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Yann BUGEAUD

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ON SIMULTANEOUS RATIONAL APPROXIMATION TO A REAL NUMBER AND ITS INTEGRAL POWERS

by Yann BUGEAUD

Abstract. — For a positive integer n and a real number ξ, let $\lambda_n(\xi)$ denote the supremum of the real numbers λ such that there are arbitrarily large positive integers q such that $||q\xi||, ||q\xi^2||, \ldots, ||q\xi^n||$ are all less than $q^{-\lambda}$. Here, $||\cdot||$ denotes the distance to the nearest integer. We study the set of values taken by the function λ_n and, more generally, we are concerned with the joint spectrum of $(\lambda_1, \ldots, \lambda_n, \ldots)$. We further address several open problems.

Résumé. — Pour un entier strictement positif n et un nombre réel ξ, on note $\lambda_n(\xi)$ le supremum des nombres réels λ pour lesquels il existe des entiers q arbitrairement grands tels que $||q\xi||, ||q\xi^2||, \ldots, ||q\xi^n||$ sont tous inférieurs à $q^{-\lambda}$. Ici, $||\cdot||$ désigne la distance à l'entier le plus proche. Nous étudions l'ensemble des valeurs prises par la function λ_n et, plus généralement, nous nous intéressons au spectre de $(\lambda_1, \ldots, \lambda_n, \ldots)$. Nous formulons également plusieurs problèmes ouverts.

1. Introduction

In 1932, in order to define his classification of real numbers, Mahler [18] introduced the exponents of Diophantine approximation w_n.

Definition 1.1. — *Let $n \geqslant 1$ be an integer and let ξ be a real number. We denote by $w_n(\xi)$ the supremum of the real numbers w such that, for arbitrarily large real numbers X, the inequalities*

$$0 < |x_n\xi^n + \ldots + x_1\xi + x_0| \leqslant X^{-w}, \qquad \max_{0\leqslant m\leqslant n} |x_m| \leqslant X,$$

have a solution in integers $x_0, \ldots, x_n$.

The Dirichlet theorem implies that $w_n(\xi)$ is at least equal to n for every real number ξ which is not algebraic of degree at most n. Sprindžuk [20]

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showed that there is equality for almost all ξ, with respect to the Lebesgue measure. Furthermore, it follows from the Schmidt Subspace Theorem that $w_n(\xi) = \min\{n, d-1\}$ for every positive integer n and every real algebraic number ξ of degree d; see [6] for an overview of the known results on the exponents w_n. In the present paper, we consider, besides w_n, several functions defined on the set of real numbers and whose values at algebraic numbers are known. Therefore, by spectrum of a function, we mean the set of values taken by this function on the set of transcendental real numbers.

By means of the theory of continued fractions, it is easy to show that the spectrum of w_1 is equal to the whole interval $[1, +\infty]$; see Section 2 below. A more precise result was proved by Jarník [13]. For $n \geqslant 2$, the determination of the spectrum of w_n is much more delicate, and the crucial tool is the theory of Hausdorff measure. It is an immediate consequence of a deep result established in 1983 by Bernik [4], on the Hausdorff dimension of the set of real numbers ξ such that $w_n(\xi)$ exceeds some prescribed real number w, that, for any positive integer n, the exponent w_n takes any value greater than or equal to n. However, to construct explicit examples of real numbers ξ with a prescribed value w for $w_n(\xi)$ remains an open question unless $n = 1$ or w is sufficiently large compared to n. In Section 2, we give a new contribution to this problem.

Another exponent of Diophantine approximation, which measures the quality of the simultaneous rational approximation to a number and its n first integral powers, has been introduced recently [10].

Definition 1.2. — *Let $n \geqslant 1$ be an integer and let ξ be a real number. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ such that, for arbitrarily large real numbers X, the inequalities*

$$0 < |x_0| \leqslant X, \qquad \max_{1 \leqslant m \leqslant n} |x_0 \xi^m - x_m| \leqslant X^{-\lambda},$$

have a solution in integers $x_0, \dots, x_n$.

The Dirichlet theorem implies that $\lambda_n(\xi)$ is at least equal to $1/n$ for every real number ξ which is not algebraic of degree at most n. The combination of Sprindžuk's above mentioned result with a classical transference principle shows that there is equality for almost all ξ, with respect to the Lebesgue measure. Furthermore, it follows from the Schmidt Subspace Theorem that $\lambda_n(\xi) = \max\{1/n, 1/(d-1)\}$ for every positive integer n and every real algebraic number ξ of degree d.

The following question is Problem 5.5 from [11] (see also Question 1 in [8]).

PROBLEM 1.3. — *Let $n \geqslant 1$ be an integer. Is the spectrum of the function λ_n equal to $[1/n, +\infty]$?*

In the present note, we summarize what is known on this problem and establish several new related results. We begin in Section 2 by a new result on the exponents w_n. Section 3 is then devoted to the study of the spectra of the exponents λ_n. In Section 4 we address the question of determining the joint spectrum of $(\lambda_1, \dots, \lambda_n, \dots)$ and establish two partial results. In Section 5, following a recent work of Laurent [17], we introduce new Diophatine exponents, which can be viewed as intermediate exponents between λ_n and w_n, and we give partial results on their spectra. Finally, in Section 6, we restrict our attention to the set of values taken by λ_n on the triadic Cantor set.

We assume that the reader is familiar with the theory of continued fractions. Throughout this note, $\lfloor \cdot \rfloor$ denotes the integer part function. The notation $a \gg_d b$ means that a exceeds b times a constant depending only on d. When $\gg$ is written without any subscript, it means that the constant is absolute. We write $a \asymp b$ if both $a \gg b$ and $a \ll b$ hold.

2. New result for the exponents w_n

Besides the exponents w_n and λ_n defined in Section1, the exponents w_n^* which measure the quality of the approximation by algebraic numbers of degree at most n have also been extensively studied (see, e.g. [6]). Recall that the height $H(P)$ of an integer polynomial $P(X)$ is the maximum of the moduli of its coefficients, and the height $H(\alpha)$ of an algebraic number α is the height of its minimal polynomial over $\mathbf{Z}$.

DEFINITION 2.1. — *Let $n \geqslant 1$ be an integer and let ξ be a real number. We denote by $w_n^*(\xi)$ the supremum of the real numbers w^* for which the inequality*

$$0 < |\xi - \alpha| \leqslant H(\alpha)^{-w^*-1}$$

is satisfied for infinitely many algebraic numbers α of degree at most n.

Recall that $w_n^*(\xi) \leqslant w_n(\xi)$ holds for every $n \geqslant 1$ and every real number ξ, but the inequality can be strict. It is a well-known problem, often referred to as the Wirsing conjecture [22], to decide whether $w_n^*(\xi) \geqslant n$ holds for every $n \geqslant 1$ and every transcendental real number ξ.

We improve a result of Güting [12] (see Theorem 7.7 from [6]) as follows.

Theorem 2.2. — *Let $m \geqslant 1$ and $k \geqslant 0$ be integers. Let w be a real number with*

$$(w+1-m)(w+1-m-k) \geqslant m(m+k)w. \tag{2.1}$$

Then, there exist uncountably many real numbers ξ such that

$$w_m(\xi) = w_m^*(\xi) = \ldots = w_{m+k}(\xi) = w_{m+k}^*(\xi) = w.$$

Theorem 2.2 is a new contribution towards the resolution of the Main Problem investigated in [6]. It improves Theorem 7.7 from [6], where (2.1) is replaced by the inequality $(w+1-m)(w+1-m-k) \geqslant m(m+k)(w+1)$.

The key idea, which goes back to Güting [12], is to construct suitable real numbers with many very good algebraic approximations of degree exactly m. This was done in earlier works by means of lacunary series; here, we obtain an improved result by using continued fractions.

We display an immediate corollary of Theorem 2.2 obtained by taking $m = 1$ and $k = n-1$ in its statement.

Corollary 2.3. — *Let $n \geqslant 1$ be an integer. For any real number $w \geqslant 2n-1$, there exist uncountably many real numbers ξ such that*

$$w_1(\xi) = w_1^*(\xi) = \ldots = w_n(\xi) = w_n^*(\xi) = w.$$

The assumption $w \geqslant 2n-1$ in Corollary 2.3 replaces the stronger assumption $w > (2n-1+\sqrt{4n^2+1})/2$ in [6]. Note that Corollary 2.3 for $n = 1$ and the existence of Liouville numbers imply that the spectrum of w_1 is equal to $[1, +\infty]$, a result first proved by Jarník.

Proof of Theorem 2.2. — For simplicity, we only give a full proof for the case $m = 1$ (thus, we establish Corollary 2.3) and explain the modifications to be done to get the whole statement.

Let $w > 1$ be a real number. Let M be a large positive integer and consider the real number

$$\xi = [0; 2, M\lfloor q_1^{w-1} \rfloor, M\lfloor q_2^{w-1} \rfloor, M\lfloor q_3^{w-1} \rfloor, \ldots], \tag{2.2}$$

where $q_1 = 2$ and q_j is the denominator of the j-th convergent to ξ, that is, of the rational number $p_j/q_j = [0; 2, M\lfloor q_1^{w-1} \rfloor, \ldots, M\lfloor q_{j-1}^{w-1} \rfloor]$, for $j \geqslant 2$. By construction, we have

$$q_{j+1} \asymp M q_j^w \quad \text{and} \quad \left| \xi - \frac{p_j}{q_j} \right| \asymp \frac{1}{M q_j^{w+1}}, \tag{2.3}$$

for $j \geqslant 1$. Consequently, we have

$$w = w_1(\xi) \leqslant \ldots \leqslant w_d(\xi), \tag{2.4}$$

for every positive integer d.

Let d be a positive integer with $d < w$. Let $P(X)$ be an integer polynomial of degree at most d and of large height $H(P)$. Assume first that $P(X)$ does not vanish at any element of the sequence $(p_j/q_j)_{j\geqslant 1}$. Let j be defined by $q_j \leqslant H(P) < q_{j+1}$. Observe that

$$|P(p_j/q_j)| \geqslant q_j^{-d}$$

and

$$|P(p_j/q_j) - P(\xi)| \ll_d H(P)|\xi - p_j/q_j| \ll_d H(P)q_j^{-w-1}M^{-1},$$

by (2.3). Consequently, we have

$$|P(\xi)| \geqslant |P(p_j/q_j)| - |P(p_j/q_j) - P(\xi)| \geqslant q_j^{-d}/2 \geqslant H(P)^{-w}/2$$

as soon as $H(P)q_j^{-w-1}M^{-1} \ll_d q_j^{-d}$, that is, as soon as

$$H(P) \ll_d Mq_j^{w+1-d}. \tag{2.5}$$

Similarly, we observe that

$$|P(p_{j+1}/q_{j+1})| \geqslant q_{j+1}^{-d}$$

and

$$|P(p_{j+1}/q_{j+1}) - P(\xi)| \ll_d H(P)q_{j+1}^{-w-1}M^{-1} \ll_d q_{j+1}^{-w}M^{-1}.$$

Since $w > d$, this implies that, if j (that is, if $H(P)$) is large enough, we have $|P(\xi)| \geqslant q_{j+1}^{-d}/2$. We then have $|P(\xi)| \geqslant H(P)^{-w}$ if $H(P)^{-w} \leqslant q_{j+1}^{-d}/2$, that is, by (2.3), if

$$H(P) \gg_d M^{d/w}q_j^{d}. \tag{2.6}$$

Selecting M sufficiently large in terms of d, it follows from (2.5) and (2.6) that the whole range of values $q_j \leqslant H(P) < q_{j+1}$ is covered as soon as

$$d \leqslant w + 1 - d. \tag{2.7}$$

This means that, for $w \geqslant 2d - 1$ and for any polynomial $P(X)$ of degree at most d that does not vanish at p_j/q_j and whose height satisfies $q_j \leqslant H(P) < q_{j+1}$, we have

$$|P(\xi)| \geqslant H(P)^{-w}/2.$$

In particular, if the polynomial $P(X)$ of degree at most d does not vanish at any element of the sequence $(p_j/q_j)_{j\geqslant 1}$, then it satisfies

$$|P(\xi)| \gg H(P)^{-w}. \tag{2.8}$$

If there are positive integers $a_1, \dots, a_h$, distinct positive integers $n_1, \dots, n_h$ and an integer polynomial $R(X)$ such that the polynomial $P(X)$ of degree at most d can be written as

$$P(X) = (q_{n_1} X - p_{n_1})^{a_1} \cdots (q_{n_h} X - p_{n_h})^{a_h} R(X),$$

where $R(X)$ does not vanish at any element of the sequence $(p_j/q_j)_{j \geqslant 1}$, then it follows from (2.3), (2.8) and the so-called Gelfond inequality

$$H(P) \asymp_d q_{n_1}^{a_1} \cdots q_{n_h}^{a_h} H(R)$$

that

$$\begin{aligned} |P(\xi)| &\gg_{d,M} q_{n_1}^{-a_1 w} \cdots q_{n_h}^{-a_h w} |R(\xi)| \\ &\gg_{d,M} q_{n_1}^{-a_1 w} \cdots q_{n_h}^{-a_h w} H(R)^{-w} \\ &\gg_{d,M} \bigl(q_{n_1}^{a_1} \cdots q_{n_h}^{a_h} H(R)\bigr)^{-w} \gg_{d,M} H(P)^{-w}. \end{aligned}$$

We conclude that, if (2.7) is satisfied, then

$$|P(\xi)| \gg_{d,M} H(P)^{-w}$$

holds for every polynomial $P(X)$ of degree at most d and sufficiently large height, hence $w_d(\xi) \leqslant w$. Combined with (2.4), this completes the proof of Theorem 1 in the case $m = 1$, since our construction is flexible enough to yield uncountably many real numbers with the required property.

As for the general case, that is, $m \geqslant 2$, we proceed exactly as above, with ξ replaced by its m-th root $\xi^{1/m}$ and with the rational numbers p_j/q_j replaced by their m-th roots $(p_j/q_j)^{1/m}$. Note that

$$|\xi^{1/m} - (p_j/q_j)^{1/m}| \asymp_m |\xi - p_j/q_j|.$$

We follow the proof of Theorem 7.7 of [6], however, there is a slight additional difficulty; indeed, we have to ensure that $(p_j/q_j)^{1/m}$ is of degree exactly m. This can be guaranteed by choosing instead of (2.2) the real number

$$\xi = [0; 2, M\lfloor q_1^{w-1} \rfloor + f_2, M\lfloor q_2^{w-1} \rfloor + f_3, M\lfloor q_3^{w-1} \rfloor + f_4, \dots], \tag{2.9}$$

where $q_1 = 2$, q_j is the denominator of the j-th convergent to ξ, and $f_2, f_3, \dots$ are suitable non-negative integers less than m. To see this, recall that ([16], Theorem 9.1), if q is not an h-th power for $2 \leqslant h \leqslant m$ and if $q/4$ is not a fourth power, then the polynomial $qX^m - p$ is irreducible when p is coprime with q. If we have

$$(M\lfloor q_{j-1}^{w-1} \rfloor + f) q_{j-1} + q_{j-2} = y^h \tag{2.10}$$

for integers y, $0 \leqslant f \leqslant m$ and $h \geqslant 2$, then $(y+1)^h$ exceeds $(M\lfloor q_{j-1}^{w-1} \rfloor + f) q_{j-1} + q_{j-2} + q_{j-1}^{w/2}$. By (2.1) we have $w > 2$, thus the number $(M\lfloor q_{j-1}^{w-1} \rfloor +$

$\ell)q_{j-1}+q_{j-2}$, where $\ell=0,1,\dots,m$, $\ell\neq f$, cannot be a perfect h-th power, if j is large enough. This shows that at most one number of the form

$$(M\lfloor q_{j-1}^{w-1}\rfloor + f)q_{j-1}+q_{j-2}, \tag{2.11}$$

with $0\leqslant f\leqslant m$, is an h-th power with $2\leqslant h\leqslant m$. The same argument applies when y^h is replaced by $4y^4$ in (2.10). Consequently, there exists f with $0\leqslant f\leqslant m$ such that the number of the form (2.11) is neither an h-th power, for any $h\geqslant 2$ at most equal to m, nor is equal to four times a fourth power. This shows that we can construct inductively integers $f_2,f_3,\dots$ in $[0,m]$ such that the number ξ defined in (2.9), whose sequence of convergents is $(p_j/q_j)_{j\geqslant 1}$, is such that the polynomial $q_jX^m-p_j$ is irreducible for $j\geqslant 2$. □

3. The exponents λ_n

Let ξ be an irrational real number. Clearly, we have

$$\lambda_1(\xi)=w_1(\xi)\geqslant 1$$

and

$$\lambda_1(\xi)\geqslant\lambda_2(\xi)\geqslant\dots \tag{3.1}$$

Our first lemma establishes a relation between the exponents λ_n and λ_m when m divides n.

Lemma 3.1. — *For any positive integers k and n, and any transcendental real number ξ we have*

$$\lambda_{kn}(\xi)\geqslant\frac{\lambda_k(\xi)-n+1}{n}.$$

Proof. — Let v be a positive real number and q be a positive integer such that

$$\max_{1\leqslant j\leqslant k}|q\xi^j-p_j|\leqslant q^{-v},$$

for suitable integers $p_1,\dots,p_k$. Let h be an integer with $1\leqslant h\leqslant kn$. Write $h=j_1+\dots+j_m$ with $m\leqslant n$ and $1\leqslant j_1,\dots,j_m\leqslant k$. Then,

$$|q^m\xi^h-p_{j_1}\dots p_{j_m}|\ll_m q^{m-1}q^{-v}$$

and

$$||q^n\xi^h||\ll q^{n-m}||q^m\xi^h||\ll_m q^{n-1-v}\ll_m (q^n)^{-(v-n+1)/n},$$

independently of h. This proves the lemma. □

We display an immediate consequence of Lemma 3.1.

Corollary 3.2. — *Let ξ be a real irrational number. Then, $\lambda_n(\xi) = +\infty$ holds for every positive n if, and only if, $\lambda_1(\xi) = +\infty$.*

We recall two relations between the exponents w_n and λ_n deduced from Khintchine's transference principle (see e.g. Theorem 3.9 of [6]).

Proposition 3.3. — *For any positive integer n and any real number ξ which is not algebraic of degree at most n, we have*

$$\frac{w_n(\xi)}{(n-1)w_n(\xi)+n} \leqslant \lambda_n(\xi) \leqslant \frac{w_n(\xi)-n+1}{n}.$$

The real numbers ξ defined in (2.2) satisfy $w_1(\xi) = \lambda_1(\xi) = w$, thus it is easy to construct explicitly real numbers ξ having any arbitrarily prescribed value for $\lambda_1(\xi)$. The same question for any exponent λ_n with $n \geqslant 2$ is not yet solved. We start with a new contribution to this problem, which improves Theorem 5.4 of [11].

Theorem 3.4. — *Let $n \geqslant 1$ be an integer and $\lambda \geqslant 1$ be a real number. There are uncountably many real numbers ξ, which can be constructed explicitly, such that $\lambda_n(\xi) = \lambda$. In particular, the spectrum of λ_n includes the interval $[1, +\infty]$.*

Proof. — Let $n \geqslant 2$ be an integer and ξ be a transcendental real number. Lemma 3.1 with $k = 1$ implies the lower bound

$$\lambda_n(\xi) \geqslant \frac{w_1(\xi)-n+1}{n}. \tag{3.2}$$

On the other hand, Proposition 3.3 gives the upper bound

$$\lambda_n(\xi) \leqslant \frac{w_n(\xi)-n+1}{n}.$$

Now, Corollary 2.3 asserts that for any given real number $w \geqslant 2n-1$, there exist uncountably many real numbers ξ_w such that

$$w_1(\xi_w) = \ldots = w_n(\xi_w) = w.$$

Then, the equalities

$$\lambda_k(\xi_w) = \frac{w}{k} - 1 + \frac{1}{k}, \quad k = 1, \ldots, n,$$

hold; in particular,

$$\lambda_n(\xi_w) = \frac{w}{n} - 1 + \frac{1}{n},$$

and this gives the requested result. □

Unfortunately, and unlike what happens for the exponents w_n, the metrical theory is not sufficiently developed at present to solve Problem 3.5 below (which may imply a positive answer to Problem 1.3).

Problem 3.5. — *Let $n \geqslant 1$ be an integer and $\lambda \geqslant 1/n$ be a real number. To determine the Hausdorff dimension of the sets*

$$\{\xi \in \mathbf{R} : \lambda_n(\xi) \geqslant \lambda\} \quad \textit{and} \quad \{\xi \in \mathbf{R} : \lambda_n(\xi) = \lambda\}.$$

We complement Theorem 3.4 with various metrical results which give a partial answer to Problem 3.5. We begin this short survey with a statement that is an immediate consequence of seminal results of Jarník [13].

Theorem 3.6. — *For any real number $\lambda \geqslant 1$, we have*

$$\dim\{\xi \in \mathbf{R} : \lambda_1(\xi) = \lambda\} = \frac{2}{1+\lambda}.$$

Theorem 3.6 was recently extended by Budarina, Dickinson, and Levesley [5] as follows.

Theorem 3.7. — *Let $n \geqslant 2$ be an integer. Let $\lambda \geqslant n-1$ be a real number. Then, we have*

$$\dim\{\xi \in \mathbf{R} : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1+\lambda)}.$$

We point out that the inequality

$$\dim\{\xi \in \mathbf{R} : \lambda_n(\xi) \geqslant \lambda\} \geqslant \frac{2}{n(1+\lambda)}$$

is valid for any $\lambda \geqslant 1/n$. Indeed, for $n \geqslant 2$ and $\lambda \geqslant 1/n$, we infer from (3.2) and Theorem 3.6 that

$$\begin{aligned}\dim\{\xi \in \mathbf{R} : \lambda_n(\xi) \geqslant \lambda\} &\geqslant \dim\{\xi \in \mathbf{R} : \lambda_1(\xi) \geqslant n\lambda + n - 1\} \\ &\geqslant \frac{2}{1+(n\lambda+n-1)} = \frac{2}{n(1+\lambda)}.\end{aligned}$$

This was already established in [5], but with a different proof.

It follows from Theorem 3.7 that, for any $n \geqslant 2$, the spectrum of λ_n includes the interval $[n-1, +\infty]$, a weaker conclusion than Theorem 3.4.

Problem 3.5 for $n = 2$ and $\lambda \in [1/2, 1]$ was solved by Beresnevich, Dickinson, Vaughan and Velani [3, 21].

Theorem 3.8. — *For any real number λ with $1/2 \leqslant \lambda \leqslant 1$, we have*

$$\dim\{\xi \in \mathbf{R} : \lambda_2(\xi) = \lambda\} = \frac{2-\lambda}{1+\lambda}.$$

We display an immediate consequence of Theorems 3.4 and 3.8. This solves Problem 1.3 for $n = 2$.

Corollary 3.9. — *The spectrum of λ_2 is equal to $[1/2, +\infty]$.*

To conclude this section, we quote a recent result of Beresnevich [2] dealing with small values of λ_n.

Theorem 3.10. — *Let $n \geqslant 2$ be an integer. Let λ be a real number with $1/n \leqslant \lambda < 3/(2n-1)$. Then, we have*

$$\dim\{\xi \in \mathbf{R} : \lambda_n(\xi) \geqslant \lambda\} \geqslant \frac{n+1}{\lambda+1} - (n-1). \tag{3.3}$$

By Theorem 3.8, the inequality (3.3) is an equality for $n = 2$ and Beresnevich conjectures that this is also an equality for $n \geqslant 3$.

4. Prescribing simultaneously the values of all the exponents λ_n

The results stated in Section 3 show that, under a suitable (very strong) assumption, it is possible to construct real numbers ξ with prescribed values of $\lambda_n(\xi)$ for *finitely many* integers n.

Theorem 4.1. — *Let k be a positive integer and $n_1, \ldots, n_k$ be distinct positive integers. Let $\lambda_1, \ldots, \lambda_k$ and τ be positive real numbers such that $\tau \geqslant 2$,*

$$\tau = n_j(1 + \lambda_j), \quad \lambda_j \geqslant n_j - 1, \quad (j = 1, \ldots, k).$$

Then, we have

$$\dim\{\xi \in \mathbf{R} : \lambda_{n_j}(\xi) = \lambda_j \text{ for } j = 1, \ldots, k\} = \frac{2}{\tau}. \tag{4.1}$$

Proof. — We use Theorem 1 of [5] to construct a suitable dimension function f such that the Hausdorff f-measure of the set defined in (4.1) is positive, while, for every positive integer h and for $i = 1, \ldots, k$, the set

$$\{\xi \in \mathbf{R} : \lambda_{n_j}(\xi) \geqslant \lambda_j, \text{ for } j = 1, \ldots, k \text{ and } \lambda_{n_i}(\xi) \geqslant \lambda_i + 1/h\}$$

has zero Hausdorff f-measure. Then, we conclude as in the proof of Theorem 5.8 of [6]. □

Unfortunately, Theorem 4.1 cannot be extended to an infinite set of positive integers since, clearly, for any real number $\tau \geqslant 2$, there are only finitely many pairs (n, λ) in $(\mathbf{Z}_{\geqslant 1}, \mathbf{R}_{\geqslant 1})$ such that $n(1 + \lambda) = \tau$.

The Main Problem investigated in [6] asks whether, for any non-decreasing sequence $(w_n)_{n \geqslant 1}$ of real numbers such that $w_n \geqslant n$ for $n \geqslant 1$, there exists a real number ξ for which $w_n(\xi) = w_n$ for $n \geqslant 1$. We refer the reader to Section 7.8 of [6] for a summary of the known results towards the Main Problem. In view of this and of Lemma 3.1, we propose the following generalisation of Problem 1.3.

Problem 4.2. — *Let $(\lambda_n)_{n\geqslant 1}$ be a non-increasing sequence of positive real numbers such that*

$$\lambda_n \geqslant \frac{1}{n}, \quad (n \geqslant 1),$$

and

$$\lambda_{kn} \geqslant \frac{\lambda_k - n + 1}{n}, \quad (k \geqslant 1, n \geqslant 1).$$

Does there exist a real number ξ with

$$\lambda_n(\xi) = \lambda_n, \quad \textit{for } n \geqslant 1?$$

We state our first modest contribution to this problem, which is apparently much more difficult than the (still unsolved) Main Problem.

Theorem 4.3. — *There exist uncountably many real numbers ξ satisfying*

$$\lambda_n(\xi) = 1, \quad \textit{for } n \geqslant 1. \tag{4.2}$$

The idea is to construct suitable real numbers which are very well approximable by quadratic numbers. This is a natural approach, since for every quadratic number γ and every positive integer n we have $\lambda_n(\gamma) = 1$.

Proof. — Let $(m_j)_{j\geqslant 1}$ be a very rapidly increasing sequence of positive integers. Let $A_1 = 1^{m_1}2$ be the finite word composed of m_1 digits 1 and terminating by the digit 2. For $j \geqslant 2$, denote by n_j the number of digits of the word $A_{j-1}^{m_j}$ and define the word $A_j = A_{j-1}^{m_j}2$ composed of m_j copies of A_{j-1} and terminating by the digit 2. Let ξ_j be the quadratic number in $(0, 1)$ whose continued fraction is purely periodic of period A_j. We define the infinite word $\mathbf{a} = a_1a_2a_3\ldots$ as the limit of the finite words A_j when j tends to infinity. Clearly, $\mathbf{a}$ is not ultimately periodic and $\xi := [0; a_1, a_2, a_3, \ldots]$ is the limit of the quadratic numbers ξ_j as j tends to infinity.

Let γ be a quadratic real number with minimal defining polynomial $c_2X^2 + c_1X + c_0$ over $\mathbf{Z}$ and height at most H. Let q be the denominator of a convergent to γ. Then we have $||q\gamma|| < q^{-1}$, where $||\cdot||$ denotes the distance to the nearest integer. Observe that $||qc_2\gamma^2|| = ||qc_1\gamma|| < Hq^{-1}$ and that

$$||qc_2^2\gamma^3|| \leqslant ||qc_1c_2\gamma^2|| + ||qc_2c_0\gamma|| < 2H^2q^{-1}.$$

An easy induction then shows that

$$||qc_2^j\gamma^{j+1}|| < (2H)^jq^{-1}, \quad \text{for } j \geqslant 1,$$

and we get

$$||qc_2^h\gamma^{j+1}|| \leqslant c_2^{h-j}||qc_2^j\gamma^{j+1}|| \leqslant (2H)^hq^{-1}, \quad \text{for } j \geqslant 1 \text{ and } h \geqslant j. \tag{4.3}$$

Let $j \geqslant 2$. Let Q_j denote the denominator of the n_{j+1}-th convergent to ξ_j (that is, of the n_{j+1}-th convergent to ξ since, by construction, ξ and ξ_j have the same first n_{j+1} partial quotients). Let L_j be the leading coefficient of the minimal defining polynomial of ξ_j over $\mathbf{Z}$ and let H_j be its height. Classical results from the theory of continued fraction ensure that $Q_j \geqslant (3/2)^{n_{j+1}}$ and $H_j \leqslant 3^{n_j}$, since the partial quotients of ξ_j belong to $\{1, 2\}$. In view of this and of (4.3), we get

$$\begin{aligned}(4.4)\quad \max\{||Q_j L_j^{j-1}\xi_j||, ||Q_j L_j^{j-1}\xi_j^2||, \ldots, ||Q_j L_j^{j-1}\xi_j^j||\} &\leqslant (2H_j)^j Q_j^{-1} \\ &\leqslant (Q_j L_j^{j-1})^{-1} \log(Q_j L_j^{j-1}),\end{aligned}$$

if n_{j+1} is sufficiently large compared to n_j. Since

$$|\xi_j^h - \xi^h| \leqslant jQ_j^{-2}, \quad \text{for } 1 \leqslant h \leqslant j,$$

we infer from (4.4) that, provided that n_{j+1} is sufficiently large, we have

$$\begin{aligned}&||Q_j L_j^{j-1}\xi^h|| \leqslant Q_j L_j^{j-1} \cdot |\xi_j^h - \xi^h| + ||Q_j L_j^{j-1}\xi_j^h|| \\ &\leqslant jQ_j^{-1}L_j^{j-1} + (Q_j L_j^{j-1})^{-1} \log(Q_j L_j^{j-1}) \leqslant 2(Q_j L_j^{j-1})^{-1} \log(Q_j L_j^{j-1}),\end{aligned}$$

for $h = 1, \ldots, j$. This implies that $\lambda_h(\xi) \geqslant 1$ for $h \geqslant 1$. Since ξ has bounded partial quotients, it satisfies $\lambda_1(\xi) = 1$, and the requested result follows from (3.1). Finally, we observe that there are uncountably many suitable choices for the sequence $(n_j)_{j\geqslant 1}$, thus, uncountably many real numbers ξ satisfy (4.2). □

The next theorem gives new information on the joint spectrum of λ_1 and λ_2.

THEOREM 4.4. — *Let λ be a real number with $1 \leqslant \lambda \leqslant 3$. There exist uncountably many real numbers ξ with $\lambda_1(\xi) = \lambda$ and $\lambda_2(\xi) = 1$.*

Note that the assumption $\lambda \leqslant 3$ in Theorem 4.4 is necessary since $\lambda_2(\xi) \geqslant (\lambda_1(\xi) - 1)/2$, by (3.2).

The constructive proof of Theorem 4.4 depends on the following auxiliary result. Recall that a finite word $a_1 a_2 \ldots a_n$ is called a palindrome if $a_j = a_{n+1-j}$ for $j = 1, \ldots, n$.

LEMMA 4.5. — *Let $a_1 \geqslant 4$ be an integer. Let $\xi = [0; a_1, a_2, \ldots, a_k, \ldots]$ be an irrational real number. Assume that there exists $n \geqslant 4$ such that $a_1 a_2 \ldots a_n$ is a palindrome and set*

$$\frac{p}{q} = [0; a_1, \ldots, a_{n-1}, a_n] \quad \textit{and} \quad \frac{p'}{q'} = [0; a_1, \ldots, a_{n-1}].$$

Then we have $p = q'$, $\max\{||q\xi||, ||q\xi^2||\} \ll a_1 q^{-1}$, *and* $||q\xi^2|| \gg a_1^{-1} q^{-1}$. *Consequently, if there are infinitely many integers* n *such that* $a_1 a_2 \ldots a_n$ *is a palindrome, then* $\lambda_2(\xi) \geqslant 1$.

Proof. — The first two assertions are established in Section 5 of [1]. For the last one, observe that

$$\left|\xi^2 - \frac{p'}{q}\right| = \left|\xi^2 - \frac{p'}{q'} \cdot \frac{p}{q}\right| = \left|\left(\xi + \frac{p'}{q'}\right)\left(\xi - \frac{p}{q}\right) \pm \frac{\xi}{qq'}\right|.$$

Since

$$\left|\left(\xi + \frac{p'}{q'}\right)\left(\xi - \frac{p}{q}\right)\right| \leqslant \frac{3\xi}{q^2} \quad \text{and} \quad \frac{\xi}{qq'} \geqslant \frac{4\xi}{q^2},$$

we deduce that $||q\xi^2|| \geqslant \xi q^{-1}$. This completes the proof of the lemma. □

Proof of Theorem 4.4. — We give an inductive construction for the continued fraction expansion of a suitable real number $\xi = [0; a_1, a_2, \ldots]$, whose sequence of convergents is denoted by $(p_n/q_n)_{n \geqslant 1}$. Set $a_1 = \ldots = a_4 = 4$, $n_1 = 5$ and $a_5 = \lfloor q_4^{\lambda-1} \rfloor$. We construct a very rapidly increasing sequence $(n_j)_{j \geqslant 1}$ of odd integers. We describe the inductive step. Let j be a positive integer such that the word $a_1 a_2 \ldots a_{n_j-2} a_{n_j-1}$ is a palindrome and $a_{n_j} = \lfloor q_{n_j-1}^{\lambda-1} \rfloor$. Let $a_{n_j+1}, \ldots, a_{(n_{j+1}-1)/2}$ be elements of $\{1, 2\}$ and set $a_{(n_{j+1}-1)/2+h} = a_{(n_{j+1}-1)/2+1-h}$ for $h = 1, 2, \ldots, (n_{j+1} - 1)/2$. Consequently, the word $a_1 a_2 \ldots a_{n_{j+1}-2} a_{n_{j+1}-1}$ is a palindrome. We select n_{j+1} sufficiently large to secure that

$$||q_n \xi|| \geqslant q_n^{-1} (\log q_n)^{-1}, \quad \text{for } n_j \leqslant n \leqslant n_{j+1} - 2. \tag{4.5}$$

Note that (4.5) certainly holds if $q_{(n_{j+1}-1)/2}$ exceeds $\exp\{a_{n_j}\}$. Finally, we put $a_{n_{j+1}} = \lfloor q_{n_{j+1}-1}^{\lambda-1} \rfloor$, thus

$$||q_{n_{j+1}-1} \xi|| \asymp q_{n_{j+1}-1}^{-\lambda}. \tag{4.6}$$

This completes the inductive step. By construction, we have $\lambda_1(\xi) = \lambda$ and Lemma 4.5 implies that $\lambda_2(\xi) \geqslant 1$. It remains to prove that $\lambda_2(\xi)$ cannot exceed 1. Let $\nu > 1$ be a real number and let q be a (large) positive integer such that $||q\xi|| < q^{-\nu}$. We deduce from (4.5) and (4.6) that q is necessarily an integer multiple of q_{n_j-1} for some $j \geqslant 1$. Write then $q = M q_{n_j-1}$ and note that there are integers p and p' such that $p = Mp'$ and

$$\frac{1}{q_{n_j-1}^{\lambda+1}} \ll \left|\xi - \frac{p'}{q_{n_j-1}}\right| = \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{\nu+1}}.$$

This shows that $q^{\nu+1} \ll q_{n_j-1}^{\lambda+1}$, thus,

$$M \ll q_{n_j-1}^{(\lambda+1)/(\nu+1)} \, q_{n_j-1}^{-1} \ll q_{n_j-1}^{(\lambda-\nu)/(\nu+1)}.$$

Since $\nu > 1$ and $\lambda \leqslant 3$, there exists $\tau < 1$ such that $M \leqslant q_{n_j-1}^{\tau}$ and, consequently,

$$||q\xi^2|| = ||Mq_{n_j-1}\xi^2|| = M||q_{n_j-1}\xi^2|| \gg Mq_{n_j-1}^{-1} \gg q^{-1},$$

by Lemma 4.5. This shows that $\lambda_2(\xi)$ cannot exceed 1. Consequently, $\lambda_2(\xi) = 1$ and the theorem is proved, since the method is flexible enough to yield uncoutably many real numbers with the required property. □

5. Intermediate exponents

Let $n \geqslant 2$ be an integer and Θ be a point in $\mathbf{R}^n$. In [17], Laurent introduced new exponents $\omega_{n,d}(\Theta)$ (simply denoted by $\omega_d(\Theta)$ in [17], since n is fixed throughout that paper) measuring the sharpness of the approximation to Θ by linear rational varieties of dimension d. He split the Khintchine transference principle into $n-1$ intermediate estimates which connect the exponents $\omega_{n,d}(\Theta)$ for $d = 0, 1, \dots, n-1$ (see also [9]). Actually, Schmidt [19] was the first to investigate the properties of these exponents $\omega_{n,d}$, but he did not introduce them explicitly. We briefly recall their definition and we consider new exponents $w_{n,d}$ defined over $\mathbf{R}$ by restricting $\omega_{n,d}$ to the Veronese curve $(x, x^2, \dots, x^n)$. It is convenient to view $\mathbf{R}^n$ as a subset of $\mathbf{P}^n(\mathbf{R})$ via the usual embedding $(x_1, \dots, x_n) \mapsto (1, x_1, \dots, x_n)$. We shall identify $\Theta = (\Theta_1, \dots, \Theta_n)$ with its image in $\mathbf{P}^n(\mathbf{R})$. Denote by d the projective distance on $\mathbf{P}^n(\mathbf{R})$ and, for any real linear subvariety L of $\mathbf{P}^n(\mathbf{R})$, set

$$\mathrm{d}(\Theta, L) = \min_{P \in L} \mathrm{d}(\Theta, P)$$

the minimal distance between Θ and the real points P of L. When L is rational over $\mathbf{Q}$, we indicate moreover by $H(L)$ its height, that is the Weil height of any system of Plücker coordinates of L. We refer to [17, 9] for precise definitions of the projective distance, heights, etc.

Definition 5.1. — *Let $n \geqslant 2$ and d be integers with $0 \leqslant d \leqslant n-1$. Let Θ be in $\mathbf{R}^n$. We denote by $\omega_{n,d}(\Theta)$ the supremum of the real numbers ω for which there exist infinitely many rational linear subvarieties $L \subset \mathbf{P}^n(\mathbf{R})$ such that*

$$\dim(L) = d \quad \textit{and} \quad \mathrm{d}(\Theta, L) \leqslant H(L)^{-1-\omega}.$$

If there exists ξ such that $\Theta = (\xi, \xi^2, \dots, \xi^n)$, then we set $w_{n,d}(\xi) = \omega_{n,d}(\Theta)$.

We observe that the functions λ_n and $w_{n,0}$ (resp. w_n and $w_{n,n-1}$) coincide.

Let the spectrum of the function $\omega_{n,d}$ denote the set of values taken by the exponents $\omega_{n,d}(\Theta)$ when Θ ranges over $\mathbf{R}^n$, with $1, \Theta_1, \ldots, \Theta_n$ linearly independent over the rationals. Using a result of Jarník [14], Laurent [17] established that the spectrum of $\omega_{n,d}$ over $\mathbf{R}^n$ is equal to the whole interval $[(d+1)/(n-d), +\infty]$ and that $\omega_{n,d}(\Theta) = (d+1)/(n-d)$ for almost all Θ in $\mathbf{R}^n$. By means of the numbers ξ_w defined in the proof of Theorem 3.4, we get some information on the spectra of the exponents $w_{n,d}$.

THEOREM 5.2. — *For $n \geqslant 2$ and $0 \leqslant d \leqslant n-1$, the spectrum of $w_{n,d}$ contains the whole interval $[(n+d)/(n-d), +\infty]$ and $w_{n,d}(\xi) = (d+1)/(n-d)$ for almost all real numbers ξ.*

Theorem 5.2 plainly includes Theorem 3.4.

Proof. — We follow the proof of the Corollary from [17], where it is established that, for any w with $1/n \leqslant \lambda \leqslant +\infty$ and for any point Θ in $\mathbf{R}^n$ such that $\omega_{n,0}(\Theta) = \lambda$ and $\omega_{n,n-1}(\Theta) = n\lambda + n - 1$, we have

$$\omega_{n,d}(\Theta) = \frac{n\lambda + d}{n-d}, \quad (d = 0, 1, \ldots, n-1). \tag{5.1}$$

For $w \geqslant 2n-1$, the numbers ξ_w defined in the proof of Theorem 3.4 satisfy

$$n\lambda_n(\xi_w) = w_n(\xi_w) - n + 1 = w - n + 1,$$

that is,

$$\omega_{n,n-1}(\xi_w, \ldots, \xi_w^n) = n\omega_{n,0}(\xi_w, \ldots, \xi_w^n) + n - 1.$$

We then get from (5.1) that

$$w_{n,d}(\xi_w) = \frac{n\lambda_n(\xi_w) + d}{n-d}, \quad (d = 0, 1, \ldots, n-1).$$

The first assertion of the theorem follows since $\lambda_n(\xi_w)$ takes every value between 1 and $+\infty$ as w varies from $2n-1$ to $+\infty$. The second assertion is an immediate consequence of (5.1) and the fact that $n\lambda_n(\xi) = w_n(\xi) - n + 1 = 1$ holds for almost every real number ξ. $\square$

We conclude this section by stating an extension of Problem 1.3.

PROBLEM 5.3. — *Let d and n be integers with $n \geqslant 2$ and $0 \leqslant d \leqslant n-1$. Is the spectrum of the function $w_{n,d}$ equal to $[(d+1)/(n-d), +\infty]$?*

Clearly, a positive answer to Problems 1.3 and 5.3 would follow if we could prove that, for any positive integer n and for any real number w_n greater than n, there exists a real number ξ such that $w_1(\xi) = \ldots = w_n(\xi) = w_n$.

We feel that condition (2.1) is likely not best possible, but we have no conjecture to what extent it could be improved.

6. Diophantine approximation on the Cantor set

Let K denote the triadic Cantor set, that is, the set of all real numbers of the form $c_1 3^{-1} + c_2 3^{-2} + \cdots + c_i 3^{-i} + \cdots$ with $c_i = 0$ or 2 for every $i \geqslant 1$. Motivated by a question of Mahler asking whether there are algebraic irrational numbers in K, several authors have recently studied the Diophantine approximation properties of the elements of K, see the references at the end of [7].

Let us mention that Kleinbock, Lindenstrauss, and Weiss (Theorem 7.10 from [15]) proved that almost every element ξ on K (with respect to the standard measure supported on K) satisfies $w_n(\xi) = w_n^*(\xi) = n$ for every positive integer n. By Khintchine's transference principle, such a ξ also satisfies $\lambda_n(\xi) = 1/n$ for every positive integer n. Furthermore, it has been established in [7] that w_1 (that is, λ_1) takes on the Cantor set any arbitrarily given value greater than or equal to 1. The proof is constructive. It is apparently a very difficult open problem to prove that for $n \geqslant 2$ the exponent w_n (resp. λ_n) takes on K any arbitrarily given value greater than or equal to n (resp. $1/n$). The following statement, which follows from the proof of Theorem 5.4 from [11], solves partially this problem.

Theorem 6.1. — *Let $n \geqslant 2$ be an integer. The spectrum of w_n restricted to the Cantor set includes the interval $[(2n-1+\sqrt{4n^2+1})/2, +\infty]$. The spectrum of λ_n restricted to the Cantor set includes the interval $[(1+\sqrt{4n^2+1})/(2n), +\infty]$.*

Observe that the left-hand side of the first (resp. second) interval is (slightly) larger than $2n-1$ (resp. than 1).

Proof. — Theorem 7.7 from [6] asserts that for any given real number $w > (2n-1+\sqrt{4n^2+1})/2$ (actually, the strict inequality can be replaced by a large one), the real number

$$\xi'_w := 2 \sum_{j \geqslant 1} 3^{-\lfloor (w+1)^j \rfloor}.$$

satisfies

$$w_1(\xi'_w) = \ldots = w_n(\xi'_w) = w.$$

We conclude as in the proof of Theorem 3.4. $\square$

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Yann BUGEAUD
Université de Strasbourg
Mathématiques
7, rue René Descartes
67084 Strasbourg (FRANCE)
bugeaud@math.u-strasbg.fr