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BERNSTEIN POLYNOMIALS AND SPECTRAL NUMBERS FOR LINEAR FREE DIVISORS

by Christian SEVENHECK (*)

Abstract. — We discuss Bernstein polynomials of reductive linear free divisors. We define suitable Brieskorn lattices for these non-isolated singularities, and show the analogue of Malgrange's result relating the roots of the Bernstein polynomial to the residue eigenvalues on the saturation of these Brieskorn lattices.

Résumé. — Dans ce travail, nous nous intéressons aux polynômes de Bernstein d'un diviseur linéairement libre réductif. Nous définissons un réseau de Brieskorn pour ces fonctions, qui sont des exemples de singularités non-isolées. Nous démontrons un théorème analogue au résultat de Malgrange qui relate les racines du polynôme de Bernstein aux valeurs propres du résidu de la saturation de ce réseau de Brieskorn.

1. Introduction

In this note, we show that for reductive linear free divisors $D \subset \mathbb{C}^n$, which were studied in a number of recent papers (see [3] [8], [10] and [9]), the roots of the Bernstein polynomial of a defining equation h of D can be recovered as a certain set of eigenvalues of a residue endomorphism. More precisely, for a generic linear form f on $\mathbb{C}^n$, one defines a family of Gauß-Manin systems for f, seen as a function on the fibres of h. This family has a specific (logarithmic) extension over D, which gives the set of residue eigenvalues we are interested in.

Keywords: Brieskorn lattice, Bernstein polynomial, linear free divisors, spectral numbers.
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This relation to the Bernstein polynomial has a number of consequences: First, while the definition of the Bernstein polynomial is rather simple, it is in general very hard to calculate its roots in concrete examples. This is true even for linear free divisors, though the differential operator occurring in Bernstein's functional equation is more explicitly known than in the general case. On the other hand, the calculation of the residue eigenvalues alluded to above is, although not trivial, easier to carry out. We apply our result to obtain, using the calculations from [10, chapter 6], Bernstein polynomials for discriminants in representation spaces of the Dynkin quivers A_n, D_n and E_6 as well as the so-called star quiver $\star_n$, also considered in loc.cit. (which is not a Dynkin quiver for $n > 3$). We also calculate the Bernstein polynomials for two irreducible linear free divisors which are discriminants of irreducible pre-homogenous vector spaces described in [26].

Another motivation for this work comes from the fact that the residue eigenvalues of the Gauß-Manin systems give information on their limit behavior (resp., of the corresponding family of Brieskorn lattices), when approaching the zero fibre of h, i.e., the divisor D. In particular, in [10], questions about the degeneration of *Frobenius manifolds*, associated to the tame functions $f_{|D_t}$, where $D_t := h^{-1}(t)$, were related to the asymptotic behavior of a natural pairing defined on the Gauß-Manin system. In particular, the residue eigenvalues of these Gauß-Manin systems then need to be symmetric around zero. This was stated as a conjecture in loc.cit., and it follows from the relation between these eigenvalues and the roots of the Bernstein polynomial that we prove here.

Finally, the family of Brieskorn lattices associated to $f_{|D_t}$ also has logarithmic extension over the divisor D, constructed using logarithmic differential forms. We define the fibre over $t = 0$ to be the *logarithmic Brieskorn lattice* of D (in fact, it does not depend on the choice of the linear form). In contrast to the fibres at $t \neq 0$, this Brieskorn lattice is regular singular at the origin, reflecting the local situation of the pair (f, h) at the origin in $\mathbb{C}^n$. It turns out that then our result can be rephrased to give the analogue of Malgrange's classical result for isolated singularities: The roots of the Bernstein polynomial are (up to a rescaling) the residue eigenvalues on the *saturation* of this logarithmic Brieskorn lattice.

2. Linear Free Divisors and Gauß-Manin systems

In this section, we first recall from [10] the construction of the family of Gauß-Manin systems associated to a linear section of a linear free divisor.

We also give a more intrinsic definition of these Gauß-Manin systems as a direct image of a map constructed from two polynomials. Finally, we discuss the definition of the residue eigenvalues relevant for the present work, as introduced in [10].

We denote throughout this article by V the complex vector space $\mathbb{C}^n$.

Definition-Lemma 1 ([25], [3], [10]). — (1) *Let $D \subset V$ be a reduced hypersurface with defining equation h. Then D is called a free divisor, if the sheaf $\Theta_V(-\log D) := \{\vartheta \in \Theta_V \,|\, \vartheta(h) \subset (h)\}$ is a free $\mathcal{O}_V$-module. If moreover a basis (ξ_i) of $\Theta_V(-\log D)$ exists such that $\xi_i = \sum_{j=1}^n \xi_{ij}\partial_{x_j}$ where ξ_{ij} are* ***linear*** *forms on V, then D is called linear free.*

(2) *Let G be the identity component of the algebraic group $G_D := \{g \in \mathrm{Gl}(V) \,|\, g(D) \subset D\}$. Then (V, G) is a pre-homogenous vector space in the sense of Sato (see, e.g., [26]), in particular, the complement $V \backslash D$ is an open orbit of G. We call D reductive if G_D is so. A rational function $r \in \mathbb{C}(V)$ is called a semi-invariant if there is a character $\chi_r : G \to \mathbb{C}^*$ such that $g(r) = \chi_r(g) \cdot r$ for all $g \in G$. Obviously, h itself is a semi-invariant.*

(3) *G acts on V^* by the dual action, with dual discriminant $D^* \subset V^*$. If G_D is reductive, then (V^*, D^*) is pre-homogenous. We call a linear form $f \in V^*$ generic with respect to h (or simply generic, if no confusion is possible) if f lies in the open orbit $V^* \backslash D^*$ of the dual action.*

There is a basis (e_i) of V with corresponding coordinates (x_i) (called unitary) such that G appears as a subgroup of $U(n)$ in these coordinates. Then $D^* = \{h^* = 0\}$, where $h^*(y) := \overline{h(\overline{y})}$, (y_i) being the dual coordinates of (x_i).

In the sequel, we always consider linear forms which are generic with respect to h.

In order to study the behavior of the restriction of the linear function f on the fibres $D_t := h^{-1}(t)$, $t \neq 0$, but also on D itself, the following *deformation algebra* was introduced in [10].

Definition 2. — *Let D be linear free with defining equation h, seen as a morphism $h : V \to T := \mathrm{Spec}\ \mathbb{C}[t]$.*

(1) *Let $E \in \Theta_V(-\log D)$ be the Euler field $E = \sum_{i=1}^n x_i\partial_{x_i}$. Call*

$$\Theta_{V/T}(-\log D) := \{\vartheta \in \Theta_V(-\log D) \,|\, \vartheta(h) = 0\}$$

the module of relative logarithmic vector fields. $\Theta_{V/T}(-\log D)$ *is* $\mathcal{O}_V$*-free of rank* $n-1$*, and we have a decomposition*

$$\Theta_V(-\log D) = \mathcal{O}_V E \oplus \Theta_{V/T}(-\log D).$$

(2) *The ideal* $J_h(f) := df(\Theta_{V/T}(-\log D)) \subset \mathcal{O}_V$ *is called the Jacobian ideal of the pair* (f,h)*. The quotient* $\mathcal{O}_V/J_h(f)$ *is the Jacobian algebra (or deformation algebra) of* (f,h)*.*

Notice that $\Theta_{V/T}(-\log D)$ was called $\Theta_V(-\log h)$ in [10]. It was shown in loc.cit., section 3.2, that if f is generic with respect to h, then $h_*\mathcal{O}_V/J_h(f)$ is $\mathcal{O}_T$-free of rank n, and generated by (f^i) for $i = 0, \dots, n-1$. Moreover, it is interpreted as the relative tangent space $T^1_{\mathcal{R}_h/\mathbb{C}}(f)$ of the deformation theory of f with respect to the group $\mathcal{R}_h$ of right-equivalences preserving all fibres of h.

Denote by

$$(\Omega^\bullet_{V/T}(\log D), d) := (\Omega^\bullet_V(\log D)/(\Omega^{\bullet-1}_V(\log D) \wedge h^*\Omega^1(\log\{0\}), d)$$

the relative logarithmic de Rham complex of h as studied, under the name $\Omega^\bullet(\log h)$ in [10, section 2.2]). This relative logarithmic complex is used in the definition of the family of Gauß-Manin-systems resp. Brieskorn lattices in loc.cit.

Definition-Lemma 3 ([10, section 4]). — *Let h and f as above. Define*

$$G(\log D) := \frac{H^0(V, \Omega^{n-1}_{V/T}(\log D)[\theta,\theta^{-1}])}{(\theta d - df\wedge)H^0(V, \Omega^{n-2}_{V/T}(\log D)[\theta,\theta^{-1}])}$$

$$G(*D) := \frac{H^0(V, \Omega^{n-1}_{V/T}(*D)[\theta,\theta^{-1}])}{(\theta d - df\wedge)H^0(V, \Omega^{n-2}_{V/T}(*D)[\theta,\theta^{-1}])}$$

(2.1)

$$G_0(\log D) := \frac{H^0(V, \Omega^{n-1}_{V/T}(\log D)[\theta])}{(\theta d - df\wedge)H^0(V, \Omega^{n-2}_{V/T}(\log D)[\theta])}$$

$$G_0(*D) := \frac{H^0(V, \Omega^{n-1}_{V/T}(*D)[\theta])}{(\theta d - df\wedge)H^0(V, \Omega^{n-2}_{V/T}(*D)[\theta])}.$$

Then $G(*D)$ *is* $\mathbb{C}[\theta,\theta^{-1},t,t^{-1}]$*-free of rank* n *and* $G(\log D)$ *(resp.* $G_0(*D)$*,* $G_0(\log D)$*) is a* $\mathbb{C}[\theta,\theta^{-1},t]$*- (resp.* $\mathbb{C}[\theta,t,t^{-1}]$*-,* $\mathbb{C}[\theta,t]$*-) lattice inside* $G(*D)$*.*

These modules fit into the following diagram

$$\begin{array}{ccc} G(\log D) & \subset & G(*D) \\ \cup & & \cup \\ G_0(\log D) & \subset & G_0(*D). \end{array}$$

Define a connection

$$\nabla : G_0(\log D) \longrightarrow G_0(\log D) \otimes \theta^{-1}\Omega^1_{\mathbb{C}\times T}\left(\log(\{0\}\times T) \cup (\mathbb{C}\times\{0\})\right)$$

by putting, for a form $\omega \in H^0(V, \Omega^{n-1}_{V/T}(\log D))$,

$$\begin{aligned} \nabla_{\partial_\theta}([\omega]) :=& \theta^{-2}[f\cdot\omega] \\ \nabla_{\partial_t}([\omega]) :=& \frac{1}{nt}([\mathrm{Lie}_E(\omega)] - [\theta^{-1} f\cdot\omega]) \end{aligned} \tag{2.2}$$

and extending by the Leibniz-rule (for ∇_{∂_θ}*) resp.* θ*-linearly (for* ∇_{∂_t}*). We denote by* ∇ *the induced connection on* $G(\log D)$, $G_0(*D)$ *and* $G(*D)$.

One of the main results of [10] concerns the construction of various bases of the module $G_0(\log D)$ (hence, of all the other modules given above), such that the connection takes a particularly simple form. This can be summarized as follows.

Proposition 4 ([10, proposition 4.5(iii)]). — *There is a* $\mathbb{C}[\theta, t]$*-basis* $\underline{\omega}^{(1)} = (\omega_1, \ldots, \omega_n)$ *of* $G_0(\log D)$ *such that*

$$\nabla(\underline{\omega}^{(1)}) = \underline{\omega}^{(1)} \cdot \left[(A_0\frac{1}{\theta} + A_\infty)\frac{d\theta}{\theta} + (-A_0\frac{1}{\theta} + A'_\infty)\frac{dt}{nt}\right] \tag{2.3}$$

where

$$A_0 := \begin{pmatrix} 0 & 0 & \ldots & 0 & c\cdot t \\ -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & -1 & 0 \end{pmatrix},$$

$A_\infty = \mathrm{diag}(\nu_1, \ldots, \nu_n)$ *and* $A'_\infty := \mathrm{diag}(0, 1, \ldots, n-1) - A_\infty$*. The constant* $c \in \mathbb{C}$ *is defined by the equation*

$$f^n = -c\cdot h + \sum_{i=1}^{n-1} \xi_i(f)\cdot k_i,$$

where $\xi_i \in \Theta_{V/T}(\log D)$ *and* $k_i \in \mathcal{O}_V$ *are homogenous polynomials of degree* $n-1$*. Here*

$$\omega_1 = \iota_E \frac{\mathrm{vol}}{h} = n\frac{\mathrm{vol}}{dh},$$

where vol $= dx_1 \wedge \ldots \wedge dx_n$*. In particular,* $(G(*D), \nabla)$ *is flat. Moreover,* $(\nu_1, \ldots, \nu_n)$ *is the spectrum at (*$\theta =$*) infinity of the restriction of* $G_0(\log D)$ *to* $t = 0$.

The following obvious consequence will be used in the next section.

Corollary 5. — *Consider the inclusion* $j : \{1\} \times T \hookrightarrow \mathbb{C} \times T$ *and the restriction* $(G_1(*D), \nabla) := j^*(G(*D), \nabla)$*. This is a meromorphic bundle on* T *with connection (the only pole being at* $0 \in T$*), and can thus be seen as a coherent and holonomic left* $\mathbb{C}[t]\langle\partial_t\rangle$*-module. Then we have an isomorphism of left* $\mathbb{C}[t]\langle\partial_t\rangle$*-modules*

$$\varphi : \mathbb{C}[t]\langle\partial_t\rangle/(b_{G_1(\log D)}(t\partial_t) + \frac{c}{n^n} \cdot t) \xrightarrow{\cong} (G_1(*D), \nabla),$$

where $b_{G_1(\log D)}$ *is the spectral polynomial of* $G_1(\log D) := j^*(G(\log D), \nabla)$ *at* $0 \in T$*, i.e.,*

$$b_{G_1(\log D)}(s) := \prod_{i=1}^{n} \left(s - \frac{i - 1 - \nu_i}{n}\right),$$

and where $\varphi(1) = \omega_1$*. In particular,* ω_1 *satisfies the functional equation* $b_{G_1(\log D)}(t\partial_t)\omega_1 = -c/n^n \cdot t\omega_1$ *in* $G_1(*D)$.

We note the following easy consequence from the definitions, which was not stated in [10].

Lemma 2.1. — *Let* $f_1, f_2 \in V^* \backslash D^*$ *be two generic linear forms. Denote by* $(G_i(\log D), \nabla_i)$ *the family of Brieskorn lattices attached to the pair* (f_i, h)*,* $i = 1, 2$*. Then* $\varphi^*_{c'}(G_1(\log D), \nabla_1) \cong (G_2(\log D), \nabla_2)$*, where* $\varphi_{c'} : \mathbb{C} \times T \to \mathbb{C} \times T$ *is defined as* $\varphi(\theta, t) = (\theta, c' \cdot t)$ *for some* $c' \in \mathbb{C}^*$.

Proof. — By definition, the complement of D^* in V^* is an (open) orbit of the dual action of G, hence, there is $g \in G$ with $g(f_1) = f_2$. Then $g(h) = \chi_h(g) \cdot h$ and it follows that $\varphi^*_{\chi_h(g)}(G_1(\log D), \nabla_1) \cong (G_2(\log D), \nabla_2)$. □

In order to relate the above objects to the Bernstein polynomial of h, we recall how the Gauß-Manin system $G(*D)$, seen as a left $\mathbb{C}[\theta, t]\langle\partial_\theta, \partial_t\rangle$-module is obtained as a direct image of a differential system on V. A similar reasoning as in the next lemma can be found in [6, proposition 2.7]. We consider f as a morphism $f : V \to R = \operatorname{Spec} \mathbb{C}[r]$, and put $\Phi := (f, h) : V \to R \times T$.

Lemma 2.2. — *Let* $\Phi_+\mathcal{O}_V(*D)$ *be the (algebraic) direct image complex of the holonomic* $\mathcal{D}_V$*-module* $\mathcal{O}_V(*D)$*. Then*

(1) *The cohomology sheaves of* $\Phi_+\mathcal{O}_V(*D)$ *are* $\mathcal{D}_{R\times T}$*-coherent, holonomic and regular.*

(2) *$\Phi_+\mathcal{O}_V(*D)$, seen in $\mathcal{D}^b(\mathcal{D}_{R\times T/T})$ is represented by*

$$\left(\Phi_*\Omega^{\bullet+n-1}_{V/T}(*D)[\partial_r], d-(df\wedge - \otimes\partial_r)\right).$$

Under the isomorphism

$$\mathcal{H}^0(\Phi_+\mathcal{O}(*D)) \cong \frac{\Phi_*\Omega^{n-1}_{V/T}(*D)[\partial_r]}{(d-df\wedge - \otimes\partial_r)\Phi_*\Omega^{n-2}_{V/T}(*D)[\partial_r]} \tag{2.4}$$

the action of ∂_t on a class $[\omega]\in\mathcal{H}^0(\Phi_+\mathcal{O}_V(*D))$ represented by a form $\omega\in\mathcal{H}^{n-1}_{V/T}(*D)$ is given by

$$\partial_t([\omega]) := \frac{1}{n\cdot t}\left([\mathrm{Lie}_E(\omega)] - [\mathrm{Lie}_E(f)\omega\otimes\partial_r]\right) \tag{2.5}$$

(3) *Put $M := \mathbb{H}^0(R\times T, \Phi_+\mathcal{O}(*D)))$. Denote by $\widehat{M}$ the partial Fourier-Laplace transformation with respect to r of M, i.e., $\widehat{M} = M$ as $\mathbb{C}$-vector spaces, and we define an structure of a $\mathbb{C}[\tau,t]\langle\partial_\tau,\partial_t\rangle$-module on $\widehat{M}$ by $\tau\cdot := \partial_r$ and $\partial_\tau := -r\cdot$. Then $\widehat{M}$ is $\mathbb{C}[\tau,t]\langle\partial_\tau,\partial_t\rangle$-holonomic, with singularities at $\tau=\{0,\infty\}$ and $t=\{0,\infty\}$ at most, regular along $\{t=0\}\cup\{\tau=\infty\}$. Moreover, by putting $\theta=\tau^{-1}$, the localized Fourier-Laplace transformation $\widehat{M}[\tau^{-1}]$ of M is isomorphic to $G(*D)$ as a meromorphic vector bundle with connection (the one on $G(*D)$ being given by formula (2.2)).*

(4) *The restriction $(G_1(*D),\nabla)$ is isomorphic (as a left $\mathbb{C}[t]\langle\partial_t\rangle$-module) to $\mathbb{H}^0(T, h_+\mathcal{O}_V(*D)e^{-f})$, where $\mathcal{O}_V(*D)e^{-f}$ is the tensor product of $\mathcal{O}_V(*D)$ with a rank one $\mathcal{O}_V$-module formally generated by e^{-f}, i.e., $\mathcal{O}_V(*D)e^{-f}\cong\mathcal{O}_V(*D)$ as $\mathcal{O}_V$-modules, and the differential of $\mathcal{O}_V(*D)e^{-f}$ (i.e., the operator defining its $\mathcal{D}_V$-module structure) is given by $d_f := d - df\wedge$.*

Proof. —

(1) see [21, theorem 9.0-8.]

(2) It is well known that for any left $\mathcal{D}_V$-module $\mathcal{M}$, the direct image complex $\Phi_+\mathcal{M}$ is represented by

$$\left(\mathbb{R}\Phi_*\Omega^{n+\bullet}_V(\mathcal{M})[\partial_r,\partial_t], d-(df\wedge-\otimes\partial_r)-(dh\wedge-\otimes\partial_t)\right).$$

Putting $\mathcal{M}=\mathcal{O}_V(*D)$ and using that Φ is affine, we consider the double complex

$$E^{p,q} :=$$
$$\left((\Phi_*\Omega^{p+q}_V(*D)[\partial_r])\otimes\partial_t^q, d-(df\wedge-\otimes\partial_r), -(dh\wedge-\otimes\partial_t)\right)$$

whose total cohomology is $\mathcal{H}^{p+q-n}(\Phi_+(\mathcal{O}_V(*D)))$. The morphism h is smooth restricted to $V\backslash D$, hence, the Koszul complex

$$(\Omega^\bullet(*D), dh\wedge)$$

is acyclic. Therefore the second spectral sequence associated to the above double complex degenerates at the E_2-term and the isomorphism

$$\Omega^{p-1}_{V/T}(*D) \xrightarrow{dh\wedge} \mathcal{K}er\left(\Omega^p_V(*D) \xrightarrow{dh\wedge} \Omega^{p+1}_V(*D)\right)$$

yields the above quasi-isomorphism.

In order to prove the formula for the action of ∂_t, notice that given a class $[\omega]$ defined by a relative n-form $\omega \in \Omega^{n-1}_{V/T}(*D)$, the class corresponding to it in $\mathcal{H}^0(\Phi_+\mathcal{O}_V(*D))$ is $[dh\wedge\omega]$. By definition, we have $\partial_t([dh\wedge\omega]) = [dh\wedge\omega\otimes\partial_t]$. This class is equal in $\mathcal{H}^0(\Phi_+\mathcal{O}_V(*D))$ to $[d\omega - df\wedge\omega\otimes\partial_r]$. It follows that under the isomorphism (2.4), this equals

$$[\frac{d\omega}{dh} - \frac{df\wedge\omega}{dh}\otimes\partial_r] \in \Phi_*\Omega^{n-1}_{V/T}(*D)[\partial_r]/(d - df\wedge - \otimes\partial_r)\Phi_*\Omega^{n-2}_{V/T}(*D).$$

Now notice that as h is smooth outside D, there is a vector field $X\in\Theta_V(*D)$ which lifts $\partial_t\in\Theta_T$. Then we have that $d\omega/dh = \iota_X d\omega$ and $(df\wedge\omega)/dh = \iota_X(df)\wedge\omega - df\wedge\iota_X\omega$. Putting this together and using once again the relation in the quotient

$$\Phi_*\Omega^{n-1}_{V/T}(*D)[\partial_r]/(d - df\wedge - \otimes\partial_r)\Phi_*\Omega^{n-2}_{V/T}(*D)$$

one arrives at the formula

$$\partial_t[\omega] = [\mathrm{Lie}_X\,\omega] - [\mathrm{Lie}_X(f)\omega]\otimes\partial_r.$$

Now the result follows as the meromorphic vector field X can be taken to be $E/(n\cdot h)$, due to the homogeneity of h.

(3) This is obvious from the last point: Fourier-Laplace transformation and localization along $\tau = 0$ transforms formula (2.4) into the defining equation (2.1) of $G(*D)$ and formula (2.5) obviously corresponds to the second part of formula (2.2). The statements about regularity follows form the general considerations in [6, theorem 1.11].

(4) By definition, $h_+\mathcal{O}_V(*D)e^{-f}$ is represented by the complex

$$(h_*\Omega^{n+\bullet}_V(*D)[\partial_t], d - df\wedge -(dh\wedge - \otimes\partial_t)).$$

The same argument as above shows that this is quasi-isomorphic to

$$(h_*\Omega^{n-1+\bullet}_{V/T}(*D), d - df\wedge).$$

Now it is clear that

$$\mathbb{H}^0(T, h_*\Omega^{n-1+\bullet}_{V/T}(*D), d - df\wedge) = \frac{H^0(\Omega^{n-1}_{V/T}(*D))}{(d - df\wedge)H^0(V, \Omega^{n-2}_{V/T}(*D)))} = G_1(*D)$$

□

Remark. — Direct images of regular holonomic modules by a morphism consisting of two polynomials occur in the work of C. Roucairol (see [22], [24] and [23]). She also studied direct images of twisted modules, i.e., $h_+(\mathcal{M}e^{-f})$. However, we will not need her results directly as the computations from [10] (i.e. proposition 4 above) give already very precise information about these direct images for a pair (f, h), with h reductive linear free and f linear and generic.

3. Bernstein Polynomials

We first give the definition of the Bernstein polynomial through the classical functional equation. Next we recall how this can be rephrased using the general theory of V-filtrations. This will be useful in proving the main result. Finally, we state and prove the relation between the roots of the Bernstein polynomial of a defining equation h for a linear free divisor and the residue eigenvalues of the family of Gauß-Manin-systems introduced in section 2.

The following classical statement is due to Bernstein (see, [1]).

Theorem 3.1. — *Let $h \in \mathcal{O}_V$ be any function, then there is a polynomial $B \in \mathbb{C}[s]$ and a differential operator $P(x_i, \partial_{x_i}, s) \in \mathcal{D}_V[s]$ such that*

$$P(x_i, \partial_{x_i}, s)h^{s+1} = B(s)h^s$$

All polynomials $B(s) \in \mathbb{C}[s]$ having this property form an ideal in $\mathbb{C}[s]$, and we denote by $b_h(s)$ the unitary generator of this ideal. $b_h(s)$ is called the Bernstein polynomial of h.

If h defines a linear free divisor, then the theory of pre-homogenous vector spaces shows that the functional equation defining $b_h(s)$ is of a particular type.

Theorem 3.2 ([26], [13], [9]). — *Let $D = h^{-1}(0)$ be a reductive linear free divisor, then the operator P appearing in Bernstein's functional equation is given by $P := h^*(\partial_{x_1}, \ldots, \partial_{x_1})$ (remember that $h^*(\underline{y}) = \overline{h(\overline{\underline{y}})}$, where x_i are the unitary coordinates and y_i are their duals). In particular,* *it is an element of* $\mathbb{C}\langle\partial_{x_1},\dots,\partial_{x_n}\rangle$*. Moreover, the degree of* $b_h(s)$ *is equal to* n *and the roots of* $b_h(s)$ *are contained in the open interval* $(-2,0)$ *and are symmetric around* -1*. In particular,* -1 *is the only integer root.*

The following classical reformulation of the definition of the Bernstein polynomial will be useful in the sequel.

Consider the ring $\mathbb{C}[s,h^{-1}]$, and denote by $M[h^{-1}] := \mathbb{C}[s,h^{-1}]h^s$ the rank one $\mathbb{C}[s,h^{-1}]$-module generated by the symbol h^s. Define an action of $\mathcal{D}_V$, the ring of algebraic differential operators on V on $M[h^{-1}]$ by putting

$$\partial_{x_i}(g\cdot h^s) := \partial_{x_i}(g)\cdot h^s + g\cdot s\cdot h^{-1}\partial_{x_i}(h)h^s.$$

This action extends naturally to an $\mathcal{D}_V[s]$-action. Let M the $\mathcal{D}_V[s]$-submodule of $M[h^{-1}]$ generated by h^s. Define an action of t on $M[h^{-1}]$ by putting $t(g(s)\cdot h^s) := g(s+1)\cdot h\cdot h^{s+1}$. Then $b_h(s)$ is the minimal polynomial of the action of s on the quotient M/tM.

This definition can be rephrased once more using the theory of V-filtrations on $\mathcal{D}$-modules. Without reviewing the details of the theory, we recall the following facts (see, e.g. [18, section 4])

Definition-Lemma 6. — *Let* X *be any smooth algebraic variety, and* $Y\subset X$ *a smooth hypersurface defined by an ideal sheaf* $I\subset\mathcal{O}_X$*. We denote by* $t\in\mathcal{O}_X$ *a local generator of* $\mathcal{I}$*.*

(1) *Let* $\mathcal{D}_X$ *be the sheaf of algebraic differential operators, then define*

$$V_k\mathcal{D}_X := \left\{P\in\mathcal{D}_X \mid P(I^j)\subset I^{j-k}\right\}$$

For any left $\mathcal{D}_X$*-module* $\mathcal{M}$*, a V-filtration on* $\mathcal{M}$ *is an increasing filtration* $U_\bullet\mathcal{M}$ *compatible with* $V_\bullet\mathcal{D}_X$*.*

(2) *A V-filtration* $U_\bullet\mathcal{M}$ *on a left* $\mathcal{D}_X$*-module* $\mathcal{M}$ *is good iff the* Rees-*module* $\oplus z^k U_k\mathcal{M}$ *is* $\mathcal{R}_V\mathcal{D}_X$*-coherent, where* $\mathcal{R}_V\mathcal{D}_X := \oplus_k z^k V_k\mathcal{D}_X$*.*

(3) *A good V-filtration* $U_\bullet\mathcal{M}$ *is said to have a Bernstein polynomial iff there is a non-zero polynomial* $b(s)\in\mathbb{C}[s]$ *such that for all* $k\in\mathbb{Z}$*, we have* $b(-\partial_t t+k)U_k\mathcal{M}\subset U_{k-1}\mathcal{M}$*.*

(4) *A coherent* $\mathcal{D}_X$*-module* $\mathcal{M}$ *is called specializable iff locally there exists a good* V*-filtration* $U_\bullet\mathcal{M}$ *having a Bernstein polynomial. Equivalently, for any local section* $m\in\mathcal{M}$ *there is a non-zero polynomial* $b_m(s)$ *(the Bernstein polynomial of* m*) such that*

$$b_m(-\partial_t t)\,m\in V_{-1}\mathcal{D}_X\cdot m.$$

(5) *A holonomic* $\mathcal{D}_X$*-module is specializable along any smooth hypersurface* Y*.*

The following evident corollary gives an example of a V-filtration that will be used later.

Corollary 7. — *Consider the left $\mathbb{C}[t]\langle\partial_t\rangle$-module $G_1(*D)$ from above. Then*

$$U_kG_1(*D) := V_k\mathbb{C}[t]\langle\partial_t\rangle \cdot G_1(\log D)$$

defines a good V-filtration on $G_1(*D)$, whose Bernstein polynomial is exactly $b_{G_1(\log D)}(s)$. Moreover, we have

$$U_0G_1(*D) = G_1(\log D) = V_0\mathbb{C}[t]\langle\partial_t\rangle \cdot \omega_1.$$

We will also use V-filtrations for $\mathcal{D}_{T\times V}$-modules. The following result is well known, see, e.g., [20], [18, lemme 4.4-1].

Lemma 3.3. —
(1) *Let $h \in \mathcal{O}_V$ an arbitrary function, seen as a morphism $h : V \to T$. Denote by $i_h : V \hookrightarrow T \times V$ the graph embedding, with image Γ_h. Put $\mathcal{N} := (i_h)_+\mathcal{O}_V$, then $\mathcal{N} \cong \mathcal{O}_V[\partial_t] \cong \mathcal{O}_{T\times V}(*\Gamma_h)/\mathcal{O}_{T\times V} \cong \mathcal{D}_{T\times V}\delta(t-h)$. A good V-filtration with respect to the hypersurface $\{0\}\times V$ on $\mathcal{N}$ is defined by putting, for all $k \in \mathbb{Z}$, $U_k\mathcal{N} := V_k\mathcal{D}_{T\times V}\delta(t-h)$. This V-filtration admits a Bernstein polynomial (namely, a Bernstein polynomial for the section $\delta(t-h)$), which is exactly the polynomial $b_h(s)$. We denote, as in [20], by $\mathcal{M}$ the $V_0\mathcal{D}_{T\times V}$-module $U_0\mathcal{N}$.*

(2) *The direct image $(i_h)_+\mathcal{O}_V(*D)$ is the localization of both $\mathcal{N}$ and $\mathcal{M}$ along $t = 0$, and is thus denoted by $\mathcal{M}[t^{-1}]$. As $\mathcal{N}$ has no t-torsion, we have an exact sequence*

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{M}[t^{-1}] \longrightarrow C \longrightarrow 0.$$

where C is a $\mathcal{D}_{T\times V}$-module. A Bernstein polynomial for a local section $m \in \mathcal{N}$ is also a Bernstein polynomial for m, seen as a local section in $\mathcal{M}[t^{-1}]$.

We can now state and prove the main result of this paper.

Theorem 3.4. — *Let $D = h^{-1}(0)$ be reductive linear free divisor and $f \in V^*$ be generic. Consider the family of Gauß-Manin systems $G(*D)$, the logarithmic extension $G(\log D)$ and the restrictions $G_1(\log D) \subset G_1(*D)$ from above. Then we have that $b_h(s) = b_{G_1(\log D)}(s+1)$ (recall that $b_{G_1(\log D)}(s)$ is the spectral polynomial of $G_1(\log D)$).*

In order to prove this result, we start with a preliminary lemma.

Lemma 3.5. — *Let $\mathcal{M}[t^{-1}] := (i_h)_+\mathcal{O}_V(*D)$ as above. Consider the twisted module $(i_h)_+\mathcal{O}_V(*D)e^{-f}$. Then the section*

$$\delta(t-h)e^{-f} \in (i_h)_+\mathcal{O}_V(*D)e^{-f}$$

admits $b_h(s)$ as a Bernstein polynomial, with associated functional equation

$$(t \cdot h^*(\partial_{x_i} + a_i) - b_h(-\partial_t t))\,\delta(t-h)e^{-f} = 0, \tag{3.1}$$

where $f = \sum_{i=1}^n a_i x_i$.

Proof. — By lemma 3.3, $b_h(s)$ is the minimal polynomial of $-\partial_t t$ on

$$\frac{\mathcal{D}_V[t\partial_t]\delta(t-h)}{t\mathcal{D}_V[t\partial_t]\delta(t-h)}.$$

In particular, by theorem 3.2, the functional equation

$$(t \cdot h^*(\partial_{x_i}) - b_h(-\partial_t t))\,\delta(t-h) = 0$$

holds in $(i_h)_+\mathcal{O}_V(*D)$. Then it follows directly from the definition of the twisted module $\mathcal{O}_V(*D)e^{-f}$ that the functional equation (3.1) from above holds in $(i_h)_+\mathcal{O}_V(*D)e^{-f}$. Now suppose that there is another equation

$$\left(t \cdot \widetilde{P}(x_i, \partial_{x_i}, -\partial_t t) - \widetilde{B}(-\partial_t t)\right)\delta(t-h)e^{-f} = 0,$$

where $\widetilde{P} \in \mathcal{D}_V[s]$ and $\widetilde{B}(s) \in \mathbb{C}[s]$ with $\deg(\widetilde{B}) < \deg(b_h)$. Then we obtain the equation

$$\left(t \cdot \widetilde{P}(x_i, \partial_{x_i} - a_i, -\partial_t t) - \widetilde{B}(-\partial_t t)\right)\delta(t-h) = 0$$

in $(i_h)_+\mathcal{O}_V(*D)$, which contradicts the minimality of $b_h(s)$. □

Proof of the theorem. — We consider, as in the last lemma, the $\mathcal{D}_{T\times V}$-module

$$(i_h)_+\mathcal{O}_V(*D)e^{-f} \cong (i_h)_*\mathcal{O}_V(*D)e^{-f}[\partial_t]$$

and the $\mathcal{D}_V[t\partial_t]$-submodule generated (over $\mathcal{D}_V[t\partial_t]$) by $\delta(t-h)e^{-f}$. The direct image $h_+(\mathcal{O}_V(*D)e^{-f})$ is obtained in the standard way from the module $(i_h)_+\mathcal{O}_V(*D)e^{-f}$ as the relative de Rham complex of the projection $p_1 : T \times V \to T$. In other words, we have

$$\mathcal{H}^i(h_+(\mathcal{O}_V(*D)e^{-f})) = \mathcal{H}^i((p_1)_*\, DR^{n+\bullet}_{T\times V/T}((i_h)_*\mathcal{O}_V(*D)e^{-f}[\partial_t]))$$

Considering $(i_h)_+\mathcal{O}_V(*D)e^{-f}$ as a $\mathcal{D}_V$-module only, we thus have

$$\mathcal{H}^i(h_+(\mathcal{O}_V(*D)e^{-f})) = h_*\mathcal{H}^i(DR^{n+\bullet}_V\, \mathcal{O}_V(*D)e^{-f}[\partial_t]).$$

Now it is well known (see, e.g., [20, proposition 2.1] or [2, proposition 2.2.10]), that for any left $\mathcal{D}_V$-module $\mathcal{L}$, the de Rham complex $DR^\bullet_V(\mathcal{L})$ represents the (shifted) derived tensor product $\Omega^n_V \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_V} \mathcal{L}[-n]$, in particular, we have

$$\mathcal{H}^n(DR^\bullet_V(\mathcal{L})) \cong \Omega^n_V \otimes_{\mathcal{D}_V} \mathcal{L}.$$

It follows that

$$\mathcal{H}^0(h_+(\mathcal{O}_V(*D)e^{-f})) \cong h_*\left(\Omega^n_V \otimes_{\mathcal{D}_V} (\mathcal{O}_V(*D)e^{-f}[\partial_t])\right). \tag{3.2}$$

so that, taking global sections and considering again the isomorphism from lemma 2.2, 4., we obtain

$$H^0(V, \Omega^n_V \otimes_{\mathcal{D}_V} (\mathcal{O}_V(*D)e^{-f}[\partial_t])) \cong G_1(*D)$$

Notice that the section $\mathrm{vol} \otimes \delta(t-h)e^{-f}$ is mapped to the section $\omega_1/n = \mathrm{vol}/dh$ under this isomorphism.

From the equation

$$(t \cdot h^*(\partial_{x_i} + a_i) - b_h(-\partial_t t))\, \delta(t-h)e^{-f} = 0$$

in $\mathcal{O}_V(*D)e^{-f}[\partial_t]$ (equation (3.1)) we deduce that the element

$$\mathrm{vol} \otimes (t \cdot h^*(\partial_{x_i} + a_i) - b_h(-\partial_t t))\, \delta(t-h)e^{-f}$$

is zero in $h_*(\Omega^n_V \otimes_{\mathcal{D}_V} (\mathcal{O}_V(*D)e^{-f}[\partial_t]))$.

Hence

$$t \cdot (h^*(\partial_{x_i} + a_i)(\mathrm{vol})) \otimes \delta(t-h)e^{-f} = b_h(-\partial_t t)(\mathrm{vol} \otimes \delta(t-h)e^{-f})$$

holds in $h_*\left(\Omega^n_V \otimes_{\mathcal{D}_V} (\mathcal{O}_V(*D)e^{-f}[\partial_t])\right)$, where the operator $h^*(\partial_{x_i} + a_i)$ acts on vol by the right $\mathcal{D}_V$-action on Ω^n_V. Now develop the polynomial $h^*(y_i + a_i)$ as $h^*(y_i + a_i) = \sum_{1 \leqslant |I| \leqslant n} a_I y^I + h^*(a_i)$, then

$$h^*(\partial_{x_i} + a_i) = \sum_{1 \leqslant |I| \leqslant n} a_I \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + h^*(a_i)$$

and the action $h^*(\partial_{x_i} + a_i)(\mathrm{vol})$ is given by

$$\sum_{1 \leqslant |I| \leqslant n} a_I \left(\underbrace{\mathrm{Lie}_{\partial_{x_1}} \cdots \mathrm{Lie}_{\partial_{x_1}}}_{i_1} \cdots \underbrace{\mathrm{Lie}_{\partial_{x_n}} \cdots \mathrm{Lie}_{\partial_{x_n}}}_{i_n} \right) (\mathrm{vol}) + h^*(a_i) \cdot \mathrm{vol}$$

But obviously $\mathrm{Lie}_{\partial_{x_i}} \mathrm{vol} = 0$ for any $i \in \{1, \dots, n\}$, so that finally we see that the section $\mathrm{vol} \otimes \delta(t-h)e^{-f}$ of $h_*\left(\Omega^n_V \otimes_{\mathcal{D}_V} (\mathcal{O}_V(*D)e^{-f}[\partial_t])\right)$ is annihilated by $h^*(a_i) \cdot t - b_h(-\partial_t t)$. It follows that $b_h(-\partial_t t)$ sends

$$U_0 G(*D) = V_0 \mathbb{C}[t]\langle \partial_t \rangle \omega_1$$

into $U_{-1}G(*D)$, hence, we have $b_{G_1(\log D)}(s+1)|b_h(s)$. Now the theorem follows as both b_h and $b_{G_1(\log D)}$ are of degree n. □

4. Consequences and Examples

Definition 8. — *Let D be a reductive linear free divisor with defining equation $h \in \mathcal{O}_V$ and $f \in V^*$ a generic linear form. Consider, as in the last section, the logarithmic extension $G_0(\log D)$ of the family of Brieskorn lattices $G_0(*D)$ attached to (f,h). We define the* logarithmic Brieskorn lattice *of h to be the restriction $G_0(h) := i^*(G_0(\log D), \nabla)$, where $i : \mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times T$.*

Notice that it follows from lemma 2.1 that $G_0(h)$ is independent of the choice of f in $V^* \backslash D^*$, so that it makes sense to speak about *the* logarithmic Brieskorn lattice of h.

The next result, which is an easy consequence of theorem 3.4, can be considered as a variant of the corresponding classical statement of Malgrange ([20]) for the isolated singularity case.

Theorem 4.1. — *Let $(G_0(h), \nabla)$ be the logarithmic Brieskorn lattice of a reductive linear free divisor D. Then ∇ is regular singular at $\theta = 0$. Consider the saturation $\widetilde{G}_0(h) := \sum_{k \geqslant 0} (\nabla_{\theta\partial_\theta})^k G_0(h)$, which has a logarithmic pole at $\theta = 0$. Let $b_{\widetilde{G}_0(h)}(s)$ be the minimal polynomial of the residue endomorphism of ∇_θ on $\widetilde{G}_0(h)$. Then $b_{\widetilde{G}_0(h)}(n(s+1)) = b_h(s)$.*

Proof. — The regularity follows easily from the particular form of the connection matrix (2.3). Namely, $G_0(h)$ is the Fourier-Laplace transformation of a regular $\mathbb{C}[r]\langle\partial_r\rangle$-module, hence, its regularity is equivalent to the nilpotency of the polar part of the connection matrix, which is obviously the case here, by putting $t = 0$ in A_0. Now the saturation of $G_0(h)$ is easy to calculate: We put $\widetilde{\omega}_i := \theta^{1-i}\omega_i$, then $G(\log D) = \oplus_{i=1}^n \mathbb{C}[\theta, \theta^{-1}, t]\widetilde{\omega}_i$, but $G_0(\log D) \subsetneq \oplus_{i=1}^n \mathbb{C}[\theta, t]\widetilde{\omega}_i$. It is evident that $\widetilde{G}_0(h) = \oplus_{i=1}^n \mathbb{C}[\theta]\widetilde{\omega}_i$, in particular, this module is invariant under $\theta\nabla_\theta$, i.e., logarithmic at $\theta = 0$. We have

$$(\theta\partial_\theta)\underline{\widetilde{\omega}} = \underline{\widetilde{\omega}} \cdot (\widetilde{A}_0 + \mathrm{diag}(\{1 - i + \nu_i\}_{i=1,\dots,n}),$$

where $\widetilde{A}_0 := (A_0)_{|t=0}$. We see by theorem 3.4 that the residue eigenvalues of ∇_θ at $\theta = 0$ are the roots of the Bernstein polynomial of h after dividing by n and shift by -1, and moreover that the residue endomorphism is regular (i.e., its minimal and characteristic polynomial coincide), as it has a cyclic generator. This proves the theorem. □

Remark. — One may ask what the meaning of the rescaling by n occurring in $b_{\widetilde{G_0(h)}}(n(s+1))$ is. The same kind of twist occurs in [10, proposition 4.5i(v)], where it is performed on the base, i.e., where the pull-back $u^*(G(*D))$ with $u : \mathbb{C}^2 \to \mathbb{C} \times T$, $(\theta, t') \mapsto (\theta, (t')^n)$ is considered, and where it is shown that after this pull-back, the resulting bundle has the "rescaling property", i.e., that it is invariant under $\nabla_{\theta\partial_\theta - t'\partial_{t'}}$.

The following easy consequence is a somewhat reverse argumentation compared to Malgrange's result, where the rationality of the roots of the Bernstein polynomial was deduced from the (known) quasi-unipotency of the monodromy acting on the cohomology of the Milnor fibre of an isolated hypersurface singularity. In our case, the rationality of the roots of $b_h(s)$ is known, but we deduce information on the (a priori unknown) monodromy of the logarithmic Brieskorn lattice $G_0(h)$. Moreover, we can use the results of [9] to obtain a symmetry property of the spectrum at infinity of the logarithmic Brieskorn lattice, which was conjectured in [10, corollary 5.6].

COROLLARY 9. — *The monodromy of the logarithmic Brieskorn lattice, i.e. of the local system associated to $G_0(h)[\theta^{-1}] := G_0(h) \otimes_{\mathbb{C}[\theta]} \mathbb{C}[\theta, \theta^{-1}]$ is quasi-unipotent. Moreover, let $\alpha_1, \ldots, \alpha_n$ be the spectral numbers of $G_0(h)$ at infinity (i.e., the numbers ν_i from proposition 4), written as a non-decreasing sequence. Then $\alpha_i + \alpha_{n+1-i} = n - 1$.*

Proof. — The eigenvalues of this monodromy are simply the exponentials of either the numbers ν_i or $\nu_i' := i - 1 - \nu_i$ from proposition 4 (or any other integer shift of them). The numbers ν_i' are the roots of the Bernstein polynomial of h shifted by one, as shown in theorem 3.4. These are known to be rational by [17]. Similarly, if we denote the roots of b_h by $\alpha_1', \ldots, \alpha_n'$, with $\alpha_i' \leqslant \alpha_j'$ if $i \leqslant j$, then we know from [9, theorem 2.5.] that $\alpha_i' + \alpha_{n+1-i}' = -2$. From theorem 3.4 and proposition 4 we deduce that $\alpha_j = (j-1) - \alpha_j' - 1$ for any $j \in \{1, \ldots, n\}$, hence,

$$\alpha_i + \alpha_{n+1-i} = \\ ((i-1) - \alpha_i' - 1) + ((n+1-i) - 1 - \alpha_{n+1-i}' - 1) = n - 1.$$

□

We outline another consequence of the theorem 3.4. Its interest is motivated by comparing the situation considered here with the one where f is still a generic linear form, but h is supposed to by an arbitrary monomial $h = \prod x_i^{w_i}$, i.e., non-reduced. The corresponding Gauß-Manin-systems resp. Brieskorn lattices have been studied in [7], [4] and [5]. It is known that they are closely related to the *Mirror symmetry* phenomenon, i.e., one constructs a Frobenius structure on the semi-universal unfolding of $f_{|h^{-1}(t)}$, $t \neq 0$ which is known to be isomorphic to the orbifold quantum cohomology of the weighted projective spaces. For a linear free divisor D, a similar construction of a Frobenius manifold has been carried out in [10]. Although these are not a priori mirrors of some variety or orbifold, the following corollary shows an interesting similarity with the case $h = \prod x_i^{w_i}$.

Corollary 10. — *The spectrum at* $\theta = \infty$ *of* $(G_0(h), \nabla)$ *and* $(G(*D), \nabla)$ *contains a (non-trivial) block of integer numbers* $k, k+1, \dots, n-1-k$ *for some* $k \in \{0, \dots, n-1\}$.

Proof. — For the spectrum of $(G_0(h), \nabla)$, this is obvious as this block corresponds to the root -1 of the Bernstein polynomial $b_h(s)$. For the spectrum of $(G(*D), \nabla)$, one shows the same statement by analyzing the construction of a good basis of $G(*D)$ from a good basis of $G_0(h)$ using algorithm 2 of [10, lemma 4.11]. □

Notice that for the normal crossing case, the integer k from above is equal to zero, i.e., the block mentioned above is the whole spectrum. This is not true in general, hence, the Frobenius structures constructed in [10] are not, a priori, mirrors of quantum cohomology algebras of orbifolds, as zero is not, in general, an element of the spectrum. Still the analogy with the orbifold quantum cohomology, i.e., the fact that there is a block of increasing integer spectral numbers corresponding to the "untwisted sector" (see, e.g., [16, section 2.1.]) is rather intriguing.

Examples of Bernstein polynomials. — We use the main result and the computations of spectral numbers in [10] to obtain the roots of the Bernstein polynomials for the following reductive linear free divisors. The definitions of the two last discriminants can be found in [8], example in 1.4(2) (this one is also called "bracelet") and [26], proposition 11, respectively.

Notice that the examples E_6 and the last two discriminants are obtained by direct calculations in Singular ([12]). On the other hand, the closed formulas for the star quiver and the D-series follows from rather involved combinatorial arguments, the details of which will appear in [11]. The Bernstein polynomials for D_4 (which is equal to $\star_3$) and the bracelet are also calculated in [9]. The one for A_n is of course completely obvious and well known. It would be of interest to complete these calculations by the Bernstein polynomials of quiver representations for the highest roots of the Dynkin quivers E_7 and E_8, however, this seems to be out of reach of computer algebra for the moment (remember from [3] that the linear free divisors associated to these roots for E_7 resp. E_8 are of degree 46 resp. 118).

linear free divisor	**Bernstein polynomial of h**
A_n - quiver	$(s+1)^n$
D_m - quiver	$\left(s+\frac{4}{3}\right)^{m-3}\cdot(s+1)^{2m-4}\cdot\left(s+\frac{2}{3}\right)^{m-3}$
E_6 - quiver	$(s+\frac{7}{5})\cdot(s+\frac{4}{3})^4\cdot(s+\frac{6}{5})\cdot(s+1)^{10}\cdot(s+\frac{4}{5})\cdot$ $(s+\frac{2}{3})^4\cdot(s+\frac{3}{5})$
$\star_m$ - quiver	$\prod_{l=0}^{m-3}\left(s+\frac{2(m-1)-l}{m}\right)^{l+1}\cdot(s+1)^{2(m-1)}\cdot$ $\prod_{l=0}^{m-3}\left(s+\frac{m-1-l}{m}\right)^{m-l-2}$
discriminant in $S^3((\mathbb{C}^2)^*)$	$\left(s+\frac{7}{6}\right)\cdot(s+1)^2\cdot\left(s+\frac{5}{6}\right)$
discriminant of $\mathrm{Sl}(3,\mathbb{C})\times\mathrm{Gl}(2,\mathbb{C})$ action on $\mathrm{Sym}(3,\mathbb{C})\times\mathrm{Sym}(3,\mathbb{C})$	$\left(s+\frac{5}{4}\right)^2\cdot\left(s+\frac{7}{6}\right)^2\cdot(s+1)^4\cdot\left(s+\frac{5}{6}\right)^2\cdot\left(s+\frac{3}{4}\right)^2$

Table 4.1. Bernstein polynomials for some examples of linear free divisors.

Let us finish this note with a remark and a conjecture exploiting further the analogy with the case of an isolated hypersurface singularity. We have seen that the theorem of Malgrange can be adapted for reductive linear free divisors using the logarithmic Brieskorn lattice from above. The regularity of $(G_0(h),\nabla)$ at $\theta=0$ suggest to study the spectrum in the classical sense of Varchenko (i.e., at $\theta=0$) of this lattice. We recall the definition and calculate two examples, in order to show that this spectrum contains additional information not present in roots of the Bernstein polynomial, similarly to the case of isolated singularities.

Definition 11. — *Let $(\mathbb{E},\nabla)$ be a vector bundle on $\mathbb{C}=\operatorname{Spec}\mathbb{C}[\theta]$ equipped with a connection with a pole at zero of order two at most, which is regular singular. The localization $\mathbb{M}:=E\otimes_{\mathbb{C}[\theta]}\mathbb{C}[\theta,\theta^{-1}]$ has the structure of a holonomic $\mathbb{C}[\theta]\langle\partial_\theta\rangle$-module with a regular singularity at $\theta=0$. We suppose that the monodromy of its de Rham complex is quasi-unipotent. Denote by $V^\bullet\mathbb{M}$ the canonical V-filtration on $\mathbb{M}$ at $\theta=0$, indexed by $\mathbb{Q}$. As this is a filtration by free $\mathbb{C}[\theta]$-modules (and not by free $\mathbb{C}[\theta^{-1}]$-modules as the V-filtration at $\theta=\infty$), we write it as a decreasing filtration. Define the spectrum of (E,∇) to be*

$$\mathrm{Sp}(E,\nabla):=\sum_{\alpha\in\mathbb{Q}}\frac{V^\alpha\mathbb{M}\cap\mathbb{E}}{V^\alpha\mathbb{M}\cap\theta\mathbb{E}+V^{>\alpha}\mathbb{M}\cap\mathbb{E}}\alpha\in\mathbb{Z}[\mathbb{Q}]$$

where $V^{>\alpha}\mathbb{M}:=\cup_{\beta>\alpha}V^\alpha\mathbb{M}$.

As an example, we consider the case of the normal crossing divisor

$$D = \{h^{A_n} = \prod_{i=1}^{n} x_i = 0\},$$

which is the discriminant in the representation space of the quiver A_n. It was stated in [10] (but essentially well known before, due to the relation of this example to the quantum cohomology of the projective space $\mathbb{P}^{n-1}$) that we have $G_0(h^{A_n}) := \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{C}\times\{0\}}\omega_i$, and

$$\nabla(\underline{\omega}) = \underline{\omega} \cdot \left[\frac{\widetilde{A}_0}{\theta} + \mathrm{diag}(0,1,\ldots,n-1)\right] \frac{d\theta}{\theta},$$

$\widetilde{A}_0 := (A_0)_{|t=0}$. On the other hand, we take up the example of the star quiver with three exterior vertices studied in [10, example 2.3(i)]. Notice that this is exactly the quiver D_4. Here $D \subset V = \mathbb{C}^6$, and $h^{\star_3} = h_1^{\star_3} \cdot h_2^{\star_3} \cdot h_3^{\star_3}$, where

$$h_1^{\star_3} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \quad ; \quad h_2^{\star_3} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} \quad ; \quad h_3^{\star_3} = \begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

Following the various algorithms of loc.cit used to obtain good basis, we have that $G_0(h^{\star_3}) := \oplus_{i=1}^{6} \mathcal{O}_{\mathbb{C}\times\{0\}}\omega_i$, and

$$\nabla(\underline{\omega}) = \underline{\omega} \cdot \left[\frac{A_0}{\theta} + \mathrm{diag}(2,1,2,3,4,3)\right] \frac{d\theta}{\theta}$$

Notice that this is the basis called $\underline{\omega}^{(2)}$ in loc.cit.

Proposition 12. — (1) *The spectrum at $\theta = 0$ for h^{A_n} is*

$$\mathrm{Sp}(G_0(h^{A_n}), \nabla) = (0,1,\ldots,n-1) \in \mathbb{Z}[\mathbb{Q}],$$

hence, it is equal to the spectrum at $\theta = \infty$ of both $(G_0(h), \nabla)$ and $(G(*D), \nabla)$ (so that in this case we do not get more information from the spectrum at $\theta = 0$ than those contained in the roots of $b_h(s)$).

(2) *The spectrum at $\theta = 0$ for $h^{\star_3}$ is given by*

$$\mathrm{Sp}(G_0(h^{\star_3}), \nabla) = (-2,1,2,3,4,7) \in \mathbb{Z}[\mathbb{Q}],$$

hence, different from $\mathrm{Sp}(G_0(h), \nabla)$ and not directly related to

$$b_h(s) = (s+\frac{4}{3})(s+1)^4(s+\frac{2}{3}).$$

Proof. —

(1) One can calculate directly that $G_0(h^{A_n})$ can be generated by elementary sections, which implies that $\mathrm{Sp}(G_0(h^{A_n}),\nabla)$ is equal to the spectrum at $\theta=\infty$, i.e., $\mathrm{Sp}(G_0(h^{A_n}),\nabla)=(0,1,\ldots,n-1)$. However, this can also be obtained in a more abstract way: For any linear free divisor D, the analytic object corresponding to the restriction of $G_0(*D)$ to $\mathbb{C}\times(T\backslash\{0\})$ is known (after a finite ramification of order n) to be a Sabbah orbit of TERP-structures (see the remark after the proof of theorem 4.1 and [10, proposition 4.5 (v)]). In the A_n-case, it is easy to see that the extension $G_0(\log D)$ is exactly the extension ${}_0\mathcal{E}$ considered in [14, proof of theorem 7.3 and lemma 6.11] and the logarithmic Brieskorn lattice $G_0(h)$ is isomorphic to the limit $\mathcal{G}_0$ considered in loc.cit, proof of theorem 7.3 and lemma 6.12. It was shown in the proof of theorem 7.3 of loc.cit. that $\mathcal{G}_0$ is generated by elementary sections.

(2) In the $\star_3$-case, one cannot apply the previous reasoning. Hence a direct calculation is necessary. We explain parts of it, leaving the details to the reader. From the connection matrix given above we see that $(\theta\partial_\theta)\omega_6=3\omega_6$, and $(\theta\partial_\theta)\omega_5=4\omega_5-\theta^{-1}\omega_6$. We make the Ansatz

$$\omega_5=\alpha\theta^{-1}\omega_6+s_4$$

where s_4 is a section of $G_0(h)[\theta^{-1}]$ satisfying $(\theta\partial_\theta)(s_4)=4\cdot s_4$. We obtain

$$(\theta\partial_\theta)\omega_5=2\alpha\theta^{-1}\omega_6+4s_4\overset{!}{=}(4\alpha-1)\theta^{-1}\omega_6+4s_4$$

from which we conclude that $\omega_5=\frac{1}{2}\theta^{-1}\omega_6+s_4$. Similarly, the equation $(\theta\partial_\theta)\omega_4=3\omega_4-\theta^{-1}\omega_5$ is satisfied by putting

$$\omega_4=\frac{1}{8}\beta_1\cdot\theta^{-2}\omega_6+s_3$$

where $s_3\in G_0(h)[\theta^{-1}]$ is a section satisfying $(\theta\partial_\theta)s_3=3s_3+\theta^{-1}s_4$. Continuing this way we see that the elements of our basis $\underline{\omega}$ can be written as finite sums of elementary sections in the following way:

$$\begin{aligned}
\omega_1 &= \frac{1}{128}\theta^{-5}\omega_6 + \frac{1}{16}\theta^{-4}s_4 + \frac{1}{8}\theta^{-3}s_3 + \frac{1}{4}\theta^{-2}s_2 + \frac{1}{2}\theta^{-1}s_1 + \widetilde{s}_2;\\
\omega_2 &= \frac{1}{32}\theta^{-4}\omega_6 + s_1;\\
\omega_3 &= \frac{1}{16}\theta^{-3}\omega_6 - s_2;\\
\omega_4 &= \frac{1}{8}\theta^{-2}\omega_6 + s_3;\\
\omega_5 &= \frac{1}{2}\theta^{-1}\omega_6 + s_4;\\
\omega_6 &= \omega_6
\end{aligned}$$

where $s_1, s_2, s_3, s_4, \widetilde{s}_2$ are sections of $G_0(h)[\theta^{-1}]$ satisfying

$$\begin{aligned}
(\theta\partial_\theta)s_1 &= s_1 + \theta^{-1}s_2;\\
(\theta\partial_\theta)s_2 &= 2s_2 + \theta^{-1}s_3;\\
(\theta\partial_\theta)s_3 &= 3s_3 + \theta^{-1}s_4;\\
(\theta\partial_\theta)s_4 &= 4s_4;\\
(\theta\partial_\theta)\widetilde{s}_2 &= 2\widetilde{s}_4
\end{aligned}$$

Now it is easy to calculate an upper triangular base change yielding a good basis and to show that the spectrum is

$$\mathrm{Sp}(G_0(h^{\star 3}), \nabla) = (-2, 1, 2, 3, 4, 7) \in \mathbb{Z}[\mathbb{Q}],$$

as required.

□

Based on the computations of these examples, we state the following conjecture, which is related to corollary 9 as well as to [10, conjecture 5.5.].

Conjecture 13. — *Let h be the defining equation of a reductive linear free divisor $D \subset V = \mathbb{C}^n$. Then the spectrum of its logarithmic Brieskorn lattice $(G_0(h), \nabla)$ at $\theta = 0$ is symmetric around $\frac{n-1}{2}$.*

Remark. — There are several questions one may ask about the spectrum at $\theta = 0$. First, it is surprising that negative numbers (even smaller than -1) occur in this spectrum. One might want to understand the possibly range for the spectrum, as well as the difference to the roots of b_h, when multiplied by n. This should be compared to the results in [15] for isolated singularities, in particular, lemma 3.4 of loc.cit.

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