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Existence results for the prescribed Scalar curvature on S^3

EXISTENCE RESULTS FOR THE PRESCRIBED
SCALAR CURVATURE ON S^3

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ABSTRACT. — This paper is devoted to the existence of conformal metrics on S^3 with prescribed scalar curvature. We extend well known existence criteria due to Bahri-Coron.

RÉSUMÉ. — Ce papier est consacré à l'existence des métriques conforme sur S^3 avec courbure scalaire prescrite. Nous étendons les critères d'existence bien connus de Bahri-Coron.

1. Introduction and the main result

On the sphere S^3 endowed with its standard metric g_0, a well studied question is the following one:
Given a function $K \in C^2(S^3)$, does there exist a metric g, conformally equivalent to the standard one whose scalar curvature is given by K? This a mounts to solve the following nonlinear PDE

$$(1.1) \qquad \begin{cases} -Lu = Ku^5 \\ \quad u > 0 \qquad \text{on } S^3. \end{cases}$$

Where $-Lu = -8\Delta u + 6u$ is the conformal Laplacian operator of (S^3, g_0).

For the last four decades, scalar curvature problem has been continuing to be one of major subjects in nonlinear elliptic PDEs. Please see [1], [2], [4], [5], [7], [9], [8], [10], [11], [13], [23], [22], [19], [17], [18], [16] and the references therein.

Unfortunately equation (1.1) does not have always a solution, indeed one can easily notice that a necessary condition is the function K is positive

somewhere. Another deeper necessary condition is the so-called Kazdan-Warner Obstruction [15].

A sufficient condition was found by A. Bahri and J. M. Coron, through the theory of critical points at infinity (see please A. Bahri [3]) it is an Euler-Hopf type criterium, namely they prove.

THEOREM 1.1 ([5] (see also [8] and [23])). — *Under the following condition:*

$(\mathbf{H_0})$ $0 < K \in C^2(S^3)$, *having only nondegenerate critical points such that*

$$\Delta K(y) \neq 0 \text{ for each } y, \text{ critical point of } K.$$

If

$$\sum_{y \in \mathcal{K}^+} (-1)^{3-ind(K,y)} \neq 1,$$

then (1.1) *has at least one solution.*

Where $\mathcal{K}^+ = \{y, \ \nabla K(y) = 0 \text{ and } -\Delta K(y) > 0\}$ and $ind(K, y)$ denote the Morse index of K at y.

A natural question which arises when looking to the above result, is what happens if the total sum is equal to 1, but a partial one is not. Under which condition can one use this partial sum to derive an existence result?

In order to give a partial answer to this question, we introduce the following condition:

We say that an integer $k \in (\mathbf{A_1})$ if it satisfies the following

$(\mathbf{A_1})$ For each $z \in \mathcal{K}^+$, such that $3 - ind(K, z) = k + 1$ and for each $y \in \mathcal{K}^+$ such that $3 - ind(K, y) \leqslant k$, we have:

$$\frac{1}{K(z)^{\frac{1}{2}}} > \frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(y)^{\frac{1}{2}}},$$

where y_0 is an absolute maximum of the function K on S^3.

We are now ready to state our main result.

THEOREM 1.2. — *Let* $K \in C^2(S^3)$ *satisfying* $(\mathbf{H_0})$, *if*

$$\underset{k \in (\mathbf{A_1})}{Max} \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3 - ind(K,y) \leqslant k}} (-1)^{3-ind(K,y)} \right| \neq 0$$

then there exist a solution to problem (1.1).

We observe that every $k \geqslant 2$, satisfies condition $(\mathbf{A_1})$ since for every $y \in \mathcal{K}^+$, we have $ind(K, y) \in \{1, 2, 3\}$. It follows that the above mentioned celebrated result of Bahri-Coron is a corollary of our Theorem. Actually we

will give in section 3 (see Remark 3.1) a situation where the Theorem of Bahri-Coron does not give a result, while our Theorem proves the existence of at least one solution. Therefore our Theorem is a generalization of one of Bahri-Coron. Please see also Remark 3.2. Moreover, we point out that Bahri-Coron criterium has an equivalent in dimension 4, see [7], while in dimension $n \geqslant 5$, under the condition ($\mathbf{H_0}$) the corresponding sum is always equal to 1. However our idea introduced in this paper, of using partial sums has corresponding statement in all dimensions. This will be the subject of a forthcoming paper.

The remainder of this paper is organized as follows: we first recall some known facts about the variational problem and its critical points at infinity, we then give in section 3 the Proof of the main Theorem. At the end of the paper we give new related type existence results.

2. Well known facts

2.1. The variational problem

We recall the variational framework. Problem (1.1) has a variational structure. The function is

$$J(u) = \frac{\int_{S^3} -Lu.u\,dv}{\left(\int_{S^3} Ku^6 dv\right)^{\frac{1}{3}}}, \quad u \in H^1(S^3),$$

where dv is the volume element of (S^3, g_0). The space $H^1(S^3)$ is equipped with the norm

$$\|u\|^2 = \int_{S^3} -Lu.u\,dv = 8\int_{S^3} |\nabla u|^2 dv + 6\int_{S^3} |u|^2 dv.$$

Problem (1.1) is equivalent to finding critical points of J subject to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \left\{ u \in H^1(S^3), \ u > 0 \text{ and } \|u\| = 1 \right\}.$$

The functional J is known not to satisfy the Palais-Smale condition which leads to the failure of the classical existence mechanisms.

In order to characterize the sequence failing the Palais-Smale condition, we need to introduce some notation. For $a \in S^3$ and $\lambda > 0$, let

$$\delta_{a\lambda}(x) = c_0 \left(\frac{\lambda}{(\lambda^2 + 1) + (\lambda^2 - 1)\cos(d(a, x))} \right)^{\frac{1}{2}},$$

where $d(a, x)$ is the geodesic distance on (S^3, g_0) and c_0 is a positive constant chosen so that

$$-L\delta_{a\lambda} = \delta_{a\lambda}^5 \quad \text{in } S^3.$$

The failure of the Palais-Smale condition can be descibed as follows,

PROPOSITION 2.1 ([20], [24], [5]). — *Assume that J has no critical point in Σ^+ and let $(u_k)_k$ be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to 0. Then there exist an integer $p \in \mathbb{N}^\star$, a sequence $\varepsilon_k > 0$, ε_k tends to 0 and an extracted of $(u_k)_k$'s again denoted $(u_k)_k$, such that $u_k \in V(p, \varepsilon_k)$. Here $V(p, \varepsilon)$ is defined by:*

$$V(p, \varepsilon) = \left\{ u \in \Sigma^+, \text{ such that } \exists a_1, .., a_p \in S^3, \exists \lambda_1, .., \lambda_p > \varepsilon^{-1}, \exists \alpha_1, .., \alpha_p > \right.$$

$$0 \text{ with } \|u - \sum_1^p \alpha_i \delta_{a_i \lambda_i}\| < \varepsilon, |J(u)^3 \alpha_i^4 K(a_i) - 1| < \varepsilon, \ \forall i = 1, .., p \text{ and } \varepsilon_{ij} <$$

$$\left. \varepsilon \ \forall i \neq j \right\},$$

where $\varepsilon_{ij} = \left(\dfrac{\lambda_i}{\lambda_j} + \dfrac{\lambda_j}{\lambda_i} + \dfrac{\lambda_i \lambda_j}{2}(1 - \cos(d(a_i, a_j))) \right)^{\frac{-1}{2}}$.

We consider the following minimization problem for $u \in V(p, \varepsilon)$, with ε small

$$(2.1) \qquad \min = \left\{ \|u - \sum_1^p \alpha_i \delta_{a_i \lambda_i}\|, \ \alpha_i > 0, \ \lambda_i > \varepsilon^{-1}, \ a_i \in S^3 \right\}.$$

We then have the following proposition which defines a parametrization of the set $V(p, \varepsilon)$.

PROPOSITION 2.2 ([4], [5]). — *For any $p \in \mathbb{N}^\star$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p, \varepsilon)$, the minimization problem (2.1) has a unique solution $(\alpha, a, \lambda) = (\alpha_1, ..., \alpha_p, a_1, ..., a_p, \lambda_1, ..., \lambda_p)$ up to permutation. In particular we can write $u \in V(p, \varepsilon)$ as follows*

$$u = \sum_1^p \alpha_i \delta_{a_i \lambda_i} + v$$

where $v \in H^1(S^3)$ satisfying:

$(\mathbf{V_0})$ $\qquad \|v\| < \varepsilon$ and $\langle v, \phi \rangle = 0$ $\forall \phi \in \left\{ \delta_{a_i \lambda_i}, \dfrac{\partial \delta_{a_i \lambda_i}}{\partial \lambda_i}, \dfrac{\partial \delta_{a_i \lambda_i}}{\partial a_i}, i = 1, ..., p \right\}.$

DEFINITION 2.3 ([3]). — *A critical point at infinity of J on Σ^+ is a limit of a flow-line $u(s)$ of the equation:*

$$\begin{cases} \dfrac{\partial u}{\partial s} = -\partial J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$, for $s \geqslant s_0$. Here $p \in \mathbb{N}^\star$ and $\varepsilon(s)$ is some function tending to 0 when $s \longrightarrow +\infty$. Using proposition 2.2, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) \delta_{(a_i(s), \lambda_i(s))} + v(s).$$

Denoting $y_i := \lim\limits_{s \longrightarrow +\infty} a_i(s)$ and $\alpha_i := \lim\limits_{s \longrightarrow +\infty} \alpha_i(s)$, we denote by

$$\sum_{i=1}^{p} \alpha_i \delta_{(y_i, \infty)} \text{ or } (y_1, ..., y_p)_\infty$$

such a critical point at infinity.

The following proposition which is proved in [5], characterizes the critical points at infinity of the associated variational problem.

PROPOSITION 2.4 ([5]). — *Assume the function K satisfies the condition $(\mathbf{H_0})$ and assume that J has no critical point in Σ^+. then the only critical points at infinity of J are:*

$$\frac{1}{K(y)^{\frac{1}{4}}} \delta_{(y, \infty)}, \ y \in \mathcal{K}^+$$

such a critical point at infinity has a Morse index equal to 3-ind(K,y). Its level is $S^{\frac{2}{3}} \dfrac{1}{K(y)^{\frac{1}{3}}}$, where S is the best constant of sobolev on S^3.

Now, we state the following proposition which is proved for dimensions $n \geqslant 7$ in [4] and the proof still works for the lower dimensions.

PROPOSITION 2.5. — *Let a_1, $a_2 \in S^3$, α_1, $\alpha_2 > 0$ and λ large enough. For $u = \dfrac{\alpha_1}{K(a_1)^{\frac{1}{4}}} \delta_{(a_1, \lambda)} + \dfrac{\alpha_2}{K(a_2)^{\frac{1}{4}}} \delta_{(a_2, \lambda)}$, we have*

$$J(u) \leqslant \left(S \sum_{i=1}^{2} \frac{1}{K(a_i)^{\frac{1}{2}}} \right)^{\frac{2}{3}} \left(1 + o(x, \lambda) \right)$$

where the term $o(x, \lambda)$ tends to zero when λ tends to $+\infty$.

Proof. — Please see the estimates (113)-(123) of [4].   □

2.2. The unstable manifolds of critical points at infinity

At the beginning of this subsection, we give some basic definitions with will allow us to describe the unstable manifolds of the critical points at infinity in $V(1, \varepsilon)$.

DEFINITION 2.6. — *Let* $K : S^3 \longrightarrow \mathbb{R}$ *be a* C^2 *Morse function and let* $\mathcal{K}$ *the set of critical points of* K. *If* $y \in \mathcal{K}$, *let* $W_s(y)$ *designate its stable manifold and* $W_u(y)$ *designate its unstable manifold. We have*

$$\dim W_u(y) = \text{ind}(K, y)$$

$$\dim W_s(y) = 3 - \text{ind}(K, y).$$

Here, $\text{ind}(K, y)$ *denotes the Morse index of the function* K *at* y.

It is convenient to specify that the notations of stable or unstable manifolds, of flow-lines, all are relative to the C^1 vector field $-\partial K$, with respect to the standard Riemannian structure on S^3. Recall the following generic hypothesis:
All stable and unstable manifolds intersect transversely and all such intersections are smooth regularly embedded submanifolds of S^3.

DEFINITION 2.7. — *Let* $y, z \in \mathcal{K}$. z *is said to be dominated by* y, *if*

$$W_u(y) \cap W_s(z) \neq \emptyset$$

then there exists (at less) a flow line of $-\partial K$ *descending from* y *to* z. *Using dimension argument and the fact that both of* $W_u(y)$ *and* $W_s(z)$ *are invariant under the action of the flow generated by* $-\partial K$, *it is easy to see that,*

(2.2)   *if* $W_u(y) \cap W_s(z) \neq \emptyset$, *then* $\text{ind}(K, y) \geqslant \text{ind}(K, z) + 1$

DEFINITION 2.8 ((see [4] p. 356-357, see also [5], Lemma 10)). — *Let* $y \in \mathcal{K}^+ = \{y \in \mathcal{K}, -\Delta K(y) > 0\}$. *In* $V(1, \varepsilon)$, *the unstable manifold at infinity* $W_u^\infty(y)_\infty$ *for the critical point at infinity* $(y)_\infty$, *along the flow-lines of* $-\partial J$ *is defined and identified by the unstable manifold* $\widetilde{W}_u(y)$ *of the critical point* y *of the function* $\dfrac{1}{K}$, *along the flow-lines of* $-\partial\left(\dfrac{1}{K}\right)$ *multiplied by a factor corresponding to the concentration* λ. *Precisely in* $V(1, \varepsilon)$, $W_u^\infty(y)_\infty$ *have the following description*

$$W_u^\infty(y)_\infty \cap V(1, \varepsilon) = \left\{ \frac{1}{S} \delta_{a\lambda}, \ a \in \widetilde{W}_u(y) \right\}$$

where λ is a fixed constant large enough.

Remark 2.9. — Observe that $\widetilde{W}_u(y)$ correspond to $W_s(y)$, the stable manifold of the critical point y along the flow-lines of $-\partial K$. Therefore it is easy to see that if $W_u^\infty(y)_\infty \subset V(1, \varepsilon)$, then $W_u^\infty(y)_\infty$ be haves as $W_s(y)$.

The following Lemma gives a sufficient condition to insure that $W_u^\infty(y)_\infty$ is included in $V(1, \varepsilon)$.

LEMMA 2.10. — *Let $y \in \mathcal{K}^+$. If*

$$W_s(y) \cap W_u(z) = \emptyset \ \forall z \in \mathcal{K} \backslash \mathcal{K}^+,$$

(in particular, if $K(y) > K(z) \quad \forall z \in \mathcal{K} \backslash \mathcal{K}^+$), then

$$W_u^\infty(y)_\infty \ \text{diffeomorphic to } W_s(y).$$

Proof. — It follows from [2] (see assumption (A_1) and proof of Theorem 1 of [2]). The idea is that a flow-line in $W_u^\infty(y)_\infty$ cannot go out from $V(1, \varepsilon)$ unless the concentration point a(s) of the flow-line nearby a critical point z of K with $-\Delta K(z) < 0$ (see [4], proposition 2), therefore it is the case when the critical point y is dominate by $z \in \mathcal{K} \backslash \mathcal{K}^+$. Hence under the condition of the Lemma such a situation cannot occur, it follows that every flow line in $W_u^\infty(y)_\infty$ is indeed in $V(1, \varepsilon)$ and we then conclude the result of the Lemma using the Remark 2.9 □

3. Proof of the main Theorem

Arguing by contradiction, we suppose that problem (1.1) has no solutions. It follows from proposition 2.4 that the only critical points at infinity of the associated variational problem are $(y)_\infty := \dfrac{1}{K(y)^{\frac{1}{4}}} \delta_{(y,\infty)}, y \in \mathcal{K}^+$. The level of J at each $(y)_\infty$ is $c_\infty(y) := \left(S \dfrac{1}{K(y)^{\frac{1}{2}}} \right)^{\frac{2}{3}}$ and the Morse index of such critical point at infinity is $i(y)_\infty := 3 - \text{ind}(K, y)$. Let us denote by $y_0, y_1, ..., y_l$ the elements of $\mathcal{K}^+$, we suppose that y_0 is an absolute maximum of K on S^3. In the first step, we can assume the following.

$(\mathbf{H_1})$ for each $y_i \neq y_j \in \mathcal{K}^+$ we have $K(y_i) \neq K(y_j)$.

Afterwards, we prove the Theorem for the case when hypothesis $(\mathbf{H_1})$ is not satisfied.

We order the $c_\infty(y_i)$'s, under $(\mathbf{H_1})$, we can assume that

$$c_\infty(y_0) = \min_{\Sigma^+} J < c_\infty(y_1) < ... < c_\infty(y_l).$$

Let $c \in \mathbb{R}$, we denote by $J_c := \{u \in \Sigma^+, J(u) \leqslant c\}$. By using a deformation Lemma (see sections 7 and 8 of [6]), for $\sigma > 0$ small enough and for any $i = 0, ..., l$, we have

(3.1) $J_{c_\infty(y_i)+\sigma}$ retracts by deformation onto $J_{c_\infty(y_i)-\sigma} \cup W_u^\infty(y_i)_\infty$.

We derive now from (3.1), taking the Euler-Poincerè characteristic (denoted χ) of both sides that:

(3.2) $\chi\left(J_{c_\infty(y_i)+\sigma}\right) = \chi\left(J_{c_\infty(y_i)-\sigma}\right) + (-1)^{3-\text{ind}(K,y_i)}.$

Recall that 3-ind (K, y_i) is the Morse index of the critical point at infinity $(y_i)_\infty$. Let us remark that

(3.3) $\underset{k \in (\mathbf{A_1})}{\text{Max}} \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3 - \text{ind}(K,y) \leqslant k}} (-1)^{3-\text{ind}(K,y)} \right|$

is achieved for $k_0 \in \{0, 1, 2\}$, since $\text{ind}(K, y) \in \{1, 2, 3\}$ for any $y \in \mathcal{K}^+$. We then have 3 cases to discuss.

Case 1. $k_0 = 2$, in this case (3.3) is equal to

$$\left| 1 - \sum_{y \in \mathcal{K}^+} (-1)^{3-\text{ind}(K,y)} \right|.$$

Since we have assumed that (1.1) has no solutions in Σ^+, by the same arguments used in the Proof of the Theorem 1 of [5] (see p. 147 and 148), we obtain (3.3) is equal to 0 which is a contradiction of assumption of our Theorem.

Case 2. If $k_0 = 1$, let

$$c_1 = \underset{\substack{y \in \mathcal{K}^+ \\ 3 - \text{ind}(K,y) \leqslant k_0}}{\text{Max}} \left(S \frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(y)^{\frac{1}{2}}} \right)^{\frac{2}{3}}.$$

Since k_0 satisfies $(\mathbf{A_1})$, we can find $\varepsilon > 0$ satisfying $z \in \mathcal{K}^+$ such that 3- $\text{ind}(K, z) = k_0 + 1$, we have

$$c_\infty(z) > c_1 + \varepsilon.$$

Therefore the only critical points at infinity under the level $c_1 + \varepsilon$ are:

$$(y)_\infty, \; y \in \mathcal{K}^+ \text{ such that 3-ind}(K, y) \leqslant k_0.$$

Using (3.1)

(3.4) $J_{c_1+\varepsilon}$ retracts by deformation onto $\displaystyle\bigcup_{\substack{y \in \mathcal{K}^+ \\ 3 - \mathrm{ind}(K,y) \leqslant k_0}} W_u^\infty(y)_\infty.$

Let

$$X_{k_0}^\infty := \bigcup_{\substack{y \in \mathcal{K}^+ \\ 3 - \mathrm{ind}(K,y) \leqslant k_0}} W_u^\infty(y)_\infty.$$

we derive from (3.4) and (3.2),that

(3.5) $$\chi\left(X_{k_0}^\infty\right) = \sum_{\substack{y \in \mathcal{K}^+ \\ 3 - \mathrm{ind}(K,y) \leqslant k_0}} (-1)^{3-\mathrm{ind}(K,y)}.$$

From another part, we claim that

(3.6) $$X_{k_0}^\infty \text{ is contractible in } J_{c_1+\varepsilon}.$$

Indeed, let

$$X_{k_0} := \bigcup_{\substack{y \in \mathcal{K}^+ \\ 3 - \mathrm{ind}(K,y) \leqslant k_0}} W_s(y).$$

Where $W_s(y)$ designate the stable manifold of the critical point y of K along the flow-lines of $-\partial K$. Of course in this case, for every $y \in \mathcal{K}^+$ such that $3 - \mathrm{ind}(K,y) \leqslant k_0$ we have $\mathrm{ind}(K,y) = 2$ or 3, thus from (2.2) y cannot be dominated through the flow-lines of $-\partial K$ only by critical points y' of K such that $\mathrm{ind}(K,y') = 3$, thus satisfies $-\Delta K(y') > 0$. Therefore we obtain

$$W_s(y) \cap W_u(z) = \emptyset \quad \forall z \in \mathcal{K} \backslash \mathcal{K}^+.$$

Using now Lemma 2.10, we derive that

$$X_{k_0}^\infty \text{ is diffeomorphie to } X_{k_0}.$$

More precisely,

$$X_{k_0}^\infty = \left\{ \frac{1}{S} \delta_{(x,\lambda)}, \ x \in X_{k_0} \right\}$$

where λ is a fixed constant large enough.

Now, let

$$H : [0\ 1] \times X_{k_0}^{\infty} \longrightarrow \Sigma^{+}$$

$$\left(\alpha, \frac{1}{S}\delta_{(x,\lambda)}\right) \longmapsto \frac{\dfrac{\alpha}{K(y_0)^{\frac{1}{4}}}\delta_{(y_0,\lambda)} + \dfrac{(1-\alpha)}{K(x)^{\frac{1}{4}}}\delta_{(x,\lambda)}}{\left\| \dfrac{\alpha}{K(y_0)^{\frac{1}{4}}}\delta_{(y_0,\lambda)} + \dfrac{(1-\alpha)}{K(x)^{\frac{1}{4}}}\delta_{(x,\lambda)}\right\|}.$$

H is continuous and satisfies:
$H\left(0, \frac{1}{S}\delta_{(x,\lambda)}\right) = \frac{1}{S}\delta_{(x,\lambda)} \in X_{k_0}^{\infty}$ and $H\left(1, \frac{1}{S}\delta_{(x,\lambda)}\right) = \frac{1}{S}\delta_{(y_0,\lambda)}$, a fixed point of $X_{k_0}^{\infty}$. Furthermore, using proposition 2.5, we derive that

$$J\left(H\left(\alpha, \frac{1}{S}\delta_{(x,\lambda)}\right)\right) \leqslant S^{\frac{2}{3}}\left(\frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(x)^{\frac{1}{2}}}\right)^{\frac{2}{3}}\left(1 + o(x,\lambda)\right),$$

for each $(\alpha, \frac{1}{S}\delta_{(x,\lambda)}) \in [0\ 1] \times X_{k_0}^{\infty}$. Where $o(x,\lambda)$ tends to zero when λ tends to $+\infty$. Taking λ large enough, we obtain

$$J\left(H\left(\alpha, \frac{1}{S}\delta_{(x,\lambda)}\right)\right) \leqslant c_1 + \varepsilon \qquad \forall(\alpha, \frac{1}{S}\delta_{(x,\lambda)}) \in [0\ 1] \times X_{k_0}^{\infty}.$$

Since, $\forall x \in X_{k_0}$ we have

$$K(x) \geqslant \min\left\{K(y),\ y \in \mathcal{K}^{+} \text{ such that } 3 - \text{ind}(K, y) \leqslant k_0\right\}.$$

Therefore the contraction H is performed under the level $c_1 + \varepsilon$ and $X_{k_0}^{\infty}$ is then contractible in $J_{c_1+\varepsilon}$. Hence our claim follows and therefore (3.4) and (3.6) implies that $X_{k_0}^{\infty}$ is a contractible set. Thus, we derive from (3.5) that

$$1 = \sum_{\substack{y \in \mathcal{K}^{+} \\ 3 - \text{ind}(K,y) \leqslant k_0}} (-1)^{3-\text{ind}(K,y)}.$$

This is yields a contradiction with the assumption of the Theorem.

case3. $k_0 = 0$. We here use again the same argument developed in case2 keeping the same notation, except that underneath the level $c_1 + \varepsilon$, one can find beyond the critical points at infinity of Morse index $\leqslant k_0$, many other critical points at infinity of Morse index $k_0 + 2$. Let us remark at this stage, that $H\left([0\ 1] \times X_{k_0}^{\infty}\right)$ definite a contraction of the set $X_{k_0}^{\infty}$ of dimension $k_0 + 1$ in $J_{c_1+\varepsilon}$.

Using the flow of $-\partial J$, $H\left([0\ 1] \times X_{k_0}^{\infty}\right)$ can also be deformed. For dimension's reason the unstable manifold at infinity of any critical point a infinity of Morse index $k_0 + 2$ can be avoided during such a deformation

(see e.g [21]). Therefore $H\left([0\ 1] \times X_{k_0}^{\infty}\right)$ retracts by deformation on $X_{k_0}^{\infty}$ and hence $X_{k_0}^{\infty}$ is contractible since $H\left([0\ 1] \times X_{k_0}^{\infty}\right)$ is a contractible set. Using now the fact that dimension of $X_{k_0}^{\infty}$ is equal to $k_0 = 0$ (in the case), then the set $X_{k_0}^{\infty}$ consists of one single point and hence we derive that

$$X_{k_0}^{\infty} = W_u^{\infty}(y_0)_{\infty} \equiv \{y_0\}$$

i.e., $\left\{y \in \mathcal{K}^+,\ 3 - \mathrm{ind}(K, y) \leqslant k_0\right\} = \left\{y_0\right\}$. This also yields a contradiction with the assumption of the Theorem. The Proof of Theorem 1.2 is therefore completed under hypothesis $(\mathbf{H_1})$.

If $(\mathbf{H_1})$ is not satisfied (i.e., there exist $y_i \neq y_j \in \mathcal{K}^+$ s.t $K(y_i) = K(y_j)$), then we proceed as follows. We can build a family of functional K_{ε}, satisfying for each ε the following proprieties:

(i) $K_{\varepsilon} = K$ out of a neighborhood of all critical points $y \in \mathcal{K}^+$.

(ii) $K_{\varepsilon}(y) \neq K_{\varepsilon}(y')$ for each $y \neq y' \in \mathcal{K}^+$.

(iii) $-\Delta K_{\varepsilon}(y) > -\dfrac{\Delta K(y)}{2}$ for each $y \in \mathcal{K}^+$.

Then hypothesis $(\mathbf{H_1})$ is satisfied for the family K_{ε}. Thus, for each ε small enough there is a critical point w_{ε} of J_{ε}.

By properties (i), (ii) and (iii) above, the functionals J_{ε} and J have the same critical points at infinity defined only by the critical points y in $\mathcal{K}^+$. For each ε small enough, we have a Morse lemma at infinity in a neighborhood of every critical point at infinity (see [5], Lemma 10, see also [4]). These neighborhood are independent of ε and donot contain a true critical point of J_{ε} and J. Therefore, w_{ε} tends to a true critical points of J as ε tends to zero. This conclude the Proof of Theorem 1.2.

Remark 3.1. — Here, we want to consider some situations where the result of [5] does not give solution to problem (1.1), but by our result we derive that problem (1.1) admits a solution. For this, let $K : S^3 \longrightarrow \mathbb{R}$ be a C^2 More function satisfying $(\mathbf{H_0})$ such that $\mathcal{K}^+$ is reduced to 3 points y_0, y_1 and y_2 with

$$K(y_0) \geqslant K(y_1) > K(y_2).$$

Assume that

(3.7) $$\frac{1}{K(y_2)^{\frac{1}{2}}} > \frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(y_1)^{\frac{1}{2}}}.$$

We suppose also one of the two following conditions:

(i) $\operatorname{ind}(K, y_0) = \operatorname{ind}(K, y_1) = 3$ and $\operatorname{ind}(K, y_2) = 2$.
Or

(ii) $\operatorname{ind}(K, y_0) = 3, \operatorname{ind}(K, y_1) = 2$ and $\operatorname{ind}(K, y_2) = 1$.

It is easy to see that

$$\sum_{y \in \mathcal{K}^+} (-1)^{3-\operatorname{ind}(K,y)} = 1.$$

However, under assumption (i),

$$\operatorname*{Max}_{k \in (\mathbf{A_1})} \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3-\operatorname{ind}(K,y) \leqslant k}} (-1)^{3-\operatorname{ind}(K,y)} \right|$$

$$= \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3-\operatorname{ind}(K,y) \leqslant 0}} (-1)^{3-\operatorname{ind}(K,y)} \right| \neq 0,$$

so by Theorem 1.2 we derive the existence of solution of problem (1.1) and under the assumption (ii)

$$\operatorname*{Max}_{k \in (\mathbf{A_1})} \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3-\operatorname{ind}(K,y) \leqslant k}} (-1)^{3-\operatorname{ind}(K,y)} \right|$$

$$= \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3-\operatorname{ind}(K,y) \leqslant 1}} (-1)^{3-\operatorname{ind}(K,y)} \right| \neq 0,$$

and again we conclude that (1.1) has a solution from the Theorem 1.2.

Remark 3.2. — A generalization of Bahri-Coron Criterion, using the degree of a related function, has been proved by Chang, Gursky and Yang [8]. Such a result extends the result of Theorem 1.1 to functions having degenerate critical points. This degree actually computes the Leray-Schauder degree of the equation (1.1). As mentioned in the appendix of [8], (please see also page 68 of [10]), in the special case that K is a positive function having only nondegenerate critical points and satisfying $\Delta K(y) \neq 0$ at

critical points, this degree is well defined and can be expressed as

$$d = \sum_{y \in \mathcal{K}^+} (-1)^{3-\mathrm{ind}(K,y)} - 1.$$

It is easy to see that, under the assumption of the above example in 3.1, d is equal to zero. Therefore using the criterium of A. Chang and P. Yang one cannot derive an existence result, while by Theorem 1.2, we are able to obtain a solution.

4. Related Results

Our argument of using partial sums can be used to derive other existence results, for example, if we assume the following.

($\mathbf{H_2}$) $\forall y \in \mathcal{K}^+$ such that $\mathrm{ind}(K, y) = 1$ and $\forall z \in \mathcal{K} \backslash \mathcal{K}^+$ such that $\mathrm{ind}(K, z) = 2$, we have

$$W_s(y) \cap W_u(z) = \emptyset.$$

We notice that condition ($\mathbf{H_2}$) is satisfies if for example there holds.

($\mathbf{H_2'}$) $\forall y \in \mathcal{K}^+$ such that $\mathrm{ind}(K, y) = 1$ and $\forall z \in \mathcal{K} \backslash \mathcal{K}^+$ such that $\mathrm{ind}(K, z) = 2$, we have

$$K(y) > K(z).$$

We denote by $y_0, y_1, ..., y_s$ the elements of $\mathcal{K}$.

We order the $K(y_i)$'s , $y_i \in \mathcal{K}$. Without loss of generality, we assume that

$$K(y_0) \geqslant K(y_1) \geqslant ... \geqslant K(y_s).$$

We say that an integer $k \in (\mathbf{A_1'})$ if it satisfies the following

($\mathbf{A_1'}$) $\dfrac{1}{K(y_0)^{\frac{1}{2}}} + \dfrac{1}{K(y_k)^{\frac{1}{2}}} \leqslant \dfrac{1}{K(y_j)^{\frac{1}{2}}}$ $\forall j > k$ such that $y_j \in \mathcal{K}^+$.

We then have

THEOREM 4.1. — Let $K \in C^2(S^3)$ satisfying (H_0) and (H_2). If

$$\underset{k \in (\mathbf{A_1'})}{Max} \left| 1 - \sum_{\substack{y_i \in \mathcal{K}^+ \\ 0 \leqslant i \leqslant k}} (-1)^{3-ind(K,y_i)} \right| \neq 0,$$

then (1.1) has a solution.

Proof of Theorem 4.1. — The Proof proceeds exactly as the second case of the Proof of Theorem 1.2, taking

$$X_{k_0}^\infty := \bigcup_{\substack{y_i \in \mathcal{K}^+ \\ 0 \leqslant i \leqslant k}} W_u^\infty(y_i)_\infty \quad \text{and} \quad c_1 = \frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(y_k)^{\frac{1}{2}}}$$

$\square$

At the end, we give another type of existence result.

THEOREM 4.2. — *Let $K \in C^2(S^3)$ satisfying* $(\mathbf{H_0})$. *If there exist $\widetilde{y} \in \mathcal{K}^+$ such that $\mathrm{ind}(K, \widetilde{y}) = 1$ satisfying the following:*

$$K(\widetilde{y}) \geqslant K(y) \quad \forall y \in \mathcal{K}^+ \text{ such that } \mathrm{ind}(K, y) = 2.$$

Then there exist a solution to problem (1.1).

Proof of theorem 4.2. — Arguing by contradiction and assuming that J has no critical point in Σ^+. Let ∂ the boundary operator in the sense of Floef [14] (see also Milnor [21] and C. C. Conley [12]), ∂ acting on critical points at infinity of J.

For any $(y)_\infty$ critical point at infinity of Morse index l, we define $\partial(y)_\infty$ to be

$$(4.1) \qquad \partial\left(W_u^\infty(y)_\infty\right) = \sum_{(y')_\infty \text{ of index } l-1} i(y_\infty, y'_\infty) W_u^\infty(y')_\infty.$$

Where $W_u^\infty(y)_\infty$ (respectively $W_u^\infty(y')_\infty$) is the unstable manifold at infinity of the critical point at infinity $(y)_\infty$, (respectively $(y')_\infty$), along the flow-lines of $-\partial J$, viewed as a simplex of dimension l (respectively $l-1$) in Σ^+ and $i(y_\infty, y'_\infty)$ is the number of flow-lines (modulo 2) in

$$W_u^\infty(y)_\infty \cap W_s^\infty(y')_\infty.$$

The operator ∂ can be found in Bahri [4] (see Proof of Lemma 7 of [4]). Under the assumption of the Theorem $W_u^\infty(\widetilde{y})_\infty$ is a manifold of dimension 2 and satisfies

$$W_u^\infty(\widetilde{y})_\infty \cap W_s^\infty(y)_\infty = \emptyset$$

for any $(y)_\infty$ critical point at infinity of Morse index 1. Since,

$$c_\infty(\widetilde{y}) \leqslant c_\infty(y) \quad \forall y \in \mathcal{K}^+ \text{ s.t } 3 - \mathrm{ind}(K, y) = 1.$$

From (4.1), we derive

$$\partial\left(W_u^\infty(\widetilde{y})_\infty\right) = 0$$

and hence $W_u^\infty(\widetilde{y})_\infty$ define a cycle in $\mathcal{C}_2(X^\infty)$, the group of chains of dimension 2 of X^∞. Where

$$X^\infty := \bigcup_{y \in \mathcal{K}^+} W_u^\infty(y)_\infty.$$

Observe that X^∞ is a stratified set of top dimension 2, since the maximal Morse index of all critical points at infinity is less than 2. Thus $W_u^\infty(\widetilde{y})_\infty$ cannot be the boundary of chain of dimension 3 of X^∞. Therefore $W_u^\infty(\widetilde{y})_\infty$ define a homological class of dimension 2 which is non trivial in X^∞ and hence we derive that

(4.2) $$H_2(X^\infty) \neq 0.$$

Where $H_2(X^\infty)$ is the homology group of dimension 2 of X^∞.

From another part, since we have assumed that J has no critical points in Σ^+, using deformation lemma (see (3.1)), we obtain:

$$\Sigma^+ \text{ retarts by deformation on } X^\infty$$

and using the fact that Σ^+ is a contractible set, we then have

$$H_\star(X^\infty) = 0 \quad \forall \star \geqslant 1.$$

This yields a contradiction with (4.2). This conclude the proof of Theorem 4.2.   □

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