $\rho_{x_0}^{\prime\omega_i} = \dfrac{h}{h(x_0)}\,\rho_{x_0}^{\omega_i}$, provided the proper hypotheses be made on h (which replaces the constant function 1) : $h \in \overset{*}{\mathcal{H}}{}_{\mathbf{R}}^+$, or $h \in \mathcal{H}_{\mathbf{R}}^+$, which permits the development of a Dirichlet problem relative to h, parallel to the one described in I.2.

This gives relative classes $\mathcal{H}_h^p$, $\overset{*}{\mathcal{H}}{}_h^p$, $1 \leqslant p \leqslant +\infty$, of h-harmonic, positive sub-h-harmonic, functions in Ω, to which all results of the case $h = 1$ do extend. For instance: A complex-valued h-harmonic function f is in the class $\mathcal{H}_h^p$, $1 \leqslant p < +\infty$, if and only if $|f|^p$ has a h-harmonic majorant in Ω; in the class $\mathcal{H}_h^\infty$ if and only if f is bounded in Ω.

The $\mathcal{H}_h^p$ functions f, $1 < p \leqslant +\infty$, are the solutions of (relative to h) Dirichlet problems with the minimal boundary Δ_1, the fine filters in Ω, and boundary functions $\check{f} \in L^p(\mu_h)$, ($\mu_h \geqslant 0$ representing h in the integral representation

$$h(x) = \int_{\Delta_1} k(x)\, d\mu_h(k)),$$

and $\mathcal{H}_h^p$ is a Banach space isometrically isomorphic to $L^p(\mu_h)$. Every h-harmonic function $f \in \mathcal{H}_h^p$ has a finite fine limit $\check{f} \in L^p(\mu_h)$ μ_h-almost everywhere on Δ_1, and similarly for every positive sub-h-harmonic function $u \in \overset{*}{\mathcal{H}}{}_h^p$.

One can try to compare relative classes corresponding to two functions $h_1 \in \mathcal{H}_{\mathbf{R}}^+$, $h_2 \in \mathcal{H}_{\mathbf{R}}^+$, when a relation such as

$$h_1/h_2 \leqslant m$$

holds. But except in the case $p = +\infty$, this gives in general no significant property.

CHAPTER III

$\mathcal{H}^{\Phi}$ CLASSES OF HARMONIC FUNCTIONS

1. The $\mathcal{H}^{\Phi}$ classes.

Hypotheses :

$$\begin{cases} \text{Axioms 1, 2, 3.} \\ \text{Existence of a} > 0 \text{ potential in } \Omega. \\ 1 \in \overset{*}{\mathcal{H}}{}^{+}_{\mathbf{R}}. \end{cases}$$

Following an idea of Nevanlinna, Parreau [22] studied on a Riemann surface what we define here as the $\mathcal{H}^{\Phi}$ classes of harmonic functions, and used them in the general problem of classification of Riemann surfaces. The idea was that since a subharmonic function gives another one by substitution into any convex increasing function, and since the $\mathcal{H}^{p}$ spaces center around properties of the subharmonic functions $|f|^{p}$, $f \in \mathcal{H}^{p}$, one should replace the function t^{p}, $t \in [0, +\infty[$, $p \geqslant 1$, by any positive convex increasing function Φ, defined in $[0, +\infty[$, and obtain essentially the same results. This was actually the case, except for a few particular points, and we shall have similar extensions in our axiomatic setting.

First :

LEMMA 9. — *Let u be subharmonic in Ω, and Ψ convex increasing defined at least on the range of u. If $1 \in \mathcal{H}^{+}_{\mathbf{R}}$, or if $\Psi(0) \leqslant 0$, then $\Psi(u)$ is subharmonic in Ω.*

Proof. — If suffices to verify the mean-value inequality $\Psi(u(x)) \leqslant \int_{\partial\omega} \Psi(u)\, d\rho_x^{\omega}$ for each $x \in \Omega$ where u is finite, and

every regular open set $\omega \ni x$. But

$$\Psi(u(x)) \equiv \Psi\left(\int_{\partial\omega} d\rho_x^\omega \cdot \frac{u(x)}{\int_{\partial\omega} d\rho_x^\omega}\right) \leqslant \int_{\partial\omega} d\rho_x^\omega \cdot \Psi\left(\frac{u(x)}{\int_{\partial\omega} d\rho_x^\omega}\right) + \left(1 - \int_{\partial\omega} d\rho_x^\omega\right)\Psi(0),$$

$\left(\text{inequality of convexity, with } \int_{\partial\omega} d\rho_x^\omega \leqslant 1, \text{ and } = 1 \text{ if } 1 \in \mathcal{H}_{\mathbf{R}}^+\right)$, and

$$\Psi\left(\frac{u(x)}{\int_{\partial\omega} d\rho_x^\omega}\right) \leqslant \Psi\left(\frac{\int_{\partial\omega} u\, d\rho_x^\omega}{\int_{\partial\omega} d\rho_x^\omega}\right) \leqslant \frac{\int_{\partial\omega} \Psi(u)\, d\rho_x^\omega}{\int_{\partial\omega} d\rho_x^\omega}$$

(Jensen's inequality for convex functions), so

$$\Psi(u(x)) \leqslant \int_{\partial\omega} \Psi(u)\, d\rho_x^\omega + \left(1 - \int_{\partial\omega} d\rho_x^\omega\right)\Psi(0),$$

which gives the result.

In the following, a positive convex strictly increasing function Φ defined in $[0, +\infty[$, with $\Phi(0) = 0$, will simply be called « *convex* »; Φ will be called *strongly* « *convex* » if moreover $\frac{\Phi(t)}{t} \to +\infty$, $t \to +\infty$.

Note that for any « convex » Φ, $\lim_{t \to +\infty} \frac{\Phi(t)}{t}$ always exists, finite or $+\infty$, and that a « convex » Φ is necessarily continuous. $x_0 \in \Omega$, $\{\omega_\iota\}_{\iota \in \mathrm{I}}$, and $\rho_{x_0}^{\omega_\iota}$, are the same as before.

Definition. — *A harmonic function $f = u + iv$ belongs to the class $\mathcal{H}^\Phi$, Φ « convex », if and only if there exists a constant* M, *independent of ι, such that* $\int_{\partial\omega_\iota} \Phi(|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant \mathrm{M}$, $\forall\, \iota \in \mathrm{I}$.

If $f = u + iv$ is in $\mathcal{H}^\Phi$, then both u and v are, but the converse is not necessarily true, as we shall see.

As before, $\int_{\partial\omega_\iota} \Phi(|f|)\, d\rho_{x_0}^{\omega_\iota} = \mathrm{H}_{\Phi(|f|)}^{\omega_\iota}(x_0)$, with $\Phi(|f|)$ subharmonic in Ω, so it is equivalent, in the definition, to replace $\{\omega_\iota\}_{\iota \in \mathrm{I}}$ by any increasing directed subfamily $\{\omega_{\iota_k}\}$ with union Ω, and, because of axiom 3′ ($\longleftrightarrow$ 3), the class $\mathcal{H}^\Phi$ is independent of the point x_0. Like in the $\mathcal{H}^p$ case, this also comes from:

Theorem 10. — *The harmonic function f is in the class $\mathcal{H}^\Phi$, Φ « convex », if and only if $\Phi(|f|)$ has a harmonic majorant in Ω.*

COROLLARY 10.1. — *For every « convex » Φ, $\mathcal{H}^\infty \subset \mathcal{H}^\Phi \subset \mathcal{H}^1$, and if a « convex » Φ_1 dominates Φ in some interval*

$$[t_0, +\infty[\subset [0, +\infty[,$$

then $\mathcal{H}^{\Phi_1} \subset \mathcal{H}^\Phi$. In particular, if $\frac{\Phi(t)}{t^p} \to 0$, $t \to +\infty$, $(p \geqslant 1)$, then $\mathcal{H}^p \subset \mathcal{H}^\Phi$, and if $\frac{\Phi(t)}{t^p}$ has a finite > 0 limit $(t \to +\infty)$, then $\mathcal{H}^p = \mathcal{H}^\Phi$.

Proof. — We only prove the first statement. Let $f \in \mathcal{H}^\Phi$, and U a harmonic majorant of $\Phi(|f|)$ in Ω. We have

$$|f| \leqslant \Phi^{-1}(\mathrm{U}),$$

superharmonic in Ω(Φ^{-1} reciprocal of Φ), hence

$$\int_{\partial\omega_\iota} |f| \, d\rho_{x_0}^{\omega_\iota} \leqslant \int_{\partial\omega_\iota} \Phi^{-1}(\mathrm{U}) \, d\rho_{x_0}^{\omega_\iota} \leqslant \Phi^{-1}(\mathrm{U})(x_0),$$

and $f \in \mathcal{H}^1$.

COROLLARY 10.2. — *Any real harmonic function $u \in \mathcal{H}^\Phi$ is the difference of two positive harmonic functions in $\mathcal{H}^\Phi$.*

Proof. — As seen above, u real $\varepsilon \mathcal{H}^\Phi$ can be written as $u = u_1 - u_2$, with

$$u_1(x) = \lim_\iota \int_{\partial\omega_\iota} u^+ \, d\rho_x^{\omega_\iota}, \qquad u_2(x) = \lim_\iota \int_{\partial\omega_\iota} u^- \, d\rho_x^{\omega_\iota},$$

both positive harmonic in Ω. If U denotes a harmonic majorant of $\Phi(|u|)$ in Ω, we have

$$\Phi\Big(\int_{\partial\omega_\iota} u^+ \, d\rho_x^{\omega_\iota}\Big) \leqslant \int_{\partial\omega_\iota} \Phi(u^+) \, d\rho_x^{\omega_\iota} \leqslant \mathrm{U}(x), \qquad x \in \omega_\iota,$$

hence, by continuity of Φ,

$$\Phi(u_1(x)) = \lim_\iota \Phi\Big(\int_{\partial\omega_\iota} u^+ \, d\rho_x^{\omega_\iota}\Big) \leqslant \mathrm{U}(x), \qquad x \in \Omega,$$

and $u_1 \in \mathcal{H}^\Phi$; so does $u_2 \in \mathcal{H}^\Phi$, whence the corollary.

$\mathcal{H}^\Phi$ is a convex, but not in general linear, subset of the vector space $\mathcal{H}$. It is linear when Φ, for instance, satisfies for large t a relation $\Phi(2t) \leqslant \mathrm{C}\Phi(t)$, $\mathrm{C} > 0$ constant, which limitates its growth at infinity, (example: $\Phi(t) = t \log(1 + t)$).

For a general « convex » Φ, we define $\mathcal{H}_l^\Phi$ as the largest linear subspace of $\mathcal{H}$ *contained* in $\mathcal{H}^\Phi$, $\mathcal{H}_l'^\Phi$ as the smallest

containing $\mathcal{H}^\Phi$. It is easily checked that

$$\mathcal{H}_l^\Phi = \{f \in \mathcal{H}^\Phi : \quad \alpha f \in \mathcal{H}^\Phi, \quad \forall \alpha > 0\},$$

and

$$\mathcal{H}_l'^\Phi = \{f \in \mathcal{H} : \quad \alpha f \in \mathcal{H}^\Phi \quad \text{for } \textit{some} \quad \alpha > 0\},$$

both equal to $\mathcal{H}^\Phi$ when this is linear.

On each $\partial\omega_\iota$, we also consider the *Orlicz space* of complex $\rho_{x_0}^{\omega_\iota}$-measurable functions f such that $\int_{\partial\omega_\iota} \Phi(\alpha|f|)\, d\rho_{x_0}^{\omega_\iota} < +\infty$ for some $\alpha > 0$, with the Minkowski norm:

$$\|f\|_{\Phi,\iota} = \inf \left\{\frac{1}{k_\iota} : \quad k_\iota \geqslant 0 \ni \int_{\partial\omega_\iota} \Phi(k_\iota|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1\right\},$$

equivalent to the Orlicz norm, as described for instance by Luxemburg [20].

THEOREM 11. — *For each « convex »* Φ, $\mathcal{H}_l'^\Phi$ *is a Banach space under the norm* $\|f\|_\Phi = \sup_\iota \|f\|_{\Phi,\iota}$, *also equal to*

$$\inf \left\{\frac{1}{k} : \quad k \geqslant 0 \ni \sup_\iota \int_{\partial\omega_\iota} \Phi(k|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1\right\}.$$

The convergence $\lim_{n\to+\infty} \|f_n - f\|_\Phi = 0$ *is equivalent to*

$$\lim_{n\to+\infty} \left\{\sup_\iota \int_{\partial\omega_\iota} \Phi(\alpha|f_n - f|)\, d\rho_{x_0}^{\omega_\iota}\right\} = 0, \qquad \forall \alpha > 0,$$

and $\mathcal{H}_l^\Phi$ *is a closed linear subspace of this Banach space* $\mathcal{H}_l'^\Phi$.

Proof. — Let $f \in \mathcal{H}_l'^\Phi$, Φ « convex »; $\|f\|_{\Phi,\iota} < +\infty$ for each $\iota \in \mathrm{I}$, increases with ω_ι, and the limit $\|f\|_\Phi = \sup_\iota \|f\|_{\Phi,\iota}$ is finite since $\int_{\partial\omega_\iota} \Phi(\alpha|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant \mathrm{M} < +\infty$, $\forall \iota \in \mathrm{I}$ and some $\alpha > 0$, implies $\int_{\partial\omega_\iota} \Phi(\alpha|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant \mathrm{M}_1 = \sup(\mathrm{M}, 1)$, hence

$$\int_{\partial\omega_\iota} \Phi\left(\frac{\alpha}{\mathrm{M}_1}|f|\right) d\rho_{x_0}^{\omega_\iota} \leqslant \int_{\partial\omega_\iota} \frac{1}{\mathrm{M}_1} \Phi(\alpha|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1,$$

and $\|f\|_{\Phi,\iota} \leqslant \dfrac{\mathrm{M}_1}{\alpha} < +\infty$, $\forall \iota \in \mathrm{I}$.

That $\|f\|_\Phi$ is a norm in $\mathcal{H}_l'^\Phi$ comes from the same fact for each $\|f\|_{\Phi,\iota}$, $\iota \in \mathrm{I}$: for instance $\|f\|_\Phi = 0$ implies $\|f\|_{\Phi,\iota} = 0$, whence $|f| = 0$ $\rho_{x_0}^{\omega_\iota}$-almost everywhere on $\partial\omega_\iota$, $\mathrm{H}_{|f|}^{\omega_\iota} = 0$, and

$f \equiv 0$ in Ω. As $\int_{\partial\omega_\iota} \Phi\left(\frac{|f|}{\|f\|_{\Phi,\iota}}\right) d\rho_{x_0}^{\omega_\iota} \leqslant 1$, for each $\iota \in I$, we also see that $\sup_\iota \int_{\partial\omega_\iota} \Phi\left(\frac{|f|}{\|f\|_\Phi}\right) d\rho_{x_0}^{\omega_\iota} \leqslant 1$, so that

$$\|f\|_\Phi \geqslant \inf\left\{\frac{1}{k} : \quad k \geqslant 0 \ni \sup_\iota \int_{\partial\omega_\iota} \Phi(k|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1\right\},$$

and the equality follows.

The completion of $\mathcal{H}_l'^\Phi$ under the above norm is a consequence of the equivalence stated in the theorem, which we first prove: Assume $\|f_n - f\|_\Phi \to 0$, $n \to +\infty$, $(f_n, f \in \mathcal{H}_l'^\Phi)$, and let $\alpha > 0$; for each $\iota \in I$

$$\begin{aligned}\int_{\partial\omega_\iota} \Phi(\alpha|f_n - f|)\, d\rho_{x_0}^{\omega_\iota} &= \int_{\partial\omega_\iota} \Phi\left(\alpha\|f_n - f\|_\Phi \cdot \frac{|f_n - f|}{\|f_n - f\|_\Phi}\right) d\rho_{x_0}^{\omega_\iota} \\ &\leqslant \alpha\|f_n - f\|_\Phi \int_{\partial\omega_\iota} \Phi\left(\frac{|f_n - f|}{\|f_n - f\|_\Phi}\right) d\rho_{x_0}^{\omega_\iota},\end{aligned}$$

n large $(\ni \alpha\|f_n - f\|_\Phi \leqslant 1)$,

$$\leqslant \alpha\|f_n - f\|_\Phi,$$

hence $\sup_\iota \int_{\partial\omega_\iota} \Phi(\alpha|f_n - f|)\, d\rho_{x_0}^{\omega_\iota} \to 0$, $n \to +\infty$; this in turn shows that $\sup_\iota \int_{\partial\omega_\iota} \Phi(\alpha|f_n - f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1 \ \ \forall n \geqslant n_\alpha$, therefore $\|f_n - f\|_\Phi \leqslant \frac{1}{\alpha}$, arbitrarily small for α sufficiently large.

So if $\{f_n\}$ is a Cauchy sequence in $\mathcal{H}_l'^\Phi$, it must satisfy, $\forall\alpha > 0$, $\sup_\iota \int_{\partial\omega_\iota} \Phi(\alpha|f_n - f_{n'}|)\, d\rho_{x_0}^{\omega_\iota} \to 0$, n, $n' \to +\infty$; for $\alpha = 1$, this means that the smallest harmonic majorant of $\Phi(|f_n - f_{n'}|)$ in Ω converges to 0 at the point x_0, hence everywhere in Ω and uniformly locally; therefore f_n also converges, uniformly locally, to a harmonic function f in Ω, which is in $\mathcal{H}_l'^\Phi$, since

$$\int_{\partial\omega_\iota} \Phi(\alpha|f_n|)\, d\rho_{x_0}^{\omega_\iota} \to \int_{\partial\omega_\iota} \Phi(\alpha|f|)\, d\rho_{x_0}^{\omega_\iota}, \qquad n \to +\infty, \qquad \forall\alpha > 0,$$

and for some $\alpha_0 > 0$

$$\int_{\partial\omega_\iota} \Phi(\alpha_0|f_n|)\, d\rho_{x_0}^{\omega_\iota} \leqslant \int_{\partial\omega_\iota} \Phi\left(\frac{|f_n|}{\|f_n\|_\Phi}\right) d\rho_{x_0}^{\omega_\iota} \leqslant 1, \qquad \forall n,$$

and is limit in norm, since

$$\sup_{\iota} \int_{\partial\omega_\iota} \Phi(\alpha|f_n - f_{n'}|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1, \quad \forall n,\ n' \geqslant n_\alpha$$

whence

$$\sup_{\iota} \int_{\partial\omega_\iota} \Phi(\alpha|f_n - f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant 1 \quad \text{and} \quad \|f_n - f\|_\Phi \leqslant \frac{1}{\alpha},$$

again arbitrarily small for α sufficiently large.

Finally, if $f_n \in \mathcal{H}_l^\Phi$ and $\|f_n - f\|_\Phi \to 0$, $n \to +\infty$, $(f \in \mathcal{H}_l'^\Phi)$, one has

$$\int_{\partial\omega_\iota} \Phi(\alpha|f|)\, d\rho_{x_0}^{\omega_\iota} \leqslant \frac{1}{2} \int_{\partial\omega_\iota} \Phi(2\alpha|f_n - f|)\, d\rho_{0x}^{\omega_\iota} + \frac{1}{2} \int_{\partial\omega_\iota} \Phi(2\alpha|f_n|)\, d\rho_{x_0}^{\omega_\iota},\ \alpha > 0;$$

for large n, the first integral on the right is arbitrarily small, uniformly in ι, and for fixed n the second one is uniformly bounded in ι; therefore $f \in \mathcal{H}_l^\Phi$, and this completes the proof of the theorem.

2. Characterization and boundary properties of $\mathcal{H}^\Phi$ functions.

Hypotheses :

$$\begin{cases} \text{Axioms 1, 2, 3,} \\ \text{Existence of a} > 0 \text{ potential in } \Omega. \\ \text{A countable base for the open sets of } \Omega. \\ 1 \in \mathcal{H}_{\mathbf{R}}^+. \end{cases}$$

We have already noted that if $\dfrac{\Phi(t)}{t}$ has a finite limit as $t \to +\infty$, then $\mathcal{H}^\Phi = \mathcal{H}^1$; so the only case to study here is the one where $\dfrac{\Phi(t)}{t} \to +\infty$, $t \to +\infty$, i.e. Φ strongly « convex ».

THEOREM 12. — *A harmonic function f belongs to the class $\mathcal{H}^\Phi$, Φ strongly « convex », if and only if f is the solution of a Dirichlet problem with the minimal boundary Δ_1, the fine filters in Ω, and boundary function $\check{f} \in L^\Phi(\mu_1)$.*

This extends to functions $f \in \mathcal{H}_l'^{\Phi}$ *and* $\check{f} \in L_l'^{\Phi}(\mu_1)$, *the Orlicz space associated to* Φ *on* Δ_1. *Moreover*, if the extreme harmonic functions are normalized so as to be equal to 1 at the point x_0, $\|f\|_\Phi = \|\check{f}\|_\Phi$, *Minkowski norm of* $\check{f}$ *in* $L_l'^{\Phi}(\mu_1)$, *and the correspondence* $f \to \check{f}$ *is an isometric isomorphism of the Banach space* $\mathcal{H}_l'^{\Phi}$ *onto* $L_l'^{\Phi}(\mu_1)$.

Proof. — By the hypothesis Φ strongly « convex », and property I.3.(3), any $f \in \mathcal{H}^\Phi$ is again uniformly integrable with respect to the harmonic measures $\rho_{x_0}^{\omega_l}$, hence (th. 2) is the solution of a Dirichlet problem with the minimal boundary Δ_1, the fine filters in Ω, and boundary function $\check{f}$; and f has the fine limit $\check{f}$ at μ_1-almost every point of Δ_1. This $\check{f} \in L^\Phi(\mu_1)$, since $\Phi(|\check{f}|)$, fine boundary function of $\Phi(|f|)$, is dominated by the μ_1-integrable boundary function of any positive harmonic majorant of $\Phi(|f|)$ in Ω.

Let conversely $\check{f} \in L^\Phi(\mu_1)$, and $f = \mathcal{S}_{\check{f},1}$,

$$f(x) = \int_{\Delta_1} k(x)\, \check{f}(k)\, d\mu_1(k), \qquad x \in \Omega.$$

By Jensen's inequality,

$$\Phi\left(\left|\int_{\Delta_1} k(x)\check{f}(k)\, d\mu_1(k)\right|\right) \leqslant \int_{\Delta_1} k(x)\ \Phi(|\check{f}|)(k)\, d\mu_1(k),$$

i.e.

$$\Phi(|f|) \leqslant \mathcal{S}_{\Phi(|\check{f}|),1} \quad \text{in} \quad \Omega;$$

so $\Phi(|f|)$ has a harmonic majorant in Ω, hence $f \in \mathcal{H}^\Phi$.

If ${}_\Phi f$ denotes the smallest harmonic majorant of $\Phi(|f|)$ in Ω, $(f \in \mathcal{H}^\Phi)$, we thus see that $\Phi(|f|) \leqslant {}_\Phi f \leqslant \mathcal{S}_{\Phi(|\check{f}|),1}$, hence ${}_\Phi f$ is also a solution, and has fine limit $\Phi(|\check{f}|)$ μ_1-almost everywhere on Δ_1; therefore ${}_\Phi f = \mathcal{S}_{\Phi(|\check{f}|),1}$; in particular

$${}_\Phi f(x_0) = \int_{\Delta_1} \Phi(|\check{f}|)(k)\, d\mu_1(k).$$

This shows that

$$\begin{aligned}\|f\|_\Phi &= \inf\left\{\frac{1}{\lambda} : \lambda \geqslant 0 \ni {}_\Phi(\lambda f)(x_0) \leqslant 1\right\} \\ &= \inf\left\{\frac{1}{\lambda} : \lambda \geqslant 0 \ni \int_{\Delta_1} \Phi(\lambda|\check{f}|)(k)\, d\mu_1(k) \leqslant 1\right\} \\ &= \|\check{f}\|_\Phi, \ (f \in \mathcal{H}_l'^{\Phi},\ \check{f} \in L_l'^{\Phi}(\mu_1)),\end{aligned}$$

which suffices for the second part of the theorem.

We explicit the following:

COROLLARY 12.1. — *Every function* $f \in \mathcal{H}^{\Phi}$, Φ *strongly « convex », has a finite fine limit* $\check{f} \in L^{\Phi}(\mu_1)$ *at* μ_1*-almost every point of the minimal boundary* Δ_1; *the smallest harmonic majorant of* $\Phi(|f|)$ *in* Ω *is* $\mathcal{G}_{\Phi(|\check{f}|),1}$, *and* $\Phi(|f|) = \mathcal{G}_{\Phi(|\check{f}|),1} - w$, *where* w *is a positive potential in* Ω.

3. $\underset{*}{\mathcal{H}}^{\Phi}$ classes of positive subharmonic functions and relative classes $\mathcal{H}_h^{\Phi}$, $\underset{*}{\mathcal{H}}_h^{\Phi}$.

Hypotheses are first those of section 1; $\{\omega_\iota\}_{\iota \in I}$ is the same family as before.

DEFINITION. — *A positive subharmonic function* u *belongs to the class* $\underset{*}{\mathcal{H}}^{\Phi}$, Φ *« convex », if and only if there exists a constant* M *independent of* ι *such that* $\int_{\partial\omega_\iota} \Phi(u)\, d\rho_{x_0}^{\omega_\iota} \leqslant M$, $\forall \iota \in I$. We have $\int_{\partial\omega_\iota} \Phi(u)\, d\rho_{x_0}^{\omega_\iota} \geqslant \underline{H}_{\Phi(u)}^{\omega_\iota}(x_0)$, (with equality if $\Phi(u)$, subharmonic, is resolutive), whence: *a positive subharmonic function* u *is in the class* $\underset{*}{\mathcal{H}}^{\Phi}$, Φ *« convex », if and only if* $\Phi(u)$ *has a harmonic majorant in* Ω; so it is equivalent in the definition to replace $\{\omega_\iota\}_{\iota \in I}$ by any increasing directed subfamily $\{\omega_{\iota_k}\}$ with union Ω, and the class $\underset{*}{\mathcal{H}}^{\Phi}$ is independent of the point x_0; classes $\underset{*}{\mathcal{H}}^{\Phi}$ corresponding to different Φ's are compared like the $\mathcal{H}^{\Phi}$'s were, and for any « convex » Φ, $\underset{*}{\mathcal{H}}^{\infty} \subset \underset{*}{\mathcal{H}}^{\Phi} \subset \underset{*}{\mathcal{H}}^{1}$. $f = u + iv \in \mathcal{H}^{\Phi}$ if and only if $|f| \in \underset{*}{\mathcal{H}}^{\Phi}$; moreover:

THEOREM 13. — *A positive subharmonic function* u *belongs to the class* $\underset{*}{\mathcal{H}}^{\Phi}$, Φ *« convex », if and only if* u *has a harmonic majorant in the class* $\mathcal{H}^{\Phi}$.

Proof. — Similar to the one of theorem 7.

We shall denote by ${}_1u$ (resp. ${}_\Phi u$) the smallest harmonic majorant of u (resp. $\Phi(u)$) in Ω; ${}_1u \in \mathcal{H}^{\Phi}$, while ${}_\Phi u \in \mathcal{H}^{1}$,

and we have the Riesz decompositions

$$u = {}_1u - w_1, \qquad \Phi(u) = \Phi^u - w,$$

where w_1, w, are positive potentials in Ω.

$\overset{*}{\mathcal{H}}{}^{\Phi}$ is a convex set, containing the set $(\mathcal{H}^{\Phi}_{\mathbf{R}})^+$ of positive functions in $\mathcal{H}^{\Phi}$, but not a cone nor a linear space.

Let $\overset{*}{\mathcal{H}}{}'^{\Phi} = \{u \geqslant 0 \text{ subharmonic} : \alpha u \in \overset{*}{\mathcal{H}}{}^{\Phi} \text{ for some } \alpha > 0\}$, $\mathcal{E}^{\Phi}_{\mathbf{R}}$ the real vector space of differences of functions in $\overset{*}{\mathcal{H}}{}'^{\Phi}$; $u \in \overset{*}{\mathcal{H}}{}'^{\Phi}$ if and only if u has a smallest harmonic majorant ${}_1u \in \overset{*}{\mathcal{H}}{}'^{\Phi}_l$, and $\|u - u'\|^*_{\Phi} = \|{}_1u - {}_1u'\|_{\Phi}$ defines in $\mathcal{E}^{\Phi}_{\mathbf{R}}$ a semi-norm to which the development of II.3 relative to $\mathcal{E}^p_{\mathbf{R}}$ does extend immediately.

As for the boundary behavior of $\overset{*}{\mathcal{H}}{}^{\Phi}$ functions, we have, under the hypotheses of section 2:

THEOREM 14. — *Every function $u \in \overset{*}{\mathcal{H}}{}^{\Phi}$, Φ « convex », has a finite fine limit $\check{u} \in L^{\Phi}(\mu_1)$ at μ_1-almost every point of the minimal boundary Δ_1. If Φ is strongly « convex », then the smallest harmonic majorant of u* (resp. of $\Phi(u)$) *in Ω is $\mathcal{G}_{\check{u},1}$ (resp. $\mathcal{G}_{\Phi(\check{u}),1}$), and*

$$\begin{cases} u = \mathcal{G}_{\check{u},1} - w_1, \\ \Phi(u) = \mathcal{G}_{\Phi(\check{u}),1} - w, \end{cases}$$

where w_1, w, are positive potentials in Ω.

Proof. — Similar to the one of theorem 8.

To finish, we just mention here the possibility of extending, with the appropriate modifications mentioned in II.4, all preceding results to relative classes $\mathcal{H}^{\Phi}_h$, $\overset{*}{\mathcal{H}}{}^{\Phi}_h$, of h-harmonic, sub-h-harmonic functions in Ω, defined by replacing the constant function 1 by any finite > 0 continuous non-constant super-harmonic function $h \in \overset{*}{\mathcal{H}}{}^{+}_{\mathbf{R}}$, or any > 0 harmonic function $h \in \mathcal{H}^{+}_{\mathbf{R}}$. Corresponding results are easy to rephrase.

CHAPTER IV

SOME APPLICATIONS

1. Strongly subharmonic functions.

In this section, we want to show how a result given by Gärding-Hörmander in a short Note [10], and yielding in particular a very simple proof of the classical F. and M. Riesz theorem, is actually an easy consequence of our results on $\overset{*}{\mathcal{H}}{}^{\Phi}$ classes, and therefore also valid in our present set-up. At the same time, we give, for positive subharmonic functions, a general theorem which can be considered as the subharmonic version, and extension, of the Phragmen-Lindelöf principle for analytic functions in the unit disk.

Following [10], a positive subharmonic function u in Ω will be called *strongly subharmonic* if $\varphi^{-1}(u)$ is subharmonic, for some function φ, positive convex strictly increasing in $]-\infty, +\infty[$, with $\lim_{t \to -\infty} \varphi(t) = 0$ and $\lim_{t \to +\infty} \frac{\varphi(t)}{t} = +\infty$ (φ is the determining function).

For strongly subharmonic functions, theorem 8 can be complemented in the following important way:

THEOREM 15. — *Let u be a positive subharmonic function, contained in the class $\overset{*}{\mathcal{H}}{}^{1}$. If u is strongly subharmonic, then the smallest harmonic majorant of u in Ω is precisely the solution $\mathcal{S}_{\check{u},1}$, i.e. $u = \mathcal{S}_{\check{u},1} - w_1$, where w_1 is a positive potential in Ω.*

($\check{u} \in L^1(\mu_1)$ is the fine boundary function of u on Δ_1.)

Proof. — Consider in $[0, +\infty[$, the function

$$\Phi(t) = \varphi(t) - \varphi(0);$$

it is strongly « convex », and the hypothesis $u \in \mathcal{H}^1$ shows that the positive subharmonic function $\{\varphi^{-1}(u)\}^+$ is in the class $\overset{*}{\mathcal{H}}{}^{\Phi}$; therefore (theorem 14), its smallest harmonic majorant U in Ω is precisely the solution $\mathcal{S}_{\check{U},1}$ corresponding to its fine boundary function $\check{U}$ on Δ_1.

As $\varphi^{-1}(u) \leqslant U$, and $\varphi^{-1}(u)$ has the fine limit $\varphi^{-1}(\check{u})$ μ_1-almost everywhere on Δ_1, it follows first that $\varphi^{-1}(\check{u}) \in L^1(\mu_1)$; next, that the smallest harmonic majorant V of $\varphi^{-1}(u)$ in Ω, which is $\leqslant U$ and has fine limit $\check{V} = \varphi^{-1}(\check{u})$ μ_1-almost everywhere on Δ_1, satisfies $V \leqslant \mathcal{S}_{\check{V},1} = \mathcal{S}_{\varphi^{-1}(\check{u}),1}$, because $U - V \geqslant 0$ harmonic in Ω can be written as

$$U - V = \mathcal{S}_{\check{U}-\check{V},1} + W,$$

where $W \geqslant 0$ harmonic in Ω has fine limit 0 μ_1-almost everywhere on Δ_1, and $U = \mathcal{S}_{\check{U},1}$.

So $\varphi^{-1}(u) \leqslant \mathcal{S}_{\varphi^{-1}(\check{u}),1}$, whence, by the Jensen inequality for convex functions, $u \leqslant \mathcal{S}_{\check{u},1}$ in Ω, which finishes the proof of the theorem.

COROLLARY 15.1. — *If $f \in \mathcal{H}^1$, and $|f|$ is strongly subharmonic (in particular if $\log |f|$ is subharmonic), then $f = \mathcal{S}_{\check{f},1}$ in Ω.*

The particular case is precisely the extended form of the F. and M. Riesz theorem for analytic functions in the unit disk.

As for the extended Phragmen-Lindelöf principle, we have :

THEOREM 16. — *Let u be a strongly subharmonic function, with determing function φ. Assume that $u \leqslant \varphi \circ \lambda$ for some function λ in Ω, positive and uniformly integrable with respect to the harmonic measures $\rho_{x_0}^{\omega_i}$. Then:*

(i) *u has a finite fine limit $\check{u}$ at μ_1-almost every point of the minimal boundary Δ_1, and $\varphi^{-1}(\check{u}) \in L^1(\mu_1)$.*

(ii) *If $\check{u} \in L^{\Phi}(\mu_1)$, Φ « convex », then $u \in \overset{*}{\mathcal{H}}{}^{\Phi}$; if $\check{u} \in L^{\infty}(\mu_1)$, then u is bounded in $\Omega(\varepsilon \overset{*}{\mathcal{H}}{}^{\infty})$.*

Proof. — With the hypotheses made, $\{\varphi^{-1}(u)\}^+$ is subharmonic and uniformly integrable with respect to the harmonic measures $\rho_{x_0}^{\omega_i}$, therefore, by property I.3.(3), is in the class $\overset{*}{\mathcal{H}}{}^{\Phi_0}$ corresponding to some strongly « convex » function Φ_0.

As seen above, this implies that $\varphi^{-1}(u) \leqslant \mathrm{U}$, with

$$0 \leqslant \mathrm{U} = \mathcal{S}_{\check{\mathrm{U}},1},$$

whence a finite fine limit $\check{u}$ for u at μ_1-almost every point of the minimal boundary Δ_1, and $\varphi^{-1}(\check{u}) \in \mathrm{L}^1(\mu_1)$; moreover, $\varphi^{-1}(u) \leqslant \mathcal{S}_{\varphi^{-1}(\check{u}),1}$, whence $\Phi(u) \leqslant \Phi(\varphi(\mathcal{S}_{\varphi^{-1}(\check{u}),1}))$ in Ω.

If $\check{u} \in \mathrm{L}^{\Phi}(\mu_1)$, i.e. $\Phi(\check{u}) \in \mathrm{L}^1(\mu_1)$, then the Jensen inequality for convex functions, applied to $\Phi \circ \varphi$, shows that

$$\Phi(u) \leqslant \mathcal{S}_{\Phi(\check{u}),1}$$

in Ω; therefore $\Phi(u)$ has a harmonic majorant in Ω, and $u \in \mathcal{H}^{\Phi}$.

If $\check{u} \in \mathrm{L}^{\infty}(\mu_1)$, one proves similarly that $u \leqslant \mathcal{S}_{\check{u},1}$, whence obviously $u \in \overset{*}{\mathcal{H}}{}^{\infty}$.

2. Extremal functions, Reproducing kernel, and Classification.

We begin with a result in the classification of harmonic classes on Ω, obtained as an almost immediate consequence of the general properties of the $\mathcal{H}^{\Phi}$ classes.

Our *hypotheses* are, as before, axioms 1, 2, 3, existence of a > 0 potential in Ω, and $1 \in \overset{*}{\mathcal{H}}{}^{+}_{\mathbf{R}}$ with greatest harmonic minorant ϖ_0 in Ω.

We denote by $\mathcal{H}'^1$ the class of all *quasi-bounded* harmonic functions in Ω, i.e. of all functions $f = u + iv$ whose real and imaginary parts are differences of two positive harmonic functions, each of which being the limit of an increasing sequence of bounded positive harmonic functions in Ω.

Following the pattern used for Riemann surfaces, we define $\mathcal{C}_{\mathcal{H}^1}$ as the class of harmonic classes $\mathcal{H}$ on Ω, which satisfy the above hypotheses, and are such that every function in $\mathcal{H}^1$ is a constant multiple of ϖ_0. We similarly define $\mathcal{C}_{\mathcal{H}'^1}$, $\mathcal{C}_{\mathcal{H}^p}$, $p \geqslant 1$, $\mathcal{C}_{\mathcal{H}^{\Phi}}$, Φ « convex », and $\mathcal{C}_{\mathcal{H}^{\infty}}$.

It is trivial that $\mathcal{C}_{\mathcal{H}^{\infty}} = \mathcal{C}_{\mathcal{H}'^1}$.

According to corollary 1.1, property I.3.(3), and corollary 10.2, the functions in $\mathcal{H}'^1$ are *all* the complex harmonic func-

tions in Ω which are uniformly integrable with respect to the harmonic measures $\rho_{x_0}^{\omega_i}$.

Therefore, (again by property I.3.(3)), for every $p > 1$, and for every strongly « convex » Φ, we have the inclusions :

$$\mathcal{H}^\infty \subset \begin{matrix}\mathcal{H}^p \\ \mathcal{H}^\Phi\end{matrix} \subset \mathcal{H}'^1 \subset \mathcal{H}^1.$$

It follows that

$$\mathcal{C}_{\mathcal{H}^1} \subset \mathcal{C}_{\mathcal{H}'^1} \subset \begin{matrix}\mathcal{C}_{\mathcal{H}^p} \\ \mathcal{C}_{\mathcal{H}^\Phi}\end{matrix} \subset \mathcal{C}_{\mathcal{H}^\infty}.$$

As $\mathcal{C}_{\mathcal{H}^\infty} = \mathcal{C}_{\mathcal{H}'^1}$, we see that, like in Parreau [22], we have :

THEOREM 17. — *For every* $p > 1$, $\mathcal{C}_{\mathcal{H}^\infty} = \mathcal{C}_{\mathcal{H}^p}$; *for every strongly « convex »* Φ, $\mathcal{C}_{\mathcal{H}^\infty} = \mathcal{C}_{\mathcal{H}^\Phi}$.

The first relation $\mathcal{C}_{\mathcal{H}^\infty} = \mathcal{C}_{\mathcal{H}^p}$, $p > 1$, has also been obtained, independently, by Loeb and Walsh [17], under the same hypotheses.

We now *assume, in addition*, a countable base for the open sets of Ω, and $1 \in \mathcal{H}_{\mathbf{R}}^+$, which permits the consideration of the minimal boundary Δ_1, and the corresponding Dirichlet problem already described. We want to show how in that case the above result can also be obtained by making use of a certain *kernel function*, whose idea for Riemann surfaces actually goes back to Bader, as mentioned by Parreau [22], and which has been used by the latter for the same purpose of classification as we do here.

As usual, x_0 is a fixed point $\varepsilon\Omega$, and the extreme harmonic functions are normalized at x_0. By theorem 2, the functions in $\mathcal{H}'^1$ are now *all* the solutions of Dirichlet problems (of the described type) with boundary functions $\varepsilon L^1(\mu_1)$.

Let x be another fixed point $\varepsilon\Omega$. By Harnack inequality, $k(x)$, considered as a *function of* $k \in \Delta_1$, is bounded, hence $\varepsilon L^\infty(\mu_1)$. The corresponding solution

$$K_x(z) \equiv \mathcal{S}_{k(x),1}(z) = \int_{\Delta_1} k(z)k(x)\, d\mu_1(k), \qquad z \in \Omega,$$

is therefore bounded > 0 harmonic in Ω, and has fine limit $k(x)$ μ_1-almost everywhere on Δ_1. For another $y \in \Omega$, we have, similarly,

$$K_y(z) \equiv \mathcal{S}_{k(y),1}(z) = \int_{\Delta_1} k(z)k(y)\, d\mu_1(k), \qquad z \in \Omega.$$

It is clear that

$$K_x(y) = \int_{\Delta_1} k(x)\ k(y)\ d\mu_1(k) = K_y(x), \qquad x,\ y \in \Omega.$$

The common value $K_x(y) = K_y(x)$ is by definition the *kernel function*

$$K(x, y) = \int_{\Delta_1} k(x)\ k(y)\ d\mu_1(k), \qquad x, \quad y \in \Omega.$$

It is bounded > 0 harmonic in each variable when the other one is fixed; at the point x_0,

$$K_x(x_0) = K(x, x_0) = \int_{\Delta_1} k(x)\, d\mu_1(k) = 1,$$

and

$$\| K_x \|_\infty = \| k(x) \|_\infty$$

while

$$\| K_x \|_p = \| k(x) \|_p, \qquad \text{for every} \qquad p \geqslant 1,$$

and

$$\| K_x \|_\Phi = \| k(x) \|_\Phi,$$

(the Minkowski norm), for every « convex » Φ.

THEOREM 18. — *Let x be a fixed point $\varepsilon\Omega$. The following two conditions are equivalent:*

(i) *The function K_x is identically 1 in Ω (equivalently, $k(x) = 1$ μ_1-almost everywhere on Δ_1).*

(ii) *For every $f \in \mathcal{H}'^1$, $f(x) = f(x_0)$.*

Proof. — We use the last mentioned characterization of $\mathcal{H}'^1$ functions. If $K_x \equiv 1$ in Ω, then $k(x) = 1$ μ_1-almost everywhere on Δ_1, therefore, for every $f \in \mathcal{H}'^1$:

$$f(x) = \int_{\Delta_1} k(x) \check{f}(k)\, d\mu_1(k) = \int_{\Delta_1} \check{f}(k)\, d\mu_1(k) = f(x_0).$$

Conversely, if $f(x) = f(x_0)$ for every $f \in \mathcal{H}'^1$, one has

$$K_y(x) = K_y(x_0) = 1\ \forall y \in \Omega,$$

hence $K_x \equiv 1$ in Ω.

COROLLARY 18.1. — *Under the present assumptions, the results of theorem 17 hold.*

Proof. — It suffices to prove the inclusion $\mathcal{C}_{\mathcal{H}^\infty} \subset \mathcal{C}_{\mathcal{H}'^1}$ with the characterization of $\mathcal{H}'^1$ functions as solutions.

But if every function in $\mathcal{H}^\infty$ is a constant, then $K_x \equiv 1$ for every $x \in \Omega$, therefore $f(x) = f(x_0)$ for every $f \in \mathcal{H}'^1$ and every $x \in \Omega$, which shows that f is constant in Ω.

We shall now study further the structure of the Banach space $\mathcal{H}^2$, and, following Parreau [22], solve a certain minimum problem relative to its norm, leading very naturally to the kernel function $K(x, y)$.

Let u, v, real $\varepsilon \mathcal{H}^2$; $\forall \iota \in I$,

$$\int_{\partial\omega_\iota} uv \, d\rho_{x_0}^{\omega_\iota} = \frac{1}{2}\left\{\int_{\partial\omega_\iota} (u+v)^2 \, d\rho_{x_0}^{\omega_\iota} - \int_{\partial\omega_\iota} u^2 \, d\rho_{x_0}^{\omega_\iota} - \int_{\partial\omega_\iota} v^2 \, d\rho_{x_0}^{\omega_\iota}\right\}.$$

When ω_ι increases to Ω, each of the three integrals on the right has a finite limit, hence $\lim_\iota \int_{\partial\omega_\iota} uv \, d\rho_{x_0}^{\omega_\iota}$ exists and is finite, $\left(\text{actually equal to } \frac{1}{2}\{{}_2(u+v)(x_0) - {}_2u(x_0) - {}_2v(x_0)\}\right)$. The same is therefore true for $\lim_\iota \int_{\partial\omega_\iota} f\bar{g} \, d\rho_{x_0}^{\omega_\iota}$, f, g, complex $\varepsilon \mathcal{H}^2$. If we set

$$(f, g) = \lim_\iota \int_{\partial\omega_\iota} f\bar{g} \, d\rho_{x_0}^{\omega_\iota},$$

we see immediately that this is an inner product in $\mathcal{H}^2$, that its quadratic form is the $\mathcal{H}^2$-norm, $\|f\|_2^2 = (f, f)$, and that

$$(f, g) = \int_{\Delta_1} \check{f}(k)\bar{\check{g}}(k) \, d\mu_1(k),$$

and

$$(f, 1) = f(x_0).$$

For each $x \in \Omega$, the evaluation at $x : f \to f(x)$ is continuous in $\mathcal{H}^2$, hence, according to the general theory of Hilbert function spaces, there exists for $\mathcal{H}^2$ a unique reproducing kernel, i.e. a unique function $\theta(x, y)$ defined in $\Omega \times \Omega$, such that

(i) for each $x \in \Omega$, $\theta_x(y) = \theta(x, y)$ is in $\mathcal{H}^2$;

(ii) for each $f \in \mathcal{H}^2$, and each $x \in \Omega$, $f(x) = (f, \theta_x)$, (reproducing property).

Again, let x be a fixed point $\varepsilon\Omega$. Let

$$\mathcal{U}_x^2 = \{f \in \mathcal{H}^2 : f(x) = 1\},$$

which is a closed convex subset of $\mathcal{H}^2$, and let

$$m_0 = \inf\{\|f\|_2 : f \in \mathcal{U}_x^2\},$$

which is $\leqslant 1$, since $1 \in \mathcal{U}_x^2$, and > 0 as we shall see.

Consider in $\mathcal{U}_x^2$ a sequence $\{f_n\}$ such that

$$\|f_n\|_2 \to m_0, \qquad n \to +\infty.$$

By the elementary equality,

$$\left\|\frac{f_n - f_{n'}}{2}\right\|_2^2 = \frac{1}{2}\{\|f_n\|_2^2 + \|f_{n'}\|_2^2\} - \left\|\frac{f_n + f_{n'}}{2}\right\|_2^2,$$

$\left(\text{where } \dfrac{f_n + f_{n'}}{2} \in \mathcal{U}_x^2, \text{ whence } \left\|\dfrac{f_n + f_{n'}}{2}\right\|_2 \geqslant m_0\right)$, it is a Cauchy sequence in $\mathcal{U}_x^2$, therefore converges in $\mathcal{H}^2$ to a function f_0, which is in $\mathcal{U}_x^2$ and has norm $\|f_0\|_2 = m_0$. Moreover, any $f_1 \in \mathcal{U}_x^2$ such that $\|f_1\|_2 = m_0$ is equal to f_0, since

$$\left\|\frac{f_1 - f_0}{2}\right\|_2^2 = \frac{1}{2}\{\|f_1\|_2^2 + \|f_0\|_2^2\} - \left\|\frac{f_1 + f_0}{2}\right\|_2^2 \leqslant m_0^2 - m_0^2 = 0.$$

So the problem of finding among the functions of $\mathcal{H}^2$ with value 1 at the point x, a function minimizing the $\mathcal{H}^2$-norm, has a unique solution $f_0 \not\equiv 0$, which we relate below to the reproducing kernel in $\mathcal{H}^2$, and the kernel function $K(x, y)$.

It is immediate that f_0 is real-valued for, if $f_0 = u_0 + iv_0$, then $u_0 \in \mathcal{U}_x^2$, $\|u_0\|_2 \leqslant \|f_0\|_2$, whence $u_0 = f_0$ by the uniqueness. It is also clear, by an elementary computation, that f_0 is orthogonal, in $\mathcal{H}^2$, to any $f \in \mathcal{H}^2$ which vanishes at x. Therefore, for any $f \in \mathcal{H}^2$,

$$(f - f(x), f_0) = 0,$$

whence

$$(f, f_0) = (f(x), f_0) = f(x).(1, f_0) = f(x).f_0(x_0);$$

in particular

$$f_0(x_0) = (f_0, f_0) = \|f_0\|_2^2 \neq 0.$$

So, finally,

$$f(x) = \frac{(f, f_0)}{\|f_0\|_2^2} = \left(f, \frac{f_0}{\|f_0\|_2^2}\right),$$

or, setting

$$\theta_x = \frac{f_0}{\|f_0\|_2^2}, \qquad \theta_x \in \mathcal{H}^2,$$
$$f(x) = (f, \theta_x), \qquad \forall f \in \mathcal{H}^2,$$

(reproducing property). To another $y \in \Omega$, one associates similarly $\theta_y \in \mathcal{H}^2$ such that

$$f(y) = (f, \theta_y), \qquad \forall f \in \mathcal{H}^2.$$

In particular

$$\theta_x(y) = (\theta_x, \theta_y) = (\theta_y, \theta_x) = \theta_y(x).$$

So there is symmetry, and the function $\theta(x, y) = \theta_x(y) = \theta_y(x)$ of $(x, y) \in \Omega \times \Omega$ is the reproducing kernel in $\mathcal{H}^2$.

By the reproducing property,

$$\int_{\Delta_1} \check{f}(k)\, k(x)\, d\mu_1(k) = \int_{\Delta_1} \check{f}(k)\, \theta_x(k)\, d\mu_1(k),$$

for every $\check{f} \in L^2(\mu_1)$, in particular uniformly continuous on Δ_1; hence $\check{\theta}_x(k) = k(x)$ μ_1-almost everywhere on Δ_1, and $\theta_x = K_x$: $\theta(x, y)$ coincides with the kernel function $K(x, y)$.

The *minimizing function* f_0 *has* $\mathcal{H}^2$*-norm*

$$\|f_0\|_2 = \frac{1}{\|K_x\|_2} = \frac{1}{\|k(x)\|_2},$$

and boundary function

$$\check{f}_0(k) = \frac{k(x)}{\|k(x)\|_2^2}\ \mu_1\text{-almost everywhere on } \Delta_1.$$

THEOREM 19. — *Let x be a fixed point $\varepsilon\Omega$. The following two conditions are equivalent, and equivalent to conditions* (i) *and* (ii) *of theorem* 18:

(iii) *The constant function* 1 *minimizes* $\|f\|_2$, $f \in \mathfrak{U}_x^2$.

(iv) *The minimum of* $\|f\|_2$, $f \in \mathfrak{U}_x^2$, *is equal to* 1.

The same minimum problem can be solved in each $\mathcal{H}^p$, for $1 < p < +\infty$, using the fact that $L^p(\mu_1)$, hence $\mathcal{H}^p$, $1 < p < +\infty$, is *uniformly convex*, i.e.:

given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$, such

that the inequalities

$$\|f\|_p \leqslant 1, \qquad \|g\|_p \leqslant 1, \qquad \left\|\frac{f+g}{2}\right\|_p \geqslant 1 - \delta(\varepsilon), \qquad f,\ g \in \mathcal{H}^p,$$

imply

$$\|f - g\|_p \leqslant \varepsilon.$$

Let x be a fixed point $\varepsilon\Omega$. Let

$$\mathfrak{U}_x^p = \{f \in \mathcal{H}^p : f(x) = 1\},$$

which is again a closed convex subspace of $\mathcal{H}^p$, and let

$$m_0 = \inf\{\|f\|_p : f \in \mathfrak{U}_x^p\},$$

which is $\leqslant 1$, and > 0.

Consider in $\mathfrak{U}_x^p$ a sequence $\{f_n\}$ such that

$$\|f_n\|_p \to m_0, \qquad n \to +\infty.$$

For $n,\ n' \geqslant n_0$, $\|f_n\|_p \leqslant \dfrac{m_0}{1-\delta(\varepsilon)}$, $\|f_{n'}\|_p \leqslant \dfrac{m_0}{1-\delta(\varepsilon)}$, while $\left\|\dfrac{f_n + f_{n'}}{2}\right\|_p \geqslant m_0$; therefore $\|f_n - f_{n'}\|_p \leqslant \dfrac{\varepsilon m_0}{1-\delta(\varepsilon)}$: $\{f_n\}$ is a Cauchy sequence in $\mathcal{H}^p$, and converges in $\mathcal{H}^p$ to a function f_0 which is in $\mathfrak{U}_x^p$ and has norm $\|f_0\|_p = m_0$.

The same type of argument shows that this minimizing function is unique, and real-valued.

Apply now the Hölder inequality to the right integral in

$$1 = f(x) = \int_{\Delta_1} k(x)\check{f}(k)\, d\mu_1(k), \qquad f \in \mathfrak{U}_x^p.$$

This gives

$$1 \leqslant \|\check{f}\|_p . \|k(x)\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

hence

$$\frac{1}{\|k(x)\|_q} \leqslant \|\check{f}\|_p, \qquad f \in \mathfrak{U}_x^p.$$

But it is elementary to verify that the $\mathcal{H}^p$-function with boundary values $\dfrac{k(x)^{q/p}}{\|k(x)\|_q^q}$ is in $\mathfrak{U}_x^p$, and has $\mathcal{H}^p$-norm equal to $\dfrac{1}{\|k(x)\|_q}$, hence is equal to the minimizing function f_0.

In conclusion, the *minimizing function* f_0 *has* $\mathcal{H}^p$*-norm*

$$\|f_0\|_p = \frac{1}{\|k(x)\|_q},$$

and *boundary function*

$$\check{f}_0(k) = \frac{k(x)^{q/p}}{\|k(x)\|_q^q}$$

μ_1-almost everywhere on Δ_1.

Preceding results can be extended to the space $\mathcal{H}^\Phi$, Φ strongly « convex », when this is uniformly convex (with the Minkowski norm we considered). Sufficient conditions on Φ (implying among others that $\mathcal{H}^\Phi$ is linear) have been given by Luxemburg [20].

For such Φ, we consider

$$\mathcal{U}_x^\Phi = \{f \in \mathcal{H}^\Phi : f(x) = 1\}, \qquad x \text{ fixed } \varepsilon\Omega,$$

and

$$m_0 = \inf\{\|f\|_\Phi : f \in \mathcal{U}_x^\Phi\};$$

$m_0 \leqslant \|1\|_\Phi$, and > 0.

There exists in $\mathcal{U}_x^\Phi$ a unique real-valued function f_0, minimizing the norm.

The explicit determination of f_0 is more delicate. (We refer to Luxemburg [20] for all results used below.)

We first apply to the right integral in

$$1 = f(x) = \int_{\Delta_1} k(x)\check{f}(k)\, d\mu_1(k), \qquad f \in \mathcal{U}_x^\Phi,$$

a generalization of the Hölder inequality to Orlicz spaces. We obtain

$$1 \leqslant \|\check{f}\|_\Phi . \|k(x)\|_\Psi^*,$$

where

$$\|\check{f}\|_\Phi = \|f\|_\Phi, \qquad \check{f} \in L^\Phi(\mu_1),$$

while

$$\|k(x)\|_\Psi^* = \sup\left\{\int_{\Delta_1} |\check{f}(k)| k(x)\ d\mu_1(k) : \check{f} \in L^\Phi(\mu_1), \quad \|\check{f}\|_\Phi \leqslant 1\right\},$$

(the Orlicz norm), and Ψ is the complementary Young function of Φ, determined by the conditions

$$\Phi(\xi) = \int_0^\xi \varphi(t)\, dt, \qquad \Psi(\eta) = \int_0^\eta \psi(t)\, dt,$$

and the functions φ, ψ, are reciprocal.

So $\dfrac{1}{\|k(x)\|_{\Psi}^{*}} \leqslant \|f\|_{\Phi}$, $\forall f \in \mathcal{U}_x^{\Phi}$; if we show that there exists in $\mathcal{U}_x^{\Phi}$ a function with norm equal to $\dfrac{1}{\|k(x)\|_{\Psi}^{*}}$, this will necessarily be the minimizing f_0.

But, as

$$1 = \sup \left\{ \int_{\Delta_1} |\check{f}(k)| \frac{k(x)}{\|k(x)\|_{\Psi}^{*}} d\mu_1(k) : \quad \check{f} \in \mathrm{L}^{\Phi}(\mu_1), \quad \|\check{f}\|_{\Phi} \leqslant 1 \right\},$$

there exists $0 \leqslant \check{f}_n \in \mathrm{L}^{\Phi}(\mu_1)$, such that $\|\check{f}_n\|_{\Phi} \leqslant 1$, and

$$\int_{\Delta_1} \check{f}_n(k) \frac{k(x)}{\|k(x)\|_{\Psi}^{*}} d\mu_1(k) \to 1, \qquad n \to +\infty.$$

Given $\varepsilon > 0$, it follows that, for n, $n' \geqslant n_0$,

$$1 \geqslant \left\| \frac{\check{f}_n + \check{f}_{n'}}{2} \right\|_{\Phi} \geqslant \int_{\Delta_1} \left(\frac{\check{f}_n + \check{f}_{n'}}{2} \right)(k) \frac{k(x)}{\|k(x)\|_{\Psi}^{*}} d\mu_1(k) \geqslant 1 - \delta(\varepsilon),$$

whence, by the uniform convexity of $\mathrm{L}^{\Phi}(\mu_1)$,

$$\|\check{f}_n - \check{f}_{n'}\|_{\Phi} \leqslant \varepsilon, \qquad n,\ n' \geqslant n_0.$$

So $\{\check{f}_n\}$ converges, in $\mathrm{L}^{\Phi}(\mu_1)$, to a function $\check{f}_0' \in \mathrm{L}^{\Phi}(\mu_1)$, $\geqslant 0$, satisfying

$$\|\check{f}_0'\|_{\Phi} = 1,$$

and

$$\int_{\Delta_1} \check{f}_0'(k) \frac{k(x)}{\|k(x)\|_{\Psi}^{*}} d\mu_1(k) = 1,$$

(because $f_n = \mathcal{S}_{\check{f}_n,1}$ converges uniformly locally to $f_0' = \mathcal{S}_{\check{f}_0',1}$). The function $\dfrac{f_0'}{\|k(x)\|_{\Psi}^{*}}$ is in $\mathcal{U}_x^{\Phi}$, and has norm $\dfrac{1}{\|k(x)\|_{\Psi}^{*}}$, therefore equals the minimizing f_0 (which is thus > 0).

On the other hand, for every $f \in \mathcal{H}^{\Phi}$ which vanishes at x, and every λ complex, $f_0 + \lambda f \in \mathcal{U}_x^{\Phi}$, therefore

$$\left\| \frac{\check{f}_0 + \lambda \check{f}}{m_0} \right\|_{\Phi} \geqslant 1, \qquad \left(m_0 = \frac{1}{\|k(x)\|_{\Psi}^{*}} \right),$$

and $\left\| \dfrac{\check{f}_0 + \lambda \check{f}}{m_0} \right\|_{\Phi} = 1$ if and only if $\lambda = 0$.

This is equivalent to

$$\int_{\Delta_1} \Phi\left(\frac{|\check{f}_0 + \lambda\check{f}|}{m_0}\right)(k)\, d\mu_1(k) \geqslant 1,$$

and $\int_{\Delta_1} \Phi\left(\frac{|\check{f}_0 + \lambda\check{f}|}{m_0}\right)(k)\, d\mu_1(k) = 1$ if and only if $\lambda = 0$.

So $F(\lambda) = \int_{\Delta_1} \Phi\left(\frac{|\check{f}_0 + \lambda\check{f}|}{m_0}\right)(k)\, d\mu_1(k)$ is minimum for $\lambda = 0$.

As $F'(\lambda) = \int_{\Delta_1} \frac{\check{f}(k)}{m_0} \varphi\left(\frac{|\check{f}_0 + \lambda\check{f}|}{m_0}\right)(k)$ sign $(\check{f}_0 + \lambda\check{f})(k)\, d\mu_1(k)$, we see that for every $f \in \mathcal{H}^\Phi$, which vanishes at x,

$$\int_{\Delta_1} \frac{\check{f}(k)}{m_0} \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k) = 0,$$

and for every $f \in \mathcal{H}^\Phi$,

$$\int_{\Delta_1} \check{f}(k)\, \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k) = \int_{\Delta_1} \check{f}_0(k)\, \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k) \cdot \int_{\Delta_1} \check{f}(k) k(x)\, d\mu_1(k).$$

This gives necessarily

$$\int_{\Delta_1} \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k) = \int_{\Delta_1} \check{f}_0(k)\, \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k),$$

and
$$\varphi\left(\frac{\check{f}_0}{m_0}\right)(k) = \left(\int_{\Delta_1} \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k)\right).k(x),$$

whence

$$\frac{\check{f}_0}{m_0}(k) = \varphi^{-1}\left(\left(\int_{\Delta_1} \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k)\right).k(x)\right),$$

μ_1-almost everywhere on Δ_1.

As there exists a unique $\alpha_0 > 0$ such that

$$\int_{\Delta_1} \varphi^{-1}(\alpha_0 k(x)) k(x)\, d\mu_1(k) = \frac{1}{m_0},$$

we see that necessarily

$$\int_{\Delta_1} \varphi\left(\frac{\check{f}_0}{m_0}\right)(k)\, d\mu_1(k) = \alpha_0,$$

and finally $\check{f}_0(k) = m_0 \varphi^{-1}(\alpha_0 k(x))$, μ_1-almost everywhere on Δ_1.

In conclusion, the *minimizing function* f_0 *has* $\mathcal{H}^\Phi$*-norm*

$$\|f_0\|_\Phi = \frac{1}{\|k(x)\|_{\overset{*}{\Psi}}},$$

and *boundary function*

$$\check{f}_0(k) = \frac{\varphi^{-1}(\alpha_0 k(x))}{\|k(x)\|_{\overset{*}{\Psi}}}$$

μ_1-almost everywhere on Δ_1, α_0 being determined by the equation

$$\int_{\Delta_1} \varphi^{-1}(\alpha_0 k(x))k(x)\, d\mu_1(k) = \|k(x)\|_{\overset{*}{\Psi}}.$$

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