ANNALES

DE

L'INSTITUT FOURIER

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Perturbations of the metric in Seiberg-Witten equations

PERTURBATIONS OF THE METRIC IN SEIBERG-WITTEN EQUATIONS

by Luca SCALA

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ABSTRACT. — Let M a compact connected oriented 4-manifold. We study the space Ξ of Spin^c-structures of fixed fundamental class, as an infinite dimensional principal bundle on the manifold of riemannian metrics on M. In order to study perturbations of the metric in Seiberg-Witten equations, we study the transversality of universal equations, parametrized with all Spin^c-structures Ξ. We prove that, on a complex Kähler surface, for an hermitian metric h sufficiently close to the original Kähler metric, the moduli space of Seiberg-Witten monopoles relative to the metric h is smooth of the expected dimension.

RÉSUMÉ. — Soit M une variété riemannienne compacte connexe orientée de dimension 4. On étudie l'espace Ξ des structures Spin^c de classe fondamentale fixée, comme fibré principal de dimension infinie sur la variété des métriques riemanniennes de M. Afin d'étudier les perturbations de la métrique dans les équations de Seiberg-Witten, on étudie la transversalité des équations universelles, paramétrées par l'espace Ξ de toutes les structures Spin^c. On montre que, sur une surface de Kähler, pour une métrique hermitienne h suffisamment proche à la métrique de Kähler de départ, l'espace de modules de monopôles de Seiberg-Witten relatif à la métrique h est lisse de la dimension attendue.

1. Introduction

Let (M, g) a compact connected oriented riemannian 4-manifold. Chosen on (M, g) a Spin^c-structure ξ of spinor bundle $W = W_+ \oplus W_-$ and of determinant line bundle $L \simeq \det W_\pm$, consider the Seiberg-Witten equations:

$$(SW_\xi a) \qquad\qquad D_A^\xi \psi = 0$$

$$(SW_\xi b) \qquad\qquad \rho_\xi(F_A^+) = [\psi^* \otimes \psi]_0,$$

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in the unknowns $(A, \psi) \in \mathcal{A}_{U(1)}(L) \times \Gamma(W_+)$, where $\mathcal{A}_{U(1)}(L)$ denotes the affine space of $U(1)$-connections on L. The aim of this article is the study of the behaviour of Seiberg-Witten equations (see [21], [17], [18], [14]) under perturbations of the metric g.

In Donaldson's theory of $SU(2)$-istantons, deeply related to Seiberg-Witten theory, the behaviour of ASD equations $F_A^+ = 0$ when changing the metric is well understood, and metric perturbations are the main tool to obtain transversality results: the celebrated Freed-Uhlenbeck theorem ([4], [7]) states that, for a generic metric, the functional defining ASD equations is transversal to the zero section at irreducible connections: consequently, for a generic metric, the moduli space of irreducible istantons is smooth of the expected dimension.

On the other hand, an analogue result in Seiberg-Witten theory is unknown; more generally, no much is known on the dependence on the metric of Seiberg-Witten equations, one of the reasons being probably the fact that the transversality for equations (SW_ξ) on an irreducible monopole can be very easily obtained by perturbing the second equation adding a generic selfdual imaginary 2-form η. The dependence of the metric in Seiberg-Witten equations has been studied by Maier in [13], but always for a generic connection A and no transversality issue is addressed. The problem of transversality with perturbation just of the metric appears in the work of Eichhorn and Friedrich (see [5], reported also in [8] and cited in [1] and more recently in [16]): the authors claim to give a positive answer, but their proof is not correct: we will discuss the reason in remark 5.12. The purpose of this article is to establish an analogue of Freed-Uhlenbeck theorem in Seiberg-Witten theory, giving a correct proof of the fact that, for generic metric, the Seiberg-Witten functional is transversal to the zero section and hence that the moduli space is smooth of the expected dimension (at least on irreducible monopoles). We succeed in proving this for Kähler surfaces; in all generality our aim is not completely achieved, but reduced to the vanishing of solutions of a specific system of PDEs.

In order to write Seiberg-Witten equations on the oriented riemannian 4-manifold (M, g), we have to fix a Spin^c-structure, that is, an equivariant lifting $\xi : Q_{\mathrm{Spin}^c(4)} \longrightarrow P_{SO(g)}$ of the $SO(4)$-principal bundle of equioriented orthonormal frames for the metric g to a $\mathrm{Spin}^c(4)$-principal bundle $Q_{\mathrm{Spin}^c(4)}$. Since the Spin^c-structure is a metric concept, that is, it actually determines the metric, when changing the metric on M we are forced to change Spin^c-structure; however, for different metrics the $SO(4)$-bundles of equioriented orthonormal frames are isomorphic and can be

lifted to the same principal bundle $Q_{\mathrm{Spin}^c(4)}$ (of course by means of different morphisms ξ): consequently, we can fix the bundle $Q_{\mathrm{Spin}^c(4)}$ once for all. Therefore it turns out that the right setting to study perturbations of the metric in Seiberg-Witten equations is considering *universal equations*, parametrized by all Spin^c-structures Ξ of fundamental class c and Spin^c-bundle $Q_{\mathrm{Spin}^c(4)}$; we will then characterize only in a second step the variations of Spin^c-structures coming from a variation just of the metric. The space Ξ can be given the structure of a (trivial infinite dimensional) principal bundle over the space of riemannian metrics Met(M), of structural group $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)} \times_{\mathrm{Spin}^c(4)} SO(4))$; on the principal bundle Ξ there can now be defined a *natural connection, the horizontal distribution being characterized by consisting precisely of variations of* Spin^c*-structures coming from variations just of the metric.* The connection thus defined turns out to have nontrivial curvature: it is therefore impossible to find (even locally) a parallel section Met$(M) \longrightarrow \Xi$, by means of which parametrizing correctly Seiberg-Witten equations with just the metric. This is however not a difficult issue, since the universal Seiberg-Witten moduli space $\mathcal{M}$ admits a $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$-equivariant fibration $\mathcal{M} \longrightarrow \Xi$ over the space of Spin^c-structures Ξ; hence the transversality of equations (SW_ξ) at the point ξ does not depend on the Spin^c-structure ξ, but only of the metric g_ξ compatible with ξ: consequently the problem of transversality of Seiberg-Witten equations for generic metrics is equivalent to the problem of transversality of these equations for generic Spin^c-structures.

This formalism (appeared first in [15]) allows us to reduce the problem — after completing Fréchet spaces to Sobolev ones in a standard way — to the proof of the surjectivity of the differential $D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)}$ of the functional

$$\mathbb{F}_{\mathrm{Met}(M)}(A, \psi, \xi) = (D_A^\xi \psi \ , \ \rho_\xi(F_A^{+,g_\xi}) - [\psi^* \otimes \psi]_0),$$

defining universal Seiberg-Witten equations, at the solution $(A, \psi, \xi) \in \mathcal{A}_{U(1)}(L) \times \Gamma(W_+) \times \Xi$. To compute this operator, we need to compute the variation of the Dirac operator, performed first by Bourguignon and Gauduchon [3]. We present here a simple alternative proof: our approach has the advantage of fixing once for all the bundle $Q_{\mathrm{Spin}^c(4)}$ and consequently the bundle of spinors W, and is particularly adapted to the transversality problem: indeed in this way all Dirac operators act on the same space of global sections, without any need of delicate identifications or transmutations operators.

The surjectivity of the differential $D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)}$ is equivalent to the injectivity of the formal adjoint $(D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)})^*$: consequently nontrivial

solutions of the kernel equations $(D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)})^*u = 0$ represent the obstruction to the transversality of the functional $\mathbb{F}_{\mathrm{Met}(M)}$. In the general case the equations are intricate and we still do not have the answer.

When (M, g, J) is a complex Kähler surface with complex structure J and with canonical line bundle K_M, the Seiberg-Witten equations admit an interpretation in terms of holomorphic couples $(\bar{\partial}_A, \alpha)$, where $\bar{\partial}_A$ is a holomorphic $(0, 1)$-semiconnection on a line bundle N such that $K_M^* \otimes N^{\otimes 2} \simeq L$, and α is a holomorphic section of $(N, \bar{\partial}_A)$. This facts allow a drastic simplification of the Seiberg-Witten equations and consequently of the problem of transversality for generic metrics. After interpreting all the preceding objects in the context of complex geometry, and thanks to the splitting of the symmetric endomorphisms with respect to the metric into hermitian and anti-hermitian ones, the kernel equations $(D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)})^*u = 0$ become much simpler. Indicating with $\mathcal{M}_{H_J(M)}$ (with $\mathcal{M}_{K_J(M)}$) the moduli space of hermitian (kählerian) monopoles — that is, monopoles $[A, \psi, \xi]$ such that g_ξ is an hermitian (kählerian) metric on M — *we proved that the moduli space $\mathcal{M}_{H_J(M)}$ is smooth at irreducible kählerian monopoles $\mathcal{M}_{K_J(M)}^*$.* In other words, we get that Seiberg-Witten equations are transversal for a generic hermitian metric sufficiently close to the Kähler metric g. We precisely proved:

THEOREM. — *Let (M, g, J) a Kähler surface. Let N a hermitian line bundle on M such that $2\deg(N) - \deg(K_M) \neq 0$. Consider the Spin^c-structure ξ given by the canonical Spin^c-structure on M twisted by the hermitian line bundle N. For a generic metric h in a small open neighbourhood of $g \in \mathrm{Met}(M)$ and for all Spin^c-structure ξ', compatible with h, the Seiberg-Witten moduli space $\mathcal{M}_{\xi'}^{SW}$ is smooth. Actually, the statement holds for a generic hermitian metric h in a small open neighbourhood of g.*

2. Spin^c-structures and metrics

The aim of this section is to recall the basics on Spin^c-structures from a point of view adapted to the study of metric perturbations, and to describe the set of all Spin^c-structures of fixed type and fundamental class as a principal fibration over the space of metrics on the manifold. See also [15, section 2].

2.1. Spin^c-structures

Let $n \in \mathbb{N}$, $n \geqslant 1$. Recall the fundamental central extensions of groups:

$$(2.1a) \qquad 0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\nu := (\mu, \lambda)} SO(n) \times S^1 \longrightarrow 1.$$

$$(2.1b) \qquad 1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\ \mu\ } SO(n) \longrightarrow 1$$

Let now M be a compact connected oriented manifold of dimension n and let $P_{GL_+(n)}$ the principal $GL_+(n)$-bundle of oriented frames of the tangent bundle TM.

DEFINITION 2.1. — *A Spin^c-structure on M (of type $Q_{\mathrm{Spin}^c(n)}$) is the data of a $\mathrm{Spin}^c(n)$-principal bundle $Q_{\mathrm{Spin}^c(n)}$ over M and of a μ-equivariant morphism $\xi : Q_{\mathrm{Spin}^c(n)} \longrightarrow P_{GL_+(n)}$. The line bundle $L := Q_{\mathrm{Spin}^c(n)} \times_\lambda \mathbb{C}$ is called the determinant line bundle and its first Chern class $c := c_1(L)$ is called the fundamental class of the Spin^c-structure ξ. Two Spin^c-structures $\xi : Q_{\mathrm{Spin}^c(n)} \longrightarrow P_{GL_+(n)}$ and $\xi' : Q'_{\mathrm{Spin}^c(n)} \longrightarrow P_{GL_+(n)}$ are isomorphic if there exist a $\mathrm{Spin}^c(n)$-equivariant morphism $f : Q_{\mathrm{Spin}^c(n)} \longrightarrow Q'_{\mathrm{Spin}^c(n)}$ such that $\xi' \circ f = \xi$.*

It is well known that a Spin^c-structure of fundamental class c exists if and only if $c \equiv w_2(M) \mod 2$.

Remark 2.2. — Given a $\mathrm{Spin}^c(n)$-principal bundle $Q_{\mathrm{Spin}^c(n)}$, we can form the $SO(n)$-principal bundle $Q_{SO(n)} := Q_{\mathrm{Spin}^c(n)} \times_\mu SO(n)$ and the $U(1)$-principal bundle $Q_{U(1)} := Q_{\mathrm{Spin}^c(n)} \times_\lambda U(1)$. Every μ-equivariant morphism $\xi : Q_{\mathrm{Spin}^c(n)} \longrightarrow P_{GL_+(n)}$ factors through the composition of the S^1-fibration $\eta : Q_{\mathrm{Spin}^c(n)} \longrightarrow Q_{SO(n)}$, followed by the $SO(n)$-equivariant embedding $\gamma_\xi : Q_{SO(n)} \hookrightarrow P_{GL_+(n)}$. It is thus clear that, once fixed a $\mathrm{Spin}^c(n)$-bundle $Q_{\mathrm{Spin}^c(n)}$, the data of a Spin^c-structure of principal bundle $Q_{\mathrm{Spin}^c(n)}$ is equivalent to the data of a $SO(n)$-equivariant embedding $Q_{SO(n)} \hookrightarrow P_{GL_+(n)}$.

Remark 2.3. — It is a fundamental fact that a Spin^c-structure ξ is a *metric* concept. Indeed the embedding γ_ξ, induced by a Spin^c-structure ξ, provides a $SO(n)$-reduction of the principal bundle $P_{GL_+(n)}$, corresponding to the choice of a riemannian metric g_ξ on TM. We will denote with $P_{SO(g_\xi)}$ the image of γ_ξ, that is, the principal $SO(n)$-subbundle of $P_{GL_+(n)}$ consisting of g_ξ-orthonormal oriented frames and with α_ξ the lifting $\alpha_\xi :$ $Q_{\mathrm{Spin}^c(n)} \longrightarrow P_{SO(g_\xi)}$. We will say that the metric g_ξ is compatible with the Spin^c-structure ξ.

2.2. Spin^c-structures of fixed type and fundamental class

Let now $c \in H^2(M,\mathbb{Z})$ such that $c \equiv w_2(M) \mod 2$. The long non-abelian cohomology sequences associated to the central extensions (2.1) read, for the H^1-level, identifying $H^1(M,S^1)$ with $H^2(M,\mathbb{Z})$:

$$(2.2) \quad \begin{array}{ccccccccc}
H^0(M,SO(n)) \times H^0(M,S^1) & \to & H^1(M,\mathbb{Z}_2) & \xrightarrow{b} & H^1(M,\mathrm{Spin}^c(n)) & \to & H^1(M,SO(n)) \times H^2(M,\mathbb{Z}) & \to & H^2(M,\mathbb{Z}_2) \\
\downarrow & & e\downarrow & & =\downarrow & & \downarrow & & \downarrow \\
H^0(M,SO(n)) & \longrightarrow & H^2(M,\mathbb{Z}) & \xrightarrow{a} & H^1(M,\mathrm{Spin}^c(n)) & \longrightarrow & H^1(M,SO(n)) & \longrightarrow & H^2(M,S^1)
\end{array}$$

Setting $\mathfrak{t} := \mathrm{Tors}_2\, H^2(M,\mathbb{Z}) = \mathrm{Im}\, e$ and $\mathfrak{c} := \ker a$, we have immediately that $\mathfrak{c} \subseteq \mathfrak{t}$ and hence the exact sequence:

$$0 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{t} \xrightarrow{a} H^1(M,\mathrm{Spin}^c(n)) \longrightarrow H^1(M,SO(n)) \times H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{Z}_2).$$

Remark 2.4. — Since $SO(n)$ is a maximal compact subgroup of $GL_+(n)$, $H^1(M,SO(n)) \simeq H^1(M,GL_+(n))$ [12, appendix B]; hence we can replace part of the first long exact sequence in (2.2) with

$$H^1(M,\mathbb{Z}_2) \longrightarrow H^1(M,\mathrm{Spin}^c(n)) \longrightarrow H^1(M,GL_+(n)) \times H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{Z}_2).$$

Fix now a $\mathrm{Spin}^c(n)$-bundle $Q_{\mathrm{Spin}^c(n)}$ such that its isomorphism class $[Q_{\mathrm{Spin}^c(n)}] \in H^1(M,\mathrm{Spin}^c(n))$ lifts the couple $([P_{GL_+(n)}],c) \in H^1(M, GL_+(n)) \times H^2(M,\mathbb{Z})$. For any metric g on M the element $[Q_{\mathrm{Spin}^c(n)}]$ lifts the couple $([P_{SO(g)}],c) \in H^1(M,SO(n)) \times H^2(M,\mathbb{Z})$, where $P_{SO(g)}$ is the principal bundle of oriented frames in TM, orthonormal for the metric g.

We denote with Ξ the space of all μ-equivariant morphisms: $\xi : Q_{\mathrm{Spin}^c(n)} \longrightarrow P_{GL_+(n)}$ or, equivalently, by remark 2.2, all $SO(n)$-equivariant maps: $\gamma : Q_{SO(n)} \hookrightarrow P_{GL_+(n)}$:

$$\Xi := \mathrm{Mor}_\mu(Q_{\mathrm{Spin}^c(n)}, P_{GL_+(n)}) \simeq \mathrm{Mor}_{SO(n)}(Q_{SO(n)}, P_{GL_+(n)}).$$

The space Ξ parametrizes all the Spin^c structures on M of fixed type $Q_{\mathrm{Spin}^c(n)}$ and fundamental class c. Since every Spin^c-structure determines a metric, the space Ξ is fibered over the space of riemannian metrics:

$$(2.3) \qquad\qquad\qquad \Xi \longrightarrow \mathrm{Met}(M).$$

Two Spin^c-structures compatible with the same metric differ by the action of $\mathrm{Aut}(Q_{SO(n)})$, hence Ξ has the structure of $\mathrm{Aut}(Q_{SO(n)})$-principal bundle; however, two Spin^c-structures compatible with the same metric need not to be isomorphic: indeed, they are isomorphic if and only if they differ by the action of $\mathrm{Aut}(Q_{\mathrm{Spin}^c(n)})$. The long exact sequence:

$$1 \longrightarrow C^\infty(M,S^1) \longrightarrow \mathrm{Aut}(Q_{\mathrm{Spin}^c(n)}) \xrightarrow{\hat{\eta}} \mathrm{Aut}(Q_{SO(n)}) \longrightarrow H^1(M,S^1)$$

induced by the central extension (2.1b), implies that the group $\Gamma := \operatorname{Im} \hat{\eta}$ acts in a free and transitive way on each isomorphism class of Spin^c-structures inside the fibers of (2.3); moreover, the group coker $\hat{\eta} \simeq \mathfrak{c}$ parametrizes the set of isomorphism classes of Spin^c-structures of fixed type $Q_{\operatorname{Spin}^c(n)}$ and fundamental class c over a fixed metric. The quotient Ξ/Γ is isomorphic to:

$$\Xi/\Gamma \simeq \Xi/\operatorname{Aut}(Q_{\operatorname{Spin}^c(n)}) \simeq \operatorname{Met}(M) \times \pi_0(\Xi/\operatorname{Aut}(Q_{\operatorname{Spin}^c(n)}) \simeq \operatorname{Met}(M) \times \mathfrak{c},$$

because $\operatorname{Met}(M)$ is contractible. If M is simply connected, all Spin^c-structures compatible with the same metric are isomorphic.

Remark 2.5. — If $n = \dim M = 4$, the map $b : H^1(M, \mathbb{Z}_2) \longrightarrow H^1(M, \operatorname{Spin}^c(4))$ is trivial; consequently $\mathfrak{c} \simeq \mathfrak{t} = \operatorname{Tors}_2 H^2(M, \mathbb{Z})$. This fact can be seen via the equivalence between Spin^c-structures on an oriented euclidian vector bundle (E, g) of rank 4 on the 4-manifold M (that is, the data of a Spin$^c(4)$-principal bundle $Q_{\operatorname{Spin}^c(4)}$ on M together with a μ-equivariant map $Q_{\operatorname{Spin}^c(4)} \longrightarrow P_{GL_+(4)}(E)$ from $Q_{\operatorname{Spin}^c(4)}$ to the principal bundle $P_{GL_+(4)}(E)$ of oriented linear frames on E) and quadruples (W_+, W_-, i, ρ), where $W_\pm$ are rank 2 hermitian vector bundles on M, i is a prescribed unitary isomorphism $i \colon \det W+ \simeq \det W_-$ and $\rho \colon E \longrightarrow \operatorname{Hom}(W_+, W_-)$ is a morphism such that $\rho(x)^* \rho(x) = -g(x, x) \operatorname{id}_{W_+}$ (see [20]). Given such a quadruple, the principal bundle $Q_{\operatorname{Spin}^c(4)}$ is built as the bundle of quadruples $(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-)$, where $\sigma_1^\pm, \sigma_2^\pm$ is a unitary frame of $W_\pm$, respectively, such that $i(\sigma_1^+ \wedge \sigma_2^+) = \sigma_1^- \wedge \sigma_2^-$. Hence the principal bundle $Q_{\operatorname{Spin}^c(4)}$ just depends on (W_+, W_-, i) (but not on ρ). The fact that the morphism b is zero follows from the following facts.

(1) The Spin^c-structures on the oriented euclidian bundle E on the manifold M form a $H^2(M, \mathbb{Z})$-torsor; an element N in the topological Picard group $\operatorname{Pic}_{\operatorname{top}}(M) \simeq H^2(M, \mathbb{Z})$ acts on the quadruple (W_+, W_-, i, ρ) by tensorising $W_\pm$ with N; hence $\det(W_\pm \otimes N) \simeq \det W_\pm \otimes N^2$; moreover i and ρ are changed into $i \otimes \operatorname{id}_{N^2}$, $\rho \otimes \operatorname{id}_N$, respectively. Consequently the Spin^c-structures on E having fundamental class $c = c_1(\det W_+)$ are parametrized by $\mathfrak{t} = \operatorname{Tors}_2 H^2(M, \mathbb{Z})$.

(2) Given a quadruple (W_+, W_-, i, ρ), the associated principal Spin$^c(4)$ bundle $Q_{\operatorname{Spin}^c(4)}$ does not depend on i, but just on W_+, W_-. Indeed the map $\det : \operatorname{Aut}(W_+) \longrightarrow \mathcal{C}^\infty(M, S^1)$ is always surjective if $n = 4$, by obstruction theory.

(3) If $N \in \operatorname{Tors}_2 H^2(M, \mathbb{Z})$, the Chern classes of $W_\pm \otimes N$ equal those of $W_\pm$, respectively. Since in dimension 4 an hermitian vector bundle

is classified topologically by its rank and its Chern classes, we can conclude.

Remark 2.6. — As a consequence of the previous remark, for any fixed fundamental class $c \in H^2(M, \mathbb{Z})$, there is a unique isomorphism class $[Q_{\mathrm{Spin}^c(4)}]$ lifting the couple $([P_{GL_+(4)}], c) \in H^1(M, GL_+(4)) \times H^2(M, \mathbb{Z})$ in remark 2.4. In this case the space Ξ parametrizes all the Spin^c structures on M of fixed fundamental class c. The connected components of the quotient Ξ/Γ are parametrized by $\mathfrak{t}$.

Remark 2.7. — The spaces $\mathrm{Met}(M)$ and Ξ can be viewed as spaces of global (smooth) sections of fiber bundles. Indeed, considered the fiber bundles over M:

$$(2.4) \qquad Met(M) := \coprod_{x \in M} \mathrm{Met}(T_x M)$$

$$(2.5) \qquad Mor_\mu(Q_{\mathrm{Spin}^c(n)}, P_{GL_+(n)}) := \coprod_{x \in M} \mathrm{Mor}_\mu(Q_{\mathrm{Spin}^c(n),x}, P_{GL_+(n),x})$$

then $\mathrm{Met}(M) = \Gamma(M, Met(M))$ and $\Xi = \Gamma(M, Mor_\mu(Q_{\mathrm{Spin}^c(n)}, P_{GL^+(n)}))$. Moreover, for each $x \in M$, the projection

$$(2.6) \qquad \mathrm{Mor}_\mu(Q_{\mathrm{Spin}^c(n),x}, P_{GL_+(n),x}) \longrightarrow \mathrm{Met}(T_x M)$$

is a trivial finite dimensional principal bundle of structural group $SO(n)$, noncanonically isomorphic to the $SO(n)$-principal bundle $P_{GL_+(n),x} \longrightarrow$ $\mathrm{Met}(T_x M)$. The fiberwise projection (2.6) induces the global projection of fiber bundles $Mor_\mu(Q_{\mathrm{Spin}^c(n)}, P_{GL_+(n)}) \longrightarrow Met(M)$ and the infinite dimensional principal bundle $\Xi \longrightarrow \mathrm{Met}(M)$.

Remark 2.8. — The spaces $\mathrm{Met}(M)$, of riemannian metrics over M, and Ξ, of Spin^c-structures on M of type $Q_{\mathrm{Spin}^c(n)}$, are infinite dimensional Fréchet manifolds, because spaces of global sections of fiber bundles over M, as explained in [6] and, more recently, in [11]. The projection $\Xi \longrightarrow \mathrm{Met}(M)$ gives Ξ the structure of an infinite dimensional Fréchet principal bundle with regular Fréchet-Lie group $\mathrm{Aut}(Q_{SO(n)})$ as structure group (see [11], Chapter VIII, § 38-39).

Remark 2.9. — The manifold $\mathrm{Met}(M)$ of riemannian metrics on M, can be equipped with a natural riemannian metric making it a ∞-dimensional Fréchet riemannian manifold (see [7], [9]).

2.3. Changes of metric: the natural connection on Ξ

The group $\mathrm{Aut}(P_{GL_+(n)})$ acts freely and transitively (on the right) on the space Ξ, hence the choice of an element $\xi \in \Xi$ defines an isomorphism: $\mathrm{Aut}(P_{GL_+(n)}) \simeq \Xi$, defined by $\varphi \mapsto \varphi^{-1} \circ \xi$ and such that $g_{\varphi^{-1} \circ \xi} = \varphi^* g_\xi$. The choice of $\xi \in \Xi$ determines the polar decomposition:

$$\mathrm{Aut}(P_{GL_+(n)}) \simeq \mathrm{Aut}(P_{SO(g_\xi)}) \times \mathrm{Sym}^+(P_{SO(g_\xi)}),$$

where we denoted with $\mathrm{Sym}^+(P_{SO(g_\xi)})$ the space of functions $f : P_{SO(g_\xi)} \longrightarrow \mathrm{Sym}^+(n)$ such that $f(pg) = g^{-1}f(p)g = {}^tgf(p)g$ for all $g \in SO(n)$ and where $\mathrm{Sym}^+(n)$ is the space of positive symmetric automorphisms of $\mathbb{R}^n$ with respect to the standard scalar product. It is clear that $\mathrm{Sym}^+(P_{SO(g_\xi)}) \cong \mathrm{Sym}^+(TM, g_\xi)$, the space of symmetric automorphisms of TM with respect to the metric g_ξ.

Consider the section

(2.7)
$$\begin{array}{ccccc} \mathrm{Met}(M) & \simeq & \mathrm{Sym}^+(P_{SO(g_\xi)}) & \xrightarrow{\ \sigma_\xi\ } & \Xi \\ \varphi^* g_\xi & \longleftarrow\!\shortmid & \varphi \longmapsto & & \varphi^{-1} \circ \xi \end{array}$$

Taking the tangent space of the image of this section in ξ, $H_\xi := T_\xi(\mathrm{im}\,\sigma_\xi)$, defines in a natural way a $\mathrm{Aut}(Q_{SO(n)})$-equivariant horizontal distribution in $T\Xi$ and hence a connection on Ξ, which we will call the *natural connection* on Ξ. It is then natural, when changing the metric $g \in \mathrm{Met}(M)$ along a path g_t, *to change the* Spin^c*-structure lifting the path to* Ξ *in a parallel way for the natural connection on* Ξ *just defined.*

Remark 2.10. — Since $\Xi \longrightarrow \mathrm{Met}(M)$ is an infinite dimensional principal bundle with regular Lie group as structural group, the parallel transport exists and it is unique for any connection on Ξ. Moreover the curvature of a connection can be interpreted, as usual, as the obstruction of the integrability of the horizontal distribution. See [11, Chapter VIII, §39], for details.

Parallel transport on Ξ for the natural connection. Let ξ_0 a given Spin^c-structure in Ξ and $g_t = \varphi_t^* g_{\xi_0}$ a path of metrics in $\mathrm{Met}(M)$, such that $\varphi_t \in \mathrm{Sym}^+(TM, g_{\xi_0})$ and $\varphi_0 = \mathrm{id}$. To determine the equation of the parallel transport of ξ_0 along the path g_t for the natural connection, consider a path of Spin^c-structures ξ_t in Ξ starting from ξ_0, and subject to the condition $g_{\xi_t} = g_t$. Writing that the path ξ_t is parallel means that $\dot{\xi}_t = d/d\lambda|_{\lambda=t}\,(\theta_\lambda^{-1} \circ \xi_t)$, $\theta_\lambda \in \mathrm{Sym}^+(TM, g_t)$, $\theta_\lambda^* g_t = g_\lambda$; indicating with $\dot{\theta}(t) = d\theta_\lambda/d\lambda|_{\lambda=t}$, we have $\dot{\xi}_t = -\dot{\theta}(t) \circ \xi_t$. Since $2g_t\dot{\theta}(t) = \dot{g}_t$ and consequently

$\dot\theta(t) = 1/2\ g_t^{-1}\dot g_t$, the parallel transport equation reads:

$$\dot\xi_t = -\frac{1}{2}\left(g_t^{-1}\dot g_t\right)\circ\xi_t.$$

Curvature. The following proposition gives the curvature of the natural connection on the principal bundle $\Xi \longrightarrow M$. Similar computations have been made pointwise by Bourguignon and Gauduchon [3] to compute the curvature of the $O(n)$-principal bundle $L(V) \longrightarrow \mathrm{Met}(V)$ of linear frames of a real n-dimensional vector space V over its cone of metrics $\mathrm{Met}(V)$. The natural connection on Ξ induces pointwise connections on the $SO(n)$-principal bundles $\mathrm{Mor}_\mu(Q_{\mathrm{Spin}^c(n),x}, P_{GL_+(n),x}) \simeq P_{GL_+(n),x} \longrightarrow \mathrm{Met}(T_xM)$. With exactly the same proof as in [3, Lemma 3], we find that the curvature of this connection is given by: $\Omega_{g_x}(h_x, k_x) = -1/4\,[g_x^{-1}h_x,\ g_x^{-1}k_x]$ for $g_x \in \mathrm{Met}(T_xM)$, $h_x, k_x \in S^2T_x^*M$. General facts on vector fields of spaces of sections of fiber bundles (see [6, Appendix]) imply that the curvature Ω of the bundle $\Xi \longrightarrow \mathrm{Met}(M)$ is pointwise the curvature of the bundle $P_{GL_+(n),x} \longrightarrow \mathrm{Met}(T_xM)$: $(\Omega_g(h,k))_x = \Omega_{g_x}(h_x, k_x)$. Therefore we get:

PROPOSITION 2.11. — *The curvature of the natural connection of the principal bundle $\Xi \longrightarrow \mathrm{Met}(M)$ is given by:*

$$\Omega_{g_\xi}(h, k) = -\frac{1}{4}[g_\xi^{-1}h, g_\xi^{-1}k],$$

in $\mathrm{ad}(\Xi)_{g_\xi} \simeq \mathfrak{so}(TM, g_\xi)$, *for* $h, k \in S^2T^*M \simeq T_{g_\xi}\mathrm{Met}(M)$.
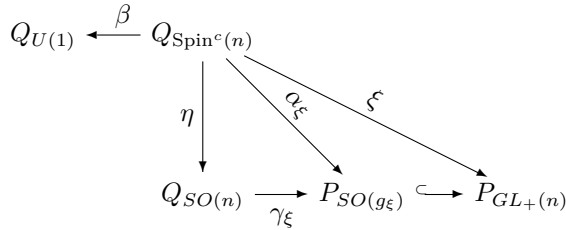
Remark 2.12. — Proposition 2.11 implies that the horizontal distribution H_ξ on $T_\xi\Xi$ is *never* integrable. Consequently *there are no parallel (smooth) sections* $\sigma : \mathrm{Met}(M) \longrightarrow \Xi$ *(even locally).*

2.4. Parametrized Dirac operators

Let $m = [n/2]$. Consider an irreducible $\mathrm{Cl}(\mathbb{R}^n)$ representation $\rho_0 : \mathrm{Cl}(\mathbb{R}^n) \longrightarrow \mathrm{End}(W_0)$ where W_0 is an hermitian vector space of complex dimension 2^m. The bundle of spinors is defined by $W := Q_{\mathrm{Spin}^c(n)} \times_{\rho_0} W_0$. The choice of an element $\xi \in \Xi$ induces a map $Q_{\mathrm{Spin}^c(n)} \times_\mu \mathrm{Cl}(\mathbb{R}^n) \longrightarrow Q_{\mathrm{Spin}^c(n)} \times_{\rho_0} \mathrm{End}(W_0)$, which is, thanks to the identifications $\mathrm{Cl}(TM) \simeq Q_{\mathrm{Spin}^c(n)} \times_\mu \mathrm{Cl}(\mathbb{R}^n)$, $Q_{\mathrm{Spin}^c(n)} \times_{\rho_0} \mathrm{End}(W_0) \simeq \mathrm{End}(W)$, the Clifford multiplication on the bundle of Clifford algebras $\mathrm{Cl}(TM)$:

$$\rho_\xi : \mathrm{Cl}(TM) \longrightarrow \mathrm{End}(W).$$

Identifying of TM and T^*M via g_ξ we get as well a Clifford multiplication on $\mathrm{Cl}(T^*M)$, which we keep on denoting ρ_ξ. We have the following diagram:

$$
\begin{array}{ccc}
Q_{U(1)} & \xleftarrow{\;\beta\;} & Q_{\mathrm{Spin}^c(n)} \\
& & \\
\Big\downarrow{\scriptstyle\eta} & & \\
& & \\
Q_{SO(n)} & \xrightarrow[\gamma_\xi]{} & P_{SO(g_\xi)} \hookrightarrow P_{GL_+(n)}
\end{array}
$$

Let $\omega_{g_\xi} \in A^1(P_{SO(g_\xi)}, \mathfrak{so}(n))$ be the Levi-Civita connection on $P_{SO(g_\xi)}$: it induces a $SO(n)$-equivariant connection form on $P_{GL_+(n)}$ which we keep on denoting ω_{g_ξ}. Let $A \in A^1(Q_{U(1)}, \mathfrak{u}(1))$ a $U(1)$-connection form on $Q_{U(1)}$. The Spin^c-*connection* $\Omega_{A,\xi}$ on $Q_{\mathrm{Spin}^c(n)}$ is defined as:

$$(2.8) \qquad\qquad \Omega_{A,\xi} := d\nu^{-1}(\alpha_\xi^* \omega_{g_\xi} + \beta^* A)$$

seen in $A^1(Q_{\mathrm{Spin}^c(n)}, \mathfrak{spin}^c(n))$, where $d\nu$ is the isomorphism: $\mathfrak{spin}^c \simeq \mathfrak{so}(n) \oplus \mathfrak{u}(1)$. The Spin^c-connection form $\Omega_{A,\xi}$ defines a connection $\nabla_A^{W,\xi}$ on the associated vector bundle of spinors W in the following standard way. If p is the projection $p : Q_{\mathrm{Spin}^c(n)} \longrightarrow M$, the vector bundle p^*W trivializes as $p^*W \simeq Q_{\mathrm{Spin}^c(n)} \times W_0$. The connection $\nabla_A^{W,\xi}$ is then characterized by:

$$(2.9) \qquad\qquad p^*\nabla_A^{W,\xi} = d + \Omega_{A,\xi},$$

where $\Omega_{A,\xi}$ is seen in $A^1(Q_{\mathrm{Spin}^c(n)}, \mathrm{End}(W_0))$ and d is the trivial connection. The Dirac operator D_A^ξ is then the composition:

$$D_A^\xi : \Gamma(W) \xrightarrow{\nabla_A^{W,\xi}} \Gamma(T^*M \otimes W) \xrightarrow{\rho_\xi} \Gamma(W).$$

Hence we have a family of first order differential operators

$$(2.10) \qquad\qquad \mathcal{D} : \Xi \longrightarrow \mathrm{Diff}^1(W),$$

given by $\mathcal{D}(\xi) = D_A^\xi$ and parametrized by Spin^c-structures $\xi \in \Xi$, *all acting on the same vector bundle of spinors W.*

2.5. Parametrized Seiberg-Witten equations

Let now be $n = 4$. The irreducible $\mathrm{Cl}(\mathbb{R}^4)$-module W_0 splits as the direct sum $W_0^+ \oplus W_0^-$ of irreducible Spin$^c(4)$-representations: consequently, the bundle of spinors W splits as well in the direct sum of positive and negative spinors: $W = W_+ \oplus W_-$. The determinants of W_+ and W_-

are canonically identified with the determinant line bundle L; we indicate with i the identification $\det W_+ \simeq \det W_-$. The Clifford multiplication ρ_ξ induces isomorphisms $\Lambda_\pm T^*M \simeq \mathfrak{su}(W_\pm)$, denoted again by ρ_ξ. Let $\mathcal{C} := \mathcal{A}_{U(1)}(L) \times \Gamma(W_+)$ be the space of unparametrized configurations and $\mathcal{D} := \Gamma(W_-) \times i\mathfrak{su}(W_+)$; we denote moreover with $\mathcal{C}^* := \mathcal{A}_{U(1)}(L) \times (\Gamma(W_+) \setminus \{0\})$ the irreducible configurations. In order to consider general parameter spaces, let T be a Fréchet splitting[1] submanifold[2] of the manifold $\mathrm{Met}(M)$ of riemannian metrics on M. Let Ξ_T be the restriction of the principal bundle Ξ to the submanifold T. The natural connection on Ξ induces a natural connection on $\Xi_T \longrightarrow T$. The *parametrized Seiberg-Witten equations* for unknowns $(A, \psi, \xi) \in \mathcal{C} \times \Xi_T$ are:

$$（2.11\text{a}) \qquad\qquad\qquad D_A^\xi \psi = 0$$

$$(2.11\text{b}) \qquad\qquad \rho_\xi(F_A^{+,g_\xi}) - [\psi^* \otimes \psi]_0 = 0$$

We denote with $\mathbb{F}_T : \mathcal{C} \times \Xi_T \longrightarrow \mathcal{D}$ the functional defining the equations and with $\mathcal{S}_T$ the space of solutions. The group $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$ acts on W via isometries, respecting the decomposition in positive and negative spinors and the identification i between the determinants $\det W_+$ and $\det W_-$; if we denote with $U(W_+) \times_0 U(W_-)$ the bundle of groups of pairs $(f_+, f_-) \in U(W_+) \times U(W_-)$ with the same determinant, we have $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)}) \simeq C^\infty(M, U(W_+) \times_0 U(W_-))$. Consequently $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$ acts (on the right) on $\mathcal{C} \times \Xi_T$ and $\mathcal{D}$, respectively, setting;

$$(A, \psi, \xi) \cdot f := (\beta(f)^* A, f^{-1}\psi, f^*\xi)$$
$$(\chi, \eta) \cdot f := (f^{-1}\chi, f^{-1}\eta f)$$

where $\beta(f)$ denotes the image of f by the morphism $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)}) \longrightarrow \mathrm{Aut}(Q_{U(1)})$ induced by the projection β. The restriction of this action to the group $\mathcal{G} := C^\infty(M, S^1) \subset \mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$ coincides with the classical action of the Seiberg-Witten gauge group on $\mathcal{C}$. The functional $\mathbb{F}_T$ is equivariant for the $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$-action:

$$(2.12) \qquad\qquad \mathbb{F}_T((A, \psi, \xi) \cdot f) = \mathbb{F}_T(A, \psi, \xi) \cdot f.$$

As a consequence the group $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$ preserves the solutions $\mathcal{S}_T$ of the equations (2.11). Moreover the actions of $\mathcal{G}$ and of $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$ commute,

---

[1] in the sense of [11, Definition 27.11]

[2] We will always consider submanifolds T given by a space of C^∞-sections of a fiber subbundle of the fiber bundle $Met(M) \longrightarrow M$ considered in remark 2.7.

so that we can form a *parametrized Seiberg-Witten moduli space* $\mathcal{M}_T :=$ $\mathcal{S}_T/\mathcal{G}$, fibered over Ξ_T:

$$(2.13) \qquad \pi_T : \mathcal{M}_T \longrightarrow \Xi_T,$$

equipped with a Γ-action, making the preceding fibration equivariant. The fiber $\mathcal{M}_\xi = \pi_T^{-1}(\xi)$ of the fibration (2.13) over ξ is exactly the standard Seiberg-Witten moduli space for the Spin^c-structure $\xi \in \Xi_T$.

3. Variation of the Dirac operator

This section is devoted to the computation of the variation of the Dirac operator with respect to the metric, by means of the formalism introduced in section 2: what we will say in this section holds for any $n \geqslant 1$. Given a particular Spin^c-structure $\xi_0 \in \Xi$, we compute the differential $D_{\xi_0}\mathcal{D}\big|_{H_{\xi_0}}$ in the point ξ_0 of the family $\mathcal{D}$ of differential operators (2.10) restricted to the horizontal direction H_{ξ_0} for the natural connection on Ξ; this amounts to compute the differential at the identity of the family of differential operators:

$$\mathcal{D} \circ \sigma_\xi : \mathrm{Sym}^+(TM, g_\xi) \longrightarrow \mathrm{Diff}^1(W)$$
$$\varphi \longmapsto D_A^{\varphi^{-1} \circ \xi}$$

Let us compute $D_{\mathrm{id}}(\mathcal{D} \circ \sigma_\xi)(s)$ for $s \in \mathrm{sym}(TM, g_\xi) \simeq T_{\mathrm{id}} \mathrm{Sym}^+(TM, g_\xi)$. Consider the path in $\mathrm{Sym}^+(TM, g_\xi)$ given by $\varphi_t = \mathrm{id} + ts$ for small t. Let g_t be the metric $g_t := \varphi_t^* g_\xi$. Let $k = dg_t/dt|_{t=0} = 2gs$. Set $\phi_t = \varphi_t^{-1}$, seen in $\mathrm{Aut}(P_{GL_+(n)})$. Since by definition of the Clifford multiplication on the cotangent bundle T^*M, we have $\rho_{\phi_t \circ \xi} = \rho_\xi \circ \phi_t^*$, the searched differential is:

$$
\begin{aligned}
D_{\mathrm{id}}(\mathcal{D} \circ \sigma_\xi)(s) &= \frac{d}{dt}\bigg|_{t=0} D_A^{\phi_t \circ \xi} \\
&= \left(\frac{d}{dt}\bigg|_{t=0} \rho_\xi \circ \phi_t^* \right) \circ \nabla_A^{W,\xi} + \rho_\xi \circ \frac{d}{dt}\bigg|_{t=0} \nabla_A^{W, \phi_t \circ \xi} \\
&= -\rho_\xi \circ s^* \circ \nabla_A^{W,\xi} + \rho_\xi \circ \dot\nabla_A^W(s),
\end{aligned}
$$

where ϕ_t^* and s^* (the transposed of ϕ_t and s, respectively) act on the first factor of $T^*M \otimes W$. We compute now the variation of the spinorial connection $\dot\nabla_A^W(s)$.

3.1. Variation of the spinorial connection

The following lemma contains the wanted result about the variation of the spinorial connection. Let ∇^{g_ξ} be the Levi-Civita connection on TM for the metric g_ξ.

LEMMA 3.1. — *The differential of the map:* $\Xi \longrightarrow \mathcal{A}(W)$ *sending* ζ *to* $\nabla_A^{W,\zeta}$ *in the point* ξ *along the horizontal direction corresponding to the variation of the metric* g_ξ *in the direction* $s \in \mathrm{sym}(TM, g_\xi)$, *is given by:*

$$\dot{\nabla}_A^W(s) = \frac{1}{2}\rho_\xi(\dot{\nabla}(s) - \nabla^{g_\xi}s) \in A^1(M, \mathfrak{su}(W))$$

where $\dot{\nabla}(s)$ *denotes the variation of the Levi-Civita connection in the point* g_ξ *along the direction* s *and where we see the form* $\dot{\nabla}(s) - \nabla^{g_\xi}s$ *in* $A^1(M, \mathfrak{so}(TM, g_\xi))$.

Proof. — We first compute the variation of the spinorial connection form $\Omega_{A,\xi}$ on $Q_{\mathrm{Spin}^c(n)}$. Let ω_{g_t} the Levi-Civita connection form on $P_{SO(g_t)}$, seen as a connection form in $A^1(P_{GL_+(n)}, \mathfrak{gl}(n))$. The spinorial connection form is defined as:

$$\Omega_{A,\phi_t \circ \xi} = \rho_0 d\nu^{-1}((\phi_t \circ \xi)^*\omega_{g_t} + \beta^*A).$$

Differentiating this relation in $t = 0$ we obtain:

$$\dot{\Omega}_{A,\xi} = \rho_0 d\nu^{-1}\frac{d}{dt}\bigg|_{t=0}(\phi_t \circ \xi)^*\omega_{g_t} = \rho_0 d\nu^{-1}\xi^*\frac{d}{dt}\bigg|_{t=0}(\phi_t^*\omega_{g_t}).$$

It is straightforward to remark that the form $\phi_t^*\omega_{g_t}$ is defined on $P_{SO(g_\xi)}$ and is there pseudotensorial [3] of type $(\mathrm{ad}, \mathfrak{so}(n))$; hence its derivative $d(\phi_t^*\omega_{g_t})/dt|_{t=0}$ is tensorial of type $((\mathrm{ad}, \mathfrak{so}(n))$. Consequently, if q is the projection $q : P_{SO(g_\xi)} \longrightarrow M$, there exists a form $\omega_{\dot{M}} \in A^1(M, \mathfrak{so}(TM, g_\xi))$ such that

$$\frac{d}{dt}(\phi_t^*\omega_{g_t})\bigg|_{t=0} = q^*(\dot{\omega}_M) \in A^1(P_{SO(g_\xi)}, \mathfrak{so}(n)).$$

Therefore:

$$\dot{\Omega}_{A,\xi} = \rho_0 d\nu^{-1}\xi^*q^*(\dot{\omega}_M) = \rho_0 d\nu^{-1}p^*(\dot{\omega}_M) = \frac{1}{2}p^*(\rho_\xi(\dot{\omega}_M)).$$

---

[3] If $P \longrightarrow M$ is a principal bundle over M of structural group G and $\rho : G \longrightarrow GL(V)$ is a representation of G, then a V-valued k-form $\sigma \in A^k(P, V)$ is pseudotensorial of type (ρ, V) if, for all $g \in G$, $R_g^*\sigma = \rho(g^{-1})\sigma$, where R_g is the right translation by g on the bundle P; the form σ is tensorial if it is pseudotensorial and horizontal, that is, $\sigma(t_1, \ldots, t_k) = 0$ if one of the vector fields $t_i \in TP$ is vertical. One has that a V-valued k-form $\sigma \in A^k(P, V)$ on P tensorial of type (ρ, V) if and only if σ is the pull back on P of a $P \times_\rho V$-valued k-form on M. See [10].

where we recall that p denotes the projection $Q_{\mathrm{Spin}^c(n)} \longrightarrow M$, and where ρ_ξ acts on $\dot\omega_M$ via the isomorphism $\mathfrak{so}(TM, g_\xi) \simeq \mathrm{Cl}_2(T^*M)$; the factor $1/2$ comes from the fact that ν is a $2 : 1$ covering. Differentiating the relation (2.9) characterizing $\nabla_A^{W, \phi_t \circ \xi}$, we get $p^*(\dot\nabla_A^W) = (1/2) \, p^*(\rho_\xi(\dot\omega_M))$, yielding:

$$\dot\nabla_A^W(s) = \frac{1}{2} \rho_\xi(\dot\omega_M).$$

It remains now to determine the form $\dot\omega_M$; but this comes from the fact that $\phi_t^* \omega_t$ is the connection form on $P_{SO(g_\xi)}$ inducing the connection $\phi_t^{-1} \nabla^{g_t} \phi_t$ on TM: consequently its derivative $p^*(\dot\omega_M)$ induces the form $\dot\omega_M = d(\phi_t^{-1} \nabla^{g_t} \phi_t)/dt|_{t=0}$, that is the form $\dot\omega_M = \dot\nabla(s) - \nabla^{g_\xi} s \in A^1(M, \mathfrak{so}(TM, g_\xi))$, because the connection $\phi_t^{-1} \nabla^{g_t} \phi_t$ is compatible with the metric g_ξ for all t.

$\square$

3.2. Variation of the Levi-Civita connection

In order to explicitly compute the variation $\dot\nabla_A^W(s)$ we only need the computation of the variation of the Levi-Civita connection $\dot\nabla(s)$ in the point g_ξ along the symmetric tensor s. This is well known (see [2, Th. 1.174]):

PROPOSITION 3.2. — *The variation of the Levi-Civita connection ∇^{g_ξ} on the tangent bundle TM along the direction $s \in \mathrm{sym}(TM, g_\xi)$ is the form $\dot\nabla(s) \in A^1(M, \mathrm{End}(TM))$ given by:*

$$(3.1) \quad g_\xi(\dot\nabla(s)_X Y, Z) = g_\xi((\nabla_X^{g_\xi} s) Y, Z) - g_\xi((\nabla_Z^{g_\xi} s) Y, X) + g_\xi((\nabla_Y^{g_\xi} s) Z, X).$$

We now express formula (3.1) in terms of a local orthonormal frame $e_i \in TM$, $i = 1, \ldots, n$ for the metric g_ξ. Let e^i its dual frame. Let τ_{ij}^k and c_{ij}^k the components of the tensors $\dot\nabla s$ and $\nabla^{g_\xi} s$, respectively, with respect to the frame e_i:

$$\tau_{ij}^k = g_\xi(\dot\nabla(s)_{e_i} e_j, e_k) \qquad c_{ij}^k = g_\xi(\nabla_{e_i}^{g_\xi} s(e_j), e_k).$$

The tensor c_{ij}^k is symmetric in j, k because the Levi-Civita connection ∇^{g_ξ} preserves the bundle of symmetric endomorphisms $\mathrm{sym}(TM, g_\xi)$ for the metric g_ξ. In the frame e_i, formula (3.1) reads:

$$\tau_{ij}^k = c_{ij}^k - c_{kj}^i + c_{jk}^i.$$

The tensor τ_{ij}^k is symmetric in i, j. The components of the tensor $\dot\omega_M = \dot\nabla(s) - \nabla^{g_\xi} s$ are:

$$\dot\omega_{ij}^k = c_{jk}^i - c_{kj}^i.$$

The tensor $\dot\omega_{ij}^k$ is skew-symmetric in j and k, and hence belongs to $A^1(M, \mathfrak{so}(TM, g_\xi))$, as expected.

3.3. Variation of the Dirac operator

We can now continue the computation began in the introduction of this section. Let us compute $\rho_\xi \circ \dot{\nabla}_A^W \phi$ for a spinor $\phi \in \Gamma(W)$. In the chosen local orthonormal frame e_i, we denote with E_j^k the endomorphism $e^j \otimes e_k - e^k \otimes e_j$ in $\mathfrak{so}(TM, g_\xi)$. We know that:

$$\dot{\omega}_M = \sum_{ijk} \dot{\omega}_{ij}^k e^i \otimes e^j \otimes e_k = \frac{1}{2} \sum_{ijk} \dot{\omega}_{ij}^k e^i \otimes E_k^j,$$

therefore:

$$\dot{\nabla}_A^W \phi = \frac{\rho_\xi(\omega_M)}{2} \phi = \frac{1}{4} \sum_{ijk} \dot{\omega}_{ij}^k e^i \otimes \rho_\xi(e^j e^k)\phi.$$

Consequently the term $\rho_\xi \circ \dot{\nabla}_A^W \phi$ is:

$$\rho_\xi \circ \dot{\nabla}_A^W \phi = \frac{1}{4} \sum_{ijk} \dot{\omega}_{ij}^k \rho_\xi(e^i e^j e^k)\phi.$$

Recalling now that $\dot{\omega}_{ij}^k = \tau_{ij}^k - c_{ij}^k$, that τ_{ij}^k is symmetric in i, j and that c_{ij}^k is symmetric in j, k:

$$\rho_\xi \circ \dot{\nabla}_A^W \phi = -\frac{1}{4} \sum_{ij} \tau_{ii}^j \rho_\xi(e^j)\phi + \frac{1}{4} \sum_{ij} c_{ji}^i \rho_\xi(e^j)\phi.$$

Recalling the definition of τ_{ij}^k and that of c_{ij}^k, we have that $\sum_{ij} \tau_{ii}^j e^j = 2\operatorname{div} s - d\operatorname{tr} s$ and $\sum_{ij} c_{ji}^i e^j = d\operatorname{tr} s$. Hence we get:

$$\rho_\xi \circ \dot{\nabla}_A^W \phi = -\frac{1}{2}\rho_\xi(\operatorname{div} s - d\operatorname{tr} s)\phi.$$

We proved the theorem:

THEOREM 3.3. — *The variation of the family $\mathcal{D}$ of Dirac operators:* $\Xi \ni \xi \longrightarrow D_A^\xi \in \operatorname{Diff}^1(W)$ *in the point ξ along the horizontal direction corresponding to the variation of the metric g_ξ in the direction $s \in \operatorname{sym}(TM, g_\xi)$ is the first order differential operator given by:*

$$(3.2) \qquad \frac{d}{dt}D_A^{\xi_t}\Big|_{t=0} = -\rho_\xi \circ s^* \circ \nabla_A^{W,\xi} - \frac{1}{2}\rho_\xi(\operatorname{div} s - d\operatorname{tr} s).$$

The result agrees with the one obtained by Bourguignon and Gauduchon (see [3, Th. 21]).

4. Variation of the Seiberg-Witten equations

The aim of this section is to compute the full differential of the *universal Seiberg-Witten functional* $\mathbb{F}_{\mathrm{Met}(M)} : \mathcal{A}_{U(1)}(L) \times \Gamma(W_+) \times \Xi \longrightarrow \Gamma(W_-) \times i\mathfrak{su}(W_+)$ on a solution (A, ψ, ξ). We will denote with $\mathbb{F}_1$ and $\mathbb{F}_2$ the components of $\mathbb{F}_{\mathrm{Met}(M)}$ with values in $\Gamma(W_-)$ and $i\mathfrak{su}(W_+)$, respectively. We will use the splitting $T\Xi \simeq V \oplus H$ defined by the natural connection on Ξ. The most interesting part of this computation is the one dealing with the variation of $\mathbb{F}_{\mathrm{Met}(M)}$ along the horizontal direction, that is, the perturbation of the metric; the difficult point was the variation of the Dirac operator, treated in the previous section.

We study here the variation of the equation (2.11b) with respect to the metric. Since the Clifford multiplication on $\Lambda^2 T^*M$ transforms as $\rho_{\varphi^{-1}\circ\xi} = \rho_\xi \circ (\Lambda^2\varphi^*)^{-1}$, we have to differentiate the map:

$$\mathbb{F}_2(A, \psi, \sigma_\xi(-)) : \mathrm{Sym}^+(TM, g_\xi) \longrightarrow i\mathfrak{su}(W_+)$$
$$\varphi \longmapsto (\rho_\xi \circ (\Lambda^2\varphi^*)^{-1})(F_A^{+,\varphi^*g_\xi}) - [\psi^* \otimes \psi]_0$$

at the identity. The map φ is an orientation-preserving isometry between $(TM, \varphi^* g_\xi)$ and (TM, g_ξ), hence the Hodge star for the metric $\varphi^* g_\xi$ can be expressed as: $*_{\varphi^* g_\xi} = \Lambda^2\varphi^* \circ *_{g_\xi} \circ (\Lambda^2\varphi^*)^{-1}$. Consequently:

$$F_A^{+,\varphi^*g_\xi} = \left(\frac{\Lambda^2\varphi^* \circ *_{g_\xi} \circ (\Lambda^2\varphi^*)^{-1} + 1}{2} \right) F_A$$

$$= \Lambda^2\varphi^* \left(\frac{*_{g_\xi} + 1}{2} \right) (\Lambda^2\varphi^*)^{-1} F_A.$$

Therefore, denoting P^{+,g_ξ} the projection onto self-dual 2-forms for the metric g_ξ, we get:

$$(\rho_\xi \circ (\Lambda^2\varphi^*)^{-1})(F_A^{+,\varphi^*g_\xi}) = \rho_\xi(P^{+,g_\xi} \circ (\Lambda^2\varphi^*)^{-1}F_A).$$

Given a path of metrics in the direction $s \in \mathrm{sym}(TM, g_\xi)$, $g_t = \varphi_t^* g_\xi$, $\varphi_t = 1 + ts$, and differentiating in $t = 0$ we get:

$$\frac{\partial \mathbb{F}_2}{\partial \varphi}(A, \psi, \mathrm{id})(s) = -\rho_\xi(P^{+,g_\xi} i(s^*)F_A),$$

where $i(s^*)$ is the derivation of degree 0 on $\Lambda^* T^*M$ that coincides with s^* on T^*M. In order to better understand the term $P^{+,g_\xi} i(s^*)F_A$ consider the splitting of symmetric endomorphisms:

$$\mathrm{sym}(TM, g_\xi) \simeq \mathrm{sym}_0(TM, g_\xi) \oplus C^\infty(M, \mathbb{R}) \cdot \mathrm{id}_{TM}$$

in traceless ones and scalar ones. It is now well known that the bundle $\mathrm{sym}(TM, g_\xi)$ embeds in $\mathrm{End}(\Lambda^2 T^* M)$ via the morphism $s \longmapsto i(s^*)$. According to the decomposition $\Lambda^2 T^* M \simeq \Lambda^2_+ T^* M \oplus \Lambda^2_- T^* M$ and indicating with s_0 the traceless part of s, we can express $i(s^*)$ as:

$$i(s^*) = \begin{pmatrix} \mathrm{tr}\, s & P^{+,g_\xi} i(s_0^*)\big|_{\Lambda^2_- T^* M} \\ P^{-,g_\xi} i(s_0^*)\big|_{\Lambda^2_+ T^* M} & \mathrm{tr}\, s \end{pmatrix}.$$

Hence there remain induced isomorphisms

(4.1) $$\delta_\pm : \mathrm{sym}_0(TM, g_\xi) \longrightarrow \mathrm{Hom}(\Lambda^2_\pm T^* M, \Lambda^2_\mp T^* M)$$

and an isomorphism between scalar endomorphisms of $T^* M$ and homotheties of $\Lambda^2 T^* M$. Therefore $P^{+,g_\xi} i(s^*) F_A = (\mathrm{tr}\, s) F_A^+ + \delta_-(s_0) F_A^-$ and

$$\frac{\partial \mathbb{F}_2}{\partial \varphi}(A, \psi, \mathrm{id})(s) = -(\mathrm{tr}\, s)\rho_\xi(F_A^+) - \rho_\xi(\delta_-(s_0) F_A^-).$$

The differential of the functional $\mathbb{F}_{\mathrm{Met}(M)}$ with respect to the connection and the spinors components is well known (see [14]): we have, for variations $\tau \in iA^1(M)$, $\phi \in \Gamma(W_+)$:

$$\frac{\partial \mathbb{F}_{\mathrm{Met}(M)}}{\partial (A, \psi)}(A, \psi, \xi)(\tau, \phi) = \begin{pmatrix} \frac{1}{2}\rho_\xi(\tau)\psi + D_A^\xi \phi \\ d^+\tau - [\phi^* \otimes \psi + \psi^* \otimes \phi]_0 \end{pmatrix}.$$

To compute the full differential of $\mathbb{F}_{\mathrm{Met}(M)}$ it remains to compute its variation along vertical direction of Ξ, that is, along the fibers. Since the group $\mathrm{Aut}(Q_{\mathrm{Spin}^c(4)})$ acts on the configuration space $\mathcal{C} \times \Xi$ preserving the solutions and since its action on Ξ is transitive on the connected component of the fibers, no contribution to the transversality can be obtained in this way, because the component of the differential we get is a linear combination of the other components.

5. The question of transversality

In this section we set up the transversality problem. Our final project is to prove that the universal Seiberg-Witten moduli space $\mathcal{M}_{\mathrm{Met}(M)}$ is smooth, at least at its irreducible points. This can be achieved with standard methods via the implicit function theorem applied to adequate Banach manifolds, once we know that the defining equations of $\mathcal{M}_{\mathrm{Met}(M)}$ inside $(\mathcal{C} \times \Xi)/\mathcal{G}$ are transversal at irreducible monopoles. In order to proceed in such a way we need, as usual, to complete our till now Fréchet manifolds to Banach ones.

5.1. Sobolev completions

Let $\mathcal{C}_p^2 = \Gamma(W_+)_p^2 \times \mathcal{A}_{U(1)}(L)_p^2$ and $\mathcal{D}_{p-1}^2 = i\mathfrak{su}(W_+)_{p-1}^2 \times \Gamma(W_-)_{p-1}^2$ be the completions of $\mathcal{C}$ and $\mathcal{D}$ in Sobolev norms $||\ ||_{2,p}$ and $||\ ||_{2,p-1}$, respectively, so that they become a Hilbert affine space and a Hilbert vector space, respectively. Consider also the Banach completions $\mathcal{C}^l$ and $\mathcal{D}^l$ of $\mathcal{C}$ and $\mathcal{D}$, respectively, in norm C^l, $l \in \mathbb{N}$. The space of metrics $\mathrm{Met}(M)$ can be completed to a Banach manifold considering the space of C^r-metrics $\mathrm{Met}^r(M)$. We suppose that the Fréchet submanifold $T \subseteq \mathrm{Met}(M)$ we are considering admits a completion[4] to a Banach splitting submanifold T^r of $\mathrm{Met}^r(M)$. Complete now Ξ with μ-equivariant morphisms of class C^r: $\Xi^r := \mathrm{Mor}_\mu^r(Q_{\mathrm{Spin}^c(4)}, P_{GL_+(4)})$; the space Ξ^r becomes then a Banach principal bundle with structural group $\mathrm{Aut}^r(Q_{SO(4)})$ (the C^r-gauge group of $Q_{SO(4)}$) over the space of C^r-metrics $\mathrm{Met}^r(M)$; the natural connection defined in subsection 2.3 extends to this setting. Now take Ξ_T^r to be the $\mathrm{Aut}^r(Q_{SO(4)})$-Banach principal bundle on T^r given by the restriction of Ξ^r to T^r. We will always suppose $r >> p >> 0$. We will complete as well the gauge group $\mathcal{G}$ to $\mathcal{G}_{p+1}^2$, in Sobolev norm $||\ ||_{2,p+1}$ in order to have a Banach-Lie group acting continuously on $\mathcal{C}_p^2$ and $\mathcal{D}_{p-1}^2$. Denote with $\mathcal{C}_p^{2,*}$ the irreducible unparametrized configurations, that is, couples (A, ψ) in $\mathcal{C}_p^2$ such that $\psi \neq 0$: such couples have trivial $\mathcal{G}_{p+1}^2$-stabilizer; denote moreover with $\mathcal{B}_p^2$ the quotient $\mathcal{B}_p^2 := \mathcal{C}_p^2/\mathcal{G}_{p+1}^2$ and with $\mathcal{B}_p^{2,*} := \mathcal{C}_p^{2,*}/\mathcal{G}_{p+1}^2$; the latter is a Hilbert manifold. Let now $(\mathfrak{C}_T)_p^{2,r} := \mathcal{C}_p^2 \times \Xi_T^r$ be the space of parametrized configurations, completed in Sobolev and C^r norm, and $(\mathfrak{C}_T^*)_p^{2,r} = \mathcal{C}_p^{2,*} \times \Xi_T^r$ the irreducible ones. The quotient $(\mathfrak{B}_T^*)_p^{2,r} := (\mathfrak{C}_T^*)_p^{2,r}/\mathcal{G}_{p+1}^2$ is isomorphic to $\mathcal{B}_p^{2,*} \times \Xi_T^r$ and hence a Banach manifold. The functional $\mathbb{F}_T$ extends to a $\mathcal{G}_{p+1}^2$-equivariant map:

$$(\mathbb{F}_T)_p^{2,r} : (\mathfrak{C}_T)_p^{2,r} \longrightarrow \mathcal{D}_{p-1}^2.$$

We indicate with $(\mathcal{M}_T)_p^{2,r} := Z((\mathbb{F}_T)_p^{2,r})/\mathcal{G}_{p+1}$ the parametrized Seiberg-Witten moduli space and with $(\mathcal{M}_T^*)_p^{2,r} = (\mathcal{M}_T)_p^{2,r} \cap (\mathfrak{B}_T^*)_p^{2,r}$ the parametrized moduli space of irreducible monopoles.

Remark 5.1. — Let $(A, \psi, \xi) \in Z((\mathbb{F}_T)_p^{2,r})$ be a solution to the parametrized Seiberg-Witten equations. Then (A, ψ, ξ) is $\mathcal{G}_{p+1}^2$-equivalent to a solution (A', ψ', ξ), with $(A', \psi') \in \mathcal{C}^{r-3} = \mathcal{A}_{U(1)}(L)^{r-3} \times \Gamma(W_+)^{r-3}$ (of class

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[4] This is always the case if T is a space of global sections of a fiber subbundle of $Met(M)$

$C^{r-3})^{(5)}$. Consequently, if ξ has C^∞-regularity, up to the $\mathcal{G}^2_{p+1}$-action, we can always suppose that (A, ψ) have C^∞-regularity.

Consider now the trivial Banach vector bundle $\mathcal{E}_T : (\mathfrak{C}^*_T)^{2,r}_p \times \mathcal{D}^2_{p-1}$ over the irreducible parametrized configurations $(\mathfrak{C}^*_T)^{2,r}_p$: it is naturally equipped with a $\mathrm{Aut}^r(Q_{\mathrm{Spin}^c(4)})$-action. Denote with Γ^r the image $\Gamma^r := \mathrm{Im}(\mathrm{Aut}^r(Q_{\mathrm{Spin}^c(4)}) \longrightarrow \mathrm{Aut}^r(Q_{SO(4)}))$. The restriction of the functional $(\mathbb{F}_T)^{2,r}_p$ to $(\mathfrak{C}^*_T)^{2,r}_p$ descends to the $\mathcal{G}^2_{p+1}$-quotients to give a Γ^r-equivariant section Ψ_T of the Γ^r-equivariant quotient bundle $\mathfrak{E}_T := \mathcal{E}_T/\mathcal{G}^2_{p+1}$:

$$\Psi_T : (\mathfrak{B}^*_T)^{2,r}_p \longrightarrow \mathfrak{E}_T.$$

The section Ψ_T is a Fredholm map between Banach manifolds; its zero set $Z(\Psi_T)$ is exactly the parametrized moduli space $(\mathcal{M}^*_T)^{2,r}_p$ of irreducible monopoles.

5.2. Remarks on the transversality statement

A metric $g \in \mathrm{Met}^r(M)$ is said c-good (for the fundamental class $c \in H^2(M, \mathbb{Z})$) if the projection of c onto the g-harmonic self-dual classes $\mathcal{H}^2_+$ is non zero. We denote with $\mathrm{Met}^r_{c\text{-good}}(M)$ the set of such metrics: it is an open set of $\mathrm{Met}^r(M)$, complementary of a closed set of codimension $b_+(M)$. Let now $b_+(M) > 0$ and suppose that $\mathrm{Met}^r_{c\text{-good}}(M) \cap T$ is a dense open set of T. Let $\Xi^{r,**}_T$ the subset of Spin^c-structures in Ξ^r_T projecting onto c-good metrics; set $\Xi^{r,**} := \Xi^{r,**}_{\mathrm{Met}(M)}$. Let finally $(\mathcal{M}^{**}_T)^{2,r}_p := Z(\Psi_T) \cap (\mathcal{B}^2_p \times \Xi^{r,**}_T)$. In the following we will always denote with $\mathrm{Met}_{c\text{-good}}(M)$, Ξ^{**}, Ξ^{**}_T, $\mathcal{M}^*_T$, $\mathcal{M}^{**}_T$ the corresponding spaces of objects of class C^∞. The projection $\pi_T \colon (\mathcal{M}^{**}_T)^{2,r}_p \longrightarrow \Xi^{r,**}_T$ is now a surjective Fredholm map with *compact*

$^{(5)}$ This can be proved with a slight modification (with C^r regularity) of the proof given in [20, Proposition 6.19]. The proof is based on two statements. The first that (A, ψ), as a solution of unparametrized Seiberg-Witten equations with fixed Spin^c-structure ξ, is $\mathcal{G}^2_{p+1}$-equivalent to (A'', ψ'') with A'' in A_0-gauge, for a fixed C^∞-connection A_0, that is $d^*(A'' - A_0) = 0$. This proof [20, Proposition 6.10] is still valid with a C^r-metric. Then, that the Seiberg-Witten equations (for fixed ξ), restricted to solutions in A_0-gauge, become the following nonlinear elliptic equations in $\tau = A'' - A_0$ and ψ:

(5.1) $$D_{A_0}\psi = -\frac{1}{2}\rho_\xi(\tau)\psi$$

(5.2) $$\begin{pmatrix} d^+ \\ d^* \end{pmatrix} \tau = \begin{pmatrix} [\psi^* \otimes \psi]_0 - F^+_{A_0} \\ 0 \end{pmatrix}$$

By a bootstrapping argument based on Sobolev multiplication and elliptic regularity (see [2, Appendix K, Theorem 40]) (the left hand term is a first order elliptic differential operator with C^{r-1}-coefficients) we get that the solution (A'', ψ) is in $\mathcal{C}^2_r$ and hence in $\mathcal{C}^{r-3} = \mathcal{A}_{U(1)}(L)^{r-3} \times \Gamma(W_+)^{r-3}$.

fibers, since no reducible monopole is allowed over a c-good metric. Denote with $(\mathcal{M}_\xi)_p^{2,r}$ its fiber over $\xi \in \Xi_T^{r,**}$. A satisfying result would be the following:

STATEMENT 5.2. — *The intrinsic differential $D_x \Psi_{\mathrm{Met}(M)}$ is surjective at all points $x \in (\mathcal{M}_{\mathrm{Met}(M)}^*)_p^{2,r}$. Consequently, the universal moduli space $(\mathcal{M}_{\mathrm{Met}(M)}^*)_p^{2,r}$ of irreducible Seiberg-Witten monopoles is a smooth Banach manifold.*

Remark 5.3. — We discuss here the consequences of transversality. If statement 5.2 is true, a standard application of Sard-Smale theorem [19] implies that *for ξ in a dense open set of $\Xi^{r,**}$, the Seiberg-Witten moduli space $(\mathcal{M}_\xi)_p^{2,r}$ is smooth of the expected dimension*. Consequently, this would be true for a generic ξ of class C^∞. These arguments can be adapted in the following case, useful in the applications to kählerian monopoles. Let S a Fréchet submanifold of $\mathrm{Met}(M)$, embedded in T, and let now $\mathcal{M}_S$ be the parametrized Seiberg-Witten moduli space[6] over S. Suppose that the intrinsic differential $D_x \Psi_T$ is surjective at all points of $\mathcal{M}_S^{**}$; in this case the parametrized moduli space $(\mathcal{M}_T)_p^{2,r}$ and the universal moduli space $(\mathcal{M}_{\mathrm{Met}(M)})_p^{2,r}$ are smooth at points in $\mathcal{M}_S^{**}$. Fix now a Spin^c-structure $\xi \in \Xi_S^{**}$ (of class C^∞). By compactness of $(\mathcal{M}_\xi)_p^{2,r} \simeq \mathcal{M}_\xi = \pi_T^{-1}(\xi)$, there exists an open neighbourhood V_ξ of ξ in $(\Xi_T^{**})_p^{2,r}$ such that $D_y \Psi_T$ is surjective for all $y \in \pi_T^{-1}(V_\xi)$: hence $\pi_T^{-1}(V_\xi)$ is a smooth Banach manifold and the projection $\pi_T^{-1}(V_\xi) \longrightarrow V_\xi$ is a Fredholm map with compact fibers. We can now apply to this map a standard argument using Sard-Smale theorem to obtain that for a generic $\xi' \in V_\xi$ the fiber $(\mathcal{M}_{\xi'})_p^{2,r}$ is smooth of the expected dimension; hence for a generic $\xi' \in V_\xi \cap \Xi_T$ of class C^∞ the moduli space $\mathcal{M}_{\xi'}$ is smooth of the expected dimension. The same can be said for an adequate neighbourhood W_ξ of ξ in $\Xi^{r,**}$.

We will now discuss the condition that the intrinsic differential $D_x \Psi_T$ is surjective at a point $x \in (\mathcal{M}_T^*)_p^{2,r}$. We recall that a local slice for the action of $\mathcal{G}_{p+1}^2$ in $\mathcal{C}_p^2$ at the point y is a smooth splitting Hilbert submanifold $\mathcal{S}_y$ of a neighbourhood of y, invariant under the stabilizer $\mathrm{Stab}(y)$, such that the natural map:

$$\mathcal{S}_y \times_{\mathrm{Stab}(y)} \mathcal{G}_{p+1}^2 \longrightarrow \mathcal{C}_p^2$$

is a diffeomorphism onto a neighbourhood of the orbit through y. Such a slice exists [14, Lemma 4.5.5] and is built, in a neighbourhood of a point

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[6] As explained above, here we consider the moduli space of C^∞-objects, introduced in section 2.5; by remark 5.1 it can be seen as embedded in $(\mathcal{M}_{\mathrm{Met}(M)})_p^{2,r}$, as being the inverse image of smooth elements Ξ_S for the projection $\pi_{\mathrm{Met}(M)}$

(A, ψ), as

$$\mathcal{S}_{(A,\psi)} = \big\{(A', \varphi) \in \mathcal{C}_p^2 \,|\, (D_{(A,\psi)}\Upsilon_{A,\psi})^*(A' - A, \varphi - \psi) = 0\,,$$
$$\|A' - A\|_p^2 < \epsilon\,,\, \|\varphi - \psi\|_p^2 < \epsilon\big\}$$

for a sufficiently small ϵ, where $\Upsilon_{A,\psi} : \mathcal{G}_{p+1}^2 \longrightarrow \mathcal{C}_p^2$ is the given by group action at (A, ψ) and where $(D_{(A,\psi)}\Upsilon_{(A,\psi)})^*$ is the formal adjoint of the differential $D_{(A,\psi)}\Upsilon_{(A,\psi)}$. As a consequence $T_{(A,\psi)}\mathcal{S}_{(A,\psi)} \simeq \ker(D_{(A,\psi)}\Upsilon_{A,\psi})^*$. A slice for the action of $\mathcal{G}_{p+1}^2$ on $(\mathfrak{C}_T^*)_p^{2,r}$ at a point (A, ψ, ξ) is then given by $\mathcal{S}_{(A,\psi,\xi)} := \mathcal{S}_{(A,\psi)} \times U_\xi$ where U_ξ is a small neighbourhood of ξ in Ξ_T^r. The slice $\mathcal{S}_{(A,\psi,\xi)}$ provides a local model for $(\mathfrak{B}_T^*)_p^{2,r}$ at an irreducible point $([A, \psi], \xi)$.

PROPOSITION 5.4. — *The section Ψ_T is transversal to the zero section at the point $x = ([A, \psi], \xi) \in (\mathcal{M}_T^*)_p^{2,r}$ if and only if the functional $(\mathbb{F}_T)_p^{2,r}$ is transversal to 0 at the point $(A, \psi, \xi) \in (\mathfrak{C}_T^*)_p^{2,r}$.*

Proof. — The section Ψ_T can be written locally on $\mathcal{S}_{(A,\psi,\xi)}$ as:

$$\mathcal{S}_{(A,\psi)} \times U_\xi \longrightarrow \mathcal{D}_{p-1}^2$$
$$((A', \psi'), \xi') \mapsto (\mathbb{F}_T)_p^{2,r}(A', \psi', \xi')$$

Since $T_{(A,\psi)}\mathcal{S}_{(A,\psi)} \simeq \ker(D_{(A,\psi)}\Upsilon_{A,\psi})^* = (\mathrm{Im}\, D_{(A,\psi)}\Upsilon_{A,\psi})^\perp$, we have that $D_x\Psi_T$ coincides with the restriction $D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r}\big|_{(\mathrm{Im}\, D_{(A,\psi)}\Upsilon_{A,\psi})^\perp \oplus T_\xi\Xi_T^r}$. Now, since $\mathrm{Im}\, D_{(A,\psi)}\Upsilon_{A,\psi} \subseteq \ker D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r}$, we can conclude. □

Remark 5.5. — Since it is Γ^r-equivariant, the section Ψ_T is transversal to zero in a point $x = ([A, \psi], \xi)$ if and only if it is transversal to zero in a point $xf = ([f^*A, \psi f], \xi f)$, for $f \in \Gamma^r$. This means that the smoothness of a standard Seiberg-Witten moduli space $(\mathcal{M}_\xi)_p^{2,r}$, $\xi \in \Xi_T^r$ does not depend on the particular Spin^c-structure ξ (chosen in a fixed connected component of Ξ_T^r) but *only* on the metric g_ξ compatible with ξ.

Remark 5.6. — Let $b_+(M) > 0$. By remark 5.5 and since the torsion subgroup $\mathfrak{t}$, counting connected components of Ξ/Γ, is finite, if statement 5.2 is true, then for a generic C^∞ metric $g \in \mathrm{Met}(M)$, the standard Seiberg-Witten moduli $\mathcal{M}_\xi$ is smooth of the expected dimension for any Spin^c-structure ξ, compatible with g. This means that, even there is no satisfactory way to parametrize Seiberg-Witten equations and moduli spaces just with metrics $\mathrm{Met}(M)$, because, by remark 2.12, there are no parallel sections $\mathrm{Met}(M) \longrightarrow \Xi$, the transversality statement depends *only* on the metric chosen and not on the particular Spin^c-structure compatible with the metric.

5.3. The adjoint operator

In this subsection we express the obstruction to the transversality of the section Ψ_T in terms of the formal adjoint of the differential of the functional $\mathbb{F}_T$. Let $([A, \psi], \xi) \in Z(\Psi_T)$. By remark 5.1 we can suppose that (A, ψ) are of class C^{r-3}. By proposition 5.4 the obstruction to the transversality of Ψ_T at the point $([A, \psi], \xi)$ is given by the cokernel of the first order differential operator with C^{r-3} coefficients:

$$D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r} : T_{(A,\psi)}\mathcal{C}_p^2 \oplus T_\xi \Xi_T^r \longrightarrow \mathcal{D}_{p-1}^2.$$

The operator $D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r}$ is the partial Sobolev completion of the first order differential operator (with C^{r-3} coefficients), given by the differential of the Seiberg-Witten functional $\mathbb{F}_T$, defined on parametrized configurations of class C^{r-3} and C^r:

$$D_{(A,\psi,\xi)}\mathbb{F}_T : T_{(A,\psi)}\mathcal{C}^{r-3} \oplus T_\xi \Xi_T^r \longrightarrow \mathcal{D}^{r-4}.$$

We indicate with $D_{(A,\psi)}F^\xi$ and with P the first and second component, respectively. The component $D_{(A,\psi)}F^\xi$ is the differential of the unparametrized Seiberg-Witten functional F^ξ, relative to the Spin^c-structure ξ: it is well known [14] that it is underdetermined elliptic, in this case with C^{r-3} coefficients; hence its Sobolev extension $(D_{(A,\psi)}F^\xi)_p^2 : T_{(A,\psi)}\mathcal{C}_p^2 \longrightarrow \mathcal{D}_{p-1}^2$ has closed image of finite codimension. Since the operator $D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r}$ can be written as the sum $D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r} = (D_{(A,\psi)}F^\xi)_p^2 + P$, it follows[7] that $D_{(A,\psi,\xi)}\mathbb{F}_T$ is underdetermined elliptic and that $D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r}$ has closed image of finite codimension. By elliptic regularity, we have firstly that

$$\operatorname{coker} D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r} \subseteq \operatorname{coker}(D_{(A,\psi)}F^\xi)_p^2 \subseteq \mathcal{D}^{r-4};$$

secondly that $\operatorname{coker} D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r}$ can be identified with the L^2-orthogonal $(\operatorname{Im} D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r})^\perp$; the latter coincides with the kernel of the formal adjoint of $D_{(A,\psi,\xi)}\mathbb{F}_T$:

(5.3)
$$\begin{aligned} \operatorname{coker} D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r} &\simeq (\operatorname{Im}(D_{(A,\psi,\xi)}(\mathbb{F}_T)_p^{2,r})^\perp \\ &\simeq \ker(D_{(A,\psi,\xi)}\mathbb{F}_T)^*\big|_{\mathcal{D}^{r-4}}. \end{aligned}$$

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[7] This follows from the following fact [20, Lemma 6.36]. Let E a Banach vector space with a continuous scalar product $< \cdot, \cdot >: E \times E \longrightarrow \mathbb{R}$ and let $F \subset E$ a closed subspace of finite codimension such that its orthogonal $F^\perp$ is a topological supplementary of F in E. If F' is a subspace of E containing F, then F' is closed of finite codimension and its orthogonal $(F')^\perp$ is a topological supplementary of F' in E.

Remark 5.7. — For $\xi \in \Xi^r$ consider the section $\sigma_\xi \colon \mathrm{Met}^r(M) \longrightarrow \Xi^r$ passing through ξ and defining the horizontal distribution H_ξ on Ξ^r. Suppose now that $g_\xi \in T^r$. Let $\sigma_{T,\xi}$ be the restriction $\sigma_{T,\xi} := \sigma_\xi\big|_{T^r}$ of this section to the submanifold T^r. We denote with $\tilde{\mathbb{F}}_T$ the composition:

$$\tilde{\mathbb{F}}_T := \mathbb{F}_T \circ (\mathrm{id}_{\mathcal{C}^{r-3}} \times \sigma_{T,\xi}) : \mathcal{C}^{r-3} \times T^r \hookrightarrow \mathcal{C}^{r-3} \times \Xi^r_T \longrightarrow \mathcal{D}^{r-4}.$$

Since, as seen in the end of section 4, there is no contribution to the transversality coming from variations of the equations along vertical directions in $T_\xi \Xi_T$, we have that

(5.4)
$$\mathrm{coker}\, D_{(A,\psi,\xi)}(\mathbb{F}_T)^{2,r}_p \simeq \ker(D_{(A,\psi,\xi)}\mathbb{F}_T)^*\big|_{\mathcal{D}^{r-4}}$$
$$\simeq \ker(D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}_T)^*\big|_{\mathcal{D}^{r-4}}.$$

Remark 5.8. — Since $\tilde{\mathbb{F}}_T$ factorizes in the composition $\mathcal{C}^{r-3} \times T^r \hookrightarrow \mathcal{C}^{r-3} \times \mathrm{Met}^r(M) \xrightarrow{\tilde{\mathbb{F}}_{\mathrm{Met}(M)}} \mathcal{D}^{r-4}$, we have:

$$(D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}_T)^* = (\mathrm{id}_{T_{(A,\psi)}\mathcal{C}^{r-3}} \oplus P_{T_{g_\xi}T^r}) \circ (D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}_{\mathrm{Met}(M)})^*,$$

where $P_{T_{g_\xi}T^r}$ denotes the orthogonal projection $T_{g_\xi}\mathrm{Met}^r(M) \longrightarrow T_{g_\xi}T^r$.

In what follows we will denote more briefly by $\tilde{\mathbb{F}}$ the functional $\tilde{\mathbb{F}}_{\mathrm{Met}(M)}$. It is clear that if ξ is of class C^∞, we can drop the superscripts, considering spaces of objects of class C^∞.

Computation of the adjoint operator

In the sequel we will always assume for simplicity's sake that the point (A,ψ,ξ), where the differential is computed, is such that A,ψ,ξ are of class[8] C^∞. In this subsection we will compute the formal adjoint of the differential of $\tilde{\mathbb{F}}$ at the point (A,ψ,g_ξ)

$$D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}} : iA^1(M) \times \Gamma(W_+) \times \mathrm{sym}(TM,g_\xi) \longrightarrow \Gamma(W_-) \times iA^2_+(M),$$

given by:

$$D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}(\tau,\phi,s) = \Big(\tfrac{1}{2}\rho_\xi(\tau)\psi + D^\xi_A\phi - \rho_\xi \circ s^* \circ \nabla^{W,\xi}_A \psi - \tfrac{1}{2}\rho_\xi(\mathrm{div}\, s - d\,\mathrm{tr}\, s)\psi,$$
$$d^+\tau - [\phi^* \otimes \psi + \psi^* \otimes \phi]_0 - (\mathrm{tr}\, s)F^+_A - \delta_-(s_0)F^-_A\Big).$$

The computation of the formal adjoint is mostly straightforward: we will just remark the less trivial steps and make clear some notations.

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[8] This will be always the case in the applications; however what we will say holds in all generality.

L^2**-norms.** We define here the L^2-norms with respect to which we are going to compute the formal adjoint; we will always indicate with $(\cdot, \cdot)$ the real inner products and with $\langle \cdot, \cdot \rangle$ the hermitian ones. On the bundle $T^*M \otimes i\mathbb{R}$ the norm is the standard one induced by the metric g_ξ. On $T^*M^{\otimes m}$ the metric g_ξ induces the inner product $(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = m! \prod_{i=0}^m (x_i, y_i)$; the decomposition $T^*M \otimes T^*M \simeq S^2 T^*M \oplus \Lambda^2 T^*M$ is then an orthogonal direct sum. We will take on $S^2 T^*M$ and on $\Lambda^2 T^*M$ the metrics induced by the metric on $T^*M \otimes T^*M$. In this way $\|e_i \otimes e_j\|^2 = 2$, $\|e_i e_j\|^2 = 1 = \|e_i \wedge e_j\|^2$, if $i \neq j$, otherwise $\|e_i \otimes e_i\|^2 = \|e_i^2\|^2 = 2$. The metric induced by $T^*M \otimes T^*M$ on $\mathrm{sym}(TM, g_\xi)$ is $(s, t) = 2\,\mathrm{tr}(st)$. The morphisms $\delta_\pm$ defined in (4.1) are isometries if we take on $\mathrm{Hom}(\Lambda_\pm^2 T^*M, \Lambda_\mp^2 T^*M)$ the metric $(u, v) = 1/2\,\mathrm{tr}(uv^*)$. On $\Gamma(W_+)$ and on $\Gamma(W_-)$ we take the real part of the hermitian metric and on $\mathrm{Hom}(W_+, W_-)$ the real part of the hermitian scalar product $\langle u, v \rangle = 1/2\ \mathrm{tr}(uv^*)$, so that the Clifford multiplication ρ_ξ is an isometry. Finally the real part of the hermitian metric on $\mathrm{End}(W)$, given by $\langle A, B \rangle = 1/4\ \mathrm{tr}(AB^*)$, induces an orthogonal direct sum $\mathfrak{u}(W) \simeq i\mathbb{R} \oplus \mathfrak{su}(W)$. We put on $i\mathfrak{su}(W)$ the real inner product induced by the real inner product just defined on $\mathrm{End}(W)$, so that the isomorphism $\rho_\xi : \Lambda^2 T^*M \longrightarrow i\mathfrak{su}(W)$ is an isometry. The isomorphism $\rho_\xi : TM \otimes \mathbb{C} \longrightarrow \mathrm{Hom}(W_+, W_-)$ allows us to identify elements in $\mathrm{Hom}(W_+, W_-)$ with complexified tangent vectors and to define a complex conjugation (and hence a real and imaginary part) for elements in $\mathrm{Hom}(W_+, W_-)$.

Computation of the adjoint operator. We express the adjoint operator $(D_{(A,\psi,g_\xi)}\widetilde{\mathbb{F}})^*$ in terms of variables $(\chi, \theta) \in \Gamma(W_-) \times iA_+^2(M)$. In the rest of the article we will identify symmetric 2-tensors $S^2 T^*M$ with symmetric endomorphisms $\mathrm{sym}(TM, g_\xi)$ by means of the metric g_ξ.

(1) To compute the adjoint of the map $j_\psi \colon A^1(M, \mathbb{C}) \longrightarrow \Gamma(W_-)$, given by $\sigma \longmapsto \rho_\xi(\sigma)\psi$, remark that, for $\chi \in \Gamma(W_-)$, we have:

(5.5)
$$\langle \rho_\xi(\sigma)\psi, \chi \rangle = \mathrm{tr}[\rho_\xi(\sigma) \circ (\psi^* \otimes \chi)^*] = 2\langle \rho_\xi(\sigma), \psi^* \otimes \chi \rangle_{\mathrm{Hom}(W_+, W_-)}$$
$$= 2\langle \sigma, \psi^* \otimes \chi \rangle_{T^*M \otimes \mathbb{C}}.$$

Therefore the *hermitian adjoint* of j_ψ is the map $\chi \longmapsto 2\,\psi^* \otimes \chi$.

(2) The adjoint of the map $q_\psi(\phi) = [\phi^* \otimes \psi + \psi^* \otimes \phi]_0$ is the operator $q_\psi^* \colon iA_+^2(M) \longrightarrow \Gamma(W_+)$ given by $q_\psi^*(\theta) = 1/2\rho_\xi(\theta)\psi$. This can be proved firstly showing that, with the taken norms:

(5.6) $\quad \langle \rho_\xi(\theta), [\varphi^* \otimes \varphi]_0 \rangle = \dfrac{1}{4}\langle \rho_\xi(\theta)\varphi, \varphi \rangle, \quad \forall \theta \in iA_+^2(M), \forall \varphi \in \Gamma(W_+)$

and, secondly, differentiating the identity (5.6) with respect to φ and identifying $iA_+^2(M)$ with $i\mathfrak{su}(W_+)$ via the isometry ρ_ξ.

(3) Recalling that δ_- is an isometry from $\mathrm{sym}_0(TM, g_\xi)$ to $\mathrm{Hom}(\Lambda_-^2 T^*M, \Lambda_+^2 T^*M)$ with the given norms, we immediately get that the adjoint of the map $s \longmapsto \delta_-(s_0)(F_A^-)$ is given by $\theta \longmapsto 2(F_A^-)^* \otimes \theta$.

(4) To compute the adjoint of the map $s \longmapsto \rho_\xi(\mathrm{div}\, s)\psi = j_\psi \circ \mathrm{div}(s)$ recall that the adjoint of the divergence operator $\mathrm{div} : \mathrm{sym}(TM, g_\xi) \longrightarrow A^1(M)$ is given by the map: $\sigma \longmapsto -(1/2)\, L_{\sigma^\sharp} g_\xi$, where we indicate with $\sigma^\sharp$ the vector field obtained from the 1-form σ by raising the indices. The adjoint of $\rho_\xi(\mathrm{div}(-))\psi$ is then: $\chi \longmapsto -L_{\mathrm{Re}(\psi^* \otimes \chi)} g_\xi$.

(5) With similar arguments one can prove that the adjoint of $s \longmapsto \rho_\xi(d\,\mathrm{tr}\, s)\psi$ is given by $\chi \longmapsto d^*(\mathrm{Re}(\psi^* \otimes \chi)) g_\xi$.

(6) Denote with $\nabla_A^W \psi^*$ the linear map $TM \longrightarrow W_+^*$ defined by: $X \longmapsto \langle -, \nabla_{A,X}^{W,\xi}\psi\rangle$ and with $\mathrm{Re}(\nabla_A^W \psi^* \otimes \chi)$ the 2-tensor defined by: $(X,Y) \longmapsto \langle Y, \mathrm{Re}(\nabla_{A,X}^{W,\xi}\psi^* \otimes \chi)\rangle$. The adjoint of the map $\mathrm{sym}(TM, g_\xi) \longrightarrow \Gamma(W_-)$, defined by $s \longmapsto \rho \circ s^* \circ \nabla_A^{W,\xi}\psi$, is the map $\chi \longmapsto \mathrm{sym}\,\mathrm{Re}(\nabla_A^W \psi^* \otimes \chi)$. This can be proved expressing everything in a local orthonormal frame e^i and recalling the identity (5.5).

We are ready to write down the formal adjoint of the operator $D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}$:

PROPOSITION 5.9. — *The formal adjoint of the operator $D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}$ is the differential operator:*

$$(D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}})^* : \Gamma(W_-) \oplus A_+^2(M, i\mathbb{R}) \longrightarrow A^1(M, i\mathbb{R}) \oplus \Gamma(W_+) \oplus \mathrm{sym}(TM, g_\xi)$$

given by $(D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}})^*(\chi,\theta) = \big(A_1(\chi,\theta), A_2(\chi,\theta), A_3(\chi,\theta)\big)$, *where the components* $A_i(\chi,\theta)$, $i = 1,\ldots,3$ *are given by*

$$A_1(\chi,\theta) := d^*\theta + i\,\mathrm{Im}(\psi^* \otimes \chi)$$
$$A_2(\chi,\theta) := D_A^\xi \chi - \frac{1}{2}\rho_\xi(\theta)\psi$$
$$A_3(\chi,\theta) := -\mathrm{sym}\,\mathrm{Re}(\nabla_A^W \psi^* \otimes \chi) + \frac{1}{2}L_{\mathrm{Re}(\psi^* \otimes \chi)} g_\xi$$
$$+ \frac{1}{2}d^*\mathrm{Re}(\psi^* \otimes \chi) g_\xi - \frac{1}{2}(F_A^+, \theta) g_\xi - 2(F_A^-)^* \otimes \theta.$$

5.4. The obstruction to transversality

By (5.4) we know that $\ker(D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)})^* = \ker(D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}})^*$; hence the equations for the kernel of $(D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)})^*$ read:

$$(5.7\text{a}) \qquad\qquad d^*\theta + i\,\mathrm{Im}(\psi^* \otimes \chi) = 0$$

$$(5.7\text{b}) \qquad\qquad D_A^\xi \chi - \frac{1}{2}\rho_\xi(\theta)\psi = 0$$

$$
(5.7\text{c}) \qquad
\begin{aligned}
&- \mathrm{sym}\,\mathrm{Re}(\nabla_A^W \psi^* \otimes \chi) + \frac{1}{2}L_{\mathrm{Re}(\psi^*\otimes\chi)}g_\xi + \frac{1}{2}d^*\,\mathrm{Re}(\psi^* \otimes \chi)g_\xi \\
&\qquad - \frac{1}{2}(F_A^+,\theta)g_\xi - 2(F_A^-)^* \otimes \theta = 0
\end{aligned}
$$

where (A, ψ, ξ) satisfies $\mathbb{F}_{\mathrm{Met}(M)}(A, \psi, \xi) = 0$, with $\psi \neq 0$. Equations (5.7) can be slightly simplified.

LEMMA 5.10. — *If (χ, θ) is a solution of equations (5.7), then $(F_A^+, \theta) = 0$ and $\mathrm{div}(\psi^* \otimes \chi) = 0$.*

Proof. — Consider the equations (5.7). Applying the operator d^* to the first equation we get $d^*\,\mathrm{Im}(\psi^*\otimes\chi) = -\mathrm{div}\,\mathrm{Im}(\psi^*\otimes\chi) = 0$; hence $\mathrm{div}(\psi^* \otimes \chi) = \mathrm{div}\,\mathrm{Re}(\psi^*\otimes\chi)$. Recall the following identity[9]: if φ is a positive spinor, and ζ is a negative one, then

$$(5.8) \qquad\qquad 2\,\mathrm{div}(\varphi^* \otimes \zeta) = \langle D_A\varphi, \zeta \rangle - \langle \varphi, D_A\zeta \rangle.$$

We now take the trace in the third equation, remembering that, for any vector field X, we have $\mathrm{tr}\,L_X g_\xi = 2\,\mathrm{div}\,X$. We get:

$$- \mathrm{tr}\,\mathrm{sym}\,\mathrm{Re}(\nabla_A^W \psi^* \otimes \chi) + \frac{1}{2}\,\mathrm{tr}\,L_{\mathrm{Re}(\psi^*\otimes\chi)}g_\xi + 2d^*\,\mathrm{Re}(\psi^*\otimes\chi) - 2(F_A^+,\theta) = 0,$$

or, equivalently, since $\mathrm{div}(\psi^* \otimes \chi)$ is real,

$$(5.9) \qquad\qquad \mathrm{div}(\psi^* \otimes \chi) + 2(F_A^+,\theta) = 0,$$

since $\mathrm{tr}\,L_{\mathrm{Re}(\psi^*\otimes\chi)}g_\xi = 2\,\mathrm{div}\,\mathrm{Re}(\psi^* \otimes \chi)$, and, by a simple computation taking a orthonormal frame, $\mathrm{tr}\,\mathrm{sym}\,\mathrm{Re}(\nabla_A^W \psi^*\otimes\chi) = 1/2\,\mathrm{Re}\langle D_A\psi, \chi \rangle = 0$. Now, taking the scalar product with ψ in the second equation we get: $\langle \psi, D_A\chi \rangle - 1/2\,\langle \psi, \rho_\xi(\theta)\psi \rangle = 0$, which becomes, using (5.6) and (5.8):

$$(5.10) \qquad\qquad \mathrm{div}(\psi^* \otimes \chi) + (F_A^+,\theta) = 0.$$

Combining (5.9) and (5.10) we get the result. $\qquad\square$

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[9] One can easily prove the equality establishing it first at the symbol level; then, showing pointwise the equality at the zero-th order terms taking an adapted orthonormal frame, that is, a local orthonormal frame e^i such that $\nabla e^i(p) = 0$ at the point p.

PROPOSITION 5.11. — *The obstruction to the transversality of the universal Seiberg-Witten functional* $(\mathbb{F}_{\mathrm{Met}(M)})^{2,r}_p$ *at the solution*[10] (A, ψ, ξ) *of the universal Seiberg-Witten equations is given by nontrivial solutions* $(\theta, \chi) \in iA^2_+(M) \oplus \Gamma(W_-)$ *to the following equations:*

(5.11a) $$d^*\theta + i\,\mathrm{Im}(\psi^* \otimes \chi) = 0$$

(5.11b) $$D^\xi_A \chi - \frac{1}{2}\rho_\xi(\theta)\psi = 0$$

(5.11c) $$-\mathrm{sym}\,\mathrm{Re}(\nabla^W_A \psi^* \otimes \chi) + \frac{1}{2}L_{\mathrm{Re}(\psi^* \otimes \chi)}g_\xi - 2(F^-_A)^* \otimes \theta = 0$$

(5.11d) $$(\theta, F^+_A) = 0$$

(5.11e) $$\mathrm{div}(\psi^* \otimes \chi) = 0$$

Proof. — By (5.3) the cokernel of the differential $D_{(A,\psi,\xi)}(\mathbb{F}_{\mathrm{Met}(M)})^{2,r}_p$ coincides with the kernel of the formal adjoint of the differential $D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)}$ on sections of class C^∞. The equations (5.7) of the kernel of $(D_{(A,\psi,\xi)}\mathbb{F}_{\mathrm{Met}(M)})^*$ are now equivalent, by lemma 5.10, to equations (5.11). □

Remark 5.12. — We discuss now the gaps in the proof of the transversality with generic metrics by Eichhorn and Friedrich. The two authors (in [5, Proposition 6.4] and Friedrich alone in [8, page 141]) try to prove directly that the differential $D_{(A,\psi,g_\xi)}\tilde{\mathbb{F}}$ of the perturbed Seiberg-Witten functional is surjective. A first source of unclearness is that they never give a precise expression of the variation of the Dirac operator, which we have seen as being a fundamental difficulty in the question; in particular no mention is made about the term $-\rho_\xi \circ s^* \circ \nabla^W_A \psi$. The authors take into account variations of the metric which are orthogonal to the orbits of the action the diffeomorphism group $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$: this condition is precisely expressed by $\mathrm{div}\,s = 0$. They now remark that the variation of the second equation involves just the traceless part of the tensor s_0: as a consequence, they now claim that they can deal with conformal perturbations separately from volume preserving ones. Thanks to this uncorrect argument, as we will see, they get to the two separate conditions, reading, in our notations:

$$\left\langle \frac{d}{dg}(*_g)(s_0)F_A, \theta \right\rangle = 0\,, \quad \langle \rho(df)\psi, \chi \rangle = 0$$

which are to be satisfied by an element (χ, θ) in the cokernel of $D_{(A,\psi,g)}\tilde{\mathbb{F}}$, for all $s_0 \in \mathrm{sym}_0(TM, g)$ and for all $f \in C^\infty(M, \mathbb{R})$ such that $\mathrm{div}\,s_0 = df$.

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[10] Here we consider A, ψ, ξ of class C^∞.

The result would follow from them (remark that they correspond[11], taking formal adjoints, to two separate equations: $(F_A^-)^* \otimes \theta = 0$; $d^* \operatorname{Re}(\psi^* \otimes \chi) = 0$). This argument is not correct for the following two reasons. Firstly, the term $-\rho \circ s^* \circ \nabla_A^W \psi$ depends on the full tensor s and not just on his trace; secondly the variation of the second equation does not involve just volume preserving perturbations, since conformal perturbations come to play a role in the identification $i\Lambda_+^2 T^* M \simeq i\mathfrak{su}(W_+)$ via ρ_ξ.

6. Transversality over Kähler monopoles

6.1. Kähler monopoles

We now consider the transversality problem on Kähler surfaces. Let (M, J) be a compact connected 4-manifold with an integrable complex structure J. We will indicate with $H_J(M)$ the space of hermitian metrics with respect to the complex structure J; it is a splitting Fréchet submanifold of the manifold of riemannian metrics $\operatorname{Met}(M)$. If $g \in H_J(M)$, we indicate with $\omega_g := g(J(-), -)$ the $(1,1)$-form associated to g. Suppose now that (M, J) is of Kähler type. Let $K_J(M)$ be the set of Kähler metrics for the complex structure J, that is, $K_J(M) := \{g \in H_J(M) \mid d\omega_g = 0\}$. A Kähler surface is by definition a 4-manifold with a $U(2)$-reduction of the structural group of the tangent bundle $P_{U(2)} \hookrightarrow P_{GL(4)}$ admitting a torsion free $U(2)$-connection. The natural morphism $i \colon U(2) \hookrightarrow SO(4) \times U(1)$ lifts to a morphism $j \colon U(2) \hookrightarrow \operatorname{Spin}^c(4)$ so that $\nu \circ j = i$. The *canonical* Spin^c-*structure* ξ_0 on a Kähler manifold M is then given by the μ-equivariant map

$$\xi_0 \colon Q_{\operatorname{Spin}^c(4)} := P_{U(2)} \times_j \operatorname{Spin}^c(4) \longrightarrow P_{GL_+(4)}$$

induced by the morphism j. Remark that $Q_{SO(4)} \simeq P_{U(2)} \times_{U(2)} SO(4)$ and that $Q_{U(1)} \simeq P_{U(2)} \times_{\det} U(1)$. As a consequence the spinor bundle is: $W := \Lambda^{0,*} T^* M$, with $W_+ \simeq \Lambda^{0,\text{even}} T^* M$, $W_- \simeq \Lambda^{0,1} T^* M$. The fundamental line bundle L is isomorphic to the anticanonical bundle $\det W_+ \simeq K_M^*$ and the fundamental class c is $c_1(M)$. The Clifford multiplication of the structure ξ_0 is given by

$$\begin{aligned}\rho_{\xi_0} \colon T^* M &\longrightarrow \operatorname{End}(\Lambda^{0,\text{even}} T^* M, \Lambda^{0,1} T^* M) \\ x &\longmapsto \sqrt{2}[x^{0,1} \wedge (\cdot) - x^{0,1} \lrcorner (\cdot)].\end{aligned}$$

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[11] without taking into account the condition $\operatorname{div} s_0 = df$

Any other Spin^c-structure ξ_N is obtained, up to isomorphism, from the canonical one by twisting the spinor bundle by a line bundle N in the topological Picard group $\mathrm{Pic}_{\mathrm{top}}(M)$ of M; the resulting bundle of spinors is $W = \Lambda^{0,*}T^*M \otimes N$, the determinant line bundle is twisted by $N^{\otimes 2}$, $L = K_M^* \otimes N^{\otimes 2}$, and the fundamental class changes as $c = c_1(M) + 2c_1(N)$. The Clifford multiplication for ξ_N is $\rho_{\xi_0} \otimes \mathrm{id}_N$. In the notations of section 2.5, let $\Xi_{H_J(M)}$ be the *hermitian* Spin^c-structures of class c and fixed type, that is, the Spin^c-structures of class c in Ξ projecting onto J-hermitian metrics, and let $\Xi_{K_J(M)}$ be the *kählerian* ones (those projecting onto Kähler metrics). We will call the parametrized Seiberg-Witten moduli space $\mathcal{M}_{H_J(M)}$ and $\mathcal{M}_{K_J(M)}$ the moduli spaces of *hermitian* and *kählerian monopoles*, respectively.

To express Seiberg-Witten equations on a Kähler surface (M, g, J) for the Spin^c-structure ξ_N, we fix the Chern connection A_{K_M} on K_M and make the change of variables $\mathcal{A}_{U(1)}(N) \simeq \mathcal{A}_{U(1)}(L)$ given by $A \longmapsto A_{K_M}^* \otimes A^{\otimes 2}$. The Dirac operator for this Spin^c-structure and for $A \in \mathcal{A}_{U(1)}(N)$ is $D_A = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*)$. The Seiberg-Witten equations on a compact Kähler surface for a spinor $(\alpha, \beta) \in A^{0,0}(N) \oplus A^{0,2}(N)$ and for a $U(1)$-connection A on N read:

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0$$

$$F_A^{0,2} = \frac{\bar{\alpha}\beta}{2}$$

$$2F_A^{1,1} - F_{K_M} = i\frac{|\alpha|^2 - |\beta|^2}{4}\omega_g$$

where we split the second equation according the splitting of self dual 2-forms in $\Lambda_+^2 T^*M \otimes \mathbb{C} \simeq \Lambda^{2,0}T^*M \oplus \Lambda^{0,2}T^*M \oplus \mathbb{C}\omega_g$. It is well known that if $\deg(L) < 0$ then (A, α, β) is a solution of the Seiberg-Witten equations if and only if $\bar{\partial}_A$ is a holomorphic structure for N, α is a non zero holomorphic section of $(N, \bar{\partial}_A)$ and $\beta = 0$. Analogously, if $\deg(L) > 0$, we have a solution whenever $\alpha = 0$, $\bar{\partial}_{A^* \otimes A_{K_M}}$ is a holomorphic structure of $N^* \otimes K_M$ and $\sharp\beta$ is a non zero holomorphic section of $(N^* \otimes K_M, \bar{\partial}_{A^* \otimes A_{K_M}})$, where $\sharp$ denotes here the complex Hodge star operator; the involution $\jmath : (A, \alpha, \beta) \longmapsto (A^* \otimes A_{K_M}, \sharp\beta, \sharp\alpha)$ exchanges solutions of Seiberg-Witten equations for the Spin^c structure ξ_N and solutions for the Spin^c-structure $\xi_{N^* \otimes K_M}$. Moreover (A, α, β) is a reducible solution if and only if $\deg(L) = 0$ and A is self-dual. As a consequence $K_J(M) \cap \mathrm{Met}_{c-\mathrm{good}}(M) = \{g \in K_J(M) \mid [\omega_g] \cup c \neq 0\}$. Therefore:

$$\mathcal{M}_{K_J(M)}^* = \mathcal{M}_{K_J(M)} \cap \mathcal{M}_{\mathrm{Met}(M)}^* = \mathcal{M}_{K_J(M)} \cap \mathcal{M}_{\mathrm{Met}(M)}^{**} = \mathcal{M}_{K_J(M)}^{**}.$$

In this section we will prove the following theorem:

THEOREM 6.1. — *The parametrized moduli space of hermitian Seiberg-Witten monopoles* $(\mathcal{M}_{H_J(M)})_p^{2,r}$, *and hence the universal moduli space* $(\mathcal{M}_{\mathrm{Met}(M)})_p^{2,r}$, *is smooth at irreducible kählerian monopoles* $\mathcal{M}_{K_J(M)}^*$.

By remarks 5.3, 5.5, 5.6, we can paraphrase this theorem as:

THEOREM 6.2. — *Let* (M, g, J) *a Kähler surface. Let* N *a hermitian line bundle on* M *such that* $2\deg(N) - \deg(K_M) \neq 0$. *Consider the* Spin^c-*structure* ξ_N, *obtained by twisting the canonical one with the hermitian line bundle* N. *For a generic hermitian metric* $h \in H_J(M)$ *in a small open neighbourhood of* g *and for any* Spin^c-*structure* ξ' *of fundamental class* $c(\xi_N) = c_1(M) + 2c_1(N)$, *compatible with* h, *the Seiberg-Witten moduli space* $\mathcal{M}_{\xi'}^{SW}$ *is smooth of the expected dimension. The statement holds as well for a generic riemannian metric* $h \in \mathrm{Met}(M)$ *in a small open neighbourhood of* g.

In the next subsection we make use of the complex structure J to split the symmetric endomorphisms $\mathrm{sym}(TM, g)$ of TM with respect to a J-hermitian metric g in hermitian and anti-hermitian ones. In subsection 6.3 we write down equations (5.11) in the Kähler context and in subsection 6.4 we will finally prove theorem 6.1.

6.2. A decomposition for symmetric 2-tensors

The endomorphisms $\mathrm{End}(TM)$ of the tangent bundle decompose, thanks to the complex structure J, in J-linear and J-antilinear ones: $\mathrm{End}(TM) \simeq \mathrm{End}(TM, J) \oplus \overline{\mathrm{End}(TM, J)}$. Consider now a metric g, *hermitian* with respect to J. The previous decomposition of $\mathrm{End}(TM)$ induces a decomposition of the symmetric endomorphism $\mathrm{sym}(TM, g)$ of TM, with respect to g, in hermitian and anti-hermitian ones:

$$\mathrm{sym}(TM, g) \simeq \mathfrak{u}(TM, J) \oplus \mathfrak{sp}(TM, J), \tag{6.1}$$

where $\mathfrak{u}(TM, J) = \mathrm{sym}(TM, g) \cap \mathrm{End}(TM, J)$ and[12] $\mathfrak{sp}(TM, J) = \mathrm{sym}(TM, g) \cap \overline{\mathrm{End}(TM, J)}$. Analogously, symmetric 2-tensors in $S^2 T^* M$ can be decomposed in the direct sum $S^2 T^* M \simeq S^{1,1} T^* M \oplus S_{AH}^2 T^* M$ of hermitian

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[12] The notation $\mathfrak{sp}(TM, J)$, suggested by the referee, is motivated by the fact that $\mathfrak{sp}(TM, J)$ coincides with the tangent space at the identity of the space of positive symmetric automorphisms of the tangent bundle (with respect to the metric g) preserving the symplectic form ω_g.

2-tensors $S^{1,1}T^*M = \{s \in S^2T^*M \mid s(JX, JY) = s(X,Y) \quad \forall\, X, Y \in TM\}$ and antihermitian ones: $S^2_{AH}T^*M = \{s \in S^2T^*M \mid s(JX, JY) = -s(X,Y) \;\; \forall X, Y \in TM\}$. The decompositions for $\mathrm{sym}(TM, g)$ and S^2T^*M identify one to the other once we identify tangent and cotangent bundle by means of the metric g.

Consider now the complexified tangent bundle $TM \otimes \mathbb{C}$. The complex symmetric 2-tensors $S^2(T^*M \otimes \mathbb{C})$ split, according to the decomposition $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ as

$$S^2(T^*M \otimes \mathbb{C}) = S^{2,0}T^*M \oplus S^{0,2}T^*M \oplus S^{1,1}_{\mathbb{C}}T^*M$$

where we indicate $S^2(\Lambda^{1,0}T^*M)$ with $S^{2,0}T^*M$, $S^2(\Lambda^{0,1}T^*M)$ with $S^{0,2}T^*M$ and with $S^{1,1}_{\mathbb{C}}T^*M$ the subbundle of $\Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M \oplus \Lambda^{0,1}T^*M \otimes \Lambda^{1,0}T^*M$ invariant by the transposition of factors τ in the tensor product; in these notations the hermitian 2-tensors $S^{1,1}T^*M$ introduced above coincide with the subspace of real tensors in $S^{1,1}_{\mathbb{C}}T^*M$, that is, tensors invariant by conjugation. Let now $s \in S^2T^*M$, extended by $\mathbb{C}$-linearity to the element $s_{\mathbb{C}} \in S^2(T^*M \otimes \mathbb{C})$; according to the above decomposition, $s_{\mathbb{C}}$ can be written as $s_{\mathbb{C}} = s_{2,0} + s_{0,2} + s_{1,1}$, with $s_{2,0} = \overline{s_{0,2}}$ and $\overline{s_{1,1}} = s_{1,1}$. It is clear that $s \in S^{1,1}T^*M$ if and only if $s_{0,2} = 0$; in this case $s_{1,1}$ defines an hermitian form on $T^{1,0}M$; hence $S^{1,1}T^*M \simeq \mathrm{Herm}(T^{1,0}M)$. On the other hand, $s \in S^2_{AH}T^*M$ if and only if $s_{1,1} = 0$; in this case $s_{2,0}$ and $s_{0,2}$ define quadratic forms on $T^{1,0}M$ and $T^{0,1}M$, respectively, one conjugated of the other. Hence $S^2_{AH}T^*M \simeq S^{2,0}T^*M \simeq S^{0,2}T^*M$.

Using the complexified metric $g_{\mathbb{C}}$ to identify $T^*M \otimes \mathbb{C}$ with $TM \otimes \mathbb{C}$, the previous considerations can be stated for $\mathbb{C}$-linear extensions of symmetric endomorphisms $f \in \mathrm{sym}(TM, g)$ to $f_{\mathbb{C}} \in \mathrm{End}(TM \otimes \mathbb{C})$. An endomorphism $f \in \mathrm{End}(TM)$ extends by $\mathbb{C}$-linearity to an endomorphism $f_{\mathbb{C}} \in \mathrm{End}(TM \otimes \mathbb{C})$ such that $f_{\mathbb{C}}(\bar{z}) = \overline{f_{\mathbb{C}}(z)}$ for all $z \in TM \otimes \mathbb{C}$. According to the decomposition $TM \otimes \mathbb{C} \simeq T^{1,0}M \oplus T^{0,1}M$ this extension can be written as:

$$(6.2) \qquad f = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}$$

The endomorphism f is then J-linear if and only if $b = 0$, J-antilinear if and only if $a = 0$. Moreover, f is symmetric with respect to g if and only if $(Z, W) \longmapsto g(a(Z), \bar{W})$ is an hermitian form on $T^{1,0}M$ and $(Z, W) \longmapsto g(b(Z), W)$ is a complex quadratic form on $T^{1,0}M$. Hence we can identify $\mathfrak{u}(TM, J) \simeq \mathrm{Herm}(T^{1,0}M)$; $\mathfrak{sp}(TM, J) \simeq S^{2,0}T^*M$. Analogously, using $\bar{a}$ and $\bar{b}$ we get identifications $\mathfrak{u}(TM, J) \simeq \mathrm{Herm}(T^{0,1}M)$; $\mathfrak{sp}(TM, J) \simeq S^{0,2}T^*M$.

Remark 6.3. — The space of hermitian 2-tensors $S^{1,1}T^*M$ is isomorphic to the space of real (1,1)-forms $\Lambda_{\mathbb{R}}^{1,1}T^*M$ via the isomorphism $s \longmapsto s(\cdot, J(\cdot))$. In local coordinates, if $a \in S^{1,1}T^*M \simeq \text{Herm}(T^{1,0}M)$ is given by $a = \sum_{i,j} a_{i\bar{j}} dz_i \otimes d\bar{z}_j$, with $a_{i\bar{j}}$ an hermitian matrix, the associated real (1,1)-form is given by $-2i \sum_{ij} a_{i\bar{j}} dz_i \wedge d\bar{z}_j$.

Remark 6.4. — Let $u \in \Lambda^{1,0}T^*M \otimes \Lambda^{0,1}T^*M$. Let $\sigma(u)$ the (1,1)-form in $\Lambda^{1,1}T^*M$ obtained by the projection of u on $\Lambda^2(T^*M \otimes \mathbb{C})$. Denote moreover with $\text{herm}\, u$ the hermitian part of the sesquilinear form on $T^{1,0}M$ defined by u: $\text{herm}\, u = 1/2(u + \overline{\tau(u)})$. Then $\text{sym}\,\text{Re}\, u \in \mathfrak{u}(TM, J)$ and coincides with $1/2\,\text{herm}\, u = 1/4\,(u + \overline{\tau(u)})$ as hermitian form on $T^{1,0}M$. By the previous remark, the associated real (1,1)-form to $\text{sym}\,\text{Re}\, u$ is $-i/2\,(\sigma(u) - \overline{\sigma(u)})$. Let now $v \in \Lambda^{0,1}T^*M \otimes \Lambda^{0,1}T^*M$. Then $\text{sym}\,\text{Re}\, v \in \mathfrak{sp}(TM, J)$ and coincides with $1/2\,\text{sym}\, v \in S^{0,2}T^*M$, in the identification $\mathfrak{sp}(TM, J) \simeq S^{0,2}T^*M$.

We need now to take into account the decomposition (6.1) in the isometry $\delta_- : \text{sym}_0(TM, g) \longrightarrow \text{Hom}(\Lambda_-^2 T^*M, \Lambda_+^2 T^*M)$. We identify $\Lambda_-^2 T^*M$ with $\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1}$, that is, with the real (1,1)-forms orthogonal to the Kahler form ω_g, and $\Lambda_+^2 T^*M$ with $\Lambda^{0,2}T^*M \oplus \mathbb{R}\omega_g$. If $f \in \text{sym}(TM, g)$, let $a(f) \in \text{End}(T^{1,0}M)$ and $b(f) \in \text{Hom}(T^{1,0}M, T^{0,1}M)$ be the components of the extension of f to $TM \otimes \mathbb{C}$ seen in (6.2). Set $\mathfrak{u}_0(TM, J) = \mathfrak{u}(TM, J) \cap \text{sym}_0(TM, g)$. With this notations we have:

LEMMA 6.5. — *For all $f \in \mathfrak{u}_0(TM, J)$ then $\delta_-(f)\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1} \subseteq \mathbb{R}\omega_g$. Therefore the isometry $\delta_- : \text{sym}_0(TM, g) \longrightarrow \text{Hom}(\Lambda_-^2 T^*M, \Lambda_+^2 T^*M)$ splits, according to the decomposition (6.1), as :*

$$\mathfrak{u}_0(TM, J) \oplus \mathfrak{sp}(TM, J) \longrightarrow \text{Hom}(\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1}, \Lambda^{0,2}T^*M) \oplus \text{Hom}(\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1}, \mathbb{R}\omega_g)$$
$$(s, t) \longmapsto (\delta_-(\bar{b}(t)),\ \delta_-(s)).$$

Proof. — Let $(s, t) \in \mathfrak{u}_0(TM, J) \oplus \mathfrak{sp}(TM, J)$. The derivation $i(s^*)$ induced by an element $s \in \mathfrak{u}_0(TM, J)$ preserves the spaces $\Lambda^{1,1}T^*M$, $\Lambda^{2,0}T^*M$ and $\Lambda^{0,2}T^*M$, because s is J-linear. Therefore, for such s, $\delta_-(s)\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1} \subseteq \Lambda^{1,1}T^*M$, but by definition $\delta_-(s)\Lambda_-^2 T^*M \subseteq \Lambda_+^2 T^*M$; as a result $\delta_-(s)\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1} \subseteq \mathbb{R}\omega_g$. Any $t \in \mathfrak{sp}(TM, J)$ is J-antilinear, hence its extension to $TM \otimes \mathbb{C}$ exchanges $T^{1,0}M$ and $T^{0,1}M$; consequently $i(t^*)\Lambda^{1,1}T^*M \subseteq \Lambda^{0,2}T^*M \oplus \Lambda^{2,0}T^*M$. We can write $t^* = b(t) + \bar{b}(t)^*$, with $b(t)^* : \Lambda^{0,1}T^*M \longrightarrow \Lambda^{1,0}T^*M$, and $\bar{b}(t)^* : \Lambda^{1,0}T^*M \longrightarrow \Lambda^{0,1}T^*M$. Therefore $i(b(t)^*)\Lambda^{1,1}T^*M \subseteq \Lambda^{2,0}T^*M$ and $i(\bar{b}(t)^*)\Lambda^{1,1}T^*M \subseteq \Lambda^{0,2}T^*M$. Therefore in the splitting

$$\text{Hom}(\Lambda_-^2 T^*M, \Lambda_+^2 T^*M) \simeq \text{Hom}(\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1}, \Lambda^{0,2}T^*M) \oplus \text{Hom}(\Lambda_{\omega_g^\perp, \mathbb{R}}^{1,1}, \mathbb{R}\omega_g)$$

the element (s, t) acts as $\delta_-(\bar{b}(t)) \oplus \delta_-(s)$. $\square$

6.3. The obstruction to the transversality on a Kähler monopole

As discussed in subsection 5.1, in order to prove theorem 6.1 we need to prove that the intrinsic differential $D_x \Psi_{H_J(M)}$ of the section $\Psi_{H_J(M)}$: $(\mathfrak{B}^*_{H_J(M)})^{2,r}_p \longrightarrow \mathfrak{E}_{H_J(M)}$ is surjective at an irreducible kählerian monopole $x = ([A, \psi], \xi)$. We can suppose that $\xi = \xi_N$ for a certain $N \in \mathrm{Pic}_{\mathrm{top}}(M)$; moreover, because of the involution $\jmath$, it is not at all restrictive to take x a kählerian monopole with negative degree. By subsection 5.3 and in particular remarks 5.7, 5.8, the obstruction to the surjectivity of $D_x \Psi_{H_J(M)}$ is given by the kernel of the operator

$$(6.3) \qquad \left(D_{(A,\psi,g_\xi)} \tilde{\mathbb{F}}_{H_J(M)}\right)^* \simeq \left(\mathrm{id}_{T_{(A,\psi)}\mathcal{C}} \oplus P_{T_{g_\xi} H_J(M)}\right) \circ \left(D_{(A,\psi,g_\xi)} \tilde{\mathbb{F}}\right)^*$$

where $P_{T_{g_\xi} H_J(M)}$ is the orthogonal projection $T_{g_\xi} \mathrm{Met}(M) \longrightarrow T_{g_\xi} H_J(M)$ onto the tangent space of hermitian metrics. Since, given the Kähler metric g_ξ, we can parametrize hermitian metrics with symmetric positive hermitian automorphisms $U^+(TM, J) = \mathrm{Sym}^+(TM, g_\xi) \cap \mathrm{End}(TM, J)$ with respect to the metric g_ξ, the tangent space to hermitian metrics is given by:

$$T_{g_\xi} H_J(M) \simeq T_{\mathrm{id}} U^+(TM, J) \simeq \mathfrak{u}(TM, J).$$

The form of the operator (6.3) implies that, to find the obstruction we want, we have to consider equations (5.11), with equation (5.11c) projected onto the component in $\mathfrak{u}(TM, J)$, according to the decomposition (6.1).

We are now going to write down the kernel equations (5.11) on the Kähler monopole of negative degree $x = ([A, \psi], \xi_N)$, where $\psi = (\alpha, 0) \in A^{0,0}(N) \oplus A^{0,2}(N)$. For brevity's sake in the sequel we will indicate the Kähler metric g_{ξ_N} just with g. In what follows we will make the following identifications:

a) we will identify imaginary 1-forms in $iA^1(M)$ with $(0, 1)$-forms in $A^{0,1}(M)$ via the isomorphism $A^{0,1}(M) \simeq iA^1(M)$ sending $\sigma \longmapsto \sigma - \bar{\sigma}$;

b) the imaginary selfdual 2-forms $iA^2_+(M)$ will be identified with forms in $A^0(M, i\mathbb{R})\omega_g \oplus A^{0,2}(M)$, since we can always write $\theta \in iA^2_+(M)$ as $\theta = \lambda\omega_g + \mu - \bar{\mu}$ for $\lambda \in A^0(M, i\mathbb{R})$, $\mu \in A^{0,2}(M)$. We can therefore express the isomorphism $iA^2_+(M) \simeq i\mathfrak{su}(W_+)$ as (*cf.* [14]):

$$\lambda\omega_g + \mu \longmapsto 2 \begin{pmatrix} \lambda & \mu_\lrcorner(-) \\ \mu \wedge (-) & -\lambda \end{pmatrix},$$

where the matrix is written according to the decomposition $W_+ \simeq \Lambda^{0,0}T^*M \otimes N \oplus \Lambda^{0,2}T^*M \otimes N$.

Remark 6.6. — Remark that the 1-form $\varphi^* \otimes \zeta$, where $\varphi \in A^{0,0}(N)$ and $\zeta \in A^{0,1}(N)$, is given by $1/\sqrt{2}\,\bar{\varphi}\zeta \in A^{0,1}(M)$.

We postpone the study of equation (5.11c) to the next paragraph. In the above identifications equations (5.11a), (5.11b), (5.11d), (5.11e) take place in the spaces $A^{0,1}(M)$, $A^{0,0}(N)$, $A^{0,2}(N)$, $A^0(M, i\mathbb{R})$, $A^0(M)$, respectively, and become easily

$$(6.4a) \qquad \bar{\partial}^*\mu + \partial^*(\lambda\omega_g) + \frac{1}{2\sqrt{2}}\bar{\alpha}\chi = 0$$

$$(6.4b) \qquad \sqrt{2}\bar{\partial}_A^*\chi - \lambda\alpha = 0$$

$$(6.4c) \qquad \sqrt{2}\bar{\partial}_A\chi - \mu\alpha = 0$$

$$(6.4d) \qquad \lambda|\alpha|^2 = 0$$

$$(6.4e) \qquad \bar{\partial}^*(\bar{\alpha}\chi) = 0$$

Since α is a nonzero holomorphic section of $(N, \bar{\partial}_A)$, we get from the fourth equation that $\lambda = 0$ on the dense open set $M \setminus Z(\alpha)$ and hence everywhere. The last equation is, consequently, dependent from the first. Hence the equations are equivalent to:

$$(6.5a) \qquad 2\sqrt{2}\bar{\partial}^*\mu + \bar{\alpha}\chi = 0$$

$$(6.5b) \qquad \bar{\partial}_A^*\chi = 0$$

$$(6.5c) \qquad \sqrt{2}\bar{\partial}_A\chi - \mu\alpha = 0$$

$$(6.5d) \qquad \lambda = 0.$$

Contribution of the metric: hermitian perturbations

We have now to write equation (5.11c) on the Kähler monopole $([A, (\alpha, 0)], \xi_N)$, according to the identifications made. We will use the decomposition (6.1) and the identifications $\mathfrak{u}(TM, J) \simeq \mathrm{Herm}(T^{0,1}M)$ and $\mathfrak{sp}(TM, J) \simeq S^{0,2}T^*M$ provided by subsection 6.2. Moreover we will identify $TM \otimes \mathbb{C}$ with $T^*M \otimes \mathbb{C}$ by means of the complexified metric $g_\mathbb{C}$ (the $\mathbb{C}$-linear extension of g to $T^*M \otimes \mathbb{C}$); it identifies $T^{1,0}M$ with $\Lambda^{0,1}T^*M$ and $T^{0,1}M$ with $\Lambda^{1,0}T^*M$.

6.4. Proof of theorem 6.1

We have to prove the surjectivity of the differential $D_{(A,\psi,g)}\tilde{\mathbb{F}}_{H_J(M)}$ on a Kählerian monopole $([A,\psi],\xi) = ([A,(\alpha,0)],\xi_N)$. The obstruction to the transversality is given by a nontrivial solution to the equations for $\ker(D_{(A,\psi,g)}\tilde{\mathbb{F}}_{H_J(M)})^*$

(6.7a) $$2\sqrt{2}\bar{\partial}^*\mu + \bar{\alpha}\chi = 0$$

(6.7b) $$\bar{\partial}_A^*\chi = 0$$

(6.7c) $$\sqrt{2}\bar{\partial}_A\chi - \mu\alpha = 0$$

(6.7d) $$\lambda = 0$$

(6.7e) $$\partial(\bar{\alpha}\chi) - \bar{\partial}(\alpha\bar{\chi}) = 0$$

taking place, in the identifications we made, in $A^{0,1}(M)$, $A^0(N)$, $A^{0,2}(N)$, $A^0(M,i\mathbb{R})$, $A^{1,1}(M)$, respectively. Denote with $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ the laplacians for the operators ∂ and $\bar{\partial}$, respectively. The system of partial differential equations (6.7) does not have any nontrivial solution: in order to see this, we apply the operator ∂^* to the last equation, obtaining:

$$\Delta_\partial(\bar{\alpha}\chi) - \partial^*\bar{\partial}(\alpha\bar{\chi}) = 0.$$

Using the Kähler identity $\partial^*\bar{\partial} + \bar{\partial}\partial^* = 0$, we get that $\partial^*\bar{\partial}(\alpha\bar{\chi}) = \bar{\partial}\partial^*(\alpha\bar{\chi}) = 0$, since we already know that $\bar{\partial}^*(\bar{\alpha}\chi) = 0$. We are left with $\Delta_\partial(\bar{\alpha}\chi) = 0$, that is, $\bar{\alpha}\chi$ is Δ_∂-harmonic. Hence it is $\Delta_{\bar{\partial}}$-harmonic and $\bar{\partial}(\bar{\alpha}\chi) = 0$. Applying now the $\bar{\partial}$ operator in the first equation we get $\Delta_{\bar{\partial}}(\mu) = 0$, which implies $\bar{\partial}^*\mu = 0$. Hence $\bar{\alpha}\chi = 0$ and $\chi = 0$. From the third equation we get $\mu = 0$. Therefore there are no nonzero solution to the kernel equations on a Kählerian monopole.

BIBLIOGRAPHY

[1] D. BENNEQUIN, "Monopôles de Seiberg-Witten et conjecture de Thom (d'après Kronheimer, Mrowka et Witten)", *Astérisque* (1997), no. 241, p. Exp. No. 807, 3, 59-96, Séminaire Bourbaki, Vol. 1995/96.

[2] A. L. BESSE, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987, xii+510 pages.

[3] J.-P. BOURGUIGNON & P. GAUDUCHON, "Spineurs, opérateurs de Dirac et variations de métriques", *Comm. Math. Phys.* **144** (1992), no. 3, p. 581-599.

[4] S. K. DONALDSON & P. B. KRONHEIMER, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1990, Oxford Science Publications, x+440 pages.

[5] J. EICHHORN & T. FRIEDRICH, "Seiberg-Witten theory", in *Symplectic singularities and geometry of gauge fields (Warsaw, 1995)*, Banach Center Publ., vol. 39, Polish Acad. Sci., Warsaw, 1997, p. 231-267.

[6] D. S. FREED & D. GROISSER, "The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group", *Michigan Math. J.* **36** (1989), no. 3, p. 323-344.

[7] D. S. FREED & K. K. UHLENBECK, *Instantons and four-manifolds*, second ed., Mathematical Sciences Research Institute Publications, vol. 1, Springer-Verlag, New York, 1991, xxii+194 pages.

[8] T. FRIEDRICH, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Providence, RI, 2000, Translated from the 1997 German original by Andreas Nestke, xvi+195 pages.

[9] O. GIL-MEDRANO & P. W. MICHOR, "The Riemannian manifold of all Riemannian metrics", *Quart. J. Math. Oxford Ser. (2)* **42** (1991), no. 166, p. 183-202.

[10] S. KOBAYASHI & K. NOMIZU, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996, Reprint of the 1963 original, A Wiley-Interscience Publication, xii+329 pages.

[11] A. KRIEGL & P. W. MICHOR, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997, x+618 pages.

[12] H. B. LAWSON, JR. & M.-L. MICHELSOHN, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989, xii+427 pages.

[13] S. MAIER, "Generic metrics and connections on Spin- and Spin^c-manifolds", *Comm. Math. Phys.* **188** (1997), no. 2, p. 407-437.

[14] J. W. MORGAN, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes, vol. 44, Princeton University Press, Princeton, NJ, 1996, viii+128 pages.

[15] C. OKONEK & A. TELEMAN, "Seiberg-Witten invariants for manifolds with $b_+ = 1$, and the universal wall crossing formula", *Internat. J. Math.* **7** (1996), no. 6, p. 811-832.

[16] A. SCORPAN, *The wild world of 4-manifolds*, American Mathematical Society, Providence, RI, 2005, xx+609 pages.

[17] N. SEIBERG & E. WITTEN, "Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory", *Nuclear Phys. B* **426** (1994), no. 1, p. 19-52.

[18] ———, "Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD", *Nuclear Phys. B* **431** (1994), no. 3, p. 484-550.

[19] S. SMALE, "An infinite dimensional version of Sard's theorem", *Amer. J. Math.* **87** (1965), p. 861-866.

[20] A. TELEMAN, "Introduction à la théorie de Jauge", 2005, Cours de D.E.A. http://www.cmi.univ-mrs.fr/~teleman/documents/cours-sw.pdf.

[21] E. WITTEN, "Monopoles and four manifolds", *Math. Res. Lett.* **3** (1994), no. 7, p. 654-675.

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