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ORDINARY p-ADIC EISENSTEIN SERIES AND p-ADIC L-FUNCTIONS FOR UNITARY GROUPS

by Ming-Lun HSIEH

Abstract. — The purpose of this work is to carry out the first step in our four-step program towards the main conjecture for $\mathrm{GL}_2 \times \mathcal{K}^\times$ by the method of Eisenstein congruence on $GU(3,1)$, where $\mathcal{K}$ is an imaginary quadratic field. We construct a p-adic family of ordinary Eisenstein series on the group of unitary similitudes $GU(3,1)$ with the *optimal* constant term which is basically the product of the Kubota-Leopodlt p-adic L-function and a p-adic L-function for $\mathrm{GL}_2 \times \mathcal{K}^\times$. This construction also provides a different point of view of p-adic L-functions of $\mathrm{GL}_2 \times \mathcal{K}^\times$.

Résumé. — Le but de ce travail est d'accomplir le premier pas de notre programme vers la conjecture principale pour $\mathrm{GL}_2 \times \mathcal{K}^\times$, par la methode de congruences entre séries d'Eisenstein sur $GU(3,1)$, où $\mathcal{K}$ est d'un corps quadratique imaginaire. Nous construisons une famille p-adique de séries d'Eisenstein ordinaires sur le groupe de similitudes unitaires avec le terme constant optimal qui est essentiellement le produit de la fonction L p-adique de Kubota-Leopoldt et d'une fonction L p-adique pour $\mathrm{GL}_2 \times \mathcal{K}^\times$. Cette construction donne ainsi un nouveau point de vue sur la fonction L p-adique de $\mathrm{GL}_2 \times \mathcal{K}^\times$.

Introduction

Iwasawa main conjecture for totally real fields was proved by Wiles in one of his celebrated papers [29]. His proof, modeled upon Ribet's proof of the converse of Herbrand's theorem, relies on the study of Eisenstein congruence for Hilbert cusp forms. One of the key ingredients in Wiles' proof is to realize the Deligne-Ribet p-adic L-function as constant terms of a particular Hida family of Eisenstein series over totally real fields, and then construct congruence between this particular family of Eisenstein series and Hida families of Hilbert cusp forms modulo the Deligne-Ribet p-adic L-function.

Keywords: Eisenstein series on unitary groups, Iwasawa-Greenberg main conjectures.
Math. classification: 11F33, 11F70, 11R23.

The first generalization of Wiles' work to Iwasawa main conjecture for non-abelian motives is due to E. Urban. In [27] and [28], he established one-sided divisibility result on Iwasawa main conjecture for adjoint representations. In particular, in [28] he constructed a Hida family of Klingen-Eisenstein series on Gsp(4) such that the constant terms at all cusps are divisible by the p-adic L-function for adjoint representations, and he deduced Eisenstein congruence on Gsp(4) by proving the non-vanishing modulo p of this Eisenstein series.

In this article, we consider certain Iwasawa main conjecture for $\mathrm{GL}_2 \times \mathcal{K}^\times$, where $\mathcal{K}$ is an imaginary quadratic field. Following Ribet, Wiles and Urban, we propose a four-step program towards a one-sided divisibility result for this main conjecture. It turns out that we will need to consider Eisenstein congruence on certain unitary groups. Our main result is the construction of a particular Hida family of Eisenstein series on the unitary group $U(3,1)$ such that the constant terms (the image of Siegel boundary operator) at all cusps are divisible by a product of Kubota-Leopodlt p-adic L-function and a p-adic L-function for $\mathrm{GL}_2 \times \mathcal{K}^\times$.

Main conjecture for p-adic Galois representations

In [7], R. Greenberg reformulates the classical Iwasawa main conjecture in the context of p-adic Galois representations, and he proposes a more general main conjecture when the Galois representation satisfies the *Panchishkin* condition (for the definition, see [7, §3 and §4]).

We now describe the Galois representation under consideration. Let p be an odd rational number. Assume p is split in $\mathcal{K}$. Fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\iota_p = \iota\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ be their composition. Let $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are primes in $\mathcal{K}$ above p, where $\mathfrak{p}$ is the prime ideal induced by ι_p. Let $G_\mathcal{K} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K})$ and $\hat{\chi} : G_\mathcal{K} \to \mathbb{C}_p^\times$ be a p-adic character. We shall regard $\hat{\chi}$ as a character of $\mathbb{A}_{\mathcal{K},f}^\times/\mathcal{K}^\times$ by the *geometrically* normalized reciprocity law. We assume $\hat{\chi}$ is locally algebraic, namely there exist two integers a and b such that $\hat{\chi}(z_p) = z_\mathfrak{p}^b z_{\overline{\mathfrak{p}}}^a$ for $z_p = (z_\mathfrak{p}, z_{\overline{\mathfrak{p}}})$ in the p-part of $\mathbb{A}_\mathcal{K}^\times$ with $z_\mathfrak{p}$ and $z_{\overline{\mathfrak{p}}}$ sufficiently close to 1 p-adically. In addition, the map $\chi : \mathbb{A}_\mathcal{K}^\times/\mathcal{K}^\times \to \mathbb{C}^\times$ defined by

$$\chi(z) = \iota_\infty \iota_p^{-1}(\hat{\chi}(z_f) z_\mathfrak{p}^{-b} z_{\overline{\mathfrak{p}}}^{-a}) z_\infty^b \overline{z}_\infty^a$$

is a well-defined Hecke character. The Hecke character χ is called the complex avatar of $\hat{\chi}$ whereas $\hat{\chi}$ is called the p-adic avatar of χ. We will say χ

(or $\hat{\chi}$) has infinity type (b,a). Let E be an elliptic curve over $\mathbb{Q}$ and let $\rho_E : G_{\mathbb{Q}} \to \mathrm{GL}(H^1_{ét}(E,\mathbb{Q}_p))$ be the p-adic Galois representation associated with E. Let $\rho_E \otimes \hat{\chi} := \rho_E|_{G_{\mathcal{K}}} \otimes \hat{\chi}$ be the p-adic Galois representation of $G_{\mathcal{K}}$. We will consider the main conjecture for $\rho_E \otimes \hat{\chi}$ when $\hat{\chi}$ varies in a p-adic family of a fixed infinity type.

The formulation of this main conjecture depends on the infinity type of χ. If the infinity type of $\hat{\chi}$ is $(1,1)$ and E has good ordinary reduction at p, the main conjecture for $\rho_E \otimes \hat{\chi}$ is the classical Iwasawa main conjecture for E over $\mathcal{K}$, which is the assertion for the equality between the p-adic L-function of $L_{\mathcal{K}}(0, E\otimes\chi)$ divided by the period of E and the characteristic power series associated to the Selmer group of $E\otimes\chi$. In this case, the conjecture has been settled down by Bertolini and Darmon in [1] if χ is further assumed to be anticyclotomic, and the general three-variable main conjecture for $\mathrm{GL}_2 \times \mathcal{K}^\times$ is studied by Skinner and Urban in [25].

We are interested in the main conjecture for $\rho_E \otimes \hat{\chi}$ when the infinity type of $\hat{\chi}$ is $(k,0)$, $k>1$. This main conjecture is quite different from the one considered in [1] and [25]. On the analytic side, the p-adic L-function is related to the complex L-value $L_{\mathcal{K}}(0, E\otimes\chi)$ divided by the CM-period attached to χ and $\mathcal{K}$ instead of the period of E. On the algebraic side, the Selmer group also has different local conditions at p. Let us make precise the L-value and the Selmer group under consideration.

L-functions and Selmer groups

L-functions

Thanks to the works of Wiles, Taylor and many other people, one can associate to E a weight two holomorphic cuspidal eigenform $f = \sum_{n=1}^{\infty} a_n(E)q^n$. We choose a prime-to-$p$ integral ideal $\mathfrak{c}$ of $\mathbb{Q}$ such that $\mathfrak{c}$ is divisible by the conductor of E. We let S be the set of prime factors of $\mathfrak{c}$ and put

$$L_{\mathcal{K}}^{S\cup\{p\}}(s, \rho_E\otimes\chi) = \sum_{(\mathfrak{n},p\mathfrak{c})=1} a_n(E)\chi(\mathfrak{n})\|\mathfrak{n}\|^{-s},\ (n) = \mathfrak{n}\cap\mathbb{Z},$$

where $\mathfrak{n}$ runs over integral ideals of $\mathcal{K}$.

Throughout we assume E has ordinary good reduction at p. In order to have nice arithmetic and p-adic properties of the L-value $L_{\mathcal{K}}(0, \rho_E\otimes\chi)$ when χ varies in a p-adic family, we need to take normalization as follows. Let α_1 and α_2 be two roots of the equation $x^2 - a_p(E)x + p$. We define δ_1

and δ_2 to be the unramified characters on $\mathbb{Q}_p^\times$ such that $\delta_i(p) = \alpha_i$, $i = 1, 2$. For a character $\mu : \mathbb{Q}_p^\times \to \mathbb{C}$, we let $L(s, \mu)$ be the local L-factor associated to μ. Following [4, Lemma 7], we define the modified Euler factor $E_p(0, \mu)$ by

$$E_p(0, \mu) = \frac{L(0, \mu)}{\epsilon(0, \mu)L(1, \mu^{-1})},$$

where $\epsilon(s, \mu)$ is Tate's ϵ-factor of μ. Now we make the following definition of the normalized L-value.

Definition 0.1. —

$$L_{\mathcal{K}}^{\mathrm{alg},S}(0, \rho_E \otimes \chi) := \frac{\Gamma(k)\Gamma(k-1)}{(2\pi i)^{2k-1}} \cdot E_p(0, \chi_{\mathfrak{p}}\delta_1)E_p(0, \chi_{\mathfrak{p}}\delta_2) \times \frac{(2\pi i)^{2k}}{\Omega_{\mathcal{K}}^{2k}} \cdot L_{\mathcal{K}}^{S\cup\{p\}}(0, \rho_E \otimes \chi),$$

where $\Omega_{\mathcal{K}}$ is the CM period associated to $\mathcal{K}$.

We remark that $L_{\mathcal{K}}^{\mathrm{alg},S}(0, \rho_E \otimes \chi)$ actually lies in $\overline{\mathbb{Q}}$ by works of Shimura and that such normalization wass suggested by J. Coates' recipe ([4, Conj. A p. 168]).

To define p-adic L-functions, we need to introduce more notations. Let $\mathcal{K}_\infty$ be the maximal $\mathbb{Z}_p^2$-extension and $\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) \cong \mathbb{Z}_p^2$. Let ψ be a *branch* character. Namely ψ is a character of $G_{\mathcal{K}}$ of finite order such that $\mathcal{K}_\psi = \mathcal{K}^{\mathrm{Ker}\,\psi}$ is linearly disjoint from $\mathcal{K}_\infty$. Let $\Delta = \mathrm{Gal}(\mathcal{K}_\psi/\mathcal{K})$. Let $\mathcal{K}'_\infty = \mathcal{K}_\psi\mathcal{K}_\infty$ and $\mathfrak{G} = \mathrm{Gal}(\mathcal{K}'_\infty/\mathcal{K})$. Then $\mathfrak{G} = \Delta \times \Gamma$. Put $\mathcal{O} = \mathbb{Z}_p[\mathrm{Im}\psi]$, the ring of values of ψ. Let Λ be $\mathcal{O}[\![\Gamma]\!]$ a two-variable Iwasawa algebra. Let $\varPsi$ be the Λ-valued character of $G_{\mathcal{K}}$ defined by

$$\varPsi : G_{\mathcal{K}} \longrightarrow \Lambda^\times$$
$$g \longrightarrow \psi(g)g|_{\mathcal{K}_\infty}.$$

Let $\mathscr{W}_0$ be the set of locally algebraic p-adic characters of Γ of infinity type $(k, 0), k \geqslant 2$. For $x \in \mathscr{W}_0$ and $\lambda \in \Lambda$, put $\lambda(x) = x \circ \lambda \in \mathbb{C}_p$. We let $\varPsi_x$ denote the complex avatar of $x \circ \varPsi$. Then $\mathscr{W}_0$ can be regarded as a Zariski-dense subset in $\mathrm{Spec}\,\Lambda(\mathbb{C}_p)$. Moreover $\mathscr{W}_0$ is an *ample* subset of the set of critical specializations for $\rho_E \otimes \varPsi$ in the sense of Greenberg ([7, §4]).

We further assume $\mathfrak{c}$ is divisible by the prime-to-p conductor of ψ. Let $L_p^S(\rho_E \otimes \varPsi)$ denote the unique element in Λ such that

$$L_p^S(\rho_E \otimes \varPsi)(x) = \iota_p\iota_\infty^{-1}(L_{\mathcal{K}}^{\mathrm{alg},S}(0, \rho_E \otimes \varPsi_x)),\ x \in \mathscr{W}_0.$$

The existence of $L_p^S(\rho_E \otimes \varPsi)$ is known. In fact it is a special case of p-adic Rankin products constructed by Hida (*cf.* [11]). We call $L_p^S(\rho_E \otimes \varPsi)$ the (non-primitive) p-adic L-function for $\mathrm{GL}_2 \times \mathcal{K}^\times$ associated to E and ψ.

Selmer groups

Let $\mathcal{K}_S$ be the maximal $S \cup \{p\}$-ramified extension of $\mathcal{K}$. Let $\Lambda^* = \mathrm{Hom}_{cont}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontragin dual of Λ. Let $T = H^1_{ét}(E, \mathbb{Z}_p)$. We consider a discrete a Λ-module $T \otimes \Lambda^*$ of corank two equipped with Galois action by $\rho_E \otimes \Psi$. According to the Panchishkin condition of $\rho_E \otimes \Psi_x$ for $x \in \mathscr{W}_0$, we define the non-primitive Λ-adic Selmer group to $\rho_E \otimes \Psi$ by

$$Sel^S_{\mathcal{K}}(\rho_E \otimes \Psi) := \ker\{H^1(\mathcal{K}_S, T \otimes \Lambda^*) \to H^1(I_{\overline{\mathfrak{p}}}, T_{\overline{\mathfrak{p}}} \otimes \Lambda^*)\}.$$

If S is empty, we shall drop S in the above definition and simply write $Sel_{\mathcal{K}}(\rho_E \otimes \Psi)$. It is well-known that $Sel^S_{\mathcal{K}}(\rho_E \otimes \Psi)$ is a discrete and cofinitely generated Λ-module. Let $\mathrm{ht}_1(\Lambda)$ denote the set of height one prime ideals of Λ. For a cofinitely generated Λ-module $\mathcal{S}$, let $\mathcal{S}^*$ be the Pontragin dual of $\mathcal{S}$. For $P \in \mathrm{ht}_1(\Lambda)$, put

$$\ell_P(\mathcal{S}) = \mathrm{length}_{\Lambda_P}(\mathcal{S}^* \otimes_\Lambda \Lambda_P).$$

The main conjecture: connection between L-functions and Selmer groups

The following main conjecture for $\rho_E \otimes \Psi$ is formulated in [7, Conj. 4.1].

CONJECTURE 0.2 (The main Conjecture). —

(1) *$Sel_{\mathcal{K}}(\rho_E \otimes \Psi)$ is cotorsion over Λ.*

(2) *For any $P \in \mathrm{ht}_1(\Lambda)$,*

$$\mathrm{ord}_P(L_p(\rho_E \otimes \Psi)) = \ell_P(Sel_{\mathcal{K}}(\rho_E \otimes \Psi)).$$

Remark 0.3. — The formulation of the main conjectures depends not only on the Galois representation itself but also the choice of an ample subset of critical specializations (or a choice of local conditions at p). The main conjectures studied in [1] and [25] are for the same Galois representation $\rho_E \otimes \Psi$ with the critical specialization at locally algebraic characters of Γ with infinity type $(1, 1)$. The p-adic L-function and Selmer group considered in this article are quite different from theirs.

We shall consider the *dual* version of Conjecture 0.2 which has the advantage of including non-primitive p-adic L-functions and Selmer groups. Let $\varepsilon : G_{\mathcal{K}} \to \mathbb{Z}_p{}^\times$ be the cyclotomic character and let c be the complex conjugation. We define the Λ-valued Galois character $\Psi^\perp$ by

$$\Psi^\perp(g) = \Psi^{-1}\varepsilon^2(cgc).$$

Conjecture 0.4. — *For every* $P \in \mathrm{ht}_1(\Lambda)$,

$$\mathrm{ord}_P(L_p^S(\rho_E \otimes \Psi)) = \ell_P(Sel_{\mathcal{K}}^S(\rho_E \otimes \Psi^\perp)).$$

We propose the following weaker conjecture.

Conjecture 0.5 (Lower bound of Selmer groups). — *For every* $P \in \mathrm{ht}_1(\Lambda)$,

$$\mathrm{ord}_P(L_p^S(\rho_E \otimes \Psi)) \leqslant \ell_P(Sel_{\mathcal{K}}^S(\rho_E \otimes \Psi^\perp)).$$

We wish to show Conjecture 0.5 by the method of Eisenstein congruences.

Eisenstein congruences

The method of congruences among modular forms provides a general strategy to construct a nontrivial Galois extension. Roughly speaking, a Galois representation over a PID which is generically irreducible and residually reducible should give rise to a nontrivial Galois extension. This basic idea was due to K. Ribet in [20].

In our project, we consider the congruences between Eisenstein series and cusp forms on the unitary group $U(3,1)$. The application of Eisenstein congruences to various main conjectures has been a success in [18], [29], [27] and [25]. Following the exposition in [24], we describe our project as follows.

(1) Construct an ordinary Λ-adic Eisenstein series $\mathcal{E}^{ord}$ on the unitary group $U(3,1)$ with the constant terms at *all* cusps divisible by $L_p^S(-2,\Psi_+) \cdot L_p^S(\rho_E \otimes \Psi)$, where $\Psi_+ = \Psi \circ V$ is the composition of Ψ and the *Verschiebung* map $V : G_{\mathbb{Q}}^{ab} \to G_{\mathcal{K}}^{ab}$, and $L_p^S(-2,\Psi_+)$ is the imprimitive p-adic L-function with the following specialization property:

$$L_p^S(-2,\Psi_+)(x) = \iota_p(L^{\mathrm{alg},S}(-2,\Psi_+(x))),\ x \in \mathscr{W}_0.$$

(2) Show that $\mathcal{E}^{ord} \not\equiv 0 \bmod \mathfrak{m}_\Lambda$, where $\mathfrak{m}_\Lambda$ is the maximal ideal of Λ.

(3) For $P \in \mathrm{ht}_1(\Lambda)$, we let $r_e = \mathrm{ord}_P(L_p^S(-2,\Psi_+))$ and $r_s = \mathrm{ord}_P(L_p^S(\rho_E \otimes \Psi))$. Using Hida theory of ordinary p-adic modular forms, we can show that there is a p-adic ordinary cusp form F such that

$$\mathcal{E}^{ord} \equiv F \bmod P^{r_e+r_s}.$$

(4) We use the Galois representations associated to cuspidal automorphic representations of $U(3,1)$ together with the above congruences to construct elements in the Selmer group $Sel_{\mathcal{K}}^S(\rho_E \otimes \Psi^\perp)$. The key

here is to perform the lattice construction ([27] and [25]). Assuming the existence of the Galois representation attached to F, in *favorable* cases for instance $r_e = 0$, we can construct a Λ-lattice $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ in $K^4 = (\mathrm{Frac}\,\Lambda)^4$ with $\dim \mathscr{L}_1 \otimes K = \dim \mathscr{L}_3 \otimes K = 1$ and a representation ρ_F.

$$\rho_F : G_{\mathcal{K}} \to \mathrm{GL}(\mathscr{L}), \quad \rho_F(\sigma) = \begin{pmatrix} \Psi^{-1}(c\sigma c) & * & * \\ * & \rho_E \varepsilon^{-1}(\sigma) & c_\sigma \\ 0 & 0 & \Psi\varepsilon^{-3}(\sigma) \end{pmatrix} \bmod P^{r_s},$$

which is unramified away from $S \cup \{p\}$. Moreover under the assumption that $r_e = 0$ we can show

$$\rho_F^1(\sigma) = \begin{pmatrix} \rho_E \varepsilon^{-1}(\sigma) & c_\sigma \\ 0 & \Psi\varepsilon^{-3}(\sigma) \end{pmatrix} \bmod P^{r_s}$$

is indeed a non-split representation of $G_{\mathcal{K}}$, but $\rho_F^1|_{D_\mathfrak{p}}$ is split. Then it can be shown that the cocycle c_σ provides a Λ_P-submodule of length at least r_s in $Sel_{\mathcal{K}}^S(\rho_E \otimes \Psi^{\perp})_P$.

The main result

Our main result in this paper is to fulfill Step (1) of this program when E has good ordinary reduction at p, namely $p \not| a_p(E)$. Let $\pi = \pi_f$ be the automorphic representation of GL_2 generated by f. We further assume that the conductor N_E of E has a decomposition $N_E = N_+N_-$ such that $(N_+, N_-) = 1$ and N_- is a product of an *odd* number of distinct primes. This is equivalent to saying that $\pi = JL(\pi^B)$ is in the image of Jacquet-Langlands correspondence of the definite quaternion algebra B ramified exactly at infinity and N_-. Let $\mathbf{f}$ be a $p-$primitive new form in π^B and choose ξ an auxiliary Dirichlet character of $G_{\mathcal{K}}$ such that $\xi_+ = 1$. Thus $\mathbf{F} := \mathbf{f} \otimes \xi$ can be regarded as a modular form of $GU(2)$ (See §4.1). Let $D_{\mathcal{K}}$ be the discriminant of $\mathcal{K}$. We further assume $\mathfrak{c}$ is divisible by the conductors of ψ and ξ, $D_{\mathcal{K}}$ and N_E.

Our main result is as follows.

Theorem 0.6. — *Under the above assumptions there exists a measure $d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}$ with values in the space of p-adic modular forms on $GU(3,1)$ such that for any $\hat{\chi}$ a p-adic algebraic character of $\mathfrak{G}$ with infinity type $(k, 0)$, $k \geqslant 4$, then we have*

$$\int_{\mathfrak{G}} \hat{\chi} d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}} = \widehat{E}^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c}),$$

where $\widehat{E}^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ *is an ordinary* p*-adic Eisenstein series. Let* $d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}(x)$ *be the* p*-adic measure induced by the constant term of* $\widehat{E}^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ *at a cusp* x*, which is a measure with value in the space of* p*-adic modular forms of* $GU(2)$*. Then there exists a well-chosen cusp* x_0 *and a* p*-adic period* $\Omega_p \in (\overline{\mathbb{Z}}_p)^\times$ *depending on* x_0 *such that*

$$\frac{1}{\Omega_p^k} \cdot \int_{\mathfrak{G}} \hat{\chi} d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}(x_0) = |D_{\mathcal{K}}|_{\mathbb{R}}^{\frac{3}{2}} (p-1) \cdot (\hat{\chi}\xi^{-1})_{\overline{\mathfrak{p}}}(-\det\theta \cdot p) \times L^{alg,S}(-2, \chi_+) \cdot L^{alg,S}_{\mathcal{K}}(0, \rho_E \otimes \chi) \cdot \mathbf{F}.$$

Moreover $\mathscr{C}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$*, the ideal generated by the constant term at* all *cusps, is*

$$\mathscr{C}(\chi \mid \mathbf{f}, \xi, \mathfrak{c}) = L^{alg,S}(-2, \chi_+) \cdot L^{alg,S}_{\mathcal{K}}(0, \rho_E \otimes \chi).$$

Let us put a few words on the current progress about Step (2). To carry out Step (2), one usually needs

- an explicit computation of Fourier-Jacobi coefficients of an Eisenstein series, and
- a non-vanishing result of L-values mod p that are related to the above one.

Bei Zhang in her thesis [30] has made a preliminary computation of Fourier-Jacobi coefficients of an Eisenstein series on our unitary group. Her computation does relate the Fourier-Jacobi coefficients of Eisenstein series to certain L-values for $\mathrm{GL}_2 \times \mathcal{K}^\times$, but the desired non-vanishing modulo p result of the corresponding L-values is not available yet. Nonetheless a wishing thinking (if we believe the main conjecture) is that since our Eisenstein series has the optimal constant term, Step (2) should hold as well. We hope to work out this problem in the future.

Sketch of the construction

The easiest way to construct an Eisenstein measure is to interpolate its Fourier coefficients as we have seen in the examples of classical p-adic Eisenstein series for GL_2. However it is difficult to do so for our non-quasi-split unitary group $GU(3,1)$, since modular forms for this group only have Fourier-Jacobi expansion, which is difficult to interpolate directly. To bypass this difficulty, our approach is as follows.

- Construct a *nice* p-adic Siegel-Eisenstein series on the quasi-split unitary group $GU(3,3)$.

- Apply the pull-back formula to obtain a p-adic Eisenstein series on $GU(3,1)$.
- Construct an ordinary p-adic Eisenstein series on $GU(3,1)$ by taking the ordinary projection.

To study congruences, we further require that this ordinary Eisenstein series have *optimal* constant terms.

Our construction of Siegel-Eisenstein series on $GU(3,3)$ is inspired by [15], [14] and [10]. In particular, Harris, Li and Skinner in [10] generalize Katz's construction in [15] to obtain a p-adic Siegel-Eisenstein series of several variables with which they can construct p-adic L-functions for general unitary groups. However the ordinary projection of the pull-back of their Eisenstein series is *zero*. A heuristic reason is that the Fourier coefficients of the Eisenstein in [10] are only supported in the matrices which are non-degenerate modulo p. Therefore, we need to modify their section to fit our purpose. The new ingredients in our paper are the choice of this modified section at p (3.15) and also the calculation of the ordinary projection of its local pull back section in §6.4. Here is the summary of this paper:

In §2, we review the theory of p-adic modular forms on unitary groups. This theory is due to Hida ([12] and [13]).

In §3, following the same spirit in [10], we construct the Siegel-Eisenstein series on $GU(3,3)$ and compute its Fourier coefficients explicitly.

In §4, we review the pull-back formula in [22], and in §5, we study the constant term of the pull back of our Siegel-Eisenstein series. It turns out that the constant term is simply the pull back section itself. We also calculate the local pull back section at places other that p in this section.

In §6, we calculate the ordinary projection of the pull back section at the place p by employing Jacquet's functor. We find that indeed it has correct modified Euler factors at p as suggested by J. Coates. The main result of this section is Prop. 6.8.

In §7, we give the explicit formula for the constant term (Theorem 7.3).

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1. Notation and definitions

1.1.

Throughout $\mathcal{K}$ is an imaginary quadratic field with the ring of integers $\mathscr{O}_\mathcal{K}$. Let $D_\mathcal{K}$ (resp. $\mathcal{D}_\mathcal{K}$) be the discriminant (resp. different) of $\mathcal{K}$.

Fix an odd rational prime p split in $\mathcal{K}$. Fix an embedding once and for all $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\iota_p = \iota\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ be their composition. Write $p\mathscr{O}_\mathcal{K} = \mathfrak{p}\overline{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal induced by ι_p.

1.2.

For a finite set $\square$ of rational primes, we define $\mathbb{Z}_{(\square)}$ by

$$\mathbb{Z}_{(\square)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b\mathbb{Z} + q\mathbb{Z} = \mathbb{Z} \text{ for all } q \in \square \right\}.$$

By definition, $\mathbb{Z}_{(\square)} = \mathbb{Q}$ if $\square$ is empty. Write $\mathbb{Z}_{(p)}$ for $\mathbb{Z}_{(\square)}$ if $\square = \{p\}$. Let $\mathbb{Z}_{(\square),+} = \left\{ a \in \mathbb{Z}_{(\square)} \mid a > 0 \right\}$ and $\mathcal{O} := \mathscr{O}_\mathcal{K} \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$.

1.3.

Denote by SET the category of sets and by $SCH_{/R}$ the category of R-schemes for a ring R. The complex conjugation c induces a natural involution on $R \otimes_\mathbb{Z} \mathcal{K}$ by $r \otimes x \mapsto r \otimes c(x)$. Define the $n \times n$ the Hermitian matrices $\mathcal{H}_n(R)$ over $R \otimes_\mathbb{Z} \mathcal{K}$ by

$$\mathcal{H}_n(R) = \{ g \in M_n(R \otimes_\mathbb{Z} \mathcal{K}) \mid g = g^* \},$$

where $g^* = c(g^t)$, ^{t} is the transpose of g.

1.4.

We write

$$\mathscr{O}_\mathcal{K} \otimes_\mathbb{Z} \mathscr{O}_\mathcal{K} = \mathscr{O}_\mathcal{K} e^+ \oplus \mathscr{O}_\mathcal{K} e^-,$$

where e^+ (resp. e^-) is the idempotent corresponding to the identity map $\Sigma : \mathcal{K} \to \mathcal{K}$ (resp. the complex conjugation $\Sigma^c = \Sigma \circ c : \mathcal{K} \to \mathcal{K}$). If M is an $\mathscr{O}_\mathcal{K}$-module, we define

$$M_\Sigma := e^+(M \otimes_\mathbb{Z} \mathbb{Z}_p) \text{ and } M_{\Sigma^c} = e^-(M \otimes_\mathbb{Z} \mathbb{Z}_p).$$

Let $\mathbb{C}(\Sigma)$ (resp. $\mathbb{Z}_p(\Sigma)$) be the $\mathscr{O}_\mathcal{K}$-module $\mathbb{C}$ (resp. $\mathbb{Z}_p$) on which $\mathscr{O}_\mathcal{K}$ acts via ι_∞ (resp. ι_p). Similarly let $\mathbb{C}(\Sigma^c)$ (resp. $\mathbb{Z}_p(\Sigma^c)$) be the $\mathscr{O}_\mathcal{K}$-module $\mathbb{C}$ (resp. $\mathbb{Z}_p$) on which $\mathscr{O}_\mathcal{K}$ acts through c.

1.5. Unitary groups

Let $r \geqslant s$ be two non-negative integers. Let (W, θ) be a skew-Hermitian space of dimension $r - s$ such that $\delta^{-1}\theta$ is positive definite, where $\delta = -\sqrt{-D_{\mathcal{K}}}$ and $D_{\mathcal{K}}$ is the discriminant of $\mathcal{K}$. We fix a $\mathcal{K}$-basis $\left\{w^i\right\}_{i=1}^{r-s}$ and regard θ as a $(r-s)\times(r-s)$ matrix according to this basis. We further assume $\iota_p(\det\theta) = \iota_p(\det\theta(w^i, w^j))$ is a p-adic unit. Let $V = \oplus_{i=1}^{s}\mathcal{K}y^i \oplus W \oplus_{i=1}^{s}\mathcal{K}x^i$ and $\theta_{r,s}$ be the skew-Hermitian form on V such that according to the basis $\left\{y^i, w^i, x^i\right\}$, we have

$$\theta_{r,s} = \begin{bmatrix} & & -\mathbf{1}_s \\ & \theta & \\ \mathbf{1}_s & & \end{bmatrix}.$$

Let $\langle\, ,\, \rangle_{r,s} : V \times V \to \mathbb{Q}$ be the alternating pairing defined by $\langle v, v'\rangle_{r,s} = \mathrm{Tr}_{\mathbb{Q}}^{\mathcal{K}}(\theta_{r,s}(v, v,'))$.

Let $G = GU(r, s)$ be the group of unitary similitudes associated to $(V, \theta_{r,s})$, i.e. for a $\mathbb{Q}$-algebra R,

$$G(R) = \{g \in \mathrm{End}_{\mathscr{O}_{\mathcal{K}}}(V \otimes R) \mid \theta_{r,s}(xg, yg) = \nu(g)\theta_{r,s}(x, y),\ x, y \in V \otimes R\}.$$

Let $GU(0, r-s)$ be the group of unitary similitudes attached to $(W, -\theta)$. The unitary group $U(V)$ is defined by

$$U(V)(R) := \{g \in G(R) \mid \nu(g) = 1\}.$$

1.6. Lattices and polarization

In what follows we make the specific choice of the lattice M in V and define the standard polarization Pol_p^0 of M_p. We let $X^\vee = \mathscr{O}_{\mathcal{K}}x^1 \oplus \cdots \mathscr{O}_{\mathcal{K}}x^s = \mathscr{O}_{\mathcal{K}}^s$ and $Y = \mathscr{O}_{\mathcal{K}}y^1 \oplus \cdots \mathscr{O}_{\mathcal{K}}y^s = \mathscr{O}_{\mathcal{K}}^s$ be the standard $\mathscr{O}_{\mathcal{K}}$-lattices in I^X and I^Y respectively. We choose an $\mathscr{O}_{\mathcal{K}}$-lattice L in W such that L is $\mathbb{Z}$-*maximal* with respect to the Hermitian form $\delta^{-1}\theta$. Let $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \sum_{i=1}^{r-s}(\mathscr{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)w^i = (\mathscr{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{r-s}$. Then we define the $\mathscr{O}_{\mathcal{K}}$-lattice M in V by

$$M := Y \oplus L \oplus X^\vee. \tag{1.1}$$

Let $M_p = M \otimes \mathbb{Z}_p$. A pair of sublattices $\mathrm{Pol}_p = \left\{N^{-1}, N^0\right\}$ of M_p is called an ordered polarization of M_p if N^{-1} and N^0 are maximal isotropic direct summands in M_p and they are dual to each other with respect to $\langle\, ,\, \rangle_{r,s}$. Moreover, we require that

$$\mathrm{rank}\, N_\Sigma^{-1} = \mathrm{rank}\, N_{\Sigma^c}^0 = r,\ \mathrm{rank}\, N_{\Sigma^c}^{-1} = \mathrm{rank}\, N_\Sigma^0 = s.$$

We endow M_p with the standard polarization as follows. Put

$$M^{-1} = Y_\Sigma \oplus L_\Sigma \oplus Y_{\Sigma^c} \text{ and } M^0 = X^\vee_{\Sigma^c} \oplus L_{\Sigma^c} \oplus X^\vee_\Sigma.$$

We call $\mathrm{Pol}^0_p = \left\{M^{-1}, M^0\right\}$ the *standard* polarization of M_p. We make the following identification according to the basis $\left\{y^i, w^i, x^i\right\}$,

$$\begin{aligned} M^0_\Sigma &= X^\vee_\Sigma = \mathbb{Z}_p(\Sigma)^s \\ M^{-1}_\Sigma &= Y_\Sigma \oplus L_\Sigma = \mathbb{Z}_p(\Sigma)^r \end{aligned} \quad \text{and} \quad \begin{aligned} M^{-1}_{\Sigma^c} &= Y_{\Sigma^c} = \mathbb{Z}_p(\Sigma^c)^s \\ M^0_{\Sigma^c} &= X^\vee_{\Sigma^c} \oplus L_{\Sigma^c} = \mathbb{Z}_p(\Sigma^c)^r. \end{aligned} \tag{1.2}$$

1.7.

Let n be a positive integer. Denote by $\mathbf{1}_n$ the identity matrix in GL_n. Denote by B_n the upper unipotent subgroup of GL_n, by T_n the diagonal matrices and by N_n the unipotent radical of B_n.

Since p splits in $\mathcal{K}$, $G(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{GL}_{r+s}(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ via the map $g \mapsto (g|_{V_\Sigma}, \nu(g))$. For $v \in \mathbf{h}$, we put

$$K^0_v = \{g \in G(\mathbb{Q}_v) \mid M_v g = M_v\} \text{ and } K^0 = \prod_{v\in\mathbf{h}} K^0_v.$$

For $g_p \in K^0_p \cong GL(M_\Sigma)$, we write $g_p = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ according to the decomposition $M_\Sigma = M^{-1}_\Sigma \oplus M^0_\Sigma$. For an open-compact subgroup K in K^0 with $K_p = K^0_p$, we put

$$\begin{aligned} K^n &= \left\{ g \in K^0 \mid g_p \equiv \begin{bmatrix} \mathbf{1}_r & * \\ 0 & \mathbf{1}_s \end{bmatrix} \bmod p^n \right\}, \\ K^n_1 &= \left\{ g \in K^0 \mid g_p \equiv N_{r+s}(\mathbb{Z}_p) \bmod p^n \right\}. \end{aligned}$$

2. Modular forms on unitary groups

2.1.

In this section, we give a brief account of the theory of p-adic modular forms on unitary groups. This theory is due to Katz for GL(2) [15] and to Hida for general reductive groups [13].

Definition 2.1 (S-quadruples). — *Let $\square$ be a finite set of rational primes. Let $U \subset K^0$ be an open-compact subgroup in $G(\mathbb{A}_f^{(\square)})$. Let S be a connected, locally noetherian $\mathcal{O}$-scheme and $\overline{s}$ be a geometric point. A* S-quadruple *of level $U^{(\square)}$ is a quardruple $\underline{A} = (A, \overline{\lambda}, \iota, \overline{\eta}^{(\square)})_S$ consisting of the following data:*

- *A is an abelian scheme of dimension $(r+s)d$ over S.*
- *λ is a prime-to-$\Box$ polarization of A over S and $\overline{\lambda}$ is the $\mathbb{Z}_{(\Box),+}$-orbit of λ. Namely*

$$\overline{\lambda} = \mathbb{Z}_{(\Box),+}\lambda := \left\{\lambda' \in \mathrm{Hom}(A, A^t) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)} \mid \lambda' = \lambda \circ a,\, a \in \mathbb{Z}_{(\Box),+}\right\}.$$

- *$\iota\colon \mathscr{O}_{\mathcal{K}} \hookrightarrow \mathrm{End}_S\, A \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$.*
- *$\overline{\eta}^{(\Box)} = U\eta^{(\Box)}$ is a $\pi_1(S, \overline{s})$-invariant U-orbit of isomorphisms of $\mathscr{O}_{\mathcal{K}}$-modules $\eta^{(\Box)}\colon M \otimes \widehat{\mathbb{Z}}^{(\Box)} \xrightarrow{\sim} T^{(\Box)}(A_{\overline{s}})$.*

Furthermore, the quadruple $(A, \overline{\lambda}, \iota, U\eta^{(\Box)})_S$ satisfies the following conditions (K1)-(K3):

(K1) *Let ^{t} denote the Rosati involution induced by λ on $\mathrm{End}_S\, A \otimes \mathbb{Z}_{(\Box)}$. Then $\iota(b)^t = \iota(c(b))$, $\forall\, b \in \mathscr{O}_{\mathcal{K}}$.*

(K2) *Let e^λ be the Weil pairing induced λ. We fix an isomorphism ζ : $\mathbb{A}_f \cong \mathbb{A}_f(1)$ once and for all, with which we regard e^λ as a skew-Hermitian form $e^\lambda : T^{(\Box)}(A_{\overline{s}}) \times T^{(\Box)}(A_{\overline{s}}) \to \mathcal{D}_{\mathcal{K}}^{-1} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(\Box)}$. Let e^η denote the skew-Hermitian form on $T^{(\Box)}(A)$ induced by $e^\eta(x, x') = \theta_{r,s}(\eta(x), \eta(x'))$. We require that*

$$e^\lambda = u \cdot e^\eta \text{ for some } u \in \mathbb{A}_f^{(\Box)}.$$

(K3) *The determinant condition:*

$$\det(X - \iota(b) | \,\mathrm{Lie}\, A) = (X - (\sigma c)(b))^r (X - \sigma(b))^s \in \mathcal{O}_S[X],\ \forall\, b \in \mathscr{O}_{\mathcal{K}}. \tag{2.1}$$

Define the fibered category $\mathfrak{C}_U^{(\Box)}$ over $SCH_{\mathcal{O}_{(\Box)}}$ as follows. Objects over S are S-quadruples. For $\underline{A} = (A, \overline{\lambda}, \iota, U\eta^{(\Box)})_S$ and $\underline{A'} = (A', \overline{\lambda'}, \iota', U(\eta')^{(\Box)})_S$, we define

$$\mathrm{Hom}_{\mathfrak{C}_U^{(\Box)}}(\underline{A}, \underline{A'}) = \left\{\phi \in \mathrm{Hom}_{\mathscr{O}_{\mathcal{K}}}(A, A') \mid \phi^*\overline{\lambda'} = \overline{\lambda},\, U\phi(\eta')^{(\Box)} = U\eta^{(\Box)}\right\}.$$

We say $\underline{A} \cong \underline{A'}$ if there exists an isomorphism in $\mathrm{Hom}_{\mathfrak{C}_U^{(\Box)}}(\underline{A}, \underline{A'})$.

2.2. Shimura varieties

2.2.1.

Let $\Box = \emptyset$ be the empty set and U be an open-compact subgroup in $G(\mathbb{A}_f)$. We define the functor $\mathfrak{C}_U : SCH_{/\mathcal{K}} \to SET$ by

$$\mathfrak{C}_U(S) = \left\{\underline{A} = (A, \overline{\lambda}, \iota, U\eta)_S \in \mathfrak{C}_U(T)\right\} / \cong .$$

By the theory of Shimura-Deligne, $\mathfrak{C}_U$ is represented by $S_G(U)_{/\mathcal{K}}$ which is a quasi-projective scheme over $\mathcal{K}$. We call $S_G(U)_{/\mathcal{K}}$ the Shimura variety associated to G.

2.2.2. Kottwitz model

Let $\square = \{p\}$. Let K be an open-compact subgroup such that $K_p = K_p^0$. we define functor $\mathfrak{E}^{(p)}_{K^{(p)}} : SCH_{/\mathcal{O}} \to SET$ by

$$\mathfrak{E}^{(p)}_{K^{(p)}}(S) = \left\{ \underline{A} = (A, \overline{\lambda}, \iota, \overline{\eta}^{(p)})_S \in \mathfrak{E}^{(p)}_{K^{(p)}} \right\} / \cong .$$

In [16], Kottwitz shows $\mathfrak{E}^{(p)}_{K^{(p)}}$ is representable by a quasi-projective scheme $S_G(K)_{/\mathcal{O}}$ over $\mathcal{O}$ if K is neat.

2.3. Igusa schemes

2.3.1.

For a quadruple $\underline{V} = (V, \theta_{r,s}, M, \mathrm{Pol}_p)$, where $(V, \theta_{r,s})$ is the skew-Hermitian space defined in §1.5, M is the $\mathscr{O}_\mathcal{K}$-lattice in (1.1) and $\mathrm{Pol}_p = \left\{N^{-1}, N^0\right\}$ is a polarization of M_p, we review the (open) Igusa schemes associated to $\underline{V}$) following the exposition in [10, 2.1].

Definition 2.2 (S-quintuples). — *Let n be a positive integer. We define the fibered category $\mathfrak{E}^{(p)}_{K,n,\mathrm{Pol}_p}$ whose objects over a base scheme S are S-quintuples $(\underline{A}, j)_S = (A, \overline{\lambda}, \iota, \overline{\eta}^{(p)}, j)_S$ of level K^n, where $\underline{A}_S \in \mathfrak{E}^{(p)}_{K^{(p)}}$ is a S-quadruple and*

$$j : \boldsymbol{\mu}_{p^n} \otimes_\mathbb{Z} N^0 \hookrightarrow A[p^n]$$

is a monomorphism as $\mathscr{O}_\mathcal{K}$-group schemes over S. We call j a level p^n structure of A. Morphisms between S-quintuples are

$$\mathrm{Hom}_{\mathfrak{E}^{(p)}_{K,n,\mathrm{Pol}_p}}((\underline{A}, j), (\underline{A'}, j')) = \left\{ \phi \in \mathrm{Hom}_{\mathfrak{E}^{(p)}_{K^{(p)}}}(\underline{A}, \underline{A'}) \mid \phi j = j' \right\}.$$

Define the functor $\mathfrak{E}^{(p)}_{K,n,\mathrm{Pol}_p} : SCH_{/\mathcal{O}} \to SET$ by

$$\mathfrak{E}^{(p)}_{K,n,\mathrm{Pol}_p}(S) = \left\{ (\underline{A}, j) = (A, \overline{\lambda}, \iota, \overline{\eta}^{(p)}, j)_S \in \mathfrak{E}^{(p)}_{K,n} \right\} / \cong .$$

Let $\underline{\mathcal{A}}$ be the universal quadruple of level $K^{(p)}$ over $S_G(K)$. Then equivalently, we have

$$\underline{I_G}(K^n) = \underline{\mathrm{Inj}}_{\mathscr{O}_\mathcal{K}}(\boldsymbol{\mu}_{p^n} \otimes_\mathbb{Z} M^0, \mathcal{A}).$$

It is known that $\mathfrak{E}^{(p)}_{K,n,\mathrm{Pol}_p}$ are relatively representable over $S_G(K)_{/\mathcal{O}}$ (*cf.* [10, Lemma(2.1.6.4)] and [8, Prop. 3.12]), and thus it is represented by a scheme. We denote by $I_G(K^n)_{/\mathcal{O}}$ the scheme that represents $\mathfrak{E}^{(p)}_{K,n,\mathrm{Pol}^0_p}$ for the standard polarization $\mathrm{Pol}^0_p = \left\{M^{-1}, M^0\right\}$ defined in (1.2).

2.3.2. p-adic one forms

Suppose p is nilpotent in R and $p^m R = 0$ for some $m \geqslant 1$. Let $(\underline{A}, j)$ be a R-quintuple of level K^n. Identity $M^0 = M_\Sigma^{-1} \oplus M_{\Sigma^c}^0$ with the basis in (1.2). Then if $n \geqslant m$, the level p^n structure j over R induces an trivialization of $\mathrm{Lie}\, A$:

$$j_*^+ : M_\Sigma^0 \otimes R \xrightarrow{\sim} e^+ \mathrm{Lie}\, A[p^n] = e^+ \mathrm{Lie}\, A\,;\ j_*^- : M_{\Sigma^c}^0 \otimes R \xrightarrow{\sim} e^- \mathrm{Lie}\, A[p^n] = e^- \mathrm{Lie}\, A.$$

Let $\underline{\omega}_A = \mathrm{Hom}(\mathrm{Lie}\, A, R)$ be the R-module of invariant one forms of A. Then via the identification in (1.2), j_* induces an isomorphism:
(2.2)

$$\boldsymbol{\omega}(j)^+ = \boldsymbol{\omega}(j^-) \colon \mathbb{Z}_p{}^r \otimes R \xrightarrow{\sim} e^+ \underline{\omega}_A\,;\ \boldsymbol{\omega}(j)^- = \boldsymbol{\omega}(j^+) \colon \mathbb{Z}_p{}^s \otimes R \xrightarrow{\sim} e^- \underline{\omega}_A.$$

2.3.3. Change of the polarization

It is clear that the notion of level p^n structures depends on the choice of the polarization of M_p. Choose $\gamma \in K_p^0$ such that $N^{-1} = M^{-1}\gamma$ and $N^0 = M^0\gamma$. Then we see that $j \mapsto \gamma j$ is an isomorphism from the level-p^n structures with respect to Pol_p^0 to those of Pol_p. Therefore the map $(\underline{A}, j) \to (\underline{A}, \gamma j)$ induces an isomorphism between $\mathfrak{C}^{(p)}_{K,n,\mathrm{Pol}_p^0}$ and $\mathfrak{C}^{(p)}_{K,n,\mathrm{Pol}_p}$.

2.4. Complex uniformization

2.4.1.

Let $G = GU(V)$ and $U \subset K^0$ be an open compact subgroup in $G(\mathbb{A}_f)$. We recall the description of the complex points $S_G(U)(\mathbb{C})$ following [23].

We begin with the Hermitian symmetric domain attached to unitary groups with signature (r, s). We treat two cases $r \geqslant s > 0$ and $rs = 0$ individually. If $r \geqslant s > 0$, we put

$$X_{r,s} = \{\tau = \begin{bmatrix} x \\ y \end{bmatrix} \mid x \in M_s(\mathbb{C}), y \in M_{(r-s)\times s}(\mathbb{C}),\, i(x^* - x) > -iy^*\theta^{-1}y\}.$$

For $\alpha \in G(\mathbb{R})$, we write

$$\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix}$$

according to the standard basis of V. The action of $\alpha \in G(\mathbb{R})^+$ on $X_{r,s}$ is defined by

$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by+c \\ gx+ey+f \end{bmatrix} \cdot (hx+ly+d)^{-1}. \tag{2.3}$$

If $rs = 0$, $X_{r,s}$ consists of a single point written $\mathbf{x}_0$ with the trivial action of G.

Then $X^+ = X_{r,s}$ is the Hermitian symmetric domain associated to G. Put

$$M_G(X^+, U) := G(\mathbb{Q})^+ \backslash X^+ \times G(\mathbb{A}_f)/U,$$

where $G(\mathbb{Q})^+ = \{g \in G(\mathbb{Q}) \mid \nu(g) > 0\}$. Then $M_G(X^+, U)$ is a complex manifold when U is neat. The group $G = GU(r,s)$ satisfies the Hasse principle ([13, 7.1.5, p. 319]). Hence we have

$$M_G(X^+, U) \overset{\sim}{\to} S_G(U)(\mathbb{C}). \tag{2.4}$$

2.4.2. Analytic construction of the universal abelian scheme over $\mathbb{C}$

Let $\mathcal{A}(V)_{/\mathbb{C}}$ be the universal quadruple of level U over $S_G(U)_{\mathbb{C}}$. After introducing some notations, we shall recall a construction of $\mathcal{A}(V)_{\mathbb{C}}$. Define $\mathbb{C}^{r,s}$ a vector space over $\mathbb{C}$ of dimension $r+s$ with $\mathcal{K}$-action by

$$\mathbb{C}^{r,s} = \mathbb{C}^s(\Sigma^c) \oplus \mathbb{C}^{r-s}(\Sigma^c) \oplus \mathbb{C}^s(\Sigma). \tag{2.5}$$

Here $\mathbb{C}^{r-s}$ and $\mathbb{C}^s$ are regarded as spaces of row vectors. According to the above decomposition (2.5), we define $c_{r,s} : \mathbb{C}^{r,s} \to \mathbb{C}^{r,s}$ by

$$(u_1, u_2, u_3)c_{r,s} = (\overline{u_1}, \overline{u_2}, u_3),$$

where $\overline{u}$ means the complex conjugation of $u \in \mathbb{C}$. We denote by $z_V^{(1)}(\Sigma), \cdots, z_V^{(r)}(\Sigma)$ the first r complex coordinates of $\mathbb{C}^{r,s}$, where $z_V^{(i)}(\Sigma) = (z_{V,\sigma}^{(i)})_{\sigma\in\Sigma}$. Similarly, we denote by $z_V^{(r+1)}(\Sigma^c), \cdots, z_V^{(r+s)}(\Sigma^c)$ the rest of s complex coordinates. If $s > 0$, we put

$$B(\tau) = \begin{bmatrix} x^* & y^* & x \\ 0 & \theta & y \\ \mathbf{1}_s & 0 & \mathbf{1}_s \end{bmatrix}. \in M_{r+s}(\mathbb{C}).$$

If $rs = 0$, we put $B(\mathbf{x}_0) = \theta$. Via the isomorphism $\mathcal{K} \otimes \mathbb{R} \overset{\iota_\infty}{\underset{\sim}{\to}} \mathbb{C}$, we regard $V_{\mathbb{R}}$ as a $\mathbb{C}$-space of row vectors according to the $\mathcal{K}$-basis $\{y^i, w^i, x^i\}$. For each $\tau \in X^+$, we define the map $p(\tau) : V_{\mathbb{R}} \to \mathbb{C}^{r,s}$ by $p(\tau)v = vB(\tau)c_{r,s}$. Then V acts on $(\tau, z) \in X^+ \times \mathbb{C}^{r,s}$ by

$$v \cdot (\tau, z) = (\tau, p(\tau)v + z).$$

We define a left action of G on V by

$$g * v := vg^{\vee} = vg^{-1}\nu(g) \tag{2.6}$$

and put

$$M_{[g]} = g * M = Mg^{\vee} \text{ and } M_{[g]}(\tau) = p(\tau)(M_{[g]}). \tag{2.7}$$

To each point $(\tau, g) \in X^{+}\times G(\mathbb{A}_f)$ we can attach a $\mathbb{C}$-quadruple $\mathcal{A}(V)_g(\tau) = (\mathcal{A}(V)_g(\tau), \overline{\langle\, ,\, \rangle}_{can}, [\cdot], U\eta_g)$ of level K^n defined by the following data

- Abelian variety: $\mathcal{A}(V)_g(\tau) = \mathbb{C}^{r,s}/M_{[g]}(\tau) \xrightarrow{\sim} V_{\mathbb{R}}/M_{[g]}$,
- Polarization: $\overline{\langle\, ,\, \rangle}_{can}$ is the $\mathbb{Q}_{+}$-orbit of the polarization induced by the Riemann form defined by the pull back of $\langle\, ,\, \rangle_{r,s}$ via $p(\tau)$,
- Endomorphism: $[\cdot] : \mathscr{O}_{\mathcal{K}} \to \operatorname{End} \mathcal{A}_g(\tau)\otimes_{\mathbb{Z}}\mathbb{Q}$ is the $\mathscr{O}_{\mathcal{K}}$-action induced by the action on V via $p(\tau)$,
- Prime-to-p level structure: $\eta_g^{(p)} : M \otimes \hat{\mathbb{Z}}^p \xrightarrow{\sim} M_{[g]} = H_1(\mathcal{A}_g(\tau), \hat{\mathbb{Z}}^p)$ is defined by

$$\eta_g^{(p)}(x) = g * x \text{ for } x \in M.$$

- Level structure at p: Taking a primitive p^n-th root $\zeta = e^{2\pi i/p^n}$, we have an isomorphism $\zeta : \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^n}$. Similarly as above we have

$$\begin{aligned} j_\zeta : \boldsymbol{\mu}_{p^n} \otimes M^0 \cong \mathbb{Z}/p^n\mathbb{Z} \otimes M^0 \hookrightarrow \mathcal{A}_g(\tau)[p^n] \\ = \mathbb{Z}/p^n\mathbb{Z} \otimes M_{[g]},\ j(x^0) = g * x^0 \text{ for } x^0 \in M^0. \end{aligned}$$

Let η_g be the full level structure $\eta_g^{(p)}\times\eta_p(j_\zeta)$. Thus the isomorphism in (2.4) can be described explicitly as follows.

$$\begin{aligned} &M_G(X^{+}, K^n) \xrightarrow{\sim} S_G(K^n)(\mathbb{C}) \\ &(\tau, g) \to \underline{\mathcal{A}}(V)_g(\tau) := \Big[(\mathcal{A}(V)_g(\tau), \overline{\langle\, ,\, \rangle}_{can}, [\cdot], K^n\eta_g)\Big]. \end{aligned} \tag{2.8}$$

Note that for $u \in K^0$, we have

$$[\tau, gu] = \Big[(\mathcal{A}_g(\tau), \overline{\langle\, ,\, \rangle}_{can}, [\cdot], Ku^{\vee}\eta^{(p)}, u_p^{\vee} j_\zeta)\Big].$$

We put

$$\begin{aligned} d\underline{z}_V(\Sigma) &= \left\{dz_V^{(1)}(\Sigma), \cdots, dz_V^{(r)}(\Sigma)\right\} \\ \text{and } d\underline{z}_V(\Sigma^c) &= \left\{dz_V^{(r+1)}(\Sigma^c), \cdots, dz_V^{(r+s)}(\Sigma^c)\right\}. \end{aligned} \tag{2.9}$$

Then $d\underline{z}_V := (d\underline{z}_V(\Sigma), d\underline{z}_V(\Sigma^c))$ form a basis of $\Omega_{\mathcal{A}_g(\tau)}$. We define a r-form $\boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma)$ and a s-form $\boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma^c)$ by

$$\boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma) = \bigwedge_{i=1}^{r} dz_V^{(i)}(\Sigma)\,;\, \boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma^c) = \bigwedge_{i=r+1}^{r+s} dz_V^{(i)}(\Sigma^c). \tag{2.10}$$

2.5. Igusa schemes associated to $U(V)$

In the later application of the pull-back formula, we have to consider the Igusa schemes attached to the unitary group $U(V)$ as well. Let $\mathbb{Z}^{ab}$ be the ring of integers of $\mathbb{Q}^{ab}$ and $\mathcal{O}^{ab}$ be the ring generated by $\mathcal{O}$ and $\mathbb{Z}^{ab}$. Let $Cl_{\mathbb{Q}}(K) = \mathbb{Q}\backslash\mathbb{A}_f^{\times}/\nu(K)$ and choose a set of representatives C_K of $\mathrm{Cl}_{\mathbb{Q}}^{+}(K)$ in $\mathbb{A}_f^p$. For $\mathbf{c} \in C_K$, we consider the functor $I_{U(V)}(K^n;\mathbf{c}) : SCH_{/\mathcal{O}^{ab}} \to SET$

$$I_{U(V)}(K^n;\mathbf{c})(S) = \left\{(A,\lambda,\iota,\eta^{(p)}K^{(p)},j)_S \mid (A,\overline{\lambda},\iota,\eta^{(p)}K^{(p)}) \in \mathfrak{C}_{K,n}^{(p)}\right\}/\cong,$$

where λ is a polarization in the class $\overline{\lambda}$ such that

$$e^{\lambda} = u \cdot e^{\eta},\ u \in \mathbf{c}\nu(K).$$

It is shown in [13, p. 136] below that the isomorphism class

$$[(A,\lambda,\iota,\eta^{(p)}K,j)_S]$$

is independent of the choice of λ in $\overline{\lambda}$. Pick $g_{\mathbf{c}} \in G(\mathbb{A}_f^p)$, $\nu(g_{\mathbf{c}}) = \mathbf{c}$ and let ${}^{\mathbf{c}}K^n = g_{\mathbf{c}}K^n g_{\mathbf{c}}^{-1} \cap U(V)(\mathbb{A}_f)$. Then over $\mathbb{C}$ we have an isomorphism

$$M_{U(V)}(X^+, {}^{\mathbf{c}}K) \stackrel{\sim}{\to} I_{U(V)}(K^n;\mathbf{c})_{/\mathbb{C}}.$$

As explained in [13, §4.2.1] for the Hilbert modular varieties, we have

$$\bigsqcup_{\mathbf{c}\in Cl_{\mathbb{Q}}(K)} I_{U(V)}(K^n;\mathbf{c}) = I_G(K^n).$$

When $\mathbf{c} = 1$, we write $I_{U(V)}(K^n)$ for $I_{U(V)}(K^n;\mathbf{c})$.

2.6. Morphisms between Igusa schemes

Let (W,θ,L) and $(V,\theta_{r,s},M)$ be as before. Let $L^{-1} = L_{\Sigma} := e^{+}(L\otimes_{\mathbb{Z}}\mathbb{Z}_p)$ and $L^0 = L_{\Sigma^c} := e^{-}(L\otimes_{\mathbb{Z}}\mathbb{Z}_p)$. Then $\left\{L^{-1}, L^0\right\}$ is a polarization of L_p. Recall that the standard polarization of M_p in §1.6 is

$$M^{-1} = \sum_{i=1}^{s}(\mathscr{O}_{\mathcal{K}}\otimes_{\mathbb{Z}}\mathbb{Z}_p)y^i \oplus L^{-1} \text{ and } M^0 = L^0 \oplus \sum_{i=1}^{s}(\mathscr{O}_{\mathcal{K}}\otimes_{\mathbb{Z}}\mathbb{Z}_p)x^i.$$

Let $\underline{V} = (V, \theta_{r,s}, M, M^{-1} \oplus M^0)$ and $-\underline{W} = (W, -\theta, L, L^0 \oplus L^{-1})$. Let $(\mathbf{W}, \eta_r) = (V \oplus (-W), \theta_{r,s} \oplus (-\theta))$. Define an $\mathscr{O}_\mathcal{K}$-lattice $\mathbf{L} = M \oplus L$ in $\mathbf{W}$ and a polarization $(\mathbf{L}^{-1}, \mathbf{L}^0) = (M^{-1} \oplus L^0, M^0 \oplus L^{-1})$ of $\mathbf{L}_p$. Put

$$\underline{\mathbf{W}} = \underline{V} \oplus (-\underline{W}) = (\mathbf{W}, \eta_r, \mathbf{L}, \mathbf{L}^{-1} \oplus \mathbf{L}^0).$$

Let $G_1 = U(V)$, $G_2 = U(W)$ and $G_3 = U(\mathbf{W})$ be the associated unitary groups. For open compact subgroups $K_i \subset G_i(\mathbb{A}_f)$ such that $K_1 \times K_2 \subset K_3$ and a fixed $\mathbf{c} \in (\mathbb{A}_f^p)^\times$, we write $I^i(K_i(p^n))$ for the Igusa schemes $I_{G_i}(K_i(p^n), \mathbf{c})$ associated to unitary groups with additional data. Then we have a natural morphism

$$i_{V,W} : I^1(K_1(p^n))_{/\mathcal{O}^{ab}} \times I^2(K_2(p^n))_{/\mathcal{O}^{ab}} \longrightarrow I^3(K_3(p^n))_{/\mathcal{O}^{ab}}$$

defined by

$$\begin{aligned}(2.11)\quad & i_{V,W}\left([(A_1, \lambda_1, \iota_1, \eta_1^p K_1, j_1)], [(A_2, \lambda_2, \iota_2, \eta_2^p K_2, j_2)]\right) \\ &\qquad = [(A_1 \times A_2, \lambda_1 \times \lambda_2, \iota_1 \times \iota_2, (\eta_1^p \times \eta_2^p)K_3, j_1 \times j_2)].\end{aligned}$$

Now we consider a different polarization of $\mathbf{L}_p$. Let $w^{+,i}$ (resp. $w^{-,i}$) be the image of w^i in W (resp. $-W$) as a subspace in $\mathbf{W}$. We define a basis $\{\mathbf{y}^i, \mathbf{x}^i\}_{i=1}^r$ of $\mathbf{W}$ by $\mathbf{y}^i = y^i$, $\mathbf{x}^i = x^i$ if $1 \leqslant i \leqslant s$ and

$$\begin{aligned}\mathbf{y}^i &= \frac{1}{2}w^{+,i-s} - \frac{1}{2}w^{-,i-s}, \\ \mathbf{x}^i &= w^{+,i-s}\theta^{-1} + w^{-,i-s}\theta^{-1}.\end{aligned}$$

if $s < i \leqslant r$. We put $\mathbf{Y} = \sum_{i=1}^{2r} (\mathscr{O}_\mathcal{K} \otimes_\mathbb{Z} \mathbb{Z}_p)\mathbf{y}^i$ and $\mathbf{X} = \sum_{i=1}^{2r} (\mathscr{O}_\mathcal{K} \otimes_\mathbb{Z} \mathbb{Z}_p)\mathbf{x}^i$. Then $\{\mathbf{Y}, \mathbf{X}\}$ is another polarization of $\mathbf{L}_p = M_p \oplus (-L_p)$. We have

$$\mathbf{L}_\Sigma = M_\Sigma \oplus (-L)_\Sigma = \mathbf{Y}_\Sigma \oplus \mathbf{X}_\Sigma.$$

We define $\Upsilon \in U(\mathbf{W})(\mathbb{Q}_p) \cong \mathrm{GL}(\mathbf{W}_\Sigma) \cong \mathrm{GL}_{2r}(\mathbb{Q}_p)$ by $y^i\Upsilon = \mathbf{y}^i$, $x^i\Upsilon = \mathbf{x}^i$ if $1 \leqslant i \leqslant s$ and $w_\Sigma^{+,i-s}\Upsilon = \mathbf{y}_\Sigma^i$ and $w_\Sigma^{-,i-s}\Upsilon = \mathbf{x}_\Sigma^i$ if $s < i \leqslant r$. Then $M_\Sigma^0\Upsilon = \mathbf{X}_\Sigma$ and $M_\Sigma^{-1}\Upsilon = \mathbf{Y}_\Sigma$. The matrix representation of Υ according the basis $\mathbf{y}_\Sigma^i$ and $\mathbf{x}_\Sigma^i$ is

$$\Upsilon = \begin{bmatrix} \mathbf{1}_s & & & \\ & \frac{1}{2}\cdot\mathbf{1}_{r-s} & & -\frac{1}{2}\cdot\mathbf{1}_{r-s} \\ & & \mathbf{1}_s & \\ & \theta^{-1} & & \theta^{-1} \end{bmatrix}.$$

We give an explicit expression of the morphism in (2.11) in terms of the complex coordinates defined in (2.8). Notations are as in §2.4. By definition of $\mathbb{C}^{r,r}$ we can decompose

$$\mathbb{C}^{r,r} = \mathbb{C}^s(\Sigma^c) \oplus \mathbb{C}^{r-s}(\Sigma^c) \oplus \mathbb{C}^s(\Sigma) \oplus \mathbb{C}^{r-s}(\Sigma) = \mathbb{C}^{r,s} \oplus \mathbb{C}^{0,r-s}.$$

Then for $\tau = \begin{bmatrix} x \\ y \end{bmatrix} \in X_{r,s}$, we let $i_\tau \in \mathrm{GL}(\mathbb{C}^{r,r})$ be the matrix such that according to the above decomposition,

$$i_\tau = \begin{bmatrix} \mathbf{1}_s & & & \\ & \mathbf{1}_{r-s} & & \\ & & \mathbf{1}_s & \\ & & \theta^{-1}y & \mathbf{1}_{r-s} \end{bmatrix}$$

and put

$$Z_\tau = \begin{bmatrix} x & 0 \\ y & \varsigma \end{bmatrix} \in X_{r,r}, \tag{2.12}$$

where $\varsigma = -2^{-1}\theta$. For $g \in U(V)$ and $h \in U(W)$, a straightforward computation shows that

$$\left(\mathbf{L}_{(g,h)}(Z_\tau)\right) i_\tau = M_g(\tau) \oplus L_h(\mathbf{x}_0),$$

hence i_τ induces an isomorphism

$$\begin{aligned} \mathcal{A}(\mathbf{W})_{(g,h)}(Z_\tau) = \frac{\mathbb{C}^{r,r}}{\mathbf{L}_{(g,h)}(Z_\tau)} &\overset{i_\tau}{\underset{\sim}{\to}} \frac{\mathbb{C}^{r,s}}{M_g(\tau)} \bigoplus \frac{\mathbb{C}^{0,r-s}}{L_h(\mathbf{x}_0)} \\ &= \mathcal{A}(V)_g(\tau) \times \mathcal{A}(W)_h(\mathbf{x}_0). \end{aligned} \tag{2.13}$$

Note that

$$i_\tau^*(\boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma), \boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma^c) \wedge \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma^c)) = (\boldsymbol{\omega}_{\mathbf{W}/\mathbb{C}}(\Sigma), \boldsymbol{\omega}_{\mathbf{W}/\mathbb{C}}(\Sigma^c)). \tag{2.14}$$

Taking into account the change of the polarization, we can deduce that the morphism in (2.11) over $\mathbb{C}$ is simply given by

$$i_{V,W}\left([\tau, g], [\mathbf{x}_0, h]\right) = [Z_\tau, (g,h)\Upsilon]. \tag{2.15}$$

2.7. CM abelian varieties and periods

2.7.1.

We consider a special case where $r = 2$ and $s = 0$. Then $G = GU(W)$ is a definite unitary group of degree two. It follows that $S_G(K)$ is finite over $\mathcal{O}$ and $S_G(K)(\mathbb{C}) = M_G(X^+, K)$ is a finite set. Let $\underline{\mathcal{B}}$ be the universal quadruple over $S_G(K)$. Then $\underline{\mathcal{B}} = \bigsqcup_{[h] \in S_G(K)(\mathbb{C})} \underline{\mathcal{B}}_h$ and each $\underline{\mathcal{B}}_h$ is defined over a discrete valuation ring $\mathfrak{o} \subset \overline{\mathbb{Z}}_{(p)}$. We consider the quintuple $(\underline{\mathcal{B}}, j)$ over $S_G(K^n)_{/\mathbb{C}}$. Since $\mathcal{B}_h$ is an abelian varietiy with CM by $\mathscr{O}_\mathcal{K}$ and p is split in $\mathcal{K}$, it follows that $\mathcal{B}_h \otimes_\mathcal{O} \overline{\mathbb{F}}_p$ is an ordinary abelian variety, and j_ζ descends to a level p^n-structure over a finite unramified extension of $\mathfrak{o}$. In short, we can enlarge $\mathfrak{o}$ so that the quintuple $(\underline{\mathcal{B}}, j)$ is defined over $\mathfrak{o}$. Note that by the complex uniformization constructed in §2.4.2, we have

$$\mathcal{B}_h(\mathbb{C}) = \frac{\mathbb{C}(\Sigma^c)}{\overline{L_{[h]}}\theta}. \tag{2.16}$$

2.7.2.

We introduce the CM-period and its p-adic avatar for the CM-algebra $\mathcal{K} \oplus \mathcal{K}$. Let $E_{\mathcal{K}}$ be an elliptic curve with CM by $\mathcal{K}$ together with a complex uniformization $i : E_{\mathcal{K}}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\mathscr{O}_{\mathcal{K}}$ (i is unique to ± 1). It is well known that $E_{\mathcal{K}}$ is defined over $\overline{\mathbb{Q}}$, and extends to an abelian scheme $\widetilde{E}$ over $\overline{\mathbb{Z}}_{(p)}$. We choose a Néron differential $\boldsymbol{\omega}_{\mathcal{K}} \in H^0(\widetilde{E}, \Omega_{\widetilde{E}/\overline{\mathbb{Z}}_{(p)}})$ such that $H^0(\widetilde{E}, \Omega_{\widetilde{E}/\overline{\mathbb{Z}}_{(p)}}) = \overline{\mathbb{Z}}_{(p)} \cdot \boldsymbol{\omega}_{\mathcal{K}}$. On the other hand over $\mathbb{C}$ i induces a holomorphic one form of first kind dz on $E_{\mathcal{K}}$, so there exists a complex number $\Omega_{\mathcal{K}}$ such that

$$\omega_{\mathcal{K}} = \Omega_{\mathcal{K}} dz.$$

We call $\Omega_{\mathcal{K}}$ the CM period of $\mathcal{K}$ which is well-defined up to $\overline{\mathbb{Z}}_{(p)}^{\times}$.

By the isomorphism (2.16), the holomorphic one forms $d\underline{z}_W(\Sigma)$ defined in (2.9) give an $\mathscr{O}_{\mathcal{K}}$-basis of $\Omega_{\mathcal{B}_{h/\mathbb{C}}}$. By the assumption on L and θ, $\mathcal{B}_h$ is $\mathbb{Z}_{(p)}$-isogenous to $E_{\mathcal{K}} \times E_{\mathcal{K}}$. Then it is easy to show that the top form $\boldsymbol{\omega}_{\mathcal{B}_h} := \Omega_{\mathcal{K}}^2 \cdot \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma)$ over $\mathbb{C}$ induced by complex uniformization actually descend to $\overline{\mathbb{Z}}_{(p)}$.

On the other hand, the level p^∞ level structure $j : \widehat{\mathbb{G}_m}^2 \cong \widehat{\mathcal{B}_h}$ induces an formal top form $\boldsymbol{\omega}(j)$ of $\widehat{\mathcal{B}_h}$ as in (2.21) which can descend to a top form of $\mathcal{B}_h$ still denoted by $\boldsymbol{\omega}(j)$. Hence there exists $\Omega_p \in (\overline{\mathbb{Z}}_p)^\times$ such that

$$\Omega_p \boldsymbol{\omega}(j) = \boldsymbol{\omega}_{\mathcal{B}_h} = \Omega_{\mathcal{K}}^2 \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma). \tag{2.17}$$

2.8. Siegel modular forms for unitary groups

2.8.1.

We introduce the notion of Siegel modular forms for unitary groups. For a R-quintuple $x = (A, \overline{\lambda}, \iota, \overline{\eta}^{(p)}, j) \in I_G(K^n)$. We say $\boldsymbol{\omega}$ is an $\mathscr{O}_{\mathcal{K}}$-top form of x if $\boldsymbol{\omega} = (\boldsymbol{\omega}(\Sigma), \boldsymbol{\omega}(\Sigma^c))$, where $\boldsymbol{\omega}(\Sigma)$ (resp. $\boldsymbol{\omega}(\Sigma^c)$) is a generator of $\Omega_{\Sigma}^r = \bigwedge^r e^+\Omega_A$ (resp. $\Omega_{\Sigma^c}^s = \bigwedge^s e^-\Omega_A$). Let $H = \mathrm{GL}_r \times \mathrm{GL}_s$. For $h \in H$, we write $h = (h_1, h_2) \in \mathrm{GL}_r \times \mathrm{GL}_s$. For $\underline{k} = (a, b) \in \mathbb{Z}$, we let $\rho_{\underline{k}}(h) = \det(h_1)^{-a} \det(h_2)^{-b}$. We have the obvious left action of $H(R)$ on $\boldsymbol{\omega}$.

Definition 2.3 (Katz-Hida). — *A Siegel modular form of weight* $\underline{k} = (a,b)$ *is a rule* f *which assigns to a pair* $(x, \boldsymbol{\omega})$ *a* R*-quintuple* x *in* $I_G(K^n)$ *together with an* $\mathscr{O}_{\mathcal{K}}$*-top form* $\boldsymbol{\omega}$ *of* $\mathrm{Lie}\, A^\vee$*, an element* $f(x,\boldsymbol{\omega}) \in R$*, such that the following three conditions are satisfied.*

(1) *$f(x,\boldsymbol{\omega})$ depends only on the R-isomorphism class of the pair $(x,\boldsymbol{\omega})$.*
(2) *For any $h \in H(R)$, $f(x, h\boldsymbol{\omega}) = \rho_{\underline{k}}(h) f(x,\boldsymbol{\omega})$.*
(3) *The formation of $f(x,\boldsymbol{\omega})$ commutes with base change. Namely, for any $\pi : R \to R'$, $\pi(f(x,\boldsymbol{\omega})) = f(x_{R'}, \boldsymbol{\omega}_{R'})$.*

Tautologically modular forms of $\underline{k}$ can be viewed as sections in the sheaf $\omega_{\underline{k}}$ as follows.

$$\omega_{\underline{k}}(S) = \{f(x,\boldsymbol{\omega}) \in \mathcal{O}_S \mid f(x, h\cdot\boldsymbol{\omega}) = \rho_{\underline{k}}(h) f(x,\boldsymbol{\omega}),\ \forall h \in H(\mathcal{O}_S)\}.$$

Set

$$\mathbf{M}_{\underline{k}}(K^n, R) := H^0(I_G(K^n)_{/R}, \omega_{\underline{k}}).$$

We call $\mathbf{M}_{\underline{k}}(K^n, R)$ the space of geometric modular forms.

2.8.2. Automorphic forms and modular forms over $\mathbb{C}$

Let $J : G(\mathbb{R})^+ \times X^+ \longrightarrow \mathrm{GL}(r,\mathbb{C}) \times \mathrm{GL}(s,\mathbb{C})$ be the standard cocycle defined by

$$g_\infty p(\tau) = p(g_\infty \tau) J(g_\infty, \tau),\ \forall\, (g_\infty, \tau) \in G(\mathbb{R})^+ \times X^+.$$

Fix a point $\mathbf{i} \in X^+$ and let K_∞^0 be the stablizer of $\mathbf{i}$ in $G(\mathbb{R})$. Then $J : K_\infty^0 \longrightarrow H(\mathbb{C})$, $k_\infty \mapsto J(k_\infty, \mathbf{i})$ defines an algebraic representation of K_∞^0.

Definition 2.4. — *Let U be an open compact subgroup in $G(\mathbb{A}_f)$ and let χ be a Hecke character of $\mathcal{K}$ with infinity type (b,a). Let $\mathcal{A}_{\underline{k}}(G,U,\chi)$ be the space of automorphic forms of weight $\underline{k} = (a,b)$ and level U with central character χ. In other words, $\mathcal{A}_{\underline{k}}(G,U,\chi)$ consists of smooth and slowly increasing functions $F : G(\mathbb{A}) \to \mathbb{C}$ such that for all $(\alpha, k_\infty, u, z) \in G(\mathbb{Q}) \times K_\infty^0 \times U \times Z(\mathbb{A}_{\mathcal{F}})$,*

$$F(z\alpha g k_\infty u) = \rho_{\underline{k}}(J(k_\infty, \mathbf{i})) F(g) \chi^{-1}(z).$$

Define the function $\underline{\mathrm{AM}}(F)$ on $X^+ \times G(\mathbb{A}_f)$ associated to $F \in \mathcal{A}_{\underline{k}}(G,U,\chi)$ by

$$\underline{\mathrm{AM}}(F)(\tau, g) := \chi_f(\nu(g)) \rho_{\underline{k}}(J(g_\infty, \mathbf{i})^{-1}) F((g_\infty, g)), \tag{2.18}$$

where $g_\infty \in G(\mathbb{R})^+$ such that $g_\infty \mathbf{i} = \tau$. Then $\underline{\mathrm{AM}}(F)$ is a well-defined function on $X^+ \times G(\mathbb{A}_f)/U$. We put

$$\mathcal{A}_{\underline{k}}^{\mathrm{Hol}}(G,U,\chi) = \left\{F \in \mathcal{A}_{\underline{k}}(G,U,\chi) \mid \underline{\mathrm{AM}}(F) \text{ is holomorphic in } X^+\right\}.$$

We denote by $\mathbf{M}_{\underline{k}}(G,U,\chi,\mathbb{C})$ *the space of holomorphic functions* f *on* $X^+\times G(\mathbb{A}_f)$ *such that*

$$f(\alpha\tau,\alpha g)=\nu(\alpha)^{-(a+b)}\rho_{\underline{k}}(J(\alpha,Z)^{-1})f(\tau,g),\ Z\in X^+,\ \alpha\in G(\mathbb{Q})$$

and

$$f(\tau,zgu)=\chi^c(z)f(\tau,g),\ u\in U,\ z\in Z(\mathbb{A}_f).$$

$M_{\underline{k}}(G,U,\chi,\mathbb{C})$ *is the space of modular forms of weight* $\underline{k}$ *and level* U *with central character* χ. *It is easy to see that* $\underline{\mathrm{AM}}$ *induces an isomorphism*

$$\underline{\mathrm{AM}}:\mathcal{A}^{\mathrm{Hol}}_{\underline{k}}(G,U,\chi)\xrightarrow{\sim}\mathbf{M}_{\underline{k}}(G,U,\chi,\mathbb{C}).$$

By G.A.G.A we have an injection:

$$\begin{aligned}\mathbf{M}_{\underline{k}}(G,K^n,\chi,\mathbb{C})&\hookrightarrow\mathbf{M}_{\underline{k}}(K^n,\mathbb{C})\\ &\qquad f\to f([\tau,a],2\pi i\boldsymbol{\omega}_{V/\mathbb{C}}):=f(\tau,a),\end{aligned}$$

where $\boldsymbol{\omega}_{V/\mathbb{C}}=(\boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma),\boldsymbol{\omega}_{V/\mathbb{C}}(\Sigma^c))$ is the $\mathscr{O}_{\mathcal{K}}$-top form defined in (1.6).

2.8.3. Analytic Fourier-Jacobi expansion

The set of cusp labels for $S_G(K)$ is defined to be

$$C(K):=(\mathrm{GL}_s(\mathcal{K})\times GU(W)(\mathbb{A}_f)N_P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K.$$

For $g\in G(\mathbb{A}_f)$, we denote by $[g]$ the class in $C(K)$.

Holomorphic modular forms $f\in\mathbf{M}_{\underline{k}}(G,U,\mathbb{C})$ have the following analytic Fourier-Jacobi expansion:

$$\begin{aligned}f(\begin{bmatrix}x\\y\end{bmatrix},g)&=f(\begin{bmatrix}x\\y\end{bmatrix},g;2\pi i d\underline{z}_V)\\ &=\sum_{\beta\in\mathcal{H}_s(\mathbb{Q})}a_\beta(y,g;f)e^{2\pi i\,\mathrm{Tr}(\beta x)},\ (\begin{bmatrix}x\\y\end{bmatrix},g)\in X^+\times G(\mathbb{A}_f),\end{aligned}$$

where $\mathcal{H}_s(\mathbb{Q})$ is the set of $s\times s$ Hermitian matrices in $M_s(\mathcal{K})$. We put

$$\mathcal{F}_{[g]}(f)=\sum_{\beta\in\mathcal{H}_s(\mathbb{Q})}a_\beta(y,g;f)q^\beta. \tag{2.19}$$

The formal power series $\mathcal{F}_{[g]}(f)$ is called the Fourier(-Jacobi) expansion of f at the cusp $[g]$.

2.8.4. p-adic modular forms

Let $n \geqslant m$ be positive integers. Write $\mathbb{Z}_m$ for $\mathbb{Z}/p^m\mathbb{Z}$. Let $T_{n,m} = I_G(K^n)_{/\mathbb{Z}_m}$. Let $T_{\infty,m} = \varprojlim_n T_{n,m}$ and $T_{\infty,\infty} = \varinjlim_m \varprojlim_n T_{n,m}$ be Igusa towers on $S_G(K)_{/\mathcal{O}}$. Then there is a natural action of $\mathrm{GL}_r(\mathbb{Z}_p) \times \mathrm{GL}_s(\mathbb{Z}_p)$ on $T_{\infty,\infty}$. We define p-adic modular forms following [12].

Definition 2.5. — *Put*

$$V_{n,m} := H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}) \text{ and } V_{\infty,m} = H^0(T_{\infty,m}, \mathcal{O}_{T_{\infty,m}}) = \varinjlim_n V_{n,m}.$$

Let $\mathbf{N} = N_r(\mathbb{Z}_p) \times N_s(\mathbb{Z}_p)$. *We define the space of p-adic modular forms* $V_p(G,K)$ *by*

$$V_p(G,K) = \varprojlim_m V_m^{\mathbf{N}}.$$

Let R be a $\mathbb{Z}_m$-algebra. For a R-quintuple $(\underline{A}, j)$, the p^n-structure j induces the following isomorphism as $\mathscr{O}_\mathcal{K}$-modules.

$$j_* : M^0 \otimes_\mathbb{Z} R \xrightarrow{\sim} \mathrm{Lie}\, A[p^n] = \mathrm{Lie}\, A. \tag{2.20}$$

Then (2.20) induces a p-adic $\mathscr{O}_\mathcal{K}$-top form $\boldsymbol{\omega}(j) = (\boldsymbol{\omega}(j)(\Sigma), \boldsymbol{\omega}(j)(\Sigma^c)) \in H^0(A, \Omega^r_\Sigma) \times H^0(A, \Omega^s_{\Sigma^c})$, where

$$\begin{aligned} \boldsymbol{\omega}(j)(\Sigma) &: \wedge^r \mathrm{Lie}\, A_{\Sigma^c} \xrightarrow{\sim} \wedge^r (M^0_{\Sigma^c} \otimes R) \cong R, \\ \boldsymbol{\omega}(j)(\Sigma^c) &: \wedge^s \mathrm{Lie}\, A_\Sigma \xrightarrow{\sim} \wedge^s (M^0_\Sigma \otimes R) \cong R. \end{aligned} \tag{2.21}$$

The last isomorphism depends on the choice of a basis of M^0. (2.21) induces the emdedding

$$\begin{aligned} \boldsymbol{\beta} : H^0(T_{n,m}, \omega_{\underline{k}}) &\longrightarrow V_m \\ f &\longrightarrow \boldsymbol{\beta}(f)(\underline{A}, j) = f(\underline{A}, j, \boldsymbol{\omega}(j)) \end{aligned} \tag{2.22}$$

which induces the following morphism

$$\mathbf{M}_{\underline{k}}(K_1^n, \mathbb{Z}_p) \xrightarrow{res} H^0(T^{\mathbf{N}}_{\infty,\infty}, \omega_{\underline{k}}) \to V_p(G,K),\ f \mapsto \widehat{f} = \boldsymbol{\beta}(res(f)).$$

We call $\widehat{f} \in V_p(G,K)$ the *p-adic avatar* of f.

3. Siegel-Eisenstein series on $GU(n,n)$

3.1.

Let n be a positive integer. In this section, we give a construction of p-adic Siegel-Eisenstein series on the quasi-split unitary group of degree

n. We retain the notation in §2.6. Let $\underline{W}$, $\underline{V}$ and $\underline{\mathbf{W}}$ be the quadruples defined in §2.6 with $r = n$ and $s = 1$. We further assume that the lattice L is a $\mathbb{Z}$-*maximal* lattice with respect to the Hermitian form $2^{-1}\delta^{-1}\theta$ in the sense of Shimura [22, Ch. I §4.7]. Let $\mathbf{G} = GU(\mathbf{W}, \eta_n)$ denote the group of unitary similitudes attached to $(\mathbf{W}, \eta_n)$. For a $\mathbb{Q}$-algebra R, we identify $\mathbf{G}(R)$ with

$$\mathbf{G}(R) = \left\{ g \in M_{2n}(R \otimes_{\mathbb{Q}} \mathcal{K}) \mid g \begin{bmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{bmatrix} g^* = \nu(g) \begin{bmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{bmatrix} \right\}$$

according to the basis $\left\{\mathbf{y}^i, \mathbf{x}^i\right\}_{i=1}^n$. Thus $\mathbf{G}$ is the standard quasi-split group of unitary similitudes of degree n. Let $\mathbf{P}$ be the stablizer of the flag $\{0\} \subset \mathbf{X} \subset \mathbf{W}$ in $\mathbf{G}$. Then $\mathbf{P}$ is the standard Siegel parabolic subgroup of $\mathbf{G}$.

Recall that $X_{n,n}$ is the Hermitian symmetric domain associated to $\mathbf{G}$ defined by

$$X_{n,n} = \{Z \in M_n(\mathbb{C}) \mid i(Z^* - Z) > 0\}.$$

We choose a *distinguished* point $\mathbf{i}$ in $X_{n,n}$ attached to θ. Put

$$\mathbf{i} = \begin{bmatrix} \sqrt{-1} \cdot \mathbf{1}_{n-1} & 0 \\ 0 & \varsigma \end{bmatrix}, \ \varsigma = -\frac{\theta}{2}.$$

Put

$$\mathbf{K}^0_\infty = \{g \in \mathbf{G}(\mathbb{R}) \mid g\mathbf{i} = g\}.$$

Let $\mathbf{K}^0_v$ be the stablizer of $\mathbf{L}$ in $\mathbf{G}(\mathbb{Q}_v)$ and let $\mathbf{K}^0_f = \prod_{v \in \mathbf{h}} \mathbf{K}^0_v$ be a maximal open compact subgroup in $\mathbf{G}(\mathbb{A}_f)$. For $g \in \mathbf{G}(\mathbb{A})$, we define

$$\delta(g) := \left|\nu^{-n} \det(\bar{d}d)\right|^{-1}, \ g = \begin{bmatrix} \nu d^{-*} & b \\ 0 & d \end{bmatrix} k, \ k \in \mathbf{K}^0_\infty \mathbf{K}^0_f.$$

We have $\delta_{\mathbf{P}}(g) = \delta(g)^n$, where $\delta_{\mathbf{P}}$ is the modular character of $\mathbf{P}$.

3.2. Eisenstein series

3.2.1.

Let χ be a Hecke characters of $\mathcal{K}$ with infinity type $(k, 0)$. Namely $\chi : \mathbb{A}^\times_{\mathcal{K}}/\mathcal{K}^\times \to \mathbb{C}$ is a character such that $\chi_\infty(z_\infty) = z^k_\infty$. Consider the induced representation $I(\chi, s) = \mathrm{Ind}^{\mathbf{G}}_{\mathbf{P}}(\chi, \mathbf{s})$, where $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{P}}(\chi, \mathbf{s})$ is the unitary induced representation of χ and $\mathbf{s} \in \mathbb{C}$ such that $n(\mathbf{s} + \frac{1}{2}) = s$.

For a place v of $\mathbb{Q}$, let $I_v(\chi, s)$ be the induced representation attached to $\chi_v := \chi|_{(\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_v)^\times}$. In other words,

$$\begin{aligned}&I_v(\chi, s)\\ &= \left\{\text{smooth functions } f : \mathbf{G}(\mathbb{Q}_v) \to \mathbb{C} \mid f(\mathbf{p}g) = \chi_v^{-1}(\det d_{\mathbf{p}})\delta^s(\mathbf{p})f(g)\right\}.\end{aligned}$$

Then we have

$$I(\chi, s) = \otimes'_v I_v(\chi, s).$$

Define the adelic Siegel-Eisenstein series associated to a smooth section $\phi \in I(\chi, s)$ by

$$E_{\mathbb{A}}(g, \phi) := \sum_{\gamma\in\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} \phi(\gamma g),$$

whenever the sum is convergent.

3.2.2.

Denote by $\mathbf{e}$ the standard additive character of $\mathbb{A}/\mathbb{Q}$ such that $\mathbf{e}_\infty(x_\infty) = \exp(2\pi i x_\infty)$, $x_\infty \in \mathbb{R}$. Let $\mathbf{h}$ be the set of finite places of $\mathbb{Q}$. If $v \in \mathbf{h}$, let dx_v be the Haar measure on $\mathbb{Q}_v$ such that $\mathrm{vol}(\mathbb{Z}_v, dx_v) = 1$. Let dX_v be the Tamagawa measure on the $n\times n$ Hermitian matrices $\mathcal{H}_n(\mathbb{Q}_v)$ in $M_n(\mathcal{K}_v)$ such that $\mathrm{vol}(\mathcal{H}_n(\mathbb{Z}_v), dX_v) = 1$. If $\mathcal{L}$ is a lattice in $\mathcal{H}_n(\mathbb{Q}_v)$, put

$$\mathcal{L}^\vee = \{x \in \mathcal{H}_n(\mathbb{Q}_p) \mid \mathbf{e}_v(\mathrm{Tr}(xy)) = 1, \forall y \in \mathcal{L}\}.$$

If v is archimedean, let

$$dX_v = \bigwedge_j dX_{jj} \bigwedge_{j<k} 2^{-1} dX_{jk} \wedge d\overline{X_{jk}}.$$

For $\beta \in \mathcal{H}_n(\mathcal{K})$ and $X \in \mathcal{H}_n(\mathbb{A})$, define $\mathbf{e}_\beta(X) = \mathbf{e} \circ \mathrm{Tr}_{\mathcal{K}/\mathbb{Q}}(\beta X)$. We choose the normalized measure dX on $\mathcal{H}_n(\mathbb{A})$ so that

$$\mathrm{vol}(\mathcal{H}_n(\mathbb{Q})\backslash\mathcal{H}_n(\mathbb{A}), dX) = 1.$$

Then it is well-known that

$$dX = C_{\mathcal{K}}(n) \cdot \otimes_v dX_v, \text{ where } C_{\mathcal{K}}(n) = 2^{\frac{n(n-1)}{2}} |D_{\mathcal{K}}|_{\mathbb{R}}^{-n(n-1)/4}.$$

Let $c^p(\chi)$ be the prime-to-p conductor of χ. We fix a prime-to-p integral ideal $\mathfrak{c}$ of $\mathbb{Z}$ such that

$$\mathfrak{c} \subset D_{\mathcal{K}} \cdot c^p(\chi). \tag{S1}$$

Let $S = \{v \in \mathbf{h} \mid v|\mathfrak{c}\}$. Note that S is not empty.

3.2.3. Fourier expansion

Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$ given by

$$\mathbf{N} = \{\mathbf{n}(X) = \begin{bmatrix} \mathbf{1}_n & X \\ & \mathbf{1}_n \end{bmatrix} \mid X \in \mathcal{H}_n\}.$$

Then Siegel-Eisenstein series $E_{\mathbb{A}} = E_{\mathbb{A}}(g, \phi)$ has the Fourier expansion

$$\begin{aligned} E_{\mathbb{A}}(g,\phi) &= \sum_{\beta\in\mathcal{H}_n(\mathbb{Q})} W_\beta(g, E_{\mathbb{A}}), \text{ where } W_\beta(g,E_{\mathbb{A}}) \\ &= \int\limits_{\mathcal{H}_n(\mathbb{Q})\backslash\mathcal{H}_n(\mathbb{A})} E_{\mathbb{A}}(\mathbf{n}(X)g)\mathbf{e}_{-\beta}(X)dX. \end{aligned} \tag{3.1}$$

Let $\mathbf{w} = \begin{bmatrix} & -\mathbf{1}_n \\ \mathbf{1}_n & \end{bmatrix}$. It is well-known that if $\phi = \otimes\phi_v$ is decomposable and $\operatorname{supp}\phi_{v_0} \subset \mathbf{P}(\mathbb{Q}_{v_0})\mathbf{w}\mathbf{P}(\mathbb{Q}_{v_0})$ for some v_0, then we have $W_\beta(g, E_{\mathbb{A}}) = C_{\mathcal{K}}(n)W_\beta(g,\phi)$ for $g \in \mathbf{G}(\mathbb{A})$ with $g_{v_0} = 1$, where

$$W_\beta(g,\phi) = \prod_v W_\beta(g_v,\phi_v),$$

$$W_\beta(g_v,\phi_v) = \int_{\mathcal{H}_n(\mathbb{Q}_v)} \phi_v(\mathbf{w}\mathbf{n}(X_v)g_v)\mathbf{e}_{-\beta}(X_v)dX_v.$$

Thus the Fourier expansion in (3.1) can be rephrased as

$$C_{\mathcal{K}}(n)^{-1}E_{\mathbb{A}}(g,\phi) = C_{\mathcal{K}}(n)^{-1}\phi(g) + M_{\mathbf{w}}\phi(g) + \sum_{\beta\in\mathcal{H}_n(\mathbb{Q}),\beta\neq 0} W_\beta(g,\phi),$$

where $M_{\mathbf{w}}\phi(g) := W_0(g,\phi)$. Then $M_{\mathbf{w}}\phi$ is called the intertwining operator of $\mathbf{w}$, and we will write $M\phi$ for $M_{\mathbf{w}}\phi$ in the sequel. Define the constant term $E_{\mathbf{P}}(g)$ along the parabolic subgroup $\mathbf{P}$ by

$$E_{\mathbf{P}}(g) = \int_{\mathcal{H}_n(\mathbb{Q})\backslash\mathcal{H}_n(\mathbb{A})} E_{\mathbb{A}}(\mathbf{n}(X)g)dX = \phi(g) + C_{\mathcal{K}}(n)\cdot M_{\mathbf{w}}\phi(g).$$

3.3. The local section at the archimedean place

3.3.1.

To define the desired Eisentein series with good arithmetic and p-adic properties, we need to choose some special decomposable local section in $I(\chi, s)$. In this subsection, we give the choice of the local sections at the archimedean place.

For $g \in \mathbf{G}(\mathbb{R})^+$ and $Z \in X_{n,n}$, we define the standard automorphy factors $J(g,Z)$ and $J'(g,Z)$ by

$$J(g,Z) := \det(c_g Z + d_g) \text{ and } J'(g,Z) := \det(g)^{-1} J(g,Z)\nu(g)^n.$$

Then we have

$$\delta(g) = |J(g,\mathbf{i})J'(h,\mathbf{i})\nu(g)^{-n}|^{-1} = |J(g,\mathbf{i})|^{-2}\,|\det g|\,,\ g \in \mathbf{G}(\mathbb{R}).$$

Let $J_{\mu,\lambda}(g,Z)$ denote the automorphy factor of weight $(\mu,\lambda) \in \mathbb{Z}^2$,

$$J_{\mu,\lambda}(g,Z) = J'(g,Z)^\mu J(g,Z)^\lambda. \tag{3.2}$$

Define the section of minimal $\mathbf{K}^0_\infty$-type $(0,k)$ in $I_\infty(\chi,0)$ by

$$\phi^h_{\chi,\infty}(g) = J_{(0,k)}(g,Z) = J(g,\mathbf{i})^{-k}.$$

Set

$$\phi_{\chi,s,\infty}(g) = \phi^h_{\chi,\infty}(g)\delta(g)^s. \tag{3.3}$$

3.3.2. Intertwining operator

The intertwining operator of $\phi_{\chi,s,\infty}$ can be computed by Gindikin-Karpelevič formula.

$$M_{\mathbf{w}}\phi_{\chi,s,\infty}(g) = i^{-nk} \cdot 2^n \cdot \pi^{n^2} \cdot \frac{\Gamma_n(k-n-2s)}{\Gamma_n(k+s)\Gamma_n(s)} \cdot \phi_{k,n-k-s,\infty}(g)$$

Note that $M_{\mathbf{w}}\phi_{\chi,s,\infty}|_{s=0} = 0$ if $k > n$.

3.3.3. Fourier coefficients

We will compute the local Fourier coefficient of $\phi_{k,s,\infty}$. First of all, we recall several definitions and facts from [21]. Put

$$H'_n = \{x + iy \in M_n(\mathbb{C}) \mid x, y \in \mathcal{H}_n(\mathbb{C}) \text{ and } x > 0\}.$$

We define the function $\xi(y,h;q_1,q_2)$ for $(y,h;q_1,q_2) \in H'_n \times \mathcal{H}_n(\mathbb{C}) \times \mathbb{C}^2$ by

(3.4)

$$\begin{aligned}\xi(y,h;q_1,q_2) :&= \int_{\mathcal{H}_n(\mathbb{C})} \det(y+ix)^{-q_1}\det(y-ix)^{-q_2} e^{-2\pi i \operatorname{Tr}(hx)} dx \\ &= i^{n(q_2-q_1)} \cdot 2^n \cdot \pi^{n^2} \cdot \Gamma_n(q_1)^{-1}\Gamma_n(q_2)^{-1} \cdot \eta(2y,\pi h;q_1,q_2),\end{aligned}$$

where

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s-j).$$

Then ξ is well-defined when $\operatorname{Re} q_1 \gg 0$ and $\operatorname{Re} q_2 \gg 0$. Moreover ξ has meromorphic continuation to the whole $H_n' \times \mathcal{H}_n(\mathbb{R}) \times \mathbb{C}^2$.

Now we assume $h > 0$ and write $\pi h = AA^*$. Then we have

$$\eta(2y, \pi h; q_1, q_2) = \eta(2g, AA^*; q_1, q_2) = \det(\pi h)^{(q_1+q_2-n)} \cdot \eta(A^* 2yA, 1; q_1, q_2)$$

and

$$(3.5) \quad \eta(g, 1; q_1, q_2) = e^{-\operatorname{Tr}(g)} 2^{(q_1+q_2-n)n} \Gamma_n(q_2) \det(2y)^{-q_2} \omega(2g; q_1, q_2),$$

where $\omega(z; p, q)$ is the function on $H_n' \times \mathbb{C}^2$ defined in [21, p. 281]. Shimura proves the following:

Proposition 3.1 (Theorem 3.1 [21]). — *$\omega(z, q_1, q_2)$ is a holomorphic function on $H_n' \times \mathbb{C}^2$ which satisfies*

$$\omega(z; n-q_2, n-q_1) = \omega(z; q_1, q_2), \quad \omega(z, n, q_2) = \omega(z, q_1, 0) = 1.$$

Let $y = 1$ in (3.5). We have

$$\eta(2, 2\pi h; q_1, q_2) = e^{-2\pi \operatorname{Tr}(h)} 2^{(q_1+q_2-n)n} \Gamma_n(q_2) \det(4\pi h)^{-q_2} \omega(4\pi h; q_1, 1).$$

Hence

$$(3.6) \quad \xi(1, h; q_1, q_2)$$
$$= i^{n(q_2-q_1)} \cdot 2^{n(q_1-q_2-n+1)} \cdot \pi^{nq_1} \cdot \Gamma_n(q_1)^{-1} \cdot \det(h)^{q_1-n} e^{-2\pi \operatorname{Tr}(h)} \omega(4\pi h; q_1, q_2)$$

When $h = 0$, we have

$$\eta(g, 0; q_1, q_2) = \Gamma_n(q_1 + q_2 - n) \det(g)^{n-q_1-q_2}.$$

Hence

$$\xi(g, 0; q_1, q_2)$$
$$= i^{n(q_2-q_1)} \cdot 2^{n(k-2n+2)} \cdot \pi^{n^2} \cdot \Gamma_n(q_1)^{-1} \Gamma_n(q_2)^{-1} \cdot \Gamma_n(q_1+q_2-n) \det(g)^{n-q_1-q_2}.$$

Given $Z \in X_{n,n}$, we can choose $\mathbf{p}_\infty \in \mathbf{P}(\mathbb{R})$ such that $Z = \mathbf{p}_\infty \cdot \mathbf{i}$ and $\nu(\mathbf{p}_\infty) = 1$. We write

$$\mathbf{p}_\infty = \begin{bmatrix} a & b \\ & d \end{bmatrix} \in \mathbf{P}(\mathbb{R}),\ a = d^{-*} \text{ and } Z = X_0 + iY_0$$

with $X_0, Y_0 \in \mathcal{H}_n(\mathbb{C})$ and $Y_0 > 0$. From $\mathbf{p}_\infty \mathbf{i} = (a\mathbf{i} + b)d^{-1} = Z = X_0 + Y_0 i$, we have $X_0 = bd^{-1}$ and $Y_0 = a\mathbf{i}d^{-1}$. Note that $\det(\mathbf{i}) = \det(-i\varsigma)$ and $\det Y = \det(a^* a) \det(i\varsigma)$. Therefore

$$\det(a)^{-k} J_{(0,k)}(\mathbf{p}_\infty, \mathbf{i}) = (\det Y_0)^{-k} \cdot \det(i\varsigma)^k.$$

By the above formulae, a straightforward calculation shows that the local Fourier coefficient of $\phi = \phi_{\chi,s,\infty}$ is given by

$$\begin{aligned}W_{\beta,\infty}(\mathbf{p}_\infty, \phi_{\chi,s,\infty}) &= \int_{\mathcal{H}_n(\mathbb{R})} \phi(\mathbf{wn}(X_\infty)\mathbf{p}_\infty)\mathbf{e}_{-\beta}(X_\infty)dX_\infty \\ &= J_{(0,k)}(\mathbf{p}_\infty, \mathbf{i})^{-1}(\det Y_0)^{n-k-s}\det(i\varsigma)^{-s}\mathbf{e}_\beta(X_0)\xi(1, Y_0\beta, k+s, s).\end{aligned}$$

By (3.4) and (3.5), the last equation equals

(3.7)
$$\begin{aligned}&J_{(0,k)}(\mathbf{p}_\infty, \mathbf{i})^{-1} \cdot D_{p,q} \cdot \pi^{n(k+s)} \\ &\qquad \cdot \Gamma_n(k+s)^{-1}\det(\beta)^{k-n+s}e^{2\pi i \operatorname{Tr}(\beta Z)}\omega(4\pi Y_0\beta; k+s, s),\end{aligned}$$

where

$$D_{k,s} = i^{-nk} \cdot 2^{n(k-n+1)}\det(i\varsigma)^{-s}.$$

Evaluating $\phi_{\chi,s,\infty}$ at $s = 0$, by Prop. 3.1 we obtain the local Fourier coefficient at the archimedean place.

Proposition 3.2. — *Let* $\Lambda_{n,\infty}(s,\chi) = i^{-nk} \cdot 2^{-n(k-n+1)} \cdot \pi^{-n(s+k)} \cdot \Gamma_n(s+k)$. *Then*

$$\begin{aligned}&J_{(0,k)}(\mathbf{p}_\infty, \mathbf{i})W_{\beta,\infty}(\mathbf{p}_\infty, \phi_{\chi,s,\infty})|_{s=0} \\ &\qquad = \begin{cases}\Lambda_{n,\infty}(0,\chi)^{-1} \cdot \det(\beta)^{k-n} \cdot e^{2\pi i \operatorname{Tr}(\beta Z)} & \textit{if } \beta > 0, \\ 0 & \textit{otherwise.}\end{cases}\end{aligned}$$

3.4. The local section at finite $v \neq p$

3.4.1.

In this subsection, we give the choice of the local sections at finite places other than p. We first introduce some notation and definitions. For n a positive integer and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2n}(R)$ with $a, b, c, d \in M_n(R)$, we write $a = a_g$, $b = b_g$, $c = c_g$ and $d = d_g$. For $\mathfrak{a}$ a subset in R and $x \in M_{n\times m}(R)$,

$$x \prec \mathfrak{a} \iff x_{ij} \in \mathfrak{a}, \forall i, j.$$

Put $F = \mathbb{Q}_v$ and $E = \mathcal{K} \otimes \mathbb{Q}_v$ for $v \in \mathbf{h}$. Let $O = \mathbb{Z}_v$ and $\mathcal{R} = \mathscr{O}_\mathcal{K} \otimes \mathbb{Z}_v$. Put

$$\mathbf{D}_1[\mathfrak{c}_v] = \left\{ x \in \mathrm{GL}_{2n}(E) \mid \det x \in \mathcal{R}^\times, x \prec \begin{bmatrix} \mathcal{R} & \mathcal{R} \\ \mathfrak{c}_v & 1+\mathfrak{c}_v \end{bmatrix} \right\}$$

and $\mathbf{D}(\mathfrak{c}_v) = \mathbf{D}_1[\mathfrak{c}_v] \cap \mathbf{G}(F)$. We define an open-compact subgroup $\mathbf{K}$ of $\mathbf{G}(\mathbb{A}_f)$ by

$$\mathbf{K} = \prod_{v \in S} \mathbf{D}(\mathfrak{c}_v) \times \prod_{v \notin S} \mathbf{K}_v^0. \tag{3.8}$$

Since L_v is a O-maximal $\mathcal{R}$-lattice in V with respect to the Hermitian $2^{-1}\delta^{-1}\theta$, by [22, Lemma 20.2], we can find $\sigma_v \in \mathrm{GL}_{r-s}(E)$ such that $L'_v\sigma_v = L_v$ where $L'_v = \mathcal{R}w^1, \cdots, +\mathcal{R}w^{n-1}$ is the standard lattice in W and $\sigma\theta\sigma^* = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ if $v \notin S$ and v is inert in $\mathcal{K}$. Define $\mathbf{S}_v$ and $\mathbf{w}'$ in $\mathbf{G}(F)$ by

$$\mathbf{S}_v = \begin{bmatrix} 1 & & & \\ & \sigma_v & & \\ & & 1 & \\ & & & \sigma_v^{-*} \end{bmatrix}; \mathbf{w}' = \begin{bmatrix} & & -1 & \\ & & & \varsigma \\ 1 & & & \\ & \varsigma^{-1} & & \end{bmatrix}, \varsigma = -\frac{\theta}{2}.$$

For $v \in S$, define $f_{\mathfrak{c},v}$ to be the unique section in $I_v(\chi, s)$ such that
(3.9)

$$\operatorname{supp} f_{\mathfrak{c},v} \subset \mathbf{P}(F)\mathbf{D}(\mathfrak{c}_v) \text{ and } f_{\mathfrak{c},v}(\mathbf{p}u) = \delta^s(\mathbf{p})\chi^{-1}(d_{\mathbf{p}}),\ \mathbf{p} \in \mathbf{P}(F), u \in \mathbf{D}(\mathfrak{c}_v).$$

For $v \notin S$, χ_v is unramified. We let $f^\circ_{\chi,s,v}$ denote the unique section in $I_v(\chi, s)$ which is invariant by $\mathbf{K}_v^0$.

Definition 3.3. — *The local sections $\phi_{\chi,s,v}$ at $v \neq p$ are defined as follows.*

$$\phi_{\chi,s,v} = \begin{cases} f^\circ_{\chi,s}(g\mathbf{S}^{-1}) & v \notin S \cup \{p\}, \\ f_{\mathfrak{c},v}(g\mathbf{w}'\mathbf{S}^{-1}) & v \in S. \end{cases} \tag{3.10}$$

Put $\mathbf{u} = \begin{bmatrix} 1 & \\ & \varsigma\sigma^* \end{bmatrix}$. Let $\mathcal{L}_v = \mathcal{H}_n(F) \cap (\mathbf{u}M_n(2\mathfrak{c}_v)\mathbf{u}^*)$ be a lattice in $\mathcal{H}_n(E)$. Then one checks easily that $\phi_{\chi,s,v}$ for $v \in S$ is the unique section such that

$$\operatorname{supp} \phi_{\chi,s,v} = \mathbf{P}(F)\mathbf{w}\mathbf{N}(\mathcal{L}_v);$$
$$\phi_{\chi,s,S}(\mathbf{w}\mathbf{n}(u)) = \chi_v^{-1}(\det \mathbf{u}) \left|\det(\mathbf{u}\bar{\mathbf{u}})\right|_v^{-s}, u \in \mathcal{L}_v.$$

Note that $\phi_{\chi,s,v}$ for $v \in S$ is supported in the big cell.

We will define the local section at p in §3.5.

3.4.2. Fourier coefficients at $v \notin S$

When $v \notin S \cup \{p\}$, χ_v is unramified and $\phi_{\chi,s,v}(g) = f^\circ_{\chi,s,v}(g\mathbf{S}^{-1})$, where $f^\circ_{\chi,s,v}$ is the standard spherical section in $I_v(\chi, s)$. In this case, the Fourier coefficients have been calculated by Shimura.

PROPOSITION 3.4 ([22] Prop. 19.2). — *Let* rank $\beta = r$. *We have*

$$\begin{aligned} W_\beta(f^\circ_{\chi,s,v}, 1) &= \int_{\mathbf{N}(\mathbb{Q}_v)} f^\circ_{\chi,s,v}(\mathbf{wn}(X_v))\mathbf{e}_{-\beta}(X_v)dX_v \\ &= \Lambda_{n,v}(s,\chi)^{-1}\Lambda^r_{n,v}(s,\chi)R_{\beta,v}(\chi_{+,v}(\varpi_v)\,|\varpi_v|_v^{2s})\mathbb{I}_{\mathcal{H}_n(\mathbb{Z}_v)^\vee}(\beta), \end{aligned}$$

where

$$\Lambda_{n,v}(s,\chi) = \prod_{j=0}^{n-1} L_v(2s-j, \chi_+\tau^j_{\mathcal{K}/\mathbb{Q}}),$$
$$\Lambda^r_{n,v}(s,\chi) = \prod_{j=0}^{n-r-1} L_v(2s-n-j, \chi_+\tau^{n+j-1}_{\mathcal{K}/\mathbb{Q}})$$

and $R_{\beta,v}(T)$ *is a polynomial which only depends on* β *and equals to* 1 *for almost* v*.*

We only need this result when $\beta > 0$ and $\Lambda^\beta_{n,v}(s,\chi) = 1$.

3.4.3. Fourier coefficients at $v \in S$

Notations are as in §3.4. The local Fourier coefficient of $\phi_{\chi,v,s}$ is

$$\begin{aligned} W_\beta(\phi_v, 1) &= \chi^{-1}(\det u)\,|\det u\bar{u}|_v^{-2s} \int_{\mathcal{H}_n(\mathbb{Q}_v)} \mathbb{I}_{\mathcal{L}}(X_v)\mathbf{e}_{-\beta}(X_v)dX_v \\ &= \chi^{-1}(\det u)\,|\det u\bar{u}|_v^{-2s}\,\mathbb{I}_{\mathcal{L}_v^\vee}(\beta)\,\mathrm{vol}(\mathcal{L}, dX_v) \end{aligned} \tag{3.11}$$

3.5. The local section at $v = p$

3.5.1. Gauss sum and ϵ-factor

Let $d^\times x$ be the Haar measure on $\mathbb{Q}_p^\times$ such that $\mathrm{vol}(\mathbb{Z}_p^\times, d^\times x) = 1$. Let μ be a character of $\mathbb{Q}_p^\times$. We let $c_p(\mu) = |c(\mu)|_p^{-1}$ if μ is ramified and $c_p(\mu) = p$ if μ is unramified. We define the Gauss sum $G(\mu)$ by

$$G(\mu) = \int_{c_p(\mu)^{-1}\mathbb{Z}_p^\times} \mu(x)\mathbf{e}_p(x)d^\times x.$$

Then $\epsilon(s,\mu) := G(\mu|\cdot|^s, \mathbf{e}_p)^{-1}$ is Tate's ϵ-factor. It is easy to verify that $\epsilon(s,\chi) = |N|^s\,\chi(N)\tau(\mu)$, where $\tau(\mu)$ is the classial Gauss sum which only

depends on $\mu|_{\mathbb{Z}_p^\times}$. Define a Bruhat-Schwartz function Φ_μ on $\mathbb{Q}_p$ by $\Phi_\mu(x) := \mu(x)\mathbb{I}_{\mathbb{Z}_p^\times}(x)$. Then one can compute its Fourier transform easily:
(3.12)
$$\hat{\Phi}_\mu(y) = \begin{cases} \chi^{-1}(y)G(\mu)\mathbb{I}_{N^{-1}\mathbb{Z}_p^\times}(y) & \text{if } \mu \text{ is ramified with conductor } N, \\ \mathbb{I}_{\mathbb{Z}_p} - |p|\mathbb{I}_{p^{-1}\mathbb{Z}_p} & \text{if } \mu \text{ is unramified.} \end{cases}$$

We now introduce the modified p-Euler factor $E_p(s,\mu)$.

Definition 3.5 (Modified p-Euler factor). —
$$E_p(s,\mu) := Z(s,\mu,\widehat{\Phi}_\mu) = \int_{\mathbb{Q}_p} \mu(x)\,|x|^s\,\hat{\Phi}_\mu(x)d^\times x.$$

The following identity inspired by (3.2a)[14] is our key to the construction of the local section.

$$E_p(s,\mu) = \frac{L_p(s,\mu)}{L_p(1-s,\mu^{-1})\epsilon(s,\mu)}. \tag{3.13}$$

3.5.2. Some Bruhat-Schwartz functions

We introduce some special Bruhat-Schwartz functions on $M_3(\mathbb{Q}_p)$.

Definition 3.6. — *Let J, N and N_2 be three p-power integers. Let $\mathcal{K}_3(J)$ be the subset of $M_6(\mathbb{Q}_p)$ of the form*
$$\begin{bmatrix} p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ J\mathbb{Z}_p & \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p^\times & p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}.$$
Let $I_0(N,N_2)$ be the subset of $M_6(\mathbb{Q}_p)$ such that $I_0(N,N_2)^t$ the transpose of $I_0(N,N_2)$ is of the form
$$\begin{bmatrix} \mathbb{Z}_p^\times & N\mathbb{Z}_p & N\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times & N_2\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}.$$

For $\mu = (\mu_2,\mu_3)$, we let $\Phi_1^{\mu,J}$ be the Bruhat-Schwartz function in $M_3(\mathbb{Z}_p)$ such that
$$\Phi_1^{\mu,J}(Z) = \mathbb{I}_{\mathcal{K}_3(J)}(Z)\mu_2(Z_{22})\mu_3(p^{-1}Z_{31}),\ Z = (Z_{ij}).$$
Set
$$\Phi_2^{\nu,N}(Z) = \mathbb{I}_{I_0(N,N_2)^t}(Z)\prod \nu_i(Z_{ii}).$$

Given μ and ν, we let $N_2 = c_p(\nu_2)$ and choose J and N such that

$$N \geqslant J > \max\{c_p(\mu_i), c_p(\nu_i)\}. \tag{3.14}$$

Then (3.14) implies that $I_0(N, N_2)$ is a group. Let $\Phi_1 = \Phi_1^{\mu,J}$ and $\Phi_2 = \Phi_2^{\nu,N}$, which satisfy the following properties:

Lemma 3.7. —

(1) $\Phi_1(tZ) = \mu_2(t_2)\mu_3(t_3)\Phi(Z)$ *and* $\Phi_1(Zt) = \mu_2(t_2)\mu_3(t_1)\Phi_1(Z)$ *for* $t = diag(t_1, t_2, t_3)$.
(2) $\hat{\Phi}_2(Z + M_3(\mathbb{Z}_p)) = \hat{\Phi}_2(Z)$ *and* $\hat{\Phi}_2(Zt) = \hat{\Phi}_2(tZ) = \nu(t)^{-1}\hat{\Phi}_2(Z)$ *for* $t \in T_3(\mathbb{Z}_p)$.
(3) $\Phi_1(Zn) = \Phi_1(Z)$ *for* $n \in N_3(\mathbb{Z}_p)$

Proof. — Straightforward verification from the definitions and (3.14). □

3.5.3. The local section at p

In this subsection, we define the local Godement section at p. Let $(\chi_1, \chi_2) = (\chi_{\mathfrak{p}}, \chi_{\overline{\mathfrak{p}}}) = \chi_p$ and $\chi_+ = \chi_1\chi_2$. For $\underline{z} = (z_1, z_2)$ a character of $W_2 = \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ of finite order., we define $\mu = (\mu_2, \mu_3)$ and $\nu = (\nu_1, \nu_2, \nu_3)$ as follows

(1) $\nu_1 = \chi_+$, $\nu_2 = \chi_1 z_2$ and $\nu_3 = \chi_1 z_1$,
(2) $\mu_2 = \chi_2^{-1} z_2$ and $\mu_3 = \chi_2^{-1} z_1$.

Recall that the Fourier transform $\hat{\Phi}$ for $\Phi \in \mathcal{S}(M_3(\mathbb{Q}_p))$ is defined by

$$\hat{\Phi}(Z) = \int_{M_3(\mathbb{Q}_p)} \Phi(X)\mathbf{e}(\mathrm{Tr}({}^tXZ))dX.$$

Define $\Phi_{\underline{z}}^{J,\underline{N}}$ a Bruhat-Schwartz function on $M_{3\times 6}(\mathbb{Q}_p)$ by

$$\Phi_{\underline{z}}^{J,\underline{N}}(X, Y) := \Phi_1(X)\widehat{\Phi_2}(Y).$$

We consider the Godement section f_Φ associated to $\Phi = \Phi_{\underline{z}}^{J,\underline{N}}$.

$$f_\Phi(g) := \chi_2(\det g)|\det g|^s \int_{\mathrm{GL}_3(\mathbb{Q}_p)} \Phi((0, Z)g)\chi_1\chi_2(\det Z)|\det Z|^{2s}d^\times Z. \tag{3.15}$$

The section f_Φ has the following properties.

Lemma 3.8. —

(1) $f_\Phi(gt) = (z_1, z_2, x_2, x_1, x_2x_1^{-1}z_2^{-1}, x_2x_1^{-1}z_1^{-1})f_\Phi(g)$ *for* $t \in T_6(\mathbb{Z}_p)$.
(2) f_Φ *is* $N_6(\mathbb{Z}_p)$*-invariant.*

Proof. — (1) follows from Lemma 3.7. As for (2), for $A, D \in N_3(\mathbb{Z}_p)$ and $B \in M_3(\mathbb{Z}_p)$, we have $\Phi_1(XA) = \Phi_1(X)$ since $J \geqslant c_p(\mu_2)$. If $X \in \operatorname{supp}\Phi_1$, we have $XB \in M_3(\mathbb{Z}_p)$. Therefore by Lemma 3.7 (1) and (2), we have

$$\Phi(X,Y)\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}) = \Phi_1(X)\hat{\Phi}_2(YD + XB) = \Phi_1(X)\hat{\Phi}_2(YD) = \Phi(X,Y)$$

□

3.5.4. Local Fourier coefficient at p

We compute the local Fourier coefficients of f_Φ. To emphasize the dependence of the local Fourier coefficient on $\underline{z}$, χ and β, we put

$$W_p(\beta;\chi,\underline{z}) := W_\beta(f_\Phi, 1).$$

Proposition 3.9. —

$$(3.16)\quad W_p(\beta;\chi,\underline{z}) = \left|p^{-1}J\right| \operatorname{vol}(I_0(N,N_2), d^\times Z)\chi_+(\det\beta)\left|\det\beta\right|^{2s-3}\cdot \mathbb{I}_{\mathcal{K}_3(p)}(\beta)H(\beta),$$

where

$$H(\beta) = z_1 z_2^{-1}(p^{-1}\beta_{31})\cdot \chi_2^{-1}z_2(-p^{-1}\det\begin{bmatrix} \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}).$$

In particular, $W_p(\beta;\chi,\underline{z}) = 0$ *if* $\det\beta = 0$.

Proof. — By definition, we have

$$(3.17)\quad W_p(\beta;\chi,\underline{z})$$
$$= \int_{M_3(\mathbb{Q}_p)}\int_{\mathrm{GL}_3(\mathbb{Q}_p)} \Phi((0,Z)\begin{bmatrix} 0 & -1 \\ 1 & X \end{bmatrix})\chi_+(\det Z)\left|\det Z\right|^{2s} d^\times Z\mathbf{e}_{-\beta}(X)dX$$
$$= \int_{\mathrm{GL}_3(\mathbb{Q}_p)} \Phi_1(Z)\Phi_2({}^tZ^{-1t}\beta)\chi_+(\det Z)\left|\det Z\right|^{2s-3} d^\times Z.$$

As Φ_2 is supported in non-degenerate matrices, we see $W_p(\beta;\chi,\underline{z}) = 0$ if $\det\beta = 0$. If $\det\beta \neq 0$, we make a change of variable $Z \mapsto Z\beta$. Then (3.17) equals

$$\chi_+(\det\beta)\left|\det\beta\right|^{2s-3}\int_{\mathrm{GL}_3(\mathbb{Q}_p)} \Phi_1(Z\beta)\Phi_2({}^tZ^{-1})\chi_+(\det Z)|\det Z|^{2s}d^\times Z$$
$$= \chi_+(\det\beta)\left|\det\beta\right|^{2s-3}\int_{I_0(N,N_2)} \Phi_1(Z\beta)\Phi_2({}^tZ^{-1})\chi_+(\det Z)d^\times Z$$

For $Z \in I_0(N, N_2)$, we write $Z = utn_{23}(x)$, where

$$u \in \begin{bmatrix} 1 & \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & 1 & 0 \\ N\mathbb{Z}_p & N_2\mathbb{Z}_p & 1 \end{bmatrix}, t \in T_3(\mathbb{Z}_p), \text{ and } n_{23}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \; x \in \mathbb{Z}_p.$$

Then by Lemma 3.7 we have $\Phi_1(Z\beta) = \mu(t)\Phi_1(n_{23}(x)\beta)$ and $\Phi_2({}^tZ^{-1}) = \nu(t)^{-1}$. Since $\mu\nu^{-1}\chi_+ = 1$, we find that

(3.18)
$$W_p(\beta; \chi, \underline{z}) = \chi_+(\det \beta) \left|\det \beta\right|^{2s-3} \operatorname{vol}(I_0(N, N_2), d^\times Z) \int_{\mathbb{Z}_p} \Phi_1(n_{23}(x)\beta) dx.$$

We use the following lemma to compute (3.18).

Lemma 3.10. — *Let $\mathcal{K}_2(J) = \begin{bmatrix} J\mathbb{Z}_p & \mathbb{Z}_p^\times \\ p\mathbb{Z}_p^\times & p\mathbb{Z}_p \end{bmatrix}$ and define the function Q_{λ_1,λ_2} on $M_4(\mathbb{Q}_p)$ by*

$$Q_{\lambda_1,\lambda_2}(g) = \mathbb{I}_{\mathcal{K}_2(J)}(g)\lambda_1(b)\lambda_2(p^{-1}c), \; g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we have

$$\int_{\mathbb{Z}_p} Q_{\lambda_1,\lambda_2}(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g) dx = \left|p^{-1}J\right| \mathbb{I}_{\mathcal{K}_2(p)}(g)\lambda_1^{-1}\lambda_2(p^{-1}c)\lambda_1(-p^{-1}\det g).$$

Proof. — Write $Q = Q_{\lambda_1,\lambda_2}$. Then

$$\begin{aligned} &\int_{\mathbb{Z}_p} Q(n(x) \begin{bmatrix} a & b \\ c & d \end{bmatrix}) dx \\ &= \int_{\mathbb{Z}_p} Q(\begin{bmatrix} a + xc & b + xd \\ c & d \end{bmatrix}) \\ &= \lambda_2(p^{-1}c)\mathbb{I}_{p\mathbb{Z}_p^\times}(c)\mathbb{I}_{p\mathbb{Z}_p}(d) \int_{\mathbb{Z}_p} \mathbb{I}_{J\mathbb{Z}_p}(a + xc)\lambda_1(b + xd) dx \\ &= \lambda_2(p^{-1}c)\mathbb{I}_{p\mathbb{Z}_p^\times}(c)\mathbb{I}_{p\mathbb{Z}_p}(d) \int_{\mathbb{Z}_p} \mathbb{I}_{\mathbb{Z}_p^\times}(b)\lambda_1(b)\mathbb{I}_{J\mathbb{Z}_p}(pc^{-1}a + xp)\lambda_1(1 + xb^{-1}d) dx \\ &= \lambda_2(p^{-1}c)\mathbb{I}_{\mathcal{K}_2(p)}(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\lambda_1(b) \int_{\mathbb{Z}_p} \mathbb{I}_{J\mathbb{Z}_p}(pc^{-1}a + xp)\lambda_1(1 + xb^{-1}d) dx. \end{aligned}$$

Since $b^{-1}dp^{-1}J\mathbb{Z}_p \subset J\mathbb{Z}_p$, the integral $\int_{\mathbb{Z}_p} \mathbb{I}_{J\mathbb{Z}_p}(pc^{-1}a + xp)\lambda_1(1 + xb^{-1}d)dx$ equals

$$\int_{-c^{-1}a+p^{-1}J\mathbb{Z}_p} \lambda_1(1 - c^{-1}ab^{-1}d)dx$$
$$= \left|p^{-1}J\right| \lambda_1(pc^{-1}b^{-1}) \cdot \lambda_1(-p^{-1}\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}).$$

□

To proceed the computation of (3.18), we note that

$$\Phi_1(n(x)\beta) = Q_{\mu_2,\mu_3}(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \beta).$$

Therefore the formula follows from Lemma 3.10 and the fact that $\mu_3\mu_2^{-1} = z_1z_2^{-1}$. □

Lemma 3.11 (Trace computation). — *Let $u \in \mathrm{GL}_3(\mathbb{Q}_p)$. Then we have*

$$W_\beta(f_\Phi, \begin{bmatrix} \mathbf{1}_3 & \\ & u \end{bmatrix}) = \chi_2^{-1}(\det u)\,|\det u|^{s+3}\, W_p(u^{-1}\beta; \chi, \underline{z}). \tag{3.19}$$

In particular if $x \in c_p(z_1z_2^{-1})\mathbb{Z}_p$, then

$$W_\beta(f_\Phi, \begin{bmatrix} \mathbf{1}_3 & \\ & u_-(x) \end{bmatrix}) = W_\beta(f_\Phi, 1),\ u_-(x) = \begin{bmatrix} 1 & & \\ & 1 & 0 \\ & x & 1 \end{bmatrix}$$

Proof. — We have

$$u \cdot \hat{\Phi}_2(Z) = \hat{\Phi}(Zu) = \int \Phi(Y)\mathbf{e}({}^tYZu)dY = \int \Phi(Y)\mathbf{e}({}^t(Y{}^tu)Z)dY$$
$$= \int \Phi(Y{}^tu^{-1})\mathbf{e}({}^tYZ)dY = \widehat{{}^tu^{-1} \cdot \Phi}.$$

Following the same computation in Prop. 3.9, we have

$$\begin{aligned}
&W_\beta(f_\Phi, \begin{bmatrix}\mathbf{1}_3 & \\ & u\end{bmatrix})\\
&= \chi_2(\det u)|\det u|^s \cdot W_\beta((I_3 \otimes u)\cdot\Phi, 1)\\
&= \chi_2(\det u)|\det u|^s \cdot \chi_+(\det\beta)|\det\beta|^{2s-3}\\
&\qquad \int_{\mathrm{GL}_3(\mathbb{Q}_p)} \Phi_1(Z\beta)\widehat{u\hat{\Phi}_2}({}^tZ^{-1})\chi_+(\det Z)|\det Z|^{2s}d^\times Z\\
&= \chi_2(\det u)|\det u|^s \cdot \chi_+(\det\beta)|\det\beta|^{2s-3}\\
&\qquad \int_{\mathrm{GL}_3(\mathbb{Q}_p)} \Phi_1(Z\beta)\Phi_2({}^t(Zu)^{-1})\chi_+(\det Z)|\det Z|^{2s}d^\times Z\\
&= \chi_2(\det u)\,|\det u|^{s+3}\, W_p(u^{-1}\beta;\chi,\underline{z}).
\end{aligned}$$

The second assertion follows from (3.16) directly. □

3.6. Normalization

In this subsection, we take a suitable normalization of the Siegel-Eisenstein series on $GU(3,3)$ attached to the section we have defined. Put

$$NC(\chi,s) = C_{\mathcal{K}}(3)^{-1}\cdot\Lambda_3^{S\cup\{p\}}(s,\chi)\cdot\frac{1}{|p^{-1}J|_p\cdot\mathrm{vol}(I_0(N,N_2),d^\times Z)}.$$

Definition 3.12. — *Let $\Phi = \Phi_{\underline{z}}^{J,\underline{N}}$ be as in §3.5. Define the section $\phi_{\chi,s}\in I(\chi,s)$ and its normalization $\phi^*_{\chi,s}$ by*

$$\phi_{\chi,s} = \otimes_{v\neq p}\phi_{\chi,s,v}\otimes f_\Phi,\ \phi^*_{\chi,s} := NC(\chi,s)\cdot\phi_{\chi,s}.$$

Define the normalized adelic *Siegel-Eisenstein series by*

$$\begin{aligned}
E_{\mathbb{A}}(g,s,\chi,\underline{z},\mathfrak{c}) &= NC(\chi,s)\cdot E_{\mathbb{A}}(g,\phi_{\chi,s}) = E_{\mathbb{A}}(g,\phi^*_{\chi,s})\\
&= \sum_{\gamma\in\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})}\phi^*_{\chi,s}(\gamma g),\ g\in\mathbf{G}(\mathbb{A}).
\end{aligned}$$

When $k>3$, $E_{\mathbb{A}}(g,s,\chi,\underline{z},\mathfrak{c})$ converges absolutely at $s=0$ (cf. [22] *and* [9]*). Thus $E_{\mathbb{A}}(g,s,\chi,\underline{z},\mathfrak{c})|_{s=0}$ is an element in $\mathcal{A}^{\mathrm{Hol}}_{(0,k)}(\mathbf{G},\mathbf{K}_1^\infty,\chi^{-3})$, where $\mathbf{K}$ is defined in* (3.8).

Let $E(\chi,\underline{z},\mathfrak{c})$ be the associated holomorphic Siegel-Eisenstein series as in (2.18). *For $(Z,g)\in X_{3,3}\times\mathbf{G}(\mathbb{A}_f)$, we have*

$$\begin{aligned}
E(\chi,\underline{z},\mathfrak{c})(Z,g) &:= \underline{\mathrm{AM}}(E_{\mathbb{A}}(-,\phi^*_{\chi,s})|_{s=0})(Z,g)\\
&= \chi^3(\nu(g))J_{(0,k)}(g_\infty,\mathbf{i})E_{\mathbb{A}}((g_\infty,g),s,\chi,\underline{z},\mathfrak{c})|_{s=0},
\end{aligned}$$

where $g_\infty \in \mathbf{G}(\mathbb{R})^+$ *with* $g_\infty \mathbf{i} = Z$.

By the inspection on Fourier coefficients of $E(\chi, \underline{z}, \mathfrak{c})$, *this normalized Eisenstein series in fact is* independent *of the choice of* $\underline{N} = (N, N_2)$ *and* J *(See Remark 3.14).*

Proposition 3.13. — *Let* $g \in \mathbf{G}(\mathbb{A}_f)$ *such that* $g_p = g_v = 1$ *at some* $v \in S$. *Then* $E(\chi, \underline{z}, \mathfrak{c})$ *has the following Fourier expansion:*

$$\mathcal{F}_g(E(\chi, \underline{z}, \mathfrak{c})) = \sum_{0<\beta\in\mathcal{H}_3(\mathbb{Q})} a_\beta(g, \chi, \underline{z}) q^\beta,$$

where $a_\beta(g, \chi, \underline{z}) = a^h_\beta(g, \chi, \underline{z}) \det \beta^{k-3}$ *and*

$$\begin{aligned} a^h_\beta(1, \chi, \underline{z}) &= \chi_{+,p}(\det \beta) |\det \beta|_p^{-3} H(\beta) \cdot \chi_S(\det \mathbf{u}^{-1}) \cdot D(\chi, \mathcal{L}) \\ &\times \prod_{v \notin S\cup\{p\}, v\nmid \det \beta} R_{\beta,v}(\chi_v(\varpi_v)) \cdot \mathbb{I}_\Xi(\beta), \end{aligned} \tag{3.20}$$

where

$$D(\chi, \mathcal{L}) = \chi^{S\cup\{p\}}(\det \sigma) \chi_S^{-1}(\det \mathbf{u}) \, |\det \sigma\bar{\sigma}|_{\mathbb{A}^S}^{-3} \prod_{v\in S} \mathrm{vol}(\mathcal{L}_v, dx_v)$$

and

$$\Xi = \prod_{v\in S} \mathcal{L}_v^\vee \times \mathcal{K}_3(p) \times \prod_{v\notin S\cup\{p\}} \mathcal{H}_3(\mathbb{Z}_v)^\vee$$

is a compact subset in $\mathcal{H}_3(\mathbb{A}_f)$ *independent of* χ *and* $\underline{z}$. *Moreover, given* $g = \begin{bmatrix} A & \\ & A^{-*} \end{bmatrix}$, $A \in \mathrm{GL}_3(\mathbb{A}^{(p)}_{\mathcal{K},f}) \times \mathrm{GL}_3(\mathscr{O}_\mathcal{K} \otimes_\mathbb{Z} \mathbb{Z}_p)$, *we have*

$$a_\beta(g, \chi, \underline{z}) = \chi^{-1}(\det A) \, |A^* A|^3 \cdot a^h_{A^*\beta A}(1, \chi, \underline{z}) \det \beta^{k-3}. \tag{3.21}$$

Proof. — Since the support of $\phi_{\chi,s,v}$ at $v \in S$ is in the big cell, the β-th Fourier coefficient of $E_\mathbb{A}(g, \phi_{\chi,s})$ is decomposed into a product of local coefficients if $x_v = 1$ at some $v \in S$. Hence the formula (3.20) and (3.21) follows from

$$W_\beta(\begin{bmatrix} A & \\ & A^{-*} \end{bmatrix}, \phi) = \chi^{-1}(\det A) |\det A^* A|^{s+n} W_{A^*\beta A}(1, \phi) \tag{3.22}$$

together with Prop. 3.2, (3.11), Prop. 3.9 and Prop. 3.4. □

Remark 3.14. — By q-expansion principle over $\mathbb{C}$ and the formulae of the Fourier coefficients, the Eisenstein series $E(\chi, \underline{z}, \mathfrak{c})$ is independent of the choice of J and $\underline{N}$ in $\Phi_{\underline{z}}^{J,\underline{N}}$. Also, by Lemma 3.11 we have

$$E(\chi, \underline{z}, \mathfrak{c})(Z, g \begin{bmatrix} \mathbf{1}_3 & \\ & u \end{bmatrix}) = E(\chi, \underline{z}, \mathfrak{c})(Z, g), \; u = \begin{bmatrix} 1 & & \\ & 1 & 0 \\ & c_p(z_1 z_2^{-1})\mathbb{Z}_p & 1 \end{bmatrix}.$$

3.7. p-adic Siegel-Eisenstein series

In this subsection, we construct an Eisenstein measure with values in the space of p-adic Siegel-Eisenstein series. We need to introduce some notation Let $\mathcal{K}(\mathfrak{c}p^n)$ be the ray class field of $\mathcal{K}$ with conductor $\mathfrak{c}p^n$ and let $\mathfrak{G} = \varprojlim_n \mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^n/\mathcal{K})$. Then $\mathfrak{G}$ is a $\mathbb{Z}_p$-module of rank two. Let $\mathcal{C}(\mathfrak{G}, \mathbb{C}_p)$ be the set of continuous $\mathbb{C}_p$-valued functions on $\mathfrak{G}$. Let $\mathscr{W}_0$ be the set of locally algebraic p-adic characters of $\mathfrak{G}$ with infinity type $(k, 0)$, $k \geqslant 3$. Then $\mathscr{W}_0$ is a Zariski-dense subset in $\mathcal{C}(\mathfrak{G}, \mathbb{C}_p)$. We shall regard p-adic Galois characters as p-adic Hecke characters of $\mathcal{K}$ by geometrcially normalized reciprocity law. Recall that to an algebraic Hecke character χ of $\mathcal{K}$ we have associated $\hat{\chi}$ its p-adic avatar in the introduction.

To construct a p-adic measure on $\mathfrak{G}$, we recall the "Abstract Kummer congruences". ([15, Prop. 4.0.6] or [10, Lemma 3.4.1]).

Lemma 3.15 (Abstract Kummer congruences). — *Let $\mathbb{V}$ be a p-adic Banach space. We consider measures on $\mathfrak{G}$ with values in $\mathbb{V}$. Let $\chi \mapsto m_\chi$ a function from $\mathscr{W}_0$ to $\mathbb{V}$, and let $\lambda(m)$ denote the corresponding $\mathbb{V}[1/p]$-valued measure. Then $\lambda(m)$ extends a p-adic measure if and only if, for every integer m and for any finite sum $\sum_j \alpha_j \chi_j$ with $\alpha_j \in R[\frac{1}{p}]$ and $\chi_j \in \mathscr{W}_0$ such that $\sum_j \alpha_j \chi_j(t) \in p^m R$ for all $t \in \mathfrak{G}$, we have*

$$\sum_j \alpha_j m_{\chi_j} \in p^m \mathbb{V}.$$

Recall that $V_p(\mathbf{G}, \mathbf{K})$ is the space of p-adic modular forms for $\mathbf{G} = GU(3,3)$ as in §2.8.4. Let $C(\mathbf{G}) = \mathbf{P}(\mathbb{A}_f^p) \times \mathbf{P}(\mathbb{Z}_p)$. For an $\mathscr{O}_{\mathcal{K}}$-algebra $R = R_{(p)} \subset \mathbb{C}$, by q-expansion principle we have

$$\mathbf{M}_{\underline{k}}(\mathbf{K}_1^n, \xi, R) = \left\{ f \in \mathbf{M}_{\underline{k}}(\mathbf{K}_1^n, \xi, \mathbb{C}) \mid \mathcal{F}_{[g]}(f) \in \prod_{\beta \in \mathcal{H}_3(\mathcal{K})} R \cdot q^\beta, \forall g \in C(\mathbf{G}) \right\}.$$

Let $\widehat{E}(\chi, \underline{z}, \mathfrak{c})$ denote the p-adic avatar of $E(\chi, \underline{z}, \mathfrak{c})$. Note that a Siegel modular form f over $\overline{\mathbb{Q}}$ of weight $\underline{k} = (a, b)$ with Fourier expansion $\mathcal{F}_{[g]}(f)$ at $g = (g^p, g_p)$, $g_p = \begin{bmatrix} A & \\ & D \end{bmatrix}$, the Fourier expansion of the p-adic avatar $\widehat{f}$ at g is

$$\mathcal{F}_{[g]}(\widehat{f}) = (\det A)^a (\det D)^{-b} \iota_p(\mathcal{F}_{[g]}(f)).$$

Theorem 3.16. — *Let $W_2 = (\mathbb{Z}_p^\times)^2$. There exists an Eisenstein measure $d\mathcal{E}_{3,3}$ on $\mathfrak{G} \times W_2$ such that for $\hat{\chi} \in \mathscr{W}_0$ and $\underline{z} = (z_1, z_2) \in W_2$, the set of* *finite order characters of* W_2*, we have*

$$\int_{\mathfrak{G}\times W_2} (\hat{\chi}, \underline{z}) d\mathcal{E}_{3,3} = \widehat{E}(\chi, \underline{z}, \mathfrak{c}) \in V_p(\mathbf{G}, \mathbf{K})$$

Proof. — For $(\hat{\chi}, \underline{z}) \in \mathscr{W}_0 \times W_2$, we define

$$\int_{\mathfrak{G}\times W_2} (\hat{\chi}, \underline{z}) d\mathcal{E}_{3,3} = \widehat{E}(\chi, \underline{z}, \mathfrak{c}).$$

We verify the above indeed gives a well-defined measure by Lemma 3.15. Write $\chi_1 = \chi_{\mathfrak{p}}$, $\chi_2 = \chi_{\overline{\mathfrak{p}}}$ and $\chi_{+,p} = \chi_1\chi_2$. For $\beta \in \mathcal{H}_3(\mathbb{Q})$, the β-th Fourier expansion of $\widehat{E}(\chi, \underline{z}, \mathfrak{c})$ at the infinity cusp (3.20) is given by

(3.23)
$$\begin{aligned} &\iota_p(a^h_\beta(1, \chi, \underline{z}) \det \beta^{k-3}) \\ = &\ D(\widehat{\chi}, \mathscr{L}) \cdot \widehat{\chi}_{+,p}(\det \beta) \cdot \widehat{\chi}_2(u_3) \prod_{v\in S^B, v\nmid \det \beta} R_{\beta,v}(\widehat{\chi}_{+,v}(\varpi_v)) \cdot z_1(u_1) z_2(u_2) \\ \times &\ (\det \beta)^{-3} |\det \beta|_p^{-3} \cdot \mathbb{I}_{\Xi}(\beta), \end{aligned}$$

where u_1, u_2 and u_3 are some p-adic units which *only* depend on β, and $R_{\beta,v}$ is a polynomial with coefficients in $\mathbb{Z}$. In general by (3.20) for $a_1, a_2 \in \mathrm{GL}_3(\mathbb{Z}_p)$, the "$a_1\beta a_2$"-th coefficient

$$\iota_p \left[a^h_{a_1\beta a_2}(1, \chi, \underline{z}) (\det \beta)^{k-3}\right] \det(a_1 a_2)^k$$

also has the form

$$\sum_i b_i \cdot (\widehat{\chi}, \underline{z})(c_i), \text{ for some } c_i \in \mathbb{A}^\times_{\mathcal{K},f} \times W_2,\ b_i \in \mathbb{Z}_{(p)}. \tag{3.24}$$

The β-th Fourier coefficient of $\widehat{E}$ at the cusp $x = \begin{bmatrix} a_1 & \\ & a_2^{-1} \end{bmatrix} \in \mathbf{G}(\mathbb{Z}_p) \subset \mathbf{G}(\mathbb{A}_f)$ is given by

$$\begin{aligned} &\det a_2^k \cdot \iota_p \left(a_\beta(x, \chi, \underline{z})\right) \\ = &\ \det a_2^k \cdot \iota_p \left[\chi_1^{-1}(\det a_1) \chi_2^{-1}(\det a_2) \left|\det a_2 a_1\right|_p^3 a^h_{a_2\beta a_1}(1, \chi, \underline{z}) \det \beta^{k-3}\right] \\ = &\ \iota_p \left[a^h_{a_2\beta a_1}(1, \chi, \underline{z}) \det \beta^{k-3}\right] \det(a_2 a_1)^k \cdot \hat{\chi}_1(\det a_1^{-1}) \hat{\chi}_2(\det a_2^{-1}) \end{aligned}$$

which also has the form as in (3.24). Since $\mathscr{W}_0 \times W_2$ is a Zariski-dense subset in $\mathcal{C}(\mathfrak{G} \times W_2, \mathbb{C}_p)$, by q-expansion principle for unitary groups ([13] and [3]) and the abstract Kummer congruences, we obtain the desired measure. □

4. Eisenstein series on $GU(3,1)$ and the pull back formula

4.1.

Let $\underline{W}$ and $\underline{V}$ be the quadruples as in §3.1 with $n=3$. Let $\mathfrak{s}$ be a positive integer. We define the skew-Hermitian form θ by $\begin{bmatrix}\mathfrak{s}\delta & \\ & \delta\end{bmatrix}$ according to the basis w^1, w^2. Let N_P and M_P be subgroups of $G = GU(V)$ given by

$$N_P = \left\{ \mathrm{n}(x,t) = \begin{bmatrix} 1 & x & t - \frac{1}{2}x\theta x^* \\ & \mathbf{1}_2 & -\theta x^* \\ & & 1 \end{bmatrix} \mid x \in \mathcal{K}^2,\, t \in \mathbb{Q} \right\}$$

and

$$M_P = \left\{ \mathrm{m}(a,h) = \begin{bmatrix} \nu(h)\bar{a}^{-1} & & \\ & h & \\ & & a \end{bmatrix} \mid a \in \mathcal{K}^\times,\, h \in GU(W) \right\}.$$

Then $P := M_P N_P$ is the standard parabolic subgroup in G. Put

$$w = \begin{bmatrix} & & -1 \\ & -\mathbf{1}_2 & \\ 1 & & \end{bmatrix}.$$

Then w is the unique nontrivial Weyl element with respect to P. For $g \in P(\mathbb{A})$, we put

$$\delta_P(g) = |\nu(g)(d\bar{d})^{-1}|_{\mathbb{A}}.$$

4.2. Open compact subgroups

For $v \in \mathbf{h}$, put $\theta_{1,v} = \sigma_v \theta \sigma_v^*$. Define an open-compact subgroup $D_W(\mathfrak{c}) = \prod_{v\in\mathbf{h}} D_W(\mathfrak{c})_v$ of $U(W)(\mathbb{A}_f)$ by

$$D_W(\mathfrak{c})_v = \left\{ g \in K(L_v) \mid \theta_{1,v}^{-1}(\sigma_v g \sigma_v^{-1} - 1) \prec \mathfrak{c}_v \right\} \text{ if } v \neq p \tag{4.1}$$

$$D_W(\mathfrak{c})_p = I_0(p) := \left\{ g_p \in \mathrm{GL}_2(\mathbb{Z}_p) \mid g_p \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod{p} \right\}. \tag{4.2}$$

Let $X = U(W)(\mathbb{Q})\backslash U(W)(\mathbb{A}_f)$ and $X_W(\mathfrak{c}) = X/D_W(\mathfrak{c})$. We assume that $\mathfrak{c}_v$ at some $v \in S$ is sufficient small so that the right $D_W(\mathfrak{c})$-action on X is free ([22, Lemma 24.3]). As $U(W)$ is a definite unitary group, $X_W(\mathfrak{c})$ is a finite set. By the weak approximation, we can find a set of representatives $\{b\}$ in $X_W(\mathfrak{c})$ such that $b_v = 1$ at $v \in S \cup \{p\}$. We will identify $X_W(\mathfrak{c})$ with this set.

4.3. Automorphic representations and automorphic forms on $GU(2)$

4.3.1.

Put
$$B = \{g \in M_2(\mathcal{K}) \cong \mathrm{End}_{\mathcal{K}}(W) \mid g\theta g^* = \det(g)\theta\}.$$
Then it is well known that B is a definite quaternion algebra over $\mathbb{Q}$ with local invariants $\mathrm{inv}_v(B) = (-\mathfrak{s}, -D)_v$. We denote by $\mathcal{S}_2(B^\times, \mathbb{C})$ the space of automorphic forms on $B^\times$ of weight 2. Namely
$$\mathcal{S}_2(B^\times, \mathbb{C}) = \left\{ \text{ locally constant } \mathbb{C}\text{-valued functions on } B^\times\backslash B^\times(\mathbb{A}_f)\right\}.$$

Let ξ be a Hecke character of $\mathcal{K}$ of finite order and let π^B be an irreducible automorphic representation in $\mathcal{S}_2(B^\times, \mathbb{C})$ with central character $\xi_+ := \xi|_{\mathbb{A}}$. By the choice of B, we have $GU(W) = B^\times \times_{\mathbb{Q}^\times} \mathcal{K}^\times$. Hence $\Pi := \pi^B \boxtimes \xi$ can be regarded an irreducible automorphic representation on $GU(W)$. Let $\mathfrak{n}$ be the conductor of π^B. We make the following assumption:
$$p^2 \nmid \mathfrak{n}.$$
In addition to (S1), we further suppose throughout the paper that the ideal $\mathfrak{c}$ chosen in the beginning of §3.2 is sufficiently small so that

(S2) $$\mathfrak{c} \subset c(\xi)c(\pi^B).$$

DEFINITION 4.1. — *For a subring $A \subset \mathbb{C}$, we put*
$$\mathcal{S}_2(B^\times, \mathfrak{n}, \xi_+, A) = \{\mathbf{f} \in \mathcal{S}_2(B^\times, \mathbb{C}) \mid \mathbf{f}(zgu) = \mathbf{f}(g)\xi_+(z),\ \forall u \in I_1(\mathfrak{n})\},$$
the space of weight two modular forms of of level $\mathfrak{n}$ with neben type ξ_+ over A.

In the remainder of this section, having fixed $B^\times$, $\mathfrak{n}$ and ξ_+, we simply write $\mathcal{S}_2(A)$ for this space. The isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$ induces $\iota : \mathcal{S}_2(\mathbb{C}) \cong \mathcal{S}_2(\mathbb{C}_p)$. Then since π^B is ordinary at p, there is an element up to scalar $\mathbf{f}$ in $\mathcal{S}_2(\mathbb{C})$ such that $\mathbf{f}$ is an eigenform of the U_p-operator and the eigenvalue is a p-unit with respect to ι. We can further manage $\iota(\mathbf{f}) \in \mathcal{S}_2(\mathbb{C}_p)$ to be *p-primitive*, which means that $\iota(\mathbf{f})$ takes value in $\overline{\mathbb{Z}}_p$ and $\iota(\mathbf{f}) \not\equiv 0 \pmod p$. We define a distinguished element $\tau_{\mathfrak{n}} = \prod \tau_v \in B^\times(\mathbb{A}_f)$
$$\tau_v = \begin{bmatrix} & -1 \\ \mathfrak{n}_v & \end{bmatrix} \text{ if } v|\mathfrak{n}\,;\ \tau_v = 1,\ v \nmid \mathfrak{n}.$$
Then the map
$$f \mapsto f|\tau_{\mathfrak{n}}(g) = f(g\tau_{\mathfrak{n}}),\ f \in \mathcal{S}_2(A)$$
defines an involution on $\mathcal{S}_2(A)$.

4.3.2. Local representations at p

We have fixed a decomposition $\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p = \mathbb{Q}_p e^+ \oplus \mathbb{Q}_p e^-$ with the idempotents in §1.1. Then as a subgroup in $\mathrm{GL}_2(\mathcal{K}\otimes\mathbb{Q}_p)$, we identify $GU(W)(\mathbb{Q}_p)$ with $\mathrm{GL}_2(\mathbb{Q}_p)\times\mathbb{Q}_p^\times$ by $g\mapsto(e^+g,\nu(g))$, so $B^\times(\mathbb{Q}_p)$ and $U(W)(\mathbb{Q}_p)$ as subgroups in $\mathrm{GL}_2(\mathbb{Q}_p)\times\mathbb{Q}_p^\times$ are isomorphic to $\mathrm{GL}_2(\mathbb{Q}_p)$. We make these isomorphisms precise as follows.

$$\begin{aligned} &\mathrm{GL}_2(\mathbb{Q}_p) \xrightarrow{\sim} B^\times(\mathbb{Q}_p),\ g\mapsto(g,\det(g)),\\ &\mathrm{GL}_2(\mathbb{Q}_p) \xrightarrow{\sim} U(W)(\mathbb{Q}_p)\,, g\mapsto(g,1) \end{aligned} \tag{4.3}$$

We regard the local representation π_p^B as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the above identification. We further assume π^B is *ordinary* at p. Namely $\pi_p^B=\pi(\delta_1,\delta_2)$ where δ_1 and δ_2 are two characters on $\mathbb{Q}_p^\times$. Moreover we may further assume that δ_2 is unramified with $v_p(\delta_1(p))=\frac{1}{2}$ and $v(\delta_2(p))=-\frac{1}{2}$. Thus π_p^B is isomorphic to a sub-representation of the unitary induced representation $I(\delta_1,\delta_2)$.

Write $\xi_p=(\xi_{\mathfrak{p}},\xi_{\overline{\mathfrak{p}}})$. According to the identification (4.3), as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)\times\mathbb{Q}_p^\times$

$$\Pi_p = I(\delta_1\xi_{\overline{\mathfrak{p}}}^{-1},\delta_2\xi_{\overline{\mathfrak{p}}}^{-1})\boxtimes\xi_{\overline{\mathfrak{p}}}.$$

Similarly, as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$

$$\Pi_p|_{U(W)(\mathbb{Q}_p)} = I(\xi_{\overline{\mathfrak{p}}}^{-1}\delta_1,\xi_{\overline{\mathfrak{p}}}^{-1}\delta_2).$$

We will fix the choice of these models in the remainder of the paper.

4.4. The embedding $U(V)\times U(W)\hookrightarrow U(\mathbf{W})$

4.4.1.

Recall that $\varsigma=-\frac{\theta}{2}$. Let i denote the diagonal embedding

$$\begin{aligned} &i: U(V)\times U(W)\to U(\mathbf{W}),\\ &\qquad i:(\alpha,\gamma)\mapsto(\alpha,\gamma)\in\mathrm{GL}(V\oplus W)=\mathrm{GL}(\mathbf{W}). \end{aligned}$$

As an embedding between matrix groups i can written as

$$\begin{aligned} &U(V)\times U(W)\hookrightarrow U(\mathbf{W})\\ &\qquad(\alpha,\gamma)\hookrightarrow i(\alpha,\gamma)=(\alpha,\gamma)_\Delta := \Delta^{-1}\begin{bmatrix}\alpha & 0\\ 0 & \gamma\end{bmatrix}\Delta, \end{aligned}$$

where

$$\Delta = \Delta(\theta) = \begin{bmatrix} 1 & & & \\ & \mathbf{1}_2 & & -\varsigma \\ & & 1 & \\ & -\mathbf{1}_2 & & \varsigma^* \end{bmatrix}. \tag{4.4}$$

4.4.2. Imbedding of Hermitian symmetric domains

Let $X_{3,1}$ and $X_{2,0}$ be the Hermitian symmetric domains as in §2.4.1. We have the following embedding of Hermitian symmetric domains:

$$\begin{array}{rcl} X_{3,1} \times X_{2,0} & \hookrightarrow & X_{3,3} \\ (\tau, \mathbf{x}_0) & \hookrightarrow & Z_\tau, \end{array}$$

where $Z_\tau = \begin{bmatrix} x & 0 \\ y & \varsigma \end{bmatrix}$ for $\tau = \begin{bmatrix} x \\ y \end{bmatrix}$ is as in (2.12). We write for $\alpha \in G$

$$\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix}.$$

Let $\mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & \varsigma \end{bmatrix}$. Then the automorphy factors are given by

$$J((\alpha, \gamma)_\Delta, \mathbf{i}) = \det \begin{bmatrix} hi + d & 0 \\ gi + f & \hat{\gamma} \end{bmatrix} = j(\alpha, i) \det \gamma.$$

The embedding defined in (4.4) at the archimedean places is compatible with the identification $\mathbf{G}(\mathbb{R})/K_\infty \cong X_{3,3}$ in an obvious sense.

4.4.3.

We record some formulae for the future use. Put

$$\omega = \begin{bmatrix} & & -1 & \\ & \theta & & \\ 1 & & & \\ & & & -\theta \end{bmatrix}, \eta = \begin{bmatrix} & & -1 & \\ & & & -\mathbf{1}_2 \\ 1 & & & \\ & \mathbf{1}_2 & & \end{bmatrix}.$$

Then $\Delta \eta \Delta^* = \omega$.

Lemma 4.2. —

$$(4.5)\qquad (\alpha,\gamma)_\Delta = \begin{bmatrix} a & b & c & \frac{1}{2}b\theta \\ \frac{1}{2}g & \frac{1}{2}(e+\gamma) & \frac{1}{2}f & \frac{1}{4}(e-\gamma)\theta \\ h & l & d & \frac{1}{2}l\theta \\ \theta^{-1}g & \theta^{-1}(e-\gamma) & \theta^{-1}f & \frac{1}{2}(\hat{e}+\hat{\gamma}) \end{bmatrix},$$

where $\hat{x} = \theta^{-1}x\theta$

Proof. — It follows form straightforward computation. Put

$$Y = \begin{bmatrix} 1 & & & \\ & \mathbf{1}_2 & & \varsigma \\ & & 1 & \\ & & & \mathbf{1}_2 \end{bmatrix}.$$

Then

$$(4.6)\qquad (\alpha,\gamma)_\Delta = Y \begin{bmatrix} a & b & c & 0 \\ g & e & f & 0 \\ h & l & d & 0 \\ \theta^{-1}g & \theta^{-1}(e-\gamma) & \theta^{-1}f & \hat{\gamma} \end{bmatrix} Y^{-1}$$

Put

$$A = \begin{bmatrix} a & b \\ g & e \end{bmatrix}, B = \begin{bmatrix} c & 0 \\ f & 0 \end{bmatrix}, C = \begin{bmatrix} h & l \\ \theta^{-1}g & \theta^{-1}(e-\gamma) \end{bmatrix}, D = \begin{bmatrix} d & 0 \\ \theta^{-1}f & \hat{\gamma} \end{bmatrix}, L = \begin{bmatrix} 0 & 0 \\ 0 & -\varsigma \end{bmatrix}.$$

Then

$$AL = \begin{bmatrix} 0 & -b\varsigma \\ 0 & -e\varsigma \end{bmatrix}$$

$$CL = \begin{bmatrix} 0 & -l\varsigma \\ 0 & -\theta^{-1}(e-\gamma)\varsigma \end{bmatrix}$$

$$LD = \begin{bmatrix} 0 & 0 \\ -\varsigma\theta^{-1}f & -\varsigma\hat{\gamma} \end{bmatrix}$$

$$LC = \begin{bmatrix} 0 & 0 \\ -\varsigma\theta^{-1}g & -\varsigma\theta^{-1}(e-\gamma) \end{bmatrix}$$

$$LCL = \begin{bmatrix} 0 & 0 \\ 0 & \varsigma\theta^{-1}(e-\gamma)\varsigma \end{bmatrix}$$

and

$$
\begin{aligned}
&(\alpha,\gamma)_\Delta\\
&=\begin{bmatrix}1 & -L\\ 0 & 1\end{bmatrix}\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\\
&=\begin{bmatrix}A-LC & B+AL-LCL-LD\\ C & D+CL\end{bmatrix}\\
&=\begin{bmatrix}a & b & c & -b\varsigma\\ g+\varsigma\theta^{-1}g & e+\varsigma\theta^{-1}(e-\gamma) & f+\varsigma\theta^{-1}f & -e\varsigma-\varsigma\theta^{-1}(e-\gamma)\varsigma+\varsigma\hat{\gamma}\\ h & l & d & -l\varsigma\\ \theta^{-1}g & \theta^{-1}(e-\gamma) & \theta^{-1}f & \hat{\gamma}-\theta^{-1}(e-\gamma)\varsigma\end{bmatrix}
\end{aligned}
$$

□

In particular, when $\gamma=\pi(\alpha):=e$, we find that

$$(\alpha,\pi(\alpha))_\Delta=\begin{bmatrix}a & b & c & -b\varsigma\\ \frac{1}{2}g & e & \frac{1}{2}f & 0\\ h & l & d & -l\varsigma\\ \theta^{-1}g & 0 & \theta^{-1}f & \hat{e}\end{bmatrix}.$$

4.5. Eisenstein series on $GU(V)$ and the pull-back formula

4.5.1. Induced representations on $GU(V)$ and the pull-back section

Let χ be the Hecke character of $\mathcal{K}$ in §3.2. Given an automorphic representation Π of $GU(W)$, we define the induced representation $I(\Pi,\chi,s)$ of G by

$$I(\Pi,\chi,s)=\{\phi:G(\mathbb{A})\to\Pi\mid\phi(pg)=\chi^{-1}(d_p)\delta_P(p)^s\Pi(e_p)\phi(h)\}.$$

Let $\varphi=\otimes_v\varphi_v$ be a decomposable vectorin $\pi^B=\otimes\pi_v^B$. We identify φ with the section $\varphi\boxtimes\xi$ in Π. Let $f_s=\otimes_v f_{s,v}$ be a decomposable section of $I(\chi,s)$. Put

$$\varphi.f_{s,v}(g):=\int_{U(W)(\mathbb{Q}_v)}f_{s,v}((g,hh')_\Delta)\chi_v(\det(hh'))\Pi_v(hh')\varphi\,dh, \tag{4.7}$$

where $g\in G(\mathbb{Q}_v)$ and h' is any element in $GU(W)(\mathbb{Q}_v)$ such that $\nu(h')=\nu(g)$. We call $\varphi.f_{s,v}$ is the local pull-back section of $f_{s,v}$ by φ_v. It is easy to see that $\varphi.f_{s,v}(h)$ is independent of the choice of g' and $\varphi.f_{s,v}$ is an element in $I_v(\Pi,\chi,s)$.

The global pull-back section $\varphi.f_s$ of f_s by φ is defined as follows. For $g=(g_v)\in G(\mathbb{A})$,

(4.8)
$$\varphi.f_s(g) := \int_{U(W)(\mathbb{A})} f_s((g,hh')_\Delta)\chi(\det(hh'))\Pi(hh')\varphi\,dh = \otimes_v(\varphi.f_{s,v})(g_v).$$

It will follows from Prop. 5.5 that the local and global pull-back sections $f_s\cdot\varphi$ converge absolutely when $\operatorname{Re} s\gg 0$ and has meromorphic continuation to $\mathbb{C}$.

4.5.2. Eisenstein series on $GU(V)$

Fix a $B^\times(\mathbb{A})$-embedding

$$A:\ \pi^B = \bigotimes_v{}' \pi_v^B \hookrightarrow \mathcal{S}_2(B^\times,\mathbb{C}).$$

Let $A_{\varphi,\xi}=A(\varphi)\boxtimes\xi$ be an automorphic form for $GU(W)$. We observe that

$$\begin{aligned}&A(\varphi.f_s(g))(x)\\ &\quad= \int_{U(W)(\mathbb{A}_f)} f_s((g,hh')_\Delta)\chi(\det hh')A_{\varphi,\xi}(xhh')dh,\ x\in GU(W)(\mathbb{A}_f).\end{aligned}$$

Define a $\mathbb{C}$-valued function $I(\varphi.f_s)$ on $M(\mathbb{Q})N(\mathbb{A})\backslash G(\mathbb{A})$ by

$$I(\varphi.f_s)(g)=A(\varphi.f_s(g))(1),\ g\in G(\mathbb{A}).$$

By the general theory for Eisenstein series in [19], to $\varphi.f_s$ we can associate an Eisenstein series $E_{\mathbb{A}}(g,\varphi.f_s)$ for G defined by

$$E_{\mathbb{A}}(g,\varphi.f_s) := \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} I(\varphi.f_s)(\gamma g). \tag{4.9}$$

4.5.3. The pull-back formula

In [22], Shimura proves the following pull back formula:

Theorem 4.3. — *Let f_s be as above. Then*

$$E_{\mathbb{A}}(g,\varphi.f_s) = \int_{U(W)(\mathbb{Q})\backslash U(W)(\mathbb{A})} E_{\mathbb{A}}((g,h)_\Delta, f_s)\chi(\det h)A_{\varphi,\xi}(h)dh.$$

Recall that we have introduced δ_1 and δ_2 characters of $\mathbb{Q}_p^\times$ in §4.3.2. Let $\underline{z} = (\xi_{\overline{\mathfrak{p}}}^{-1}\delta_1, \xi_{\overline{\mathfrak{p}}}^{-1}\delta_2)$ be a pair of characters of $\mathbb{Q}_p^\times$, to which we attach $f_\Phi \in I_p(\chi, s)$ in §3.5.3 with auxiliary choices of integers $J, \underline{N}$. Define the section $\phi_{\chi,s}^\Upsilon \in I(\chi, s)$ by

$$\phi_{\chi,s}^\Upsilon(g) = \bigotimes_{v\neq p} \phi_{\chi,s,v}(g) \bigotimes_{v=p} f_\Phi(g\Upsilon),\ g \in \mathbf{G}(\mathbb{A})$$
$$\text{where } \Upsilon = \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & -\frac{1}{2} \\ & & 1 & \\ & \theta^{-1} & & \theta^{-1} \end{bmatrix} = \Delta^{-1}. \tag{4.10}$$

Let $\mathbf{f}$ be a primitive ordinary form as in §4.3. Then $\mathbf{f} = A(\varphi)$ for some $\varphi \in \pi^B$, and $\varphi = \otimes_v \varphi_v$ is decomposable. Now we define the Eisenstein series obtained by the pull-back of $\phi_{\chi,s}^\Upsilon$ and $\mathbf{f}|\tau_{\mathfrak{n}}$ by

(4.11)
$$E_{\mathbb{A}}(g, s, \chi \mid \mathbf{f}, \xi, \mathfrak{c}) = \frac{NC(\chi, s)}{\mathrm{vol}(D_W(\mathfrak{c}), dh)} \cdot E_{\mathbb{A}}(g, (\phi_{\chi,s}^\Upsilon)^{pb}),\ (\phi_{\chi,s}^\Upsilon)^{pb} := \phi_{\chi,s}^\Upsilon . \tau_{\mathfrak{n}}\varphi.$$

Let $E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ be the associated holomorphic Eisenstein series as in (2.18). For $(\tau, g) \in X_{3,1} \times G(\mathbb{A}_f)$, we have

$$\begin{aligned} E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})(\tau, g) &= \underline{\mathrm{AM}}(E_{\mathbb{A}}(-, s, \chi \mid \mathbf{f}, \xi, \mathfrak{c}))(\tau, g) \\ &= \chi(\nu(g)) j(g_\infty, i)^k E_{\mathbb{A}}((g_\infty, g), s, \chi \mid \mathbf{f}, \xi; \mathfrak{c})|_{s=0}, \end{aligned} \tag{4.12}$$

where $g_\infty \in G(\mathbb{R})^+$ such that $g_\infty i = \tau$. By definition, $E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ is the holomorphic Eisenstein series attached to the section

$$\frac{NC(\chi, s)}{\mathrm{vol}(D_W(\mathfrak{c}), dh)} \cdot (\phi_{\chi,s}^\Upsilon)^{pb}|_{s=0}. \tag{4.13}$$

Define an automorphic form $\mathbf{F}$ for $GU(W)$ by

$$\mathbf{F} = A(\tau_{\mathfrak{n}}.\varphi) \boxtimes \xi = \mathbf{f}|\tau_{\mathfrak{n}} \boxtimes \xi.$$

Applying the pull-back formula, we obtain

(4.14)
$$\begin{aligned} &E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})(\tau, g) \\ &= \frac{1}{\mathrm{vol}(D_W(\mathfrak{c}), dh)} \int_{U(W)(\mathbb{Q})\backslash U(W)(\mathbb{A}_f)} E(\chi, \underline{z}, \mathfrak{c})\,(Z_\tau, (g, h)_\Delta \Upsilon)\, \chi(\det h) \mathbf{F}(h) dh \\ &= \sum_{h \in X_W(\mathfrak{c})} \frac{1}{|\Gamma_b|} \cdot E(\chi, \underline{z}, \mathfrak{c})(Z_\tau, (g, h)_\Delta \Upsilon) \chi(\det h) \mathbf{F}(h), \end{aligned}$$

where $|\Gamma_h|$ is the order of the group $U(W)(\mathbb{Q}) \cap hD_W(\mathfrak{c})h^{-1}$. It is well known that $|\Gamma_h| = 1$ if $\mathfrak{c}$ is sufficiently small.

On the other hand, we regard $E(\chi, \underline{z}, \mathfrak{c})$ as geometric modular form over $\mathbb{C}$, and thus by the discussion in §2.6 and (2.14), we get

$$E(\chi, \underline{z}, \mathfrak{c})(i_{V,W}([\tau, g], [\mathbf{x}_0, h]), (\boldsymbol{\omega}_{V/\mathbb{C}}, \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma^c))) = E(\chi, \underline{z}, \mathfrak{c})\left([Z_\tau, (g,h)_\Delta \Upsilon], \boldsymbol{\omega}_{\mathbf{W}/\mathbb{C}}\right).$$

Therefore by (4.14)

$$E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})([\tau, g], \boldsymbol{\omega}_{V/\mathbb{C}}) = \sum_{h \in X_W(\mathfrak{c})} E(\chi, \underline{z}, \mathfrak{c})(i_{V,W}([\tau, g], [\mathbf{x}_0, h]), (\boldsymbol{\omega}_{V/\mathbb{C}}, \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma^c)))\chi(\det h)\mathbf{F}(h). \tag{4.15}$$

4.5.4. The measure attached to ordinary p-adic Eisenstein series

Let e be the ordinary projector on the space of modular forms for the unitary group $U(V)$ constructed by Hida in [12] and let $E^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c}) = e.E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ be the ordinary projection of $E(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$. Let $\widehat{E}^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ be the p-adic avatar of $E^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$. Then $\widehat{E}^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c}) = e.\widehat{E}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$, and by (4.15) we can deduce that

$$\widehat{E}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})(\underline{A}) = \sum_{h \in X_W(\mathfrak{c})} \widehat{E}(\chi, \underline{z}, \mathfrak{c})(\underline{A} \times \underline{\mathcal{B}_h}) \cdot \chi(\det h)\mathbf{F}(h) \tag{4.16}$$

for a test object $\underline{A} \in I_G(K_1^n)$. Now we can easily prove the following theorem.

Theorem 4.4. — *There exists a measure $d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}$ on $\mathfrak{G}$ such that for any algebraic p-adic character $\hat{\chi}$ of infinity type $(k, 0)$, we have*

$$\int_{\mathfrak{G}} \hat{\chi} d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}} = \widehat{E}^{ord}(\chi\xi^{-1} \mid \mathbf{f}, \xi, \mathfrak{c}),$$

where $\widehat{E}^{ord}(\chi\xi^{-1} \mid \mathbf{f}, \xi, \mathfrak{c})$ is an ordinary p-adic Eisenstein series of weight $(0, k)$.

Proof. — Note that ξ is of finite order, we see that $\widehat{\chi} \in \mathscr{W}_0 \iff \widehat{\chi}\xi^{-1} = \widehat{\chi\xi^{-1}} \in \mathscr{W}_0$. For $\phi \in \mathcal{C}(\mathfrak{G}, \mathbb{C}_p)$ and $h \in U(W)(\mathbb{A}_f)$, we put

$$\phi|h(x) = \phi(x \det h)\xi^{-1}(x \det h).$$

We define the measure $d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}$ by the following rule: for a test object $\underline{A} \in I_G(K_1^n)$,

$$\int_{\mathfrak{G}} \phi \, d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}(\underline{A}) := \sum_{h \in X_W(\mathfrak{c})} e. \int_{\mathfrak{G} \times \mathcal{W}_2} (\phi|h, \underline{z}) \, d\mathcal{E}_{3,3}(\underline{A} \times \underline{\mathcal{B}_h}) \cdot \mathbf{F}(h), \ \underline{z} = (\delta_1 \xi_{\overline{\mathfrak{p}}}^{-1}, \, \delta_2 \xi_{\overline{\mathfrak{p}}}^{-1}).$$

It is clear that the theorem follows from the definition of $d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}$ and (4.16). □

5. Constant term of the Eisenstein series

5.1. Constant term of modular forms and the Φ-operator

Let f be a holomorphic modular form on $G = GU(V)$ of weight $\underline{k} = (0, k)$. For the remainder of this article, we shall identify $GU(W)$ with the image G_P in M_P by $h \mapsto \mathrm{m}(1, h)$.

DEFINITION 5.1. — *Let Z_{N_P} be the center of N_P. For $g \in G(\mathbb{A}_f)$, the analytic Siegel Φ-operator at g is defined by*

$$\mathbf{M}_{\underline{k}}(G, \mathbb{C}) \to \mathcal{S}_2(G_P, \mathbb{C})$$
$$f \to \Phi^{an}_{[g]}(f)(h) = \int_{Z_{N_P}(\mathbb{Q})\backslash Z_{N_P}(\mathbb{A})} f(zh \cdot (\tau_0, g))dz,\ h \in G_P(\mathbb{A}_f).$$

The measure dz is normalized so that $\mathrm{vol}(Z_{N_P}(\mathbb{Q})\backslash Z_{N_P}(\mathbb{A}), dz) = 1$. It is well-known that this definition is independent of the choice of τ_0 and $\Phi^{an}_{[g]}(f) = \Phi^{an}_{[ng]}(f)$ for $n \in N_P(\mathbb{A})$ (See the discussion in [9, 2.2.1]*). On the other hand, the constant term for an adelic automorphic form F is defined by*

$$F_P(g) = \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} F(ng)dn.$$

The following lemma is evident.

LEMMA 5.2. — *Let $F \in \mathcal{A}^{\mathrm{Hol}}_{\underline{k}}(G)$ and $f = \underline{\mathrm{AM}}(F) \in \mathbf{M}_{\underline{k}}(G, \mathbb{C})$ be the associated holomorphic modular form, where $\underline{\mathrm{AM}}$ is defined in* (2.18)*. Then*

$$F_P(hg) = \Phi^{an}_{[g]}(f)(h).$$

It is well-known that the (adelic) constant term of $E = E(g, \phi)$ is

$$E_P(g) = \phi(g) + M_w(\phi)(g),\ w = \begin{bmatrix} & & -1 \\ & -\mathbf{1}_2 & \\ 1 & & \end{bmatrix}, \tag{5.1}$$

where if $\phi = \otimes_v \phi_v$,

$$M_w(\phi)(g) = \prod_v M_w(\phi_v)\,;\ M_w(\phi_v)(g) = \int_{N_P(\mathbb{Q}_v)} \phi(wng)dn. \tag{5.2}$$

5.2. The local pull-back section at ∞

We compute the local pull back section and the intertwining operator of the section in (4.11) at the archimedean place.

Proposition 5.3. —

(1) $\phi^{pb}_{\chi,s,\infty}(g) = j(g,\mathbf{i})^{-k}\delta^s(g)\cdot\varphi_\infty$.
(2) $M\phi^{pb}_{\chi,s,\infty}|_{s=0} = 0$.

Proof. — Since π_∞ is the trivial representation, we have

$$\begin{aligned}\phi^{pb}_{\chi,s,\infty}(g) &:= \int_{G(\mathbb{R})}\phi_{\chi,s,\infty}((g,h)_\Delta)\chi_\infty(\det h)\pi_\infty(h)\varphi_\infty dh\\ &= j(g,\mathbf{i})^{-k}\delta^s(g)\cdot\varphi_\infty.\end{aligned}$$

The first assertion follows. We proceed to show the second assertion. Write $\phi_\infty = \phi^{pb}_{\chi,s,\infty}$. To prove the proposition, it suffices to show $M\phi_\infty(1)|_{s=0} = 0$. Write $n = \mathrm{n}(b,t)\in N_P(\mathbb{R})$, $(b,t)\in\mathbb{C}^2\times\in\mathbb{R}$. Then

$$\begin{aligned}M\phi_\infty(1) &= \int_{N_P(\mathbb{R})}\phi_\infty(wn)dn\\ &= \int_{\mathbb{C}^2}\xi(1+i[b],0;s+k,s)db\wedge d\bar{b},\ [b]=\frac{1}{2}b\theta b^*\\ &= 2^{-2s+2}\pi\cdot i^{-k}\cdot\frac{\Gamma(2s+k-1)}{\Gamma(s+k)\Gamma(s)}\int_{\mathbb{C}^2}(1+i[b])^{1-k-2s}db\wedge d\bar{b}.\end{aligned}$$

The last integral equals

$$\begin{aligned}&\int_{\mathbb{C}^2}(1+i[b])^{1-k-2s}db\wedge d\bar{b}\\ =&\ \det(\frac{i\theta}{2})^{-2}\int_{\mathbb{C}^2}(1+bb^*)^{1-2s}db\wedge d\bar{b}\\ =&\ 4\pi^2(\det i\theta)^{-2}4^2\int_0^\infty\int_0^\infty(1+x_1^2+x_2^2)^{1-k-2s}x_1x_2dx_1dx_2\\ =&\ 2^4\pi^2(\det i\theta)^{-2}\frac{1}{(2s+k-2)(2s+k-3)}.\end{aligned}$$

Thus $M\phi_\infty(1)|_{s=0} = 0$. □

Corollary 5.4. — *The constant term of* $E(\chi\mid\mathbf{f},\xi,\mathfrak{c})$ *is given by the section defined in* (4.13).

Proof. — This follows from the Prop. 5.3, (5.1) and (5.2). □

5.3. The local pull-back section at unramified places

For the remainder of this section, we put $F = \mathbb{Q}_v$ and $E = \mathcal{K} \otimes \mathbb{Q}_v$ for $v \in \mathbf{h}$. Let O and $\mathcal{R}$ be the rings of integers of F and E respectively. In this subsection, we calculate the local pull-back section $\phi^{pb}_{\chi,s,v}$ at $v \notin S \cup \{p\}$. For $v \notin S$, note that $U(W)(F) \cong U(1,1)(F)$ if v is inert and $U(W)(F) \cong \mathrm{GL}_2(F)$ if v is split.

Let $f^{\circ}_{\pi,\chi,s,v}$ be the unique spherical section in $I_v(\pi,\chi,s)$ such that $f^{\circ}_{\pi,\chi,s,v}(1) = \varphi^{\circ}_v$.

Proposition 5.5. — *For $v \notin S \cup \{p\}$*

$$(f^{\circ}_{\chi,s,v})^{pb} = \frac{L_v(s-\frac{1}{2},\pi,\chi)}{\Lambda_{2,v}(s,\chi_+)} f^{\circ}_{\pi,\chi,s,v}$$

Proof. — The is equivalent to

$$(f^{\circ}_v)^{pb}(1) = \frac{L_v(s-\frac{1}{2},\pi,\chi)}{\Lambda_{2,v}(s,\chi_+)}\varphi^{\circ}_v.$$

The above local integrals have been computed by the doubling method in the following form. □

Proposition 5.6. — *For $\varphi' \in \pi^{\vee}$,*

$$\begin{aligned}\langle f^{pb}(1),\varphi'\rangle &= \int_{U(W)(F)} \phi_{\chi,s,v}((1,h)_{\Delta})\chi(\det h)\langle\pi(h)\varphi,\varphi'\rangle dh \\ &= \frac{L(s-\frac{1}{2},\pi,\chi)}{\Lambda_2(s,\chi_+)}\langle\varphi,\varphi'\rangle,\end{aligned}$$

where $\Lambda_2(s,\chi) = L(2s,\chi_+)L(2s-1,\chi_+\tau_{\mathcal{K}/\mathbb{Q}})$.

Proof. — This is the classical integral in the doubling method. We refer the computation to [6] and [17]. □

5.4. The local pull back section at $v \in S$

In this subsection, we calculation the pull-back section of $\phi_{\chi,s,v}$ when $v \in S$. Let $\theta_1 = \sigma^*\theta\sigma$. Because L is a O-maximal $\mathcal{R}$-lattice with respect to $2^{-1}\delta^{-1}\theta$ and δ is chosen to be a generator of the different of $\mathcal{K}/\mathbb{Q}$., we have $\theta_1 \prec 2\mathcal{R}$ (*cf.* [22, (20.6.1)]). For $(\alpha,\gamma) \in U(V)(F) \times U(W)(F)$, we write

$$\alpha_1 = \begin{bmatrix}1 & & \\ & \sigma_v & \\ & & 1\end{bmatrix}\alpha\begin{bmatrix}1 & & \\ & \sigma_v^{-1} & \\ & & 1\end{bmatrix} = \begin{bmatrix}a & b & c\\ g & e & f\\ h & l & d\end{bmatrix} \text{ and } \gamma_1 = \sigma_v\gamma\sigma_v^{-1}.$$

Write $\mathfrak{c} = \mathfrak{c}_v$ for simplicity. Define a subset $D_V(\mathfrak{c})$ in $G(F)$ by

(5.3)
$$D_V(\mathfrak{c}) = \left\{ \alpha \in K_v^0 \mid \alpha_1 \prec \begin{bmatrix} \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \theta_1\mathfrak{c} & \mathcal{R} & \theta_1\mathcal{R} \\ \mathfrak{c} & \mathfrak{c} & \mathcal{R} \end{bmatrix}, \theta_1^{-1}(e-1) \prec \mathfrak{c},\, d-1 \prec \mathfrak{c} \right\}.$$

If $\alpha \in D_V(\mathfrak{c})$, the relation $\alpha\theta_{3,1}\alpha^* = \theta$ implies $d^*a + f^*\theta_1 - c^*h = 1$, hence $a - 1 \prec \mathfrak{c}$. From the identity $\alpha^{-1} = \theta_{3,1}\alpha^*\theta_{3,1}^{-1} \in K_v^0$, we see that

$$\alpha_1^{-1} = \begin{bmatrix} -d^* & f^*\theta_1^{-1} & -c^* \\ -\theta_1 l^* & \theta_1 e^*\theta_1^{-1} & \theta_1 b^* \\ -h^* & g^*\theta_1^{-1} & a^* \end{bmatrix} \prec \mathcal{R}. \tag{5.4}$$

This shows $D_V(\mathfrak{c})$ is an open-compact subgroup in $G(F)$.

Let $f_{\mathbf{S}}(g) = f_{\mathfrak{c},v}(g\mathbf{S}^{-1})$ be the section defined in (3.9). Now we compute the pull-back section $f_{\mathbf{S}}^{pb}$ of $f_{\mathbf{S}}$.

Proposition 5.7. — *$f_{\mathbf{S}}^{pb}$ is the unique section in $I(\pi, \chi, s)$ such that*

(1) $\operatorname{supp} f_{\mathbf{S}}^{pb} \subset P(F)D_V(\mathfrak{c})$,

(2) *For $\alpha \in D_V(\mathfrak{c})$, $f_{\mathbf{S}}^{pb}(\alpha) = \operatorname{vol}(D_W(\mathfrak{c}), dh)\tilde{\varphi}_v$, where $D_W(\mathfrak{c})$ is the group defined in* (4.1).

Proof. — Let α be in the support of $f_{\mathbf{S}}^{pb}$. To prove (1), we may assume $\alpha \in K_v^0$ by the Iwasawa decomposition. By definition, we have

$$\alpha_1 \prec \mathcal{R},\, \theta_1^{-1} f \prec \mathcal{R};\, \frac{1}{2} f^*\theta_1 f = f^*\theta_1^{-1}(\frac{1}{2}\theta_1)\theta_1^{-1} f \prec \mathcal{R}.$$

Because $f_{\mathbf{S}}((\alpha,\gamma)_\Delta) \neq 0$ for some $\gamma \in U(W)(F)$, we have $\mathbf{S}(\alpha,\gamma)_\Delta\mathbf{S}^{-1} = \Delta_1^{-1}(\alpha_1,\gamma_1)\Delta_1 \in \mathbf{P}(F)\mathbf{D}(\mathfrak{c})$, where $\Delta_1 = \mathbf{S}\Delta\mathbf{S}^{-1} = \Delta(\theta_1)$ as in (4.4). By the formula (4.6), we find that

$$\begin{aligned} &\begin{bmatrix} 1 & \\ & \theta_1^{-1} \end{bmatrix} \begin{bmatrix} d & 0 \\ f & \gamma_1 \end{bmatrix}^{-1} \begin{bmatrix} h & l \\ g & e-\gamma_1 \end{bmatrix} \\ &= \begin{bmatrix} d^{-1}h & d^{-1}l \\ \theta_1^{-1}\gamma_1^{-1}(g - fd^{-1}h) & \theta_1^{-1}(\gamma_1^{-1}(e - fd^{-1}l) - 1) \end{bmatrix} \prec \mathfrak{c}. \end{aligned}$$

Observe that

$$\begin{aligned} &\begin{bmatrix} 1 & & \\ & \gamma_1^{-1} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -f^*\theta_1^{-1} & \frac{1}{2}f^*\theta_1^{-1}f \\ & 1 & -f \\ & & 1 \end{bmatrix} \begin{bmatrix} \bar{d} & 0 & 0 \\ & 1 & 0 \\ & & d^{-1} \end{bmatrix} \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix} \\ &= \begin{bmatrix} * & * & * \\ \gamma_1^{-1}(g - fd^{-1}h) & \gamma_1^{-1}(e - fd^{-1}l) & 0 \\ d^{-1}h & d^{-1}l & 1 \end{bmatrix} \in D_V(\mathfrak{c}). \end{aligned}$$

Hence $\alpha \in P(F)D_V(\mathfrak{c})$. This completes the proof of (1).

We proceed to prove (2). If $\alpha \in D_V(\mathfrak{c})$, then by Lemma 4.2 and (5.4) together with the fact that $\theta_1 \prec 2\mathcal{R}$, we can deduce that

$$\mathbf{S}(\alpha, 1)_\Delta \mathbf{S}^{-1} \in \mathbf{D}_0(\mathfrak{c})\,;\ \frac{1}{2}(\theta_1^{-1} e_1 \theta_1 + 1) \equiv 1\,(\mathrm{mod}\ \mathfrak{c}).$$

Because $\mathfrak{c}$ is sufficiently small as in (S2), we conclude that

$$f_{\mathbf{S}}((\alpha, 1)_\Delta) = \chi^{-1}(\det \sigma^*).$$

Now we use the following lemma.

Lemma 5.8. — *Let $h \in U(W)(F)$, then*

$$(1, h)_\Delta \mathbf{S}^{-1} \in \mathbf{P}(F)\mathbf{D}(\mathfrak{c}) \Leftrightarrow h \in D_W(\mathfrak{c}).$$

Proof. — Since

$$Y_1 \mathbf{S}(1, h_1)_\Delta \mathbf{S}^{-1} Y_1^{-1} = \begin{bmatrix} 1 & & & \\ & \mathbf{1}_2 & & \\ & & 1 & \\ & \theta_1^{-1}(1 - h_1) & & \theta_1 h_1 \theta_1^{-1} \end{bmatrix}$$

with $h_1 = \sigma h \sigma^{-1}$, we have

$$\begin{aligned} (1, h_1)_\Delta \in \mathbf{P}(F)\mathbf{D}(\mathfrak{c}) &\Leftrightarrow \theta^{-1}(h_1^{-1} - 1) = y \text{ for some } y \prec \mathfrak{c} \\ &\Leftrightarrow h \in D_W(\mathfrak{c}) \end{aligned}$$

□

By Prop. 5.7,

$$f_{\mathbf{S}}((1, h)_\Delta) = \chi^{-1}(\det h)\mathbb{I}_{D_W(\mathfrak{c})}(h).$$

Therefore

$$\begin{aligned} f_{\mathbf{S}}^{pb}(\alpha) &= \int_{U(W)(F)} f_{\mathbf{S}}((\alpha, 1)_\Delta (1, h)_\Delta)\chi(h)\pi(h) \cdot \tilde{\varphi}\, dh \\ &= \chi^{-1}(\det \sigma^*) \int_{D_W(\mathfrak{c})} f_{\mathbf{S}}(1, h)_\Delta)\chi(h)\pi(h) \cdot \tilde{\varphi} dh \\ &= \chi^{-1}(\det \sigma^*)\,\mathrm{vol}(D_W(\mathfrak{c}), dh)\tilde{\varphi}. \end{aligned}$$

This completes the proof of (2) □

Prop. 5.7 shows that $f_{\mathbf{S}}^{pb}$ is the unique section in $I(\pi, \chi, s)$ such that

$$\begin{aligned} &\mathrm{supp}\, f_{\mathbf{S}}^{pb} = P(F)D_V(\mathfrak{c}) \\ &f_{\mathbf{S}}^{pb}(\alpha) = \chi^{-1}(\det \sigma^*) \cdot \mathrm{vol}(D_W(\mathfrak{c}), dh)\tilde{\varphi},\ \alpha \in D_W(\mathfrak{c}). \end{aligned}$$

Now we are ready to determine $\phi^{pb}_{\chi,s,v}$. Note that $\phi_{\chi,s,v} = f_{\mathbf{S}}|\mathbf{w}'$, $\mathbf{w}' = (w, \mathbf{1}_2)_\Delta$. Thus

$$\begin{aligned}\phi^{pb}_{\chi,s,v} = (f_{\mathbf{S}}|\mathbf{w}')^{pb}(g) &= \int_{U(W)(F)} f_{\mathbf{S}}((g,h)_\Delta \mathbf{w}')\chi(\det h)\pi(h)\tilde{\varphi} dh \\ &= f^{pb}_{\mathbf{S}}(gw),\ w = \begin{bmatrix} & & -1 \\ & -\mathbf{1}_2 & \\ 1 & & \end{bmatrix}.\end{aligned}$$

Now the following proposition is straightforward.

Proposition 5.9. —

$$\phi^{pb}_{\chi,s,v} = \mathrm{vol}(D_W(\mathfrak{c}), dh_v) f^{\mathfrak{c}}_{\chi,\pi,s,v},$$

where $f^{\mathfrak{c}}_{\chi,\pi,s,v}$ is the unique section supported in $P(F)D_V(\mathfrak{c})w^{-1}$ and

$$f^{\mathfrak{c}}_{\chi,\pi,s,v}(uw^{-1}) = \chi_v^{-1}(\det \sigma^*)\tilde{\varphi}_v,\ u \in D_V(\mathfrak{c}).$$

6. The ordinary projection of the local pull-back section

6.1.

We have computed the local pull back section $\phi^{pb}_{\chi,s,v}$ at places other than p. In this section, we compute the ordinary projection $e.\phi_{\chi,s,p}$ of the pull-back section $(f_\Phi|\Upsilon)^{pb}$ by using the ordinary linear functional for a regular principal series. This idea is inspired by Hida's proof of multiplicity one theorem for ordinary vectors ([13, Theorem 5.3]).

6.2. The ordinary linear functional

6.2.1.

In this subsection, we let B denote the standard Borel subgroup of GL_n, T and N denote the diagonal matrices and the unipotent radical of B respectively. We let W be the Weyl group of GL_n with respect to T. Consider the (unitary) induced representation $\mathscr{I} = \mathrm{Ind}_B^{\mathrm{GL}_n}(\lambda_1, \cdots, \lambda_n)$. We assume $\mathscr{I}$ is *regular*, which means the p-adic valuation of $\lambda_i(p)$ are distinct. We call these numbers $v_p(\lambda_i(p))$ p-adic *weights*.

Let $D = \{d \in T(\mathbb{Q}_p) \mid d^{-1}N(\mathbb{Z}_p)d \subset N(\mathbb{Z}_p)\} = \{\mu(p) \mid \langle \mu, R_+\rangle \geqslant 0\}$. Let $\mathcal{A}_p := \mathbb{Z}_p[t_1, t_2, \cdots, t_n, t_n^{-1}]$ be the Atkin-Lehner ring of $G(\mathbb{Q}_p)$, where t_i is defined by

$$t_i = [N(\mathbb{Z}_p)\alpha_i N(\mathbb{Z}_p)],\ \alpha_i = \begin{bmatrix} \mathbf{1}_{n-i} & \\ & p\cdot \mathbf{1}_i \end{bmatrix}.$$

$t_i \in \mathcal{A}_p$ acts on $\mathscr{I}^{N(\mathbb{Z}_p)}$ by

$$v|t_i = \sum_{x \in N/\alpha_i^{-1}N\alpha_i} x_i \alpha_i^{-1} \cdot v.$$

We also define the action of $\mathcal{A}_p$ on the Jacquet module $J(\mathscr{I}) = \mathscr{I}_N$ of $\mathscr{I}$ by

$$\bar{v}|t_i = \delta(\alpha_i)\alpha_i^{-1} \cdot \bar{v}$$

so that the natural projection $\mathscr{I} \twoheadrightarrow J(\mathscr{I})$ is a $\mathcal{A}_p$-module homomorphism. Hida proves the following theorem in [13].

Theorem 6.1. — *Let $\mathscr{I}^\circ$ be the maximal subspace of $\mathscr{I}^N$ on which the action of $\mathcal{A}_p$ is semisimple. Then the natural projection induces an isomorphism as $\mathcal{A}_p$-modules*

$$\mathscr{I}^\circ \cong J(\mathscr{I}).$$

The linear functionals l_w on $J(\mathscr{I})$ for $w \in W$ are defined by $l_w(f) := M_w(f)(1_n)$, the evaluation of the intertwining operator at the identity. It is well-known that l_w enjoys the following properties:

$$l_w(u \cdot f) = l_w(f) \text{ and } l_w(t \cdot f) = \lambda(wtw^{-1})\delta^{\frac{1}{2}}(t) l_w(f)$$
$$\text{for } u \in U(\mathbb{Q}_p),\ t \in T(\mathbb{Q}_p).$$

By definition,

$$l_w(f) = \int_{U_w^- \backslash U} f(wu)du,$$
$$\text{where } U_w^- = \prod_{\alpha>0, w\alpha>0} U_\alpha(\mathbb{Q}_p),\ w\alpha(t) := \alpha(w^{-1}tw)$$

whenever the integral is convergent. These l_w's induce a $T(\mathbb{Q}_p)$-equivariant map

$$\bigoplus_{w\in W} l_w : J(\mathscr{I}) \longmapsto \bigoplus_{w \in W} \mathbb{C}\cdot \lambda^w \delta^{\frac{1}{2}}, \lambda^w(t) := \lambda(wtw^{-1}).$$

Now we define the ordinary function l_{w_3} as follows. Let $\mu_i \in X_*(T)$ be a cocharacter such that

$$\mu_i(x) = \begin{bmatrix} \mathbf{1}_{i-1} & & \\ & x & \\ & & \mathbf{1}_{n-i} \end{bmatrix} \in x \in \mathbb{Q}_p^\times.$$

For $w \in W$, we put

$$a_i = v_p(\lambda(\mu_i(p))) \in \mathbb{Z}.$$

Since $\mathscr{I}$ is regular, these a_i are distinct integers. Hence we can let W act on $\{a_i\}_{i=1}^n$ by $a_{w.i} = v_p(\lambda^w(\mu_i(p)))$. We let w_3 be the unique element in W such that

$$a_{w_3.1} < a_{w_3.2} < \cdots < a_{w_3.n}.$$

Let $\alpha = \sum_{i=1}^n (i-1)\mu_i$ and let $t_\alpha = [N(\mathbb{Z}_p)\alpha(p)N(\mathbb{Z}_p)] = t_1 t_2 \cdots t_{n-1}$. Then the normalized u_p-operator on $\mathscr{I}$ is defined by

$$v||t_i := \delta(\alpha_i)^{-\frac{1}{2}}\lambda^{w_3}(\alpha_i)v|t_i \text{ and } u_p.v = v||t_\alpha. \tag{6.1}$$

The Hida's idempotent e attached u_p is defined by

$$e = \lim_{n\to\infty}(u_p)^{n!}. \tag{6.2}$$

Now we prove the following key lemma.

Lemma 6.2. —

$$l_w(e.v) = \begin{cases} l_{w_3}(v) & w = w_3, \\ 0 & w \neq w_3. \end{cases}$$

Proof. — Put $b_{w.i} = \sum_{j=i}^n a_{w.j}$. Then

$$v_p(\lambda(\alpha(p))) = \sum_{i=1}^n i \cdot a_{w.i} = \sum_{i=2}^n b_{w.i}.$$

It is easy to see that if $w \neq w_3$,

$$b_{w_3.i} \geqslant b_{w.i} \text{ and } b_{w_3.i_0} > b_{w.i_0} \text{ for some } i_0.$$

Put $D_w = \sum_i b_{w_3.i} - b_{w.i} \geqslant 0$ and $D_w = 0$ only if $w = w_3$. From the identity

$$l_w(v|t_\alpha) = \delta(\alpha)l_w(\alpha^{-1}v) = \delta^{\frac{1}{2}}\lambda^w(\alpha)^{-1}l_w(v),$$

we see that $l_w(u_p.v) = p^{rD_w}l_w(v)$, hence the assertion follows. □

Remark 6.3. — From this lemma together with Theorem 6.1, we see that

$$\dim e.\mathscr{I} \leqslant 1,$$

which has been proved in [13, Theorem. 5.3] by essentially the same argument.

6.2.2. The functional l_{w_3}

We apply the above discussion to our case $I(s) = I_p(\pi, \chi, s)$. Notations are as in §4.3.2. For brevity, let $(\xi_{\mathfrak{p}}, \xi_{\overline{\mathfrak{p}}}) = (\xi_1, \xi_2)$ and $(\chi_1, \chi_2) = (\chi_{\mathfrak{p}}, \chi_{\overline{\mathfrak{p}}})$. Put

$$\mathscr{I}(s) = I(\chi_2|\cdot|^{s-\frac{3}{2}}, \xi_2^{-1}\delta_1, \xi_2^{-1}\delta_2, \chi_1^{-1}|\cdot|^{\frac{3}{2}-s}) \boxtimes \xi_2\chi_2^{-1}.$$

We identify these representations of $\mathrm{GL}_4(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ with each other via the following map L:

$$\begin{aligned} L : I_p(\pi, \chi, s) &\cong I(\chi_2|\cdot|^{s-\frac{3}{2}}, \xi_2^{-1}\delta_1, \xi_2^{-1}\delta_2, \chi_1^{-1}|\cdot|^{\frac{3}{2}-s}) \\ f &\mapsto g :\to L(f(g)) := f(g)(1_n) \end{aligned} \tag{6.3}$$

Let $\mathscr{I} := \mathscr{I}(0)$. Then the p-adic weights of $\mathscr{I}$ are $(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, k - \frac{3}{2})$. Since $k \geqslant 4$, $\mathscr{I}$ is a regular principal series. We put

$$w_3 = \begin{bmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix}, s_1 = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{bmatrix} \text{ and } w_l = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Let $\phi^{w_\ell} \in I_p(\chi_1^{-1}|\cdot|^{\frac{3}{2}}, \chi_2|\cdot|^{-\frac{3}{2}}, \xi_2^{-1}\delta_2, \xi_2^{-1}\delta_1)$ denote the unique section which is supported in $Bw_lN(\mathbb{Z}_p)$ and invariant by $N(\mathbb{Z}_p)$. Then we define $\phi^{ord} := L^{-1}(M_{s_1}\phi^{w_\ell}) \in I(\pi, \chi, 0)$. Thanks to the following lemma, we can cut off the ordinary projection from a given section by using the ordinary functional.

Lemma 6.4. — *ϕ^{ord} is an ordinary section in $I_p(\Pi, \chi, 0)$ as in* [13], *and*

$$e.f = l_{w_3}(f)\phi^{ord}, \quad \forall f \in I_p(\Pi, \chi, 0). \tag{6.4}$$

Proof. — First we observe that the section ϕ^{w_ℓ} is supported in the big cell, and then it is an eigenvector of the u_p-operator. By (6.1),

$$\phi^{ord}||t_i(x) = \delta(\alpha_i)^{-\frac{1}{2}}\lambda^{w_3}(\alpha_i) \sum_{u \in U/U_{\alpha_i}} \phi(xu\alpha_i^{-1}).$$

Straightforward computation shows $\phi^{ord}||t_i = \phi^{ord}$, hence $u_p.\phi^{ord} = \phi^{ord}$ is an ordinary section. This proves the first assertion.

Now we prove the second assertion. First $M_{s_1}\phi^{w_\ell} \in \mathscr{I}$ and $l_{w_3}(M_{s_1}\phi^{w_\ell}) = M_{w_l}\phi^{w_\ell}(1) = 1$, so ϕ^{ord} is a non-zero ordinary vector. The space of ordinary forms has at most one dimensional (Remark 6.3), so $e.f = \beta \cdot \phi^{ord}$ for some β. By Lemma 6.2 we see $\beta = l_{w_3}(f)$. □

6.3. The ordinary section in π_p^B

We study the ordinary section in π_p^B. As in §4.3.2, π_p^B is ordinary and is regarded as a sub-representation of $I(\delta_1,\delta_2)$.

Proposition 6.5. — *Then there exists φ a unique section in $\pi_p^B \subset I(\delta_1,\delta_2)$ such that φ is invariant by $I_1(p)$ and $\varphi|U_p = a_p(\mathbf{f})\varphi$, where $a_p(\mathbf{f}) = \delta_2(p)^{-1}|p|^{-\frac{1}{2}}$. Moreover $\varphi(1)=1$. φ is called the ordinary section of π_p^B.*

Proof. — This is well-known. We recall the proof here, as we need the construction of φ later. There are three cases:

π_p^B is a ramified principal series In this case, δ_1 is ramified with the conductor p. Thus φ is in fact the new vector in $I(\delta_1,\delta_2)$. Namely, φ is the unique function such that $\varphi(1)=1$ and

$$\varphi(h\begin{bmatrix} a & b \\ pc & d\end{bmatrix}) = \delta_1(a)\varphi(h),\ \forall \begin{bmatrix} a & b \\ pc & d\end{bmatrix} \in I_0(p).$$

π_p^B is a unramified principal series. We have $I(\delta_1,\delta_2)^{I_0(p)} = \mathbb{C}\phi^1 \oplus \mathbb{C}\phi^w$, where ϕ^1 is the function with $\operatorname{supp}\phi^1 = B(\mathbb{Q}_p)I_0(p)$ and $\phi^1(1)=1$, and ϕ^w is the function with $\operatorname{supp}\phi^w = B(\mathbb{Q}_p)wN(\mathbb{Z}_p)$ and $\phi^w(w)=1$. Let $\alpha = \delta_1\,|p|^{\frac{1}{2}}$ and $\beta = \delta_2(p)\,|p|^{\frac{1}{2}}$. By a simple calculation, we find that

$$\begin{aligned} \phi^1|U_p &= \beta^{-1}\phi^1 + (1-|p|)\beta^{-1}\phi^w \\ \phi^w|U_p &= \alpha^{-1}\phi^w. \end{aligned}$$

From the above, we can solve for φ easily:

$$\varphi = \phi^1 + \frac{1-|p|}{1-\delta_1^{-1}\delta_2(p)}\phi^w. \tag{6.5}$$

π_p^B is special. In this case, δ_2 is unramified and $\delta_1\delta_2^{-1} = |\cdot|$, and we have

$$0 \longrightarrow \pi_p^B \longrightarrow I(\delta_1,\delta_2) \longrightarrow \mathbb{C}\cdot\delta_1|\cdot|^{-\frac{1}{2}} \longrightarrow 0.$$

Hence φ must be of the form as in (6.5). As U_p acts on $\mathbb{C}\cdot\delta_1$ as a scalar α^{-1}, we conclude that φ is in π_p^B.

In either of the above three case, we find that $\varphi(1)=1$. □

When no confusion arises, we will identify φ with $\varphi \boxtimes \xi_p$ the ordinary section of $\Pi_p = \pi^B \boxtimes \xi_p$, and its restriction to $U(W)(\mathbb{Q}_p)$ as a section in $\pi_p = \Pi|_{U(W)(\mathbb{Q}_p)}$ is still denoted by φ Note that when φ_p is viewed as a section of $\pi_p = I(\delta_1,\delta_2)\otimes\xi_2^{-1}$, the eigenvalue of U_p-operator is $\xi_2(p)a_p(\mathbf{f})$.

The following lemma roughly says that the ordinary section is controlled by the evaluation at w_3.

Lemma 6.6. — *Let W be the Weyl group of $\mathrm{GL}_4(\mathbb{Q}_p)$. For $w \in W$, we have*

$$\phi^{ord}(w) = \varphi \text{ if } w = w_3, \text{ and } \phi^{ord}(w) = 0 \text{ for } w \neq w_3,$$

$$\text{if } \chi_+, \chi_1\xi_2^{-1}\delta_1 \text{ are ramified.}$$

Proof. — Define a function on $U(W)(\mathbb{Q}_p)$ by

$$f(h) = L(\phi^{ord}(\mathrm{m}(1,h)\, s_1^{-1}w_l))$$

for $h \in U(W)(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}_p)$. Then $f \in I(\delta_1, \delta_2)$. We find that $w_3 = s_1^{-1}w_l$ and $f(1) = M_{s_1}\phi^{w_\ell}(s_1^{-1}w_l) = 1$. It is straightforward to verify that for $k = \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in I_0(p)$, $f(hk) = f(g)\xi_2^{-1}(\det k)\delta_1(a)$. Hence, if δ_1 is ramified, then $f(h) = \varphi_p$. If δ_1 is unramified, we can compute $f(w)$ as follows.

$$f(w) = \phi^{ord}(s_1^{-1}w_l)(w) = \int_{U_{s_1}} \phi^{w_\ell}(s_1 u \mathrm{m}(1,w)\, s_1^{-1}w_l)du \tag{6.6}$$

$$= \int_{\mathbb{Q}_p} \phi^{w_\ell} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & a \end{bmatrix} w_l)da \tag{6.7}$$

$$= \int_{\mathbb{Q}_p} f(\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix})da, \tag{6.8}$$

where

$$f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := \phi^{w_\ell} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & a & b \\ & & c & d \end{bmatrix} w_l).$$

Note that $f(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}) = \mathbb{I}_{\mathbb{Z}_p}(x)$. By the identity

$$\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix} = \begin{bmatrix} -a^{-1} & 1 \\ & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a^{-1} & 1 \end{bmatrix},$$

the last integral in (6.6) equals

$$\int_{\mathbb{Q}_p^\times} f(\begin{bmatrix} -a & 1 \\ & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} |a|^{-1}d^\times a = \sum_{n \geqslant 0} \int_{p^n\mathbb{Z}_p{}^\times} \delta_2\delta_1^{-1}(a)d^\times a$$
$$= \frac{1}{1 - \delta_1^{-1}\delta_2(p)} \cdot (1 - |p|).$$

Hence in the case where δ_1 is unramified, we find that $f(m) = \varphi(m)$ by (6.5).

Next we show the second assertion. Recall that $z_1 = \xi_2\delta_1^{-1}$ and $z_2 = \xi_2\delta_1^{-1}$. We identify with S_4 and w be an element in W. Let λ' be the character of $T_4(\mathbb{Z}_p)$ defined by $\lambda'(diag(t_1,t_2,t_3,t_4)) = z_1(t_1)z_2(t_2)\chi_2(t_3)\chi_1^{-1}(t_4)$. Then we have $\phi^{ord}(gt) = \lambda'(t)\phi^{ord}(g)$ for $t \in T_4(\mathbb{Z}_p)$.

Now we regard W the Weyl group of $\mathrm{GL}_4(\mathbb{Q}_p)$ as the permutation group S_4 acting on the standard basis of $\mathbb{Q}_p^4$. Suppose $M_{s_1}\phi^{w_\ell}(w) \neq 0$ and $w \neq (123) = w_3$. Since $\chi_1\chi_2$, $\chi_1\eta_1$ are ramified and $\phi^{ord}(wt) = \lambda'(t)\phi^{ord}(w) = \lambda(wtw^{-1})\phi^{ord}(w)$, it follows that w can only be (13) or (243) and (1243). And s_1Bw contains the big cell, which implies the reduced decomposition of w contains $(123) = (12)(23)$, so $w = (13) = (12)(23)(12)$ or $(1243) = (12)(34)(23)$. By direct computation, $M_{s_1}\phi^{w_\ell}((13)) = M_{s_1}\phi^{w_\ell}((1243)) = 0$. (since $\chi_1\chi_2$ is ramified). □

6.4. The computation of $l_{w_3}(f^{pb}_{\Phi_\Upsilon})$

6.4.1.

We calculate the ordinary projection of the local pull back section at p in this subsection. Recall that

$$(\phi^{\Upsilon}_{\chi,s,p})^{pb} = \chi_2(\det\Upsilon)\,|\det\Upsilon|^s\, f^{pb}_{\Phi_\Upsilon}$$

for $\Phi = \Phi^{J,N}_{\underline{z}}$ and $\underline{z} = (\xi_2^{-1}\delta_1, \xi_2^{-1}\delta_2)$. Then $f^{pb}_{\Phi_\Upsilon}$ is $N(\mathbb{Z}_p)$-invariant by Lemma 3.8. It suffices to compute $l_{w_3}(f_{\Phi_\Upsilon})$ by Lemma 6.4. For $(x,y) \in \mathbb{Q}_p^2$, put

$$u(x,y) = \begin{bmatrix} 1 & & y & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

We have $U^-_{w_3}\backslash U = u(\mathbb{Q}_p^2)$. Recall that in (6.3) we identify $I(s) = I_p(\pi,\chi,s)$ with $\mathscr{I}(s)$ via L, and we have the following commutative diagram when $\mathrm{Re}\, s \gg 0$,

$$\begin{array}{ccc} I(s) & \xrightarrow[\sim]{L} & \mathscr{I}(s) \\ \downarrow{\scriptstyle l_{w_3}} & & \downarrow{\scriptstyle l_{w_3}} \\ I(\pi_p) & \xrightarrow{L} & \mathbb{C}, \end{array}$$

where l_{w_3} at left hand side is defined by

$$l_{w_3}(f) := \int_{U^-_{w_3}\backslash U} \pi(wn) f dn,$$

and the bottom map L is the evaluation at the identity $\mathbf{1}_2$. Put $H = U(W)$. Then by definition,

$$l_{w_3}(f^{pb}_{\Phi_\Upsilon}) = \int_{\mathbb{Q}_p^2} dxdy \int_{H(\mathbb{Q}_p)} dh f_\Phi((w_3 u(x,y), h)_\Delta \Upsilon)\chi_1\chi_2^{-1}(\det h)\pi(h)\tau_p\varphi.,$$

where φ is the ordinary section in prop 6.5. To compute $l_{w_3}(f^{pb}_{\Phi_\Upsilon})$, we begin with

$$(0,Z)(w_3 u(x,y),h)_\Delta \Upsilon = (0,Z)\cdot \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & -\frac{1}{2} \\ & & 1 & \\ & \theta^{-1} & & \theta^{-1} \end{bmatrix} \begin{bmatrix} 0 & {}^t v & & \\ v & C_{x,y} & & \\ & & 1 & 0 \\ & & 0 & h \end{bmatrix}$$
$$= (Z\begin{bmatrix} 0 & 0 \\ \theta^{-1}v & \theta^{-1}C_{x,y}\end{bmatrix}, Z\begin{bmatrix} 1 & \\ & \theta^{-1}h\end{bmatrix}),$$

where $C_{x,y} = \begin{bmatrix} 0 & y \\ 1 & x \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus

$$l_{w_3}(f^{pb}_{\Phi_\Upsilon}) = \int_{\mathbb{Q}_p^2} dydx \int_{H(\mathbb{Q}_p)} dh \int_{\mathrm{GL}_3(\mathbb{Q}_p)} d^\times Z P(h) Q(Z)\Phi$$
$$(Z\begin{bmatrix} 0 & 0 \\ \theta^{-1}v & \theta^{-1}C_{x,y}\end{bmatrix}, Z\begin{bmatrix} 1 & \\ & \theta^{-1}h\end{bmatrix})\pi(h)\tau_p\varphi,$$

where $P(h) = \chi_1(\det h)|\det h|^s$ and $Q(Z) = \chi_1\chi_2(\det Z)|\det Z|^{2s}$. We make change of variable

$$Z \to Z\begin{bmatrix} 1 & 0 \\ 0 & C_{x,y}^{-1}\theta\end{bmatrix} \text{ and } h \to C_{x,y}h.$$

The above integral becomes

$$\chi_+(\det\theta)\int_{\mathbb{Q}_p^2} \chi_2^{-1}(-y)|y|^{-s} dydx \int_{H(\mathbb{Q}_p)} P(h)\pi(C_{x,y}h)\tau_p\varphi dh$$

(6.9)

$$\times \int_{\mathrm{GL}_3(\mathbb{Q}_p)} Q(Z)\Phi_1(Z\begin{bmatrix} 0 & 0 \\ -C_{x,y}^{-1}v & I_2\end{bmatrix})\hat{\Phi}_2(Z\begin{bmatrix} 1 & 0 \\ 0 & h\end{bmatrix})d^\times Z,$$

We first compute the last integral. Write $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$. We find that

$$\Phi_1(Z \begin{bmatrix} 0 & 0 \\ C_{x,y}^{-1}v & I_2 \end{bmatrix})\hat{\Phi}_2(Z \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}) = \Phi_1(\begin{bmatrix} Z_2C_{x,y}^{-1}v & Z_2 \\ Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix})\hat{\Phi}_2(\begin{bmatrix} Z_1 & Z_2h \\ Z_3 & Z_4h \end{bmatrix})$$

Considering the support of Φ_1 and Φ_2, we see that

$$Z_2 \in \mathbb{Z}_p^2,\ Z_4 \in I_0(p),\ Z_3 \in \mathbb{Z}_p^2.$$

Since $Z_4 \in I_0(p)$, we can write

$$Z = \begin{bmatrix} 1 & Y \\ & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ Z_3 & Z_4 \end{bmatrix},\ Y \in \mathbb{Z}_p^2, Z_4 \in I_0(p)$$

and we have $d^\times Z = |a|^{-2}\, dY d^\times a d^\times Z_4 dZ_3$. The last integral in (6.9) equals

$$\begin{aligned}(6.10)\qquad & |N|^2 \int_{\mathbb{Q}_p^\times} \chi_+(a)\, |a|^{s-2}\, \hat{\Phi}_{\nu_1}(a) d^\times a \\ & \qquad \int_{I_0(p)} \chi_+(\det Z_4)\Phi_1 \begin{bmatrix} 0 & 0 \\ Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix} \hat{\Phi}_{\nu_2,\nu_3}(Z_4h)d^\times Z_4 \\ & = |N|^2\, E_p(\nu_1, 2s-2) \cdot \int_{I_0(p)} \Phi_1 \begin{bmatrix} 0 & 0 \\ Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix} \hat{\Phi}_{\nu_2,\nu_3}(Z_4h)d^\times Z_4\end{aligned}$$

Make change of variable $h \mapsto Z_4^{-1}h$. The integral (6.9) equals

(6.11)

$$\chi_+(\det\theta)\, |N|^2\, E_p(2s-2, \nu_1) \int_{\mathbb{Q}_p^2} \chi_2^{-1}(-y)|y|^{-s}dydx \int_{I_0(p)} d^\times Z_4$$

$$\times\chi_2(\det Z_4)\Phi_1(\begin{bmatrix} 0 & 0 \\ Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix})\pi(C_{x,y}Z_4^{-1}) \int_{H(\mathbb{Q}_p)} P(h)\hat{\Phi}_{\nu_2,\nu_3}(h)\pi(h)\tau_p\varphi dh,$$

Put

$$R_{\nu_2,\nu_3} = \int_{H(\mathbb{Q}_p)} P(h)\hat{\Phi}_{\nu_2,\nu_3}(h)\pi(h)\tau_p\varphi dh \in \pi.$$

We claim

$$(6.12)\qquad R_{\nu_2,\nu_3} = \mathrm{vol}(I_0(N_2), dh)E_p(s-\frac{1}{2}, \nu_2)E_p(s-\frac{1}{2}, \nu_3)\tau_p\varphi.$$

We assume (6.12) first. Then the integral (6.11) can be rewritten as

(6.13)

$$\chi_+(\det\theta)\,|N|^2\,E_p(\nu_1,2s-2)\,\mathrm{vol}(I_0(N_2),dh)E_p(s-\frac{1}{2},\nu_2)E_p(s-\frac{1}{2},\nu_3)$$
$$\times\,\mathrm{vol}(I_0(p),d^\times Z_4)\int_{\mathbb{Q}_p^2}\chi_2^{-1}(-y)|y|^{-s}\pi(C_{x,y})dydx\cdot\int_{\mathbb{Z}_p}\Phi_1(\begin{bmatrix}0 & 0\\ n(z)C_{x,y}^{-1}v & I_2\end{bmatrix}\tau_p\varphi dz.$$

The last integral by Lemma 3.10 equals

$$\int_{\mathbb{Q}_p^2}\chi_2^{-1}(-y)\,|y|^{-s}\,\left|p^{-1}J\right|\mathbb{I}_{S(p)}(\begin{bmatrix}-y^{-1}x & 1\\ y^{-1} & 0\end{bmatrix})\mu_3(p^{-1}y^{-1})\pi(\begin{bmatrix}0 & y\\ 1 & x\end{bmatrix})\tau_p\varphi$$
$$=\left|p^{-1}J\right|\,|p|^s\int_{p^{-1}\mathbb{Z}_p^\times}dy\int_{\mathbb{Z}_p}dx\chi_2^{-1}(y)\mu_3^{-1}(py)\pi(\begin{bmatrix}1 & 0\\ xy^{-1} & 1\end{bmatrix}\begin{bmatrix}py & \\ & 1\end{bmatrix})\varphi$$
$$=\left|p^{-1}J\right|\,|p|^s\,\mathrm{vol}(p^{-1}\mathbb{Z}_p^\times,dy)\chi_2(p)$$

Combining the above equation with (6.13) and rearranging terms, we obtain finally

(6.14)

$$l_{w_3}(f_{\Phi_\Upsilon}^{pb})=|N|^2\,\mathrm{vol}(I_0(p),dZ_4)\,\mathrm{vol}(I_0(N_2),dh)\,\left|p^{-1}J\right|(p-1)$$
$$\times\,E_p(2s-2,\chi_+)\cdot E_p(s-\frac{1}{2},\chi_1\xi_2^{-1}\delta_1)\cdot E_p(s-\frac{1}{2},\chi_1\xi_2^{-1}\delta_2)$$
$$\times\,\chi_+(\det\theta)\cdot(\chi_2|\cdot|^s)(-p)\cdot\varphi.$$

6.4.2. Computation of R_{ν_2,ν_3}

It remains to do the calculation of R_{ν_2,ν_3} in (6.12).

Lemma 6.7. — *We have*

$$R_{\nu_2,\nu_3}=\mathrm{vol}(I_0(N_2),dh)E_p(s-\frac{1}{2},\nu_2)E_p(s-\frac{1}{2},\nu_3)\tau_p\varphi. \tag{6.15}$$

Proof. — There are two cases:

Case(1) ν_2 and ν_3 are ramified: In this case, since $\nu_2\nu_3^{-1}=\delta_1\delta_2^{-1}$ and $c_p(\delta_1\delta_2^{-1})=p$, $c_p(\nu_3)=c_p(\nu_2)=N_2$. For $h\in\mathrm{supp}\,\hat{\Phi}_{\nu_2,\nu_3}$, we have $h=N^{-1}u$, $u=\begin{bmatrix}a & b\\ Nc & d\end{bmatrix}\in I_0(N)$. We recall that

$$\pi(u)\tau_p\varphi=\xi_2^{-1}(\det u)\delta_1(d)\tau_p\varphi\text{ for }u=\begin{bmatrix}a & b\\ cN & d\end{bmatrix}\in I_0(p), \tag{6.16}$$

and $\hat{\Phi}_{\nu_2,\nu_3}(hk) = \hat{\Phi}_{\nu_2,\nu_3}(h)\nu(k)$, so we have

$$\begin{aligned}R_{\nu_2,\nu_3} &= \mathrm{vol}(I_0(N),dh)\chi_1(N^2)\left|N\right|^{-2s}\hat{\Phi}_{\nu_2,\nu_3}(\begin{bmatrix}N^{-1} & \\ & N^{-1}\end{bmatrix})\xi_1\xi^{-2}(N^{-1})\tau_p\varphi\\ &= \mathrm{vol}(I_0(N),dh)\left|N\right|G(\nu_2)G(\nu_3)|N|^{-2s}\tau_p\varphi\\ &= \mathrm{vol}(I_0(N),dh)\epsilon(s-\frac{1}{2},\nu_2)^{-1}\epsilon(s-\frac{1}{2},\nu_3)^{-1}\tau_p\varphi\\ &= \mathrm{vol}(I_0(N),dh)E_p(s-\frac{1}{2},\nu_2)E_p(s-\frac{1}{2},\nu_3)\tau_p\varphi\end{aligned}$$

Case(2) Either ν_2 or ν_3 is unramified: In this case, we have $N_2 = p$, and we can verify R_{ν_2,ν_3} also satisfies (6.16). Now we view φ as the ordinary section in the model $I(\delta_1,\delta_2)\boxtimes\xi_2^{-1}$. To prove the lemma, it suffices to show $R_{\nu_2,\nu_3}(e) = \tau_p\varphi(e)$ and $R_{\nu_2,\nu_3}(\tau_p^{-1}) = \tau_p\varphi(\tau_p^{-1})$. We choose the measure

$$dh = \left|a\right|^{-1}dyd^\times ad^\times bdk \text{ for } h = \begin{bmatrix}a & y\\ 0 & b\end{bmatrix}k,\ k\in \mathrm{GL}_2(\mathbb{Z}_p).$$

We have

$$\begin{aligned}R_{\nu_2,\nu_3}(e) &= \iiint \chi_1(ab)\left|ab\right|^s\left|a\right|^{-1}\hat{\Phi}_{\nu_2,\nu_3}(\begin{bmatrix}a & y\\ 0 & b\end{bmatrix})\tau_p\varphi(\begin{bmatrix}a & y\\ 0 & b\end{bmatrix})\,dyd^\times ad^\times b\\ &= \mathrm{vol}(I_0(p),dh)E_p(s-\frac{1}{2},\nu_2)E_p(s-\frac{1}{2},\nu_3)\tau_p\varphi(e).\end{aligned}$$

Similarly

$$\begin{aligned}R_{\nu_2,\nu_3}(\tau_p^{-1}) &= \int_{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1|\cdot|^s(\det h)\hat{\Phi}_{\nu_2,\nu_3}(h)\varphi(\tau_p^{-1}h\tau_p)dh\\ &= \int_{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1|\cdot|^s(\det h)\hat{\Phi}_{\nu_2,\nu_3}(\tau_p h\tau_p^{-1})\varphi(h)dh.\end{aligned}$$

We find that

$$\begin{aligned}&R_{\nu_2,\nu_3}(\tau_p^{-1})\\ =\ &\mathrm{vol}(I_0(p),dh)\iiint\chi_1(ab)\left|ab\right|^s\left|a\right|^{-1}\hat{\Phi}_{\nu_2,\nu_3}(\begin{bmatrix}b & 0\\ py & a\end{bmatrix})\varphi(\begin{bmatrix}a & y\\ 0 & b\end{bmatrix})\,dyd^\times ad^\times b\\ =\ &\mathrm{vol}(I_0(p),dh)\iint\nu_2(b)\nu_3(a)\left|ab\right|^{s-\frac{1}{2}}\hat{\Phi}_{\nu_2}(b)\hat{\Phi}_{\nu_3}(a)d^\times ad^\times b\,\varphi(e)\\ =\ &\mathrm{vol}(I_0(p),dh)E_p(s-\frac{1}{2},\nu_2)E_p(s-\frac{1}{2},\nu_3)\tau_p\varphi(\tau_p^{-1})\end{aligned}$$

□

Now we summarize our calculations as the following proposition.

Proposition 6.8. — *We have*

$$\begin{aligned} l_{w_3}(f^{pb}_{\Phi_\Upsilon}) &= \mathrm{vol}(I_0(p), dh)\,\mathrm{vol}(I_0(N, N_2), d^\times Z)\left|p^{-1}J\right|(p-1) \\ &\times E_p(2s-2, \chi_+)\cdot E_p(s-\frac{1}{2}, \chi_1\xi_2^{-1}\delta_1)\cdot E_p(s-\frac{1}{2}, \chi_1\xi_2^{-1}\delta_2) \\ &\times \chi_+(\det\theta)\cdot(\chi_2|\cdot|^s)(p) \end{aligned}$$

Proof. — By (6.14) and note that

$$\begin{aligned} &|N|^2\,\mathrm{vol}(I_0(p), d^\times Z_4)\,\mathrm{vol}(I(N_2), dh) \\ &= \mathrm{vol}(I_0(N, N_2), d^\times Z)\,\mathrm{vol}(I_0(N_2), dh)\left|pN_2^{-1}\right| \\ &= \mathrm{vol}(I_0(N, N_2), d^\times Z)\,\mathrm{vol}(I_0(p), dh) \end{aligned}$$

□

7. Proof of the main result

7.1. Fourier-Jacobi expansion and Φ-operators

In this subsection, we give a brief description of arithmetic Fourier-Jacobi expansion of modular forms for the unitary group $G = GU(3,1)$. The purpose is to relate the constant term of our Eisenstein series $E^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ and that of its p-adic avatar.

7.1.1. Local charts and Mumford families

We begin with some notations. Let $g \in G(\mathbb{A}_f^{(p)})$ and write $g^\vee = kg_i^\vee\gamma$ with $\gamma \in G(\mathbb{Q})^+$ and $k \in K$. Let $Y_g = Yg^\vee$ and $X_g^\vee = X^\vee g^\vee$. Let $X_g = \left\{y \in I_Y \mid \langle y, X_g^\vee\rangle_{3,1} \subset \mathbb{Z}\right\}$ be the $\mathbb{Z}$-dual of $X_g^\vee$. Then we have the inclusion $i : Y_g \hookrightarrow X_g$. We let I_g be the subgroup in $X_g \otimes_\mathbb{Z} Y_g$ generated by

$$y' \otimes y - y \otimes y;\ xb \otimes y - x \otimes c(b)y,\ \forall, x \in X_g, y, y' \in Y_g, b \in \mathscr{O}_\mathcal{K}.$$

Let $\mathscr{S}_g = S(X_g \otimes_\mathbb{Z} Y_g)$ be the maximal free quotient of the group $X_g \otimes_\mathbb{Z} Y_g/I_g$. We have $Y_g\gamma = \mathfrak{b}y^1$ and $X_g^\vee\gamma = \mathfrak{a}^*x^1$ for two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathscr{O}_\mathcal{K}$. Then the dual of $\mathscr{S}_{[g]}$ is the space of integral Hermitian forms on $\mathfrak{a}\times\mathfrak{b}$ which is isomorphic to $(\mathfrak{a}\mathfrak{b}\mathcal{D}_\mathcal{K})^{-1}\cap\mathbb{Q}$, hence $\mathscr{S}_{[g]}$ is the fractional ideal $\mathfrak{a}\mathfrak{b}\mathcal{D}_\mathcal{K}\cap\mathbb{Q}$ of $\mathbb{Q}$. Let $\mathscr{S}^+_{[g]} = \left\{s \in \mathfrak{c}^{-1}\mathscr{S}_{[g]} \mid s > 0\right\}$ and let $\mathscr{S}^0_{[g]} = \mathscr{S}^+_{[g]} \cup \{0\}$.

Let $S_{[g]} := S_{G_P}(K^g_P)$ and let $\underline{\mathcal{B}} = (\mathcal{B}, \overline{\lambda}_{\mathcal{B}}, \iota_{\mathcal{B}}, \eta_{\mathcal{B}})$ be the universal quadruple over $S_{[g]}$. Let $\mathcal{P}$ be the Poincaré line bundle over $\mathcal{B}\times\mathcal{B}^t$. Define the group scheme $\mathcal{Z}_{[g]}$ over $S_{[g]}$ by

$$\begin{aligned}\mathcal{Z}_{[g]} &= \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathcal{K}}}(X_g, \mathcal{B}^t)\times_{\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathcal{K}}}(Y_g,\mathcal{B}^t)}\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathcal{K}}}(Y_g, \mathcal{B}) \\ &= \left\{(c,c^t)\in\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathcal{K}}}(X_g,\mathcal{B}^t)\times\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathcal{K}}}(Y_g,\mathcal{B})\,|\,c(i(y))=\lambda(c^t(y)), y\in Y_g\right\}\end{aligned}$$

Let $\mathcal{Z}^\circ_{[g]}$ be the connected component of Z. Each $\beta \in X\otimes Y$ induces a tautological map from $\mathbf{c}(\beta) : \mathcal{Z}^\circ_{[g]} \to \mathcal{B}\times\mathcal{B}^t$. We let $\mathcal{L}(\beta) = \mathbf{c}(\beta)^*\mathcal{P}$ be the line bundle over $\mathcal{Z}^\circ_{[g]}$ obtained by the pull back of $\mathcal{P}$ via $\mathbf{c}(\beta)$. By the symmetry of the polarization λ and the ampleness of $\mathcal{P}$, $\mathcal{L}(\beta)$ only depends on the holomorphic image of β in $\mathscr{S}_{[g]}$.

Let $K^{n,g}_\bullet = gK^n_\bullet g^{-1}\cap G_P(\mathbb{A}_f)$, $\bullet = 1$ or $\emptyset$. Let $I_{[g]}(K^n_\bullet) := I_{G_P}(K^{n,g}_\bullet)$ be the Igusa scheme over $S_{[g]}$. In our simple case, $S_{[g]}$ and $I_{[g]}(K^n_\bullet)$ are affine schemes of finite type over $\mathcal{O}$. Let $A_{[g]}$ (resp. $A^n_{[g]}$) be the coordinate ring of $S_{[g]}$ (resp. $I_{[g]}(K^n)$). Let $\mathcal{R}_{[g]}$ be the $A^n_{[g]}[\![\mathscr{S}_g]\!]$-algebra defined by

$$\mathcal{R}_{[g]} := \prod_{\beta\in\mathscr{S}^0_g} H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))q^\beta. \tag{7.1}$$

Let $\mathcal{I}_+$ be the ideal of $\mathcal{R}^\bullet_{[g]}$ generated by $\left\{q^\beta\right\}_{\beta\in\mathscr{S}^+_{[g]}}$.By Mumford's construction ([2] and [5]), there exists a semi-abelian scheme $(\mathcal{M}_{[g]}, \iota_{\mathcal{M}})$ over $\operatorname{Spec}\mathcal{R}_{[g]}$ together with an $\mathscr{O}_{\mathcal{K}}$-action $\iota : \mathscr{O}_{\mathcal{K}} \to \operatorname{End}\mathcal{M}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ such that $\mathcal{M}\otimes\mathcal{R}_{[g]}/\mathcal{I}_+ = \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is the universal $\mathscr{O}_{\mathcal{K}}$-Raynaud extension over $\mathcal{Z}^\circ_{[g]}$

$$0\longrightarrow X^\vee_g\otimes\mathbb{G}_m\longrightarrow\widetilde{\mathcal{M}}\longrightarrow\mathcal{B}\longrightarrow 0, \tag{7.2}$$

and $\mathcal{M}$ is an abelian scheme over $\operatorname{Spec}\mathcal{R}^\xi_{[g]}[1/\mathcal{I}_+]$. Moreover, $(\overline{\lambda}_{\mathcal{B}}, K^g_P\eta_{\mathcal{B}})$ induces a natural polarization and level structures $(\overline{\lambda}_{\mathcal{M}}, \overline{\eta}^{(p)}_{\mathcal{M}})$ of $\mathcal{M}$ over $\mathcal{R}_{[g]}[1/\mathcal{I}_+]$ (*cf.* [5, IV. 6.4, 6.5 and V. 2.5]). We define the $\mathcal{R}_{[g]}[1/\mathcal{I}_+]$-quadruple of level K by

$$\underline{\mathcal{M}}_{[g]} = (\mathcal{M}_{[g]}, \overline{\lambda}_{\mathcal{M}}, \iota_{\mathcal{M}}, \overline{\eta}^{(p)}_{\mathcal{M}})_{\mathcal{R}^\xi_{[g]}[1/\mathcal{I}_+]}.$$

We call $\underline{\mathcal{M}}_{[g]}$ the Mumford quadruple at the cusp $[g]$. Moreover there exists a morphism $\varphi_{\mathcal{M}_{[g]}} : \operatorname{Spec}\mathcal{R}^\xi_{[g]} \to \overline{S}_G(K)$ such that $(\varphi_{\mathcal{M}_{[g]}})^*\underline{\mathcal{G}} = \underline{\mathcal{M}}_{[g]}$.

Similarly, let $(\underline{\mathcal{B}}, j_{\mathcal{B}})$ be the universal quintuple over $I_{[g]}(K^n)$. Then $\mathbf{N}j_{\mathcal{B}}$ induces a canonical p^n-level structure $\mathbf{N}j_{\mathcal{M}})$ on $\underline{\mathcal{M}}_{[g]}$ over $\mathcal{R}_{[g]}$. Then $(\underline{\mathcal{M}}_{[g]}, \overline{j}_{\mathcal{M}})$ is the Mumford quintuple over $I_{[g]}(K^n_1)$.

7.1.2. Fourier-Jacobi expansion and the Φ-operator

Let R be an $\mathcal{O}$-algebra and let $f \in \mathbf{M}_{\underline{k}}(K_1^n, R)$ be a modular form of weight $\underline{k} = (0, k)$. We define the Fourier-Jacobi expansion $\mathcal{F}_g(f)$ of f at the cusp $[g]$ as follows. Let $g \in G(\mathbb{A}_f^{(p)})$. Then (7.2) induces the exact sequence of $\mathscr{O}_{\mathcal{K}}$-modules

$$0 \longrightarrow \Omega_{\mathcal{B}} \longrightarrow \Omega_{\mathcal{M}} \longrightarrow \Omega_{\mathcal{T}_g} \longrightarrow 0,\ \mathcal{T}_g = X_g^\vee \otimes \mathbb{G}_m.$$

Let $d^\times t$ be the canonical $\mathscr{O}_{\mathcal{K}}$-basis of $\Omega_{\mathcal{T}_g}$. Choose a lifting $d^\times t'_+$ of $e^+.d^\times t$ in $\Omega_{\mathcal{M}}$ and a canonical lifting $d^\times t_-$ of $e^-.d^\times t$ in $\Omega_{\mathcal{M}}$. Let $\boldsymbol{\omega}_{\mathcal{M}} = (d^\times t'_+ \wedge \boldsymbol{\omega}_B, d^\times t_-)$ be a $\mathscr{O}_{\mathcal{K}}$-top form of $\underline{\omega}_{\mathcal{M}_{[g]}}$. Let $u \in H(\mathbb{Z}_p) = \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$. We regard u as an element in $U(V)(\mathbb{Q}_p)$ by the embedding $H(\mathbb{Z}_p) \hookrightarrow \mathrm{GL}_4(\mathbb{Z}_p) \cong U(V)(\mathbb{Z}_p)$. If $f \in H^0(I_G(K_1^n)_{/R}, \omega_{\underline{k}})$, then evaluating f at $(\mathcal{M}_{[g]}, u^{-1}j_{\mathcal{M}}, \boldsymbol{\omega}_{\mathcal{M}})$, we obtain the Fourier-Jacobi expansion of f

$$f(\mathcal{M}_{[g]}, u^{-1}j_{\mathcal{M}}, \boldsymbol{\omega}_{\mathcal{M}}) = \sum_\beta a^u_{[g]}(\beta, f) q^\beta \in \mathcal{R}_{[g]} \otimes_{\mathcal{O}} R.$$

Suppose further that $u^{-1}N_P(\mathbb{Z}_p)u \subset N_P(\mathbb{Z}_p)$. Define the Siegel Φ-operator:

$$\Phi^u_{[g]} : H^0(I_G(K_1^n)_{/R}, \omega_{\underline{k}}) \longrightarrow H^0(I^n_{[g]/R}, \mathcal{O}_{I^n_{[g]}})$$
$$f \mapsto \Phi^u_{[g]}(f) := \mathbf{a}^u_{[g]}(0, f).$$

Now we consider the case $R = \mathbb{C}$. Recall that $I_{[g]}(K^n)(\mathbb{C}) = G_P(\mathbb{Q})^+ \backslash G_P(\mathbb{A}_f)/K^{n,g}$. Thus for every $h \in G_P(\mathbb{A}_f)$, the image $[h]$ of h in $I_{[g]}(K^n)(\mathbb{C})$ gives rise to a classifying $\mathbb{C}$-algebra homomorphism $\varphi_{[h]} : A^n_{[g]} \to \mathbb{C}$. we have the following important comparison between analytic and algebraic Fourier-Jacobi expansion

$$\mathcal{F}_{[hug]}(f) = \varphi_{[h]}\left(f(\underline{\mathcal{M}}_{[g]}, u^{-1}j_{\mathcal{M}}, (d^\times t'_+ \wedge (2\pi i)^2 \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma), d^\times t_-))\right). \tag{7.3}$$

7.2. The constant term of $E^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$

Let $E = E^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ and let $\widehat{E}$ be its p-adic avatar. We compare the constant terms of E and $\widehat{E}$. First, by (4.15) and the definition of $\widehat{E}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ together with the identity (2.17), we conclude that

$$\frac{1}{\Omega_p^k} \cdot \widehat{E} \text{ is the } p\text{-adic avatar of } (\frac{2\pi i}{\Omega_{\mathcal{K}}})^{2k} \cdot E.$$

Let $\boldsymbol{\omega}_{\mathcal{M}}(\mathbb{C}) = (d^\times t_+ \wedge (2\pi i)^2 \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma), d^\times t_-)$ and $\boldsymbol{\omega}_{\mathcal{M}}(p) = (d^\times t_+ \wedge \boldsymbol{\omega}_{\mathcal{B}_h}(j_{\mathcal{B}_h}), d^\times t_-)$. By (2.17) we have

$$\iota_\infty(E(\underline{\mathcal{M}}_{[g]}, \bar{j}_{\mathcal{M}}, \boldsymbol{\omega}_{\mathcal{M}})) = (\frac{2\pi i}{\Omega_{\mathcal{K}}^{2k}})^{2k} \cdot E(\underline{\mathcal{M}}_{[g]}, \bar{j}_{\mathcal{M}}, \boldsymbol{\omega}_{\mathcal{M}}(\mathbb{C}))$$
$$\iota_p(E(\underline{\mathcal{M}}_{[g]}, \bar{j}_{\mathcal{M}}, \boldsymbol{\omega}_{\mathcal{M}})) = \frac{1}{\Omega_p^k} \cdot E(\underline{\mathcal{M}}_{[g]}, \bar{j}_{\mathcal{M}}, \boldsymbol{\omega}_{\mathcal{M}}(p)) = \widehat{E}(\underline{\mathcal{M}}_{[g]}, \bar{j}_{\mathcal{M}}).$$

Comparing the constant terms of E and $\widehat{E}$, we get

$$(\frac{2\pi i}{\Omega_{\mathcal{K}}})^{2k} \cdot \Phi^u_{[g]}(E)(\underline{\mathcal{B}}_h, \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma)) = \frac{1}{\Omega_p^k} \cdot \Phi^u_{[g]}(\widehat{E})(\underline{\mathcal{B}}_h). \tag{7.4}$$

Comparing the constant terms on both sides in (7.3), we obtain

$$\Phi^{an}_{ug}(E)(h) = \Phi^u_{[g]}(E)(\underline{\mathcal{B}}_h, \boldsymbol{\omega}_{W/\mathbb{C}}(\Sigma)). \tag{7.5}$$

7.3. Proof of the main result

We are now in a position to prove the main result Theorem 0.6. Actually we will prove it in a more general setting. First of all we recall our set-up. We begin with a quadruple $(\psi, \mathfrak{c}, f, \xi)$, where

- ψ is a Dirichlet Hecke character of $\mathcal{K}^\times$ and $\mathfrak{c}$ is an integral ideal of $\mathbb{Z}$ which satisfies (S2).
- f is an ordinary new GL_2-cusp form in $S_2(\Gamma_0(pN_f), \epsilon_f)$.
- ξ is a Hecke character of $\mathcal{K}^\times$ of finite order such that $\xi|_{\mathbb{Q}^\times} = \xi_+ = \epsilon_f^{-1}$.

We assume N has a decomposition $N_f = N_+ N_-$ with $(N_+, N_-) = 1$ and N_- is product of an odd number of distinct primes. We choose a positive integer $\mathfrak{s}$ which is a p-unit and $(-\mathfrak{s}, -D)_v = -1$ exactly at $v \mid \infty N_-$. B is the definite quaternion algebra defined in §4.3 with the above choice of $\mathfrak{s}$. Then B is exactly ramified at the infinity and N_-, and

$$\pi_f = \mathrm{JL}(\pi^B) \text{ is in the image of Jacquet-Langlands of } B.$$

Let $\mathbf{f}$ be the corresponding p-primitive form of f for $GU(W)$.

We give the definition of the normalized L-values.

Definition 7.1 (Normalized L-values). — *Recall that we have chosen $I(\delta_1, \delta_2)$ a model of π_p such that $\delta_2|\cdot|^{\frac{1}{2}}(p)$ is a unit. Then the normalized L-values are defined by*

$$L^{\mathrm{alg},S}(-2, \chi_+) = E_p(-2, \chi_+) \cdot \frac{\Gamma(k-2)}{(2\pi i)^{k-2}} \cdot L^{S \cup \{p\}}(-2, \chi_+), \tag{7.6}$$

and

$$L^{alg,S}_{\mathcal{K}}(-\frac{1}{2},\pi,\chi)=E_p(-\frac{1}{2},\chi_{\mathfrak{p}}\xi^{-1}_{\overline{\mathfrak{p}}}\delta_1)E_p(-\frac{1}{2},\chi_{\mathfrak{p}}\xi^{-1}_{\overline{\mathfrak{p}}}\delta_2) \cdot\frac{\Gamma(k)\Gamma(k-1)}{(2\pi i)^{2k-1}}\cdot\frac{(2\pi i)^{2k}L^{S\cup\{p\}}_{\mathcal{K}}(-\frac{1}{2},\pi,\chi)}{\Omega^{2k}_{\mathcal{K}}}, \tag{7.7}$$

where $\Omega_{\mathcal{K}}$ is the CM period associated to $\mathcal{K}$. We express (7.7) *in terms of L-values attached to Galois representation $\rho=\rho_f$. Note that $\rho_f=\mathrm{rec}(\pi^\vee_f(-\frac{1}{2}))$ and $\xi_+=\epsilon^{-1}_\rho$, we see that* (7.7) *equals*

$$L^{alg,S}_{\mathcal{K}}(-\frac{1}{2},\pi,\chi)=L^{alg,S}_{\mathcal{K}}(0,\rho\otimes\chi\xi) \tag{7.8}$$

as defined in the introduction.

Remark 7.2. — According to the recipe of J. Coates in [4], a p-adic L-function has no p-Euler factor for the eigenspace of positive (geometric) Hodge-Tate weights, and we need to modify p-Euler factor for the eigenspace of negative Hodge-Tate weights.

Theorem 7.3. — *Let $(\psi,\mathfrak{c},f,\xi)$ as above. Let $d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}(g)$ denote the $\mathbb{C}_p$-valued measure induced by the evaluation of constant term at $g\in G(\mathbb{A}^{(p)}_f)$. Let $g\in G(\mathbb{A}^{(p)}_f)$ be a cusp defined by $g_v=w^{-1}$ if $v\in S$ and $g_v=1$ elsewhere and $h\in G_P(\mathbb{A}^{(p)}_f)$. We have*

$$\frac{1}{\Omega^k_p}\int_{\mathfrak{G}}\widehat{\chi}d\mathcal{E}^{ord}_{\mathbf{f},\xi,\mathfrak{c}}(hg)=|D_{\mathcal{K}}|^{\frac{3}{2}}_{\mathbb{R}}(p-1)\cdot\iota_p\iota^{-1}_\infty\left[L^{alg,S}(-2,\chi_+\epsilon_\rho)L^{alg,S}_{\mathcal{K}}(0,\rho\otimes\chi)\right] \times(\widehat{\chi}\xi^{-1})_{\overline{\mathfrak{p}}}(-p\cdot\det\theta)\iota(\mathbf{F}(h)).$$

In other words, the ordinary p-adic modular form $\Phi^{w_3}_{[g]}(\widehat{E}^{ord}(\chi\xi^{-1}\mid\mathbf{f},\xi,\mathfrak{c})$ on G_P is given by

$$\frac{1}{\Omega^k_p}\cdot\Phi^{w_3}_{[g]}(\widehat{E}^{ord}(\chi\xi^{-1}\mid\mathbf{f},\xi,\mathfrak{c}))=L^S_p(-2,\chi_+\epsilon_\rho)L^S_p(\rho\otimes\chi)\cdot\widehat{\mathbf{F}} \times|D_{\mathcal{K}}|^{\frac{3}{2}}_{\mathbb{R}}(p-1)(\chi\xi^{-1})_{\overline{\mathfrak{p}}}(-p\cdot\det\theta).$$

Moreover the constant term at other cusps is a p-integral multiple of this element in $\overline{\mathbb{Z}}_p$. Therefore the ideal generated by the constant terms at all cusps is

$$L^S_p(-2,\chi_+\epsilon_\rho)L^S_p(\rho\otimes\chi).$$

Proof. — By Cor. 5.4 the constant term of $E^{ord} := E^{ord}(\chi \mid \mathbf{f}, \xi, \mathfrak{c})$ is

$$\Phi^{an}_{[w_3 g]}(E^{ord})(h) = \frac{NC(s,\chi)}{\mathrm{vol}(D_W(\mathfrak{c}))} \cdot \left[\bigotimes_{v \nmid p} \phi^{pb}_{\chi,s,v} \bigotimes_{v=p} e.f^{pb}_{\Phi_\Upsilon} \right] (hw_3 g)|_{s=0} \cdot \chi(\det(hg)).$$

Therefore the first assertion follows from the computation in Prop. 5.3, Prop. 5.5, Prop. 5.9 and Prop. 6.8 together with (7.4) and (7.5). By [26, Lemma 4.3 and Cor. 4.1], the ordinary section is only supported in $g \in G(\mathbb{A}_f)$ such that g_p has p-depth 0 (Def. 4.1 *loc.cit.*). Hence the second assertion follows from Lemma 6.6 and p-adic continuity. □

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