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Flows of flowable Reeb homeomorphisms

FLOWS OF FLOWABLE REEB HOMEOMORPHISMS

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ABSTRACT. — We consider a fixed point free homeomorphism h of the closed band $B = \mathbb{R} \times [0,1]$ which leaves each leaf of a Reeb foliation on B invariant. Assuming h is the time one of various topological flows, we compare the restriction of the flows on the boundary.

RÉSUMÉ. — Considérons une homéomorphisme sans points fixes sur la bande fermé $B = \mathbb{R} \times [0,1]$ qui laisse un feuilletage de Reeb invariant et qui est le temps un des flots topologiques. Nous comparons les restrictions de plusieurs tels flots au bord de B.

1. Introduction

Orientation preserving and fixed point free homeomorphisms of the plane are called *Brouwer homeomorphisms*. Since the seminal work of L. E. Brouwer nearly 100 years ago, they draw attentions of many mathematicians ([3], [8], [4], [5] [6], [11]). Nowadays there still remains interesting problems about them.

Besides those Brouwer homeomorphisms which are topologically conjugate to the translation, the simplest ones are perhaps those which preserve the leaves of Reeb foliations; the main theme of the present notes. It is simpler and loses nothing to consider their restriction to the Reeb component.

Let $B = \mathbb{R} \times [0,1]$ be a closed band, and denote $\partial_i B = \mathbb{R} \times \{i\}$ ($i = 0, 1$) and $\mathrm{Int}\, B = \mathbb{R} \times (0,1)$. An oriented foliation $\mathcal{R}$ on B is called a *Reeb foliation* if $\partial_0 B$ with the positive orientation and $\partial_1 B$ with the negative orientation are leaves of $\mathcal{R}$ and the foliation restricted to the interior $\mathrm{Int}\, B$ is a bundle foliation. The leaf space of a Reeb foliation is homeomorphic to the

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non Hausdorff space obtained by glueing two copies of $[0, \infty)$ along $(0, \infty)$. This shows ([7]) that any two Reeb foliations are mutually topologically equivalent.

A homeomorphism $h\colon B \to B$ is called a *Reeb homeomorphism* if h preserves each leaf of a Reeb foliation and $h(x) > x$ for any $x \in B$, where $>$ is the total order on a leaf given by the orientation. In this paper we consider *flowable* Reeb homeomorphisms h, *i.e.* those which are the time one of topological flows. (F. Béguin and F. Le Roux constructed in [2] examples of non flowable Reeb homeomorphisms. See also Sect. 4 of the present paper.) When h is flowable, h can be embedded as the time one into a great variety of flows. The purpose of this paper is to compare the restriction of one flow to the boundary of B with that of another. This problem is motivated by a result of [9] which states that if two flows have a common orbit foliation, then their restrictions to the boundary are the same.

Let
$$P = \big\{(x, y) \mid x \geqslant 0, y \geqslant 0\big\} - \big\{(0, 0)\big\}.$$
Notice that P is homeomorphic to B. Consider a homeomorphsim $h_P\colon P \to P$ defined by
$$h_P(x, y) = (2x, 2^{-1}y)).$$
A homeomorphism of B is called a *standard Reeb homeomorphism* if it is topologically conjugate to h_P, and *nonstandard* otherwise. It is known ([2]) that there are nonstandard flowable Reeb homeomorphisms. See also Sect. 4. The main result of this paper is the following.

THEOREM 1.1.
 (1) *Assume h is a standard Reeb homeomorphsim of B. For $i = 0, 1$, let $\{\psi_i^t\}$ be an arbitrary flow on $\partial_i B$ whose time one is the restriction of h. Then there is a flow $\{\varphi^t\}$ on B, an extension of both $\{\psi_0^t\}$ and $\{\psi_1^t\}$, whose time one is h.*
 (2) *If h is a nonstandard flowable Reeb homeomorphism, there is a homeomorphism from $\partial_0 B$ to $\partial_1 B$ which commutes with any flow whose time one is h.*

Notice that (1) is immediate from the model h_P which is a product map. After we prepare some necessary prerequisites in Sect. 2, we prove Theorem 1.1 (2) in Sect. 3. Sect. 4 is devoted to two examples of nonstandard flowable Reeb homeomorphisms, one for which Theorem 1.1 (2) is the optimal, and the other for which the restriction of the flow to the boundary is unique.

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2. Preliminaries

Let $\mathcal{R}$ be a Reeb foliation on B. A topological flow on B is called an $\mathcal{R}$-flow if its oriented orbit foliation is $\mathcal{R}$. To fix the idea we assume throughout that $\partial_0 B$ is repelling and $\partial_1 B$ is attracting. Let $\mathcal{E}$ be the set of the topological conjugacy classes of $\mathcal{R}$-flows. We shall summarize a main result of [10], a classification of $\mathcal{E}$, which will play a crucial role in what follows.

For $i = 0, 1$ let $\gamma_i \colon [0, \infty) \to B$ be a continuous path such that $\gamma_i(0) \in \partial_i B$ and that γ_i intersects every interior leaf of $\mathcal{R}$ at exactly one point. Let us parametrize γ_i so that for any $y > 0$ the points $\gamma_0(y)$ and $\gamma_1(y)$ lie on the same leaf of $\mathcal{R}$. Let $\{\Phi^t\}$ be an $\mathcal{R}$-flow. Then one can define a continuous function

$$f_{\{\Phi^t\},\gamma_0,\gamma_1} \colon (0, \infty) \to \mathbb{R}$$

by setting that $f_{\{\Phi^t\},\gamma_0,\gamma_1}(y)$ is the time needed for the flow $\{\Phi^t\}$ to drift from the point $\gamma_0(y)$ to $\gamma_1(y)$. That is,

$$\Phi^t(\gamma_0(y)) = \gamma_1(y) \text{ for } t = f_{\{\Phi^t\},\gamma_0,\gamma_1}(y).$$

Then the function $f_{\{\Phi^t\},\gamma_0,\gamma_1}$ belongs to the space

$$E = \big\{ f \colon (0, \infty) \to \mathbb{R} \mid f \text{ is continuous and } \lim_{y \to 0} f(y) = \infty \big\}.$$

Of course $f_{\{\Phi^t\},\gamma_0,\gamma_1}$ depends upon the choice of γ_i. There are two umbiguities, one coming from the parametrization of γ_i, and the other coming from the positions of γ_i. Let H be the space of homeomorphisms of $[0, \infty)$ and C the space of continuous functions on $[0, \infty)$. Define an equivalence relation $\sim$ on E by

$$f \sim f' \iff f' = f \circ h + k, \quad \exists h \in H, \quad \exists k \in C.$$

Then clearly the equivalence class $[f_{\{\Phi^t\},\gamma_0,\gamma_1}]$ does not depend on the choice of γ_i. Moreover it is an invariant of the topological conjugacy classes of $\mathcal{R}$-flows. Therefore we get a well defined map

$$\iota \colon \mathcal{E} \to E/\sim .$$

It is easy to see that ι is injective. The main result of [10] states that ι is surjective as well, *i.e.* any $f \in E$ is realized as $f = f_{\{\Phi^t\},\gamma_0,\gamma_1}$ for some $\mathcal{R}$-flow $\{\Phi^t\}$ and curves γ_i.

The equivalence class $[f]$ of $f \in E$ is determined by how $f(y)$ oscillates while it tends to ∞ as $y \to 0$. For example any monotone function of E

belongs to a single equivalence class, which corresponds to a standard Reeb flow. By definition a *standard Reeb flow* is a flow which is topologically conjugate to the flow $\{\varphi_P^t\}$ on P given by

$$\varphi_P^t(x, y) = (2^t x, 2^{-t} y).$$

To measure the degree of oscillation of $f \in E$, define a nonnegative valued continuous function f^* defined on $(0, 1]$ by

$$f^*(y) = \max(f|_{[y,1]}) - f(y).$$

Then we have the following easy lemma.

LEMMA 2.1.
(1) If $h \in H$, then $(f \circ h)^* = f^* \circ h$ in a neighbourhood of 0.
(2) If $k \in C$ and $y \to 0$, then $(f + k)^*(y) - f^*(y) \to 0$.
(3) There is a sequence $\{y_n\}$ converging to 0 such that $f^*(y_n) = 0$.

For f as above, define an invariant $\sigma(f) \in [0, \infty]$ by

$$\sigma(f) = \limsup_{y \to 0} f^*(y).$$

Lemma 2.1 implies that $\sigma(f)$ is an invariant of the class $[f]$. We also have $\sigma(f) = 0$ if and only if the class $[f]$ is represented by a monotone function, that is, $[f]$ corresponds to a standard Reeb flow.

3. Proof of Theorem 1.1

Fix once and for all a nonstandard Reeb homeomorphism h of B and assume that h is the time one of a flow $\{\Phi^t\}$. Then it can be shown that the orbit foliation $\mathcal{R}$ of $\{\Phi^t\}$ is a bundle foliation in $\mathrm{Int}(B)$, and therefore $\mathcal{R}$ is a Reeb foliation. Let γ_i $(i = 0, 1)$ and $f = f_{\{\Phi^t\}, \gamma_0, \gamma_1}$ be as in Sect. 2. Notice that $\sigma(f) > 0$ since $\{\Phi^t\}$ must be nonstandard.

Our plan is to define "coordinates" of B via these data, and study the behaviour of any other flow $\{\varphi^t\}$ whose time one is h using these coordinates. But it is more convenient to work with the quotient space by h. So let

$$A = B/\langle h \rangle.$$

A is a non Hausdorff 2-manifold with two boundary cirles, $\partial_i A = \partial_i B/\langle h \rangle$ $(i = 0, 1)$. Any neighbourhood of any point of $\partial_0 A$ intersects any neighbourhood of any point of $\partial_1 A$. Denote

$$\mathrm{Int}(A) = \mathrm{Int}(B)/\langle h \rangle, \quad A_i = \mathrm{Int}(A) \cup \partial_i A, \quad i = 0, 1.$$

A_i is a Hausdorff space homeomorphic to $S^1 \times [0, \infty)$.

The flow $\{\Phi^t\}$, as well as the other flow $\{\varphi^t\}$, induces an S^1-action on A, still denoted by the same letter. The curve γ_i induces a curve in A_i, denoted by the same letter. One can use the parameter of the curve γ_i as a hight function p on A. Recall that by the convention of Sect.2, the points $\gamma_0(y)$ and $\gamma_1(y)$ lie on the same leaf of $\mathcal{R}$ if $y > 0$. Let us define a projection $p \colon A \to [0, \infty)$ as follows. For any $\xi \in A$,

$$p(\xi) = y \Longleftrightarrow \xi \text{ lies on the leaf of } \mathcal{R} \text{ passing through } \gamma_0(y) \text{ or } \gamma_1(y).$$

Of course $p(\partial_i A) = \{0\}$. The orbit foliation of the S^1 action $\{\Phi^t\}$ is now *horizontal*. On the other hand we do not know what the orbit foliation of $\{\varphi^t\}$ looks like.

Next define a projection $\pi_i \colon A_i \to S^1$ as follows. For any point $\xi \in A_i$,

$$\pi_i(\xi) = x \Longleftrightarrow \xi = \Phi^x(\gamma_i p(\xi)).$$

Since

$$\xi = \Phi^{\pi_0(\xi)} \gamma_0 p(\xi) \quad \text{and} \quad \xi = \Phi^{\pi_1(\xi)} \gamma_1 p(\xi) = \Phi^{\pi_1(\xi)} \Phi^{fp(\xi)} \gamma_0 p(\xi),$$

we have

(3.1)
$$\pi_0 - \pi_1 = f \circ p \mod \mathbb{Z} \quad \text{on } \mathrm{Int}(A).$$

There is a homeomorphism

$$\pi_0 \times p \colon A_0 \to S^1 \times [0, \infty).$$

If we use the arguments $x = \pi_0(\xi)$ and $y = p(\xi)$ on A_0, the foliation by π_0 is *vertical* i.e. given by the curves $x = $ const., while the foliation by π_1, defined on $\mathrm{Int}(A)$, is given by the curves

$$x = f(y) + \text{const.} \mod \mathbb{Z}.$$

Both foliations are invariant by the horizontal rotation $\{\Phi^t\}$.

Now choose a decreasing sequence of positive numbers

$$1 > y_1 > y_1' > y_2 > y_2' > \cdots$$

such that

(3.2)
$$f^*(y_n') > \sigma(f)/2, \quad y_n = \min\{y > y_n' \mid f^*(y) = 0\}, \quad y_n \to 0 \ (n \to \infty).$$

By Lemma 2.1 (3), the value of the function f^* is oscillating between 0 and around $\sigma(f)$. Therefore it is possible to choose such a sequence. Returning to f, (3.2) implies

(3.3)
$$f(y_n) = \max(f|_{[y_n', 1]}), \quad f(y_n') < f(y_n) - \sigma(f)/2.$$

Thus the foliation by π_1 has oscillating leaves. It is (topologically) tangent to the vertical foliation at the level set $p^{-1}(y_n)$.

To prove Theorem 1.1, it suffices to show the existence of a homeomorphism of $\partial_0 A$ to $\partial_1 A$ that conjugates $\varphi^t|_{\partial_0 A}$ to $\varphi^t|_{\partial_1 A}$ for any free S^1 action $\{\varphi^t\}$ on A. Define an S^1 action $\{\varphi_i^t\}$ on S^1 $(i = 0, 1)$ as the conjugate of $\varphi^t|_{\partial_i A}$ by π_i, *i.e.* so as to satisfy

$$\varphi_i^t \circ \pi_i = \pi_i \circ \varphi^t \text{ on } \partial_i A.$$

Then our goal is to show that φ_0^t is conjugate to φ_1^t by a homeomorphism g of S^1 which can be chosen independently of the S^1 action $\{\varphi^t\}$. But since $\{\Phi^t\}$ is one such S^1 action and Φ_i^t is just a rotation, the homeomorphism g must be a rotation itself.

Besides (3.3), we may assume

(3.4) $f(y_n) \to \alpha \mod \mathbb{Z}.$

Denote by $R_\alpha \colon S^1 \to S^1$ the rotation by α. Now our goal is to show the following proposition.

PROPOSITION 3.1. — *For any free S^1-action $\{\varphi^t\}$ on A, we have*

$$R_\alpha \circ \varphi_1^t = \varphi_0^t \circ R_\alpha, \quad \forall t.$$

This follows from the following lemma.

LEMMA 3.2. — *For any free S^1 action $\{\varphi^t\}$ on A and for any nonnegative integer k*

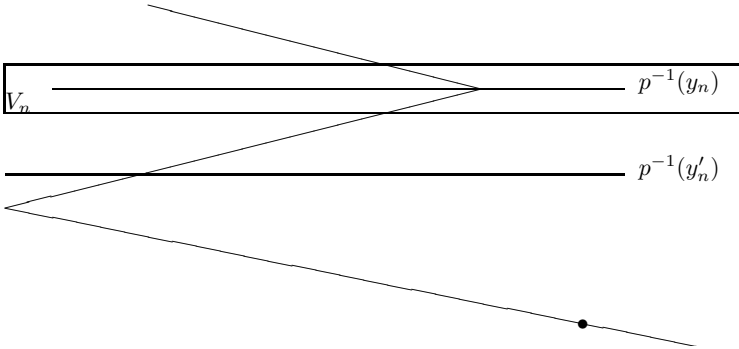$$R_\alpha \circ \varphi_1^{1/2^k} = \varphi_0^{1/2^k} \circ R_\alpha.$$

The rest of this section is devoted to the proof of Lemma 3.2. We shall first prove it for $k = 1$. Define a function $\delta \colon \mathrm{Int}(A) \to \mathbb{R}$ by

$$\delta(\xi) = (\pi_0 \varphi^{1/2}(\xi) - \pi_1 \varphi^{1/2}(\xi)) - (\pi_0(\xi) - \pi_1(\xi)).$$

We shall study the function δ on the circle $p^{-1}(y_n)$. For $\xi \in p^{-1}(y_n)$, we have by (3.1)

(3.5) $\delta(\xi) = fp\varphi^{1/2}(\xi) - f(y_n).$

The position of $\varphi^{1/2}(\xi)$ for $\delta(\xi) > 0$ is indicated by the dot in the figure. Notice that it must be below $p^{-1}(y_n')$.

There exists a *horizontally going point* $\xi_n(0)$ in $\pi^{-1}(y_n)$, *i.e.* a point such that

$$p\varphi^{1/2}(\xi_n(0)) = p(\xi_n(0)) = y_n.$$

For, otherwise $\varphi^{1/2}$ will displace the curve $p^{-1}(y_n)$, sending it, say below itself. But then $\varphi^{1/2} \circ \varphi^{1/2}$ cannot be the identity.

Notice that $\varphi^{1/2}(\xi_n(0))$ is also a horizontally going point. By (3.5), we have

(3.6) $$\delta(\xi_n(0)) = 0 \quad \text{and} \quad \delta\varphi^{1/2}(\xi_n(0)) = 0.$$

Passing to a subsequence if necessary, we may assume that

$$\pi_i(\xi_n(0)) \to \alpha_i, \quad i = 0, 1.$$

Of course we have by (3.1)

$$\alpha = \alpha_0 - \alpha_1.$$

For any $x \in S^1$, define $\xi_n(x)$ to be the point on $p^{-1}(y_n)$ such that

$$\pi_i(\xi_n(x)) = \pi_i(\xi_n(0)) + x \quad (i = 0, 1).$$

Then we have

(3.7) $$\pi_i(\xi_n(x)) \to x + \alpha_i, \quad \text{and} \quad \pi_i\varphi^{1/2}(\xi_n(x)) \to \varphi_i^{1/2}(x + \alpha_i).$$

To see the second assertion, notice that by the first assertion and the fact that $p(\xi_n(x)) \to 0$, the point $\xi_n(x)$ converges to a point $\xi_{\infty,i}(x) \in \partial_i A$. Of course

$$\pi_i(\xi_{\infty,i}(x)) = x + \alpha_i \quad \text{and} \quad \varphi^{1/2}(\xi_n(x)) \to \varphi^{1/2}(\xi_{\infty,i}(x)).$$

Therefore

$$\pi_i\varphi^{1/2}(\xi_n(x)) \to \pi_i\varphi^{1/2}(\xi_{\infty,i}(x)) = \varphi_i^{1/2}\pi_i(\xi_{\infty,i}(x)) = \varphi_i^{1/2}(x + \alpha_i),$$

as is asserted.

Now we have

(3.8)
$$\delta(\xi_n(x)) = (\pi_0\varphi^{1/2}(\xi_n(x)) - \pi_0(\xi_n(x))) - (\pi_1\varphi^{1/2}(\xi_n(x)) - \pi_1(\xi_n(x)))$$
$$\longrightarrow (\varphi_0^{1/2}(x + \alpha_0) - \alpha_0) - (\varphi_1^{1/2}(x + \alpha_1) - \alpha_1) = J_0(x) - J_1(x),$$
where
$$J_i = R_{\alpha_i}^{-1} \circ \varphi_i^{1/2} \circ R_{\alpha_i},$$
an involution on S^1. Now (3.6) and (3.8) imply that
$$J_1(0) = J_0(0).$$
By some abuse we denote by $<$ the positive circular order for two nearby points of S^1.

All we are about is to show that $J_0 = J_1$. Assume for contradiction that this is not the case. Since J_i is an involution, there is a point $0 < x_0 < J_i(0)$ such that $J_0(x_0) \neq J_1(x_0)$. Choosing the point x_0 as near 0 as we wish, we can assume

(3.9) $$|J_1(x) - J_0(x)| < \sigma(f)/4 \quad \text{if } 0 \leqslant x \leqslant x_0.$$

There are two cases, one $J_0(x_0) > J_1(x_0)$ and the other $J_0(x_0) < J_1(x_0)$. But the latter case can be reduced to the former case by replacing 0 by $J_0(0) = J_1(0)$ and x_0 by $J_0(x_0)$. Notice that the image by π_i of the horizontally going points $\varphi^{1/2}(\xi_n(0))$ converge to $J_0(0) = J_1(0)$. This is all we need in the argument that follows, and therefore we can replace 0 by $J_0(0) = J_1(0)$.

So we assume

(3.10) $$J_0(x_0) > J_1(x_0).$$

By (3.8) (3.9) and (3.10), we have for any large n,

(3.11) $$|\delta(\xi_n(x))| < \sigma(f)/3 \text{ if } 0 \leqslant x \leqslant x_0 \quad \text{and} \quad \delta(\xi_n(x_0)) > 0.$$

Let
$$W_n = \{\xi \in A \mid fp(\xi) > f(y_n) - \sigma(f)/3\}$$
and let V_n be the connected component of W_n that contains $p^{-1}(y_n)$. The subset V_n is a horizontal open annulus disjoint from $p^{-1}(y'_n)$. See the figure. By (3.3),

(3.12) $$\xi \in V_n \Longrightarrow fp(\xi) \leqslant f(y_n).$$

Now we have by (3.5) and (3.11)
$$fp\varphi^{1/2}(\xi_n(x)) - f(y_n) = \delta(\xi_n(x)) > -\sigma(f)/3$$

if $0 \leqslant x \leqslant x_0$. This shows that $\varphi^{1/2}(\xi_n(x))$ is contained in W_n. But since $\xi_n(0)$ is horizontally going, $\varphi^{1/2}(\xi_n(0))$ lies in V_n. Moreover the assignment

$$x \mapsto \varphi^{1/2}(\xi_n(x))$$

is continuous. Thus $\varphi^{1/2}(\xi_n(x))$ lies in V_n for any $0 \leqslant x \leqslant x_0$. In particular

$$\varphi^{1/2}(\xi_n(x_0)) \in V_n \quad \text{and} \quad fp\varphi^{1/2}(\xi_n(x_0)) \leqslant f(y_n).$$

But the assumption $\delta(\xi_n(x_0)) > 0$ of (3.11) implies

$$fp\varphi^{1/2}(\xi_n(x_0)) > f(y_n).$$

The contradiction shows that $J_0 = J_1$, *i.e.*

$$\varphi_1^{1/2} \circ R_\alpha = R_\alpha \circ \varphi_0^{1/2}$$

for $\alpha = \alpha_0 - \alpha_1$, as is required.

Now $J_0 = J_1$ implies that for any large n

$$\xi \in p^{-1}(y_n) \implies |\delta(\xi)| < \sigma(f)/3,$$

that is, any point in $p^{-1}(y_n)$ is *nearly horizontally going*, meaning that it is mapped by $\varphi^{1/2}$ into V_n.

To show Lemma 3.2 for $k = 2$, first choose a horizontally going point $\xi'_n(0) \in p^{-1}(y_n)$ for $\varphi^{1/4}$. Its image $\varphi^{1/4}(\xi'_n(0))$ is not horizontally going, but nearly horizontally going for $\varphi^{1/4}$. Passing to a subsequence, we may assume

$$\pi_i(\xi'_n(0)) \to \alpha'_i.$$

Clearly we have

$$\alpha = \alpha'_0 - \alpha'_1.$$

The point $\xi'_n(x)$ in $p^{-1}(y_n)$ is defined just as before by

$$\pi_i(\xi'_n(x)) = \pi_i(\xi'_n(0)) + x.$$

Define a function $\delta' \colon \text{Int}(A) \to \mathbb{R}$ by

$$\delta'(\xi) = (\pi_0\varphi^{1/4}(\xi) - \pi_1\varphi^{1/4}(\xi)) - (\pi_0(\xi) - \pi_1(\xi)),$$

and let

$$J'_i = R_{\alpha'_i}^{-1} \circ \varphi^{1/4} \circ R_{\alpha'_i}.$$

Then we have

$$\delta(\xi'_n(x)) \to J'_0(x) - J'_1(x).$$

By the previous step we have shown

$$\varphi_1^{1/2} = R_\alpha^{-1} \circ \varphi_0^{1/2} \circ R_\alpha,$$

which implies $(J'_0)^2 = (J'_1)^2$. This enables us to find a point x'_0 playing the same role as x_0 in the previous argument such that $J'_0(x'_0) > J'_1(x'_0)$ either near 0 or near $J'_0(0) = J'_1(0)$. In the latter case the point $\varphi^{1/4}(\xi'_n(0))$ is only nearly horizontally going but this is enough for our purpose. By the same argument as before, we can show $J'_0 = J'_1$.

The proof for general k is by an induction.

4. Examples

We shall construct two examples of $f \in E$. We consider the corresponding flow $\{\Phi^t\}$ and construct the non Hausdorff space A as in Sect.3. Properties of examples are stated in terms of the S^1 action on A. All the notations of Sect. 3 will be used.

Example 4.1. — There exists $f \in E$ such that $\sigma(f) = 1$ satisfying the following property: For any S^1 action $\{\psi^t\}$ on S^1, there is an S^1 action $\{\varphi^t\}$ on A such that $\varphi^t_i = \psi^t$ $(i = 0, 1)$.

The construction of f goes as follows. Let

$$y_1 > y'_1 > y_2 > y'_2 \cdots$$

be a sequence converging to 0. Define f such that

$$f(y_n) = n, \quad \text{and} \quad f(y'_n) = n - 1$$

and that f is monotone on the complementary intervals.

On the circles $p^{-1}(y_n)$ and $p^{-1}(y'_n)$, $\pi_0 = \pi_1 \mod \mathbb{Z}$. The desired flow is to preserve these circles and to be the conjugate of ψ^t by π_i there. The complementary regions are open annulus, and there the foliations by π_0 and π_1 are transverse, thanks to the monotonicity assumption on f. Therefore one can define φ^t so as to satisfy $\pi_i \circ \varphi^t = \psi^t \circ \pi_i$ $(i = 0, 1)$.

Example 4.2. — There exists $f \in E$ such that any S^1 action $\{\varphi^t\}$ on A satisfies $\varphi^t_i = R_t$ $(i = 0, 1)$.

Using the sequence of Example 4.1, define $f \in E$ such that

$$f(y_n) = n\beta \quad \text{and} \quad f(y'_n) = n\beta - 1,$$

for some irrational $\beta > 0$ and that f is monotone on the complementary intervals. Then

$$\pi_0 - \pi_1 = n\beta \quad \text{on} \quad p^{-1}(y_n).$$

Let $\{\varphi^t\}$ be an arbitrary S^1 action on A. Then any point $\tau \in S^1$ is an accumulation point of $\pi_0(\xi_n(0)) - \pi_1(\xi_n(0))$. The argument of Sect. 3 shows that

$$\varphi_1^t \circ R_\tau = R_\tau \circ \varphi_0^t, \quad \forall t, \tau.$$

This clearly shows that

$$\varphi_0^t = \varphi_1^t = R_t.$$

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