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FOURIER TRANSFORMS OF MEASURES AND ALGEBRAIC RELATIONS ON THEIR SUPPORTS

by Thomas W. KÖRNER

Abstract. — We investigate the relation between the rate of decrease of a Fourier transform and the possible algebraic relations on its support.

Résumé. — Si la transformée de Fourier d'une mesure décroît rapidement alors le support ne satisfait que très peu des relations algèbriques.

1. Non-technical introduction

This paper is fairly technical but deals with natural questions.

We work on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A well known result tells us that, if a set E has positive Lebesgue measure, then $E + E$ contains an interval. It follows that there exist x, $y \in E$ and integers m and n satisfying some non-trivial equation

$$mx + ny = 0.$$

In other words, if a set has positive Lebesgue measure, it must be rich in short algebraic relations.

A closely related argument shows that any Borel measure μ whose Fourier transform $\hat{\mu}(r)$ tends fairly rapidly to zero must have a support which is rich in fairly short algebraic relations. More specifically, if $\hat{\mu}(r) = O(|r|^{-\epsilon - q^{-1}})$, then we can find $x_j \in \operatorname{supp} \mu$ and integers m_j satisfying some non-trivial equation

$$\sum_{j=1}^{q} m_j x_j = 0.$$

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In an earlier paper, I used a fairly simple probabilistic argument to construct a Borel measure μ such that $\hat{\mu}(r) = O(|r|^{\epsilon - 2^{-1}q^{-1}})$, but there do not exist $x_j \in \operatorname{supp} \mu$ and integers m_j satisfying some non-trivial equation

$$\sum_{j=1}^{q} m_j x_j = 0.$$

There is a large gap between the results of the two paragraphs and both seem 'natural'. However, in Theorem 2.4, I show that that, by using more complicated probabilistic arguments, we can construct a Borel measure μ such that $\hat{\mu}(r) = O(|r|^{\epsilon - 2^{-1}q^{-1}})$, but there do not exist $x_j \in \operatorname{supp} \mu$ and integers m_j satisfying some non-trivial equation

$$\sum_{j=1}^{q+1} m_j x_j = 0.$$

If ϵ is small, the set

$$\left\{ \sum_{j=1}^{q+1} x_j \ : \ x_j \in \operatorname{supp} \mu \right\}$$

has positive Lebesgue measure and this suggests that the new result is close to best possible or, at least, that it will be quite hard to improve.

On the other hand, if we deal with sets, I show (in Theorem 2.6) how to construct a closed set E such that the q-fold sum $E + E + \cdots + E$ has positive Lebesgue measure but there do not exist $x_j \in \operatorname{supp} \mu$ and integers m_j satisfying some non-trivial equation

$$\sum_{j=1}^{2q-1} m_j x_j = 0.$$

2. Technical introduction

As stated earlier, we work on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. All measures will be Borel measures and m will denote the Lebesgue measure. If μ is a measure, we write

$$\mu_{[q]} = \mu * \mu * \cdots * \mu$$

for the q-fold convolution of μ with itself and

$$E_{[q]} = E + E + \cdots + E = \left\{ \sum_{j=1}^{q} x_j \ : \ x_j \in E \right\}.$$

As usual $\|f\|_1 = \int_{\mathbb{T}} |f(t)|\, dt$.

This paper, like its predecessor [4], centres round the following two simple observations.

Lemma 2.1. — *Suppose that μ is a non-zero measure with support E.*

(i) If $\sum_{r=-\infty}^{\infty} |\hat{\mu}(r)|^q$ converges, then there exists a non-trivial interval I such that every $x \in I$ can be written

$$x = x_1 + x_2 + \cdots + x_q$$

with $x_j \in E$.

(ii) If $\sum_{r=-\infty}^{\infty} |\hat{\mu}(r)|^{2q}$ converges, then there exists a set A of strictly positive Lebesgue measure such that every $x \in A$ can be written

$$x = x_1 + x_2 + \cdots + x_q$$

with $x_j \in E$.

Proof. —

(i) Observe that

$$|\hat{\mu}_{[q]}(r)| = |\hat{\mu}(r)|^q$$

so $d\mu_{[q]} = f dm$ where f has an absolutely convergent Fourier series and so is continuous. The support of f contains a non-trivial interval I and

$$\text{supp } f = \text{supp } \mu_{[q]} \subseteq \{x_1 + x_2 + \cdots + x_q \,:\, x_j \in E\}.$$

(ii) Observe that $d\mu_{[q]} = f dm$ where $f \in L^2(m)$ and argue much as in (i).

□

As I remarked earlier, part (i) of the next lemma is extremely well known, but, although part (ii) is a simple consequence, I do not know if it has been observed before.

Lemma 2.2. — *(i) If E has strictly positive Lebesgue measure, then $E + E$ contains a non-trivial interval.*

(ii) If E has strictly positive Lebesgue measure, then we can find a non-trivial interval I such that, whenever $x \in I$, the equation

$$x_1 + x_2 = x$$

has uncountably many distinct solutions with $x_1, x_2 \in E$.

Proof. —

(i) If E has strictly positive Lebesgue measure then we can find a closed set $E^* \subseteq E$ with E^* having strictly positive measure. Thus, without loss of generality, we may assume that E is closed. We now know that the indicator function $\mathbb{I}_E$ is a nontrivial $L^2(m)$ function so $\sum_{j=-\infty}^{\infty} |\hat{\mathbb{I}}_E(j)|^2$ converges and we may apply Lemma 2.1 (i).

(ii) Suppose that the result is false. Then each interval I contains a point y such that the equation

$$x_1 + x_2 = y$$

has only countably many distinct solutions with $x_1,\, x_2 \in E$. Thus we can find a countable dense sequence y_j and associated countable sets E_j such that, if

$$x_1 + x_2 = y_j$$

with $x_1,\, x_2 \in E$ then $x_1,\, x_2 \in E_j$. Now observe that $E \setminus \bigcup_{j=1}^{\infty} E_j$ is a set of strictly positive Lebesgue measure disobeying the conclusions of (i) which is impossible

□

Since every non-trivial interval contains a rational, Lemma 2.1 (i) implies the following result.

Lemma 2.3. — *Suppose that μ is a non-zero measure on $\mathbb{T}$ and q is a positive integer such that we can find an $\alpha > 1/q$ and an $A > 0$ with*

$$|\hat{\mu}(r)| \leqslant A|r|^{-\alpha}$$

for all $r \neq 0$. Then we can find distinct points $x_1,\, x_2,\, \dots,\, x_q \in \operatorname{supp} \mu$ and $m_j \in \mathbb{Z}$, not all zero, such that

$$\sum_{j=1}^{q} m_j x_j = 0.$$

In this paper we show how to prove the following result in the other direction.

Theorem 2.4. — *If q is an integer with $q \geqslant 1$ and $\psi : \mathbb{N} \to \mathbb{R}$ is a sequence of strictly positive numbers such that $\psi(r) \to \infty$ as $r \to \infty$, then there exists a probability measure μ such that*

$$|\hat{\mu}(r)| \leqslant |r|^{-1/(2q)} \big(\log(1 + |r|)\big)^{1/2} \psi(|r|)$$

for all $r \neq 0$, but, given distinct points $x_1, x_2, \ldots, x_{q+1} \in$ supp μ, the only solution to the equation

$$\sum_{j=1}^{q+1} m_j x_j = 0$$

with $m_j \in \mathbb{Z}$ is the trivial solution $m_1 = m_2 = \cdots = m_{q+1} = 0$.

In [4] we proved a similar result with the equation $\sum_{j=1}^{q+1} m_j x_j = 0$ replaced by $\sum_{j=1}^{q} m_j x_j = 0$. Earlier I explained why the new result might be substantially more difficult to prove than the old. Observe that if, for example, $\psi(r) = \big(\log(1+|r|)\big)^{1/2}$, then, by Lemma 2.1,

$$\left\{\sum_{j=1}^{q+1} x_j \;:\; x_j \in \operatorname{supp} \mu\right\}$$

must have strictly positive Lebesgue measure.

The key lemma in our proof is the following.

Lemma 2.5. — *Let q be an integer with $q \geqslant 1$ and $\psi : \mathbb{N} \to \mathbb{R}$ be a sequence of strictly positive numbers such that $\psi(r) \to \infty$ as $r \to \infty$.*

Suppose $\mathbf{m} \in \mathbb{Z}^{q+1} \setminus \{\mathbf{0}\}$ and N is a positive integer such that

$$N \geqslant 12\left(q + 1 + \sum_{j=1}^{N} |m_j|\right).$$

Then, given closed intervals $I_j = [(n_j - \frac{1}{2})/N, (n_j + \frac{1}{2})/N]$, with n_j an integer, such that

$$\left|\frac{n_j}{N} - \frac{n_k}{N}\right| \geqslant \frac{6}{N} \text{ for } 1 \leqslant j < k \leqslant q+1$$

and $\epsilon > 0$, we can find an infinitely differentiable function f with the following properties.

(i) $f(t) \geqslant 0$ for all $t \in \mathbb{T}$.

(ii) $\hat{f}(0) = 1$.

(iii) $|\hat{f}(r)| \leqslant |r|^{-1/(2q)}\big(\log(1+|r|)\big)^{1/2}\psi(|r|)$ for all $r \neq 0$.

(iv) If $x_j \in$ supp $f \cap I_j$ then

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(v) If $x \in \mathbb{T}$ we can find a $y \in$ supp f with $|x - y| < \epsilon$.

If we deal with sets rather than measures we have the following result which excludes a natural conjecture.

Theorem 2.6. — *If q is an integer with $q \geqslant 1$, then we can find a closed set E with the following properties.*

(i) $E_{[q]}$ *has strictly positive Lebesgue measure.*

(ii) The equation
$$\sum_{j=1}^{2q-1} m_j x_j = 0$$
has no non-trivial solution with $m_j \in \mathbb{Z}$ and the x_j distinct points of E.

The μ we construct in the proof of Theorem 2.4 also has the property described in the next lemma, which furnishes a complement to Lemma 2.1 (ii).

Lemma 2.7. — *If q is an integer with $q \geqslant 1$ and $\psi : \mathbb{N} \to \mathbb{R}$ is a sequence of positive numbers such that $\psi(r) \to \infty$ as $r \to \infty$, then there exists a probability measure μ such that*
$$|\hat{\mu}(r)| \leqslant |r|^{-1/(2q)} \big(\log(1+|r|)\big)^{1/2} \psi(|r|)$$
for all $r \neq 0$, but the set
$$\left\{ \sum_{j=1}^{q} m_j x_j \ :\ x_j \in \mathrm{supp}\ \mu,\ m_j \in \mathbb{Z} \right\}$$
has Lebesgue measure zero.

However, the method of [4] can be easily adapted to give a much simpler proof of this result.

Since the proof of Theorem 2.6 is substantially simpler than that of Theorem 2.4 we shall devote the next two sections to its proof. We give the fairly routine proof of Theorem 2.4 from Lemma 2.5 in section 5 and devote the rest of the paper to the proof of Lemma 2.5.

Like many others of my papers, this one owes a great deal to two remarkable papers [2] and [3] of Kaufman.

3. Sums and algebraic relations

We shall prove Theorem 2.6 by a Baire category argument. We use the Hausdorff metric $d_{\mathcal{F}}$ defined in the next lemma.

Definition 3.1. — *Consider the space $\mathcal{F}$ of non-empty closed subsets of $\mathbb{T}$. We set*
$$d_{\mathcal{F}}(E,F) = \sup_{e\in E} \inf_{f\in F} |e-f| + \sup_{f\in F} \inf_{e\in E} |e-f|.$$

It is well known that $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space. (See, for example, Chapter II §21 VII and Chapter III §33 IV of [5].)

We need he following remarks.

Lemma 3.2. — *(i) If E, F, G and H are closed then*

$$d_{\mathcal{F}}(E+F, G+H) \leqslant d_{\mathcal{F}}(E,G) + d_{\mathcal{F}}(F,H).$$

(ii) Suppose E_n, F_n, E and F are closed sets with

$$d_{\mathcal{F}}(E_n,E),\ d_{\mathcal{F}}(F_n,F) \to 0.$$

Then $d_{\mathcal{F}}(E_n+F_n, E+F) \to 0$ as $n \to \infty$.

(iii) Suppose E_n and E are closed sets with $d_{\mathcal{F}}(E_n,E) \to 0$. Then

$$m(E) \geqslant \limsup_{n\to\infty} m(E_n).$$

Proof. —

(i) Observe that, if $e \in E$, $f \in F$, $g \in G$ and $h \in H$,

$$|(e+f)-(g+h)| \leqslant |e-g| + |f-h|$$

so, if $e \in E$, $f \in F$,

$$\inf_{e\in E,\, f\in F} |(e+f)-(g+h)| \leqslant \inf_{e\in E} |e-g| + \inf_{f\in F} |f-h|$$

whence

$$\sup_{g\in G,\, h\in H} \inf_{e\in E,\, f\in F} |(e+f)-(g+h)| \leqslant \sup_{g\in G} \inf_{e\in E} |e-g| + \sup_{h\in H} \inf_{f\in F} |f-h|.$$

(ii) This follows directly from (i).

(iii) Given $\epsilon > 0$, we can find an $\eta > 0$ such that

$$m\big(E + (-\eta,\eta)\big) < m(E) + \epsilon.$$

When n is sufficiently large,

$$E_n \subseteq E + (-\eta,\eta)$$

so

$$m(E_n) < m(E) + \epsilon.$$

Thus $\limsup_{n\to\infty} m(E_n) \leqslant m(E) + \epsilon$ for all ϵ and the result follows.

□

Definition 3.3. — *If $q \geqslant 1$, we define $\mathcal{H} = \mathcal{H}_q$ to be the subspace of $\mathcal{F}$ consisting of those closed sets for which $m(E_{[q]}) \geqslant 1/2$ with the inherited metric $d_{\mathcal{H}} = d_{\mathcal{F}} | \mathcal{H} \times \mathcal{H}$.*

Lemma 3.2 tells us that $\mathcal{H}$ is a closed subspace of $\mathcal{F}$ and so $(\mathcal{H}, d_{\mathcal{H}})$ is a complete metric space. Since $(\mathbb{T}, m) \in \mathcal{H}$, the space $\mathcal{H}$ is non-empty.

We can thus deduce Theorem 2.6 from the Baire category version.

Theorem 3.4. — *The collection of $E \in \mathcal{H}$ such that the equation*

$$\sum_{j=1}^{2q-1} m_j x_j = 0$$

has a non-trivial solution with $m_j \in \mathbb{Z}$ and the x_j distinct points of E is of first category.

Since $\mathbb{Z}^{2q-1}$ is countable, Theorem 3.4 follows in turn from the simpler result.

Lemma 3.5. — *Let $\mathbf{m} \in \mathbb{Z}^{2q-1} \setminus \{\mathbf{0}\}$ and $N \geqslant 1$. Let $\mathcal{E}(\mathbf{m}, N)$ be the collection of of $E \in \mathcal{H}$ such that the equation*

$$\sum_{j=1}^{2q-1} m_j x_j = 0$$

has a solution with $x_j \in E$ $[1 \leqslant j \leqslant 2q-1]$ and

$$|x_j - x_k| \geqslant N^{-1} \text{ for } 1 \leqslant j < k \leqslant 2q-1.$$

Then $\mathcal{E}(\mathbf{m}, N)$ is closed and has dense complement.

We split the proof of Theorem 3.4 into two parts, the easy Lemma 3.6 and the harder Lemma 3.7.

Lemma 3.6. — *Suppose $\mathbf{m} \in \mathbb{Z}^{2q-1} \setminus \{\mathbf{0}\}$ and $N \geqslant 1$. Then $\mathcal{E}(\mathbf{m}, N)$ is open in $\mathcal{H}$.*

Proof. — We show that the complement of $\mathcal{E}(\mathbf{m}, N, \mathbf{I})$ is closed. Suppose that $E_n \notin \mathcal{E}(\mathbf{m}, N, \mathbf{I})$ and $d(E_n, E) \to 0$ as $n \to \infty$ Then we can find $\mathbf{x}(n) \in E_n^{2q-1}$ such that $|x_j(n) - x_i(n)| \geqslant N^{-1}$ for all $i \neq j$ and

$$\sum_{j=1}^{2q-1} m_j x_j(n) = 0.$$

The Bolzano–Weierstrass theorem tells us that, by extracting a subsequence if necessary, we may suppose that $x_j(n) \to x_j$ as $n \to \infty$ for each $1 \leqslant j \leqslant 2q-1$. Now $|x_j - x_i| \geqslant N^{-1}$ for all $i \neq j$ and

$$\sum_{j=1}^{2q-1} m_j x_j = 0.$$

Thus $E_n \notin \mathcal{E}(\mathbf{m}, N)$. □

Thus our proof of Theorem 3.4 reduces to proving the following result.

LEMMA 3.7. — *Let* $\mathbf{m} \in \mathbb{Z}^{2q-1} \setminus \{\mathbf{0}\}$*, let* $\delta > 0$ *and let* $\epsilon > 0$*. Then, given* $E \in \mathcal{H}$ *we can find an* $F \in \mathcal{H}$ *with* $d_{\mathcal{H}}(E, F) < \epsilon$ *such that the equation*

$$\sum_{j=1}^{2q-1} m_j x_j = 0$$

has no solution with $x_j \in E$ $[1 \leqslant j \leqslant 2q-1]$ *and*

$$|x_j - x_k| \geqslant \delta \text{ for } 1 \leqslant j < k \leqslant 2q-1.$$

4. Completion of the proof of the theorem on sums

The main step in the construction for Lemma 3.7 is the following.

LEMMA 4.1. — *Suppose* $E_1, E_2, \dots E_q$ *are closed subsets of* $\mathbb{T}$ *such that*

$$m(E_1 + E_2 + \cdots + E_q) \geqslant 1/2.$$

Then, given $\epsilon_1 > 0$ *and* $\mathbf{m} \in \mathbb{Z}^{2q-1} \setminus \{\mathbf{0}\}$*, we can find* $F_1, F_2, \dots F_q$ *closed subsets of* $\mathbb{T}$ *and* $\epsilon_2 > 0$ *with the following properties.*

(i) $m(F_1 + F_2 + \cdots + F_q) \geqslant 1/2$.

(ii) *The Hausdorff distance* $d_{\mathcal{F}}(E_j, F_j) < \epsilon_1$ *for* $1 \leqslant j \leqslant q$.

(iii) *If* $x_1 \in \bigcup_{s=1}^{q} F_s$ $x_r \in \bigcup_{s=2}^{q} F_s$ *for* $2 \leqslant r \leqslant 2q-1$, $|x_r - y_r| \leqslant \epsilon_2$ *for* $1 \leqslant r \leqslant 2q-1$ *and* $|y_j - y_k| \geqslant \delta$ *for* $1 \leqslant j < k \leqslant 2q-1$*, we have*

$$\sum_{j=1}^{2q-1} m_j y_j \neq 0.$$

Proof. — Choose $0 < \gamma < \epsilon_1/4$. We observe that the collection of open sets

$$\big(E_1 + (\gamma, -\gamma)\big) + E_2 + \cdots + E_q$$

form an open cover of the compact set

$$E_1 + E_2 + \cdots + E_{q-1} + E_q.$$

We can thus find a finite collection of points

$$\mathbf{e}(r) \in E_2 \times E_3 \times \cdots \times E_q \qquad [1 \leqslant r \leqslant N$$

such that

$$\bigcup_{r=1}^{N} \left(\sum_{j=2}^{q} e_j(r) + \big(E_1 + (\gamma, -\gamma)\big) \right) \supseteq E_1 + E_2 + \cdots + E_{q-1} + E_q.$$

We now choose finite subsets $\tilde{F}_j$ of E_j $[2 \leqslant j \leqslant q]$ such that

$$\mathbf{e}(r) \in \prod_{j=2}^{q} \tilde{F}_j$$

for $[1 \leqslant r \leqslant N]$ and $d_{\mathfrak{F}}(E_j, \tilde{F}_j) < \gamma$. Automatically

$$\big(E_1 + (\gamma, -\gamma)\big) + \tilde{F}_2 + \cdots + \tilde{F}_q \supseteq E_1 + E_2 + \cdots + E_{q-1} + E_q.$$

By perturbing each of the points in the $\tilde{F}_j$ in turn by an amount less than γ we can find disjoint finite sets F_j $[2 \leqslant j \leqslant q]$ such that $d_{\mathfrak{F}}(E_j, F_j) < 2\gamma$,

(iii)′ If $y_r \in \bigcup_{s=2}^{q} F_s$ for $1 \leqslant r \leqslant 2q-1$ and the y_r are distinct, we have

$$\sum_{j=1}^{2q-1} m_j y_j \neq 0.$$

and

$$\big(E_1 + (2\gamma, -2\gamma)\big) + F_2 + \cdots + F_q \supseteq E_1 + E_2 + \cdots + E_{q-1} + E_q.$$

A simple argument shows that

$$m\Big(\big(E_1 + [3\gamma, -3\gamma]\big) + F_2 + \cdots + F_q\Big) > 1/2.$$

Since F_2, F_3, …, F_q are finite, it follows that there is a finite set X such that if

$$y \notin X \text{ and } y_r \in \bigcup_{s=2}^{q} F_s \text{ for } 2 \leqslant r \leqslant 2q-1,$$

with the y_r distinct then

$$m_1 y + \sum_{j=2}^{2q-1} m_j y_j \neq 0.$$

Now set

$$F_1 = \big(E_1 + [3\gamma, -3\gamma]\big) \setminus \big(X + (-\eta, \eta)\big)$$

with $\eta > 0$. Provided we take η small enough, we have $d_{\mathfrak{F}}(E_1, F_1) < \epsilon_1$ and

$$m(F_1 + F_2 + \cdots + F_q) \geqslant 1/2.$$

Further, combining (iii)′ with the definition of X we see that

(iii)″ If $y_1 \in \bigcup_{s=1}^{q} F_s$ $y_r \in \bigcup_{s=2}^{q} F_s$ for $2 \leqslant r \leqslant 2q-1$ and y_1, y_2, …, y_{2q-1} are distinct then

$$\sum_{j=1}^{2q-1} m_j y_j \neq 0.$$

Take K to be the collection of $\mathbf{x} \in \mathbb{T}^{2q-1}$ such that $x_1 \in \bigcup_{s=1}^{q} F_s$, $x_r \in \bigcup_{s=2}^{q} F_s$ for $2 \leqslant r \leqslant 2q-1$ and $|x_j - x_k| \geqslant \delta/2$ for $1 \leqslant j < k \leqslant 2q-1$. If we set

$$L = \left\{ \mathbf{x} \in \mathbb{T}^{2q-1} \ : \ \sum_{j=1}^{2q-1} m_j x_j = 0 \right\},$$

then K and L are disjoint compact subsets of $\mathbb{T}^{2q-1}$. A standard theorem now tells us that there exists an $\epsilon_2' > 0$ such that, if $\mathbf{x} \in K$ and $|y_j - x_j| \leqslant \epsilon_2'$ for $1 \leqslant j \leqslant 2q-1$, we have

$$\sum_{j=1}^{2q-1} m_j y_j \neq 0.$$

If we take $\epsilon_2 = \min(\epsilon_2', \delta)/4$, then condition (iii) holds and the required result follows. □

We now show how to prove Lemma 3.7 from Lemma 4.1.

We first observe that, by repeated application of Lemma 4.1, with the various $2q-1$tuples obtained by permuting the entries of $\mathbf{m}$ we obtain the following version.

Lemma 4.2. — *The result of Lemma 4.1 holds with condition (iii) of the conclusion replaced by the following.*

(iii)′ Suppose $1 \leqslant p \leqslant 2q-1$. *If* $x_p \in \bigcup_{s=1}^{q} F_s$, $x_r \in \bigcup_{s=2}^{q} F_s$ *for* $r \neq p$, $|x_r - y_r| \leqslant \epsilon_2$ *for* $1 \leqslant r \leqslant 2q-1$, *and* $|y_j - y_k| \geqslant \delta$ *for* $1 \leqslant j < k \leqslant 2q-1$, *we have*

$$\sum_{j=1}^{2q-1} m_j y_j \neq 0.$$

Next observe that, by repeated application of Lemma 4.2 we can obtain the following version.

Lemma 4.3. — *Suppose* $E_1, E_2, \ldots E_q$ *are closed subsets of* $\mathbb{T}$ *such that*

$$m(E_1 + E_2 + \cdots + E_q) \geqslant 1/2.$$

Then, given $\epsilon > 0$, $\delta > 0$ *and* $\mathbf{m} \in \mathbb{Z}^{2q-1} \setminus \{\mathbf{0}\}$, *we can find* $F_1, F_2, \ldots F_q$ *closed subsets of* $\mathbb{T}$ *and* $\eta > 0$ *with the following properties.*

(i) $m(F_1 + F_2 + \cdots + F_q) \geqslant 1/2$.

(ii) The Hausdorff distance $d_{\mathfrak{F}}(E_j, F_j) < \epsilon$ *for* $1 \leqslant j \leqslant q$.

(iii) Suppose $1 \leqslant p_1 \leqslant 2q-1$ *and* $1 \leqslant q_1 \leqslant q$. *If* $x_{p_1} \in \bigcup_{s=1}^{q} F_s$ $x_r \in \bigcup_{s \neq q_1} F_s$ *for* $r \neq p_1$ *and* $|x_j - x_k| \geqslant \delta$ *for* $1 \leqslant j < k \leqslant 2q-1$, *we have*

$$\sum_{j=1}^{2q-1} m_j x_j \neq 0.$$

We now disentangle condition (iii) of Lemma 4.3.

Lemma 4.4. — *Condition (iii) of Lemma 4.3 can be rewritten as follows.*

(iii)″ If $x_j \in \bigcup_{s=1}^{q} F_s$ for $1 \leqslant j \leqslant 2q-1$ and $|x_j - x_k| \geqslant \delta$ for $1 \leqslant j < k \leqslant 2q-1$, we have

$$\sum_{j=1}^{2q-1} m_j x_j \neq 0.$$

Proof. — By a simple counting argument there exists a $1 \leqslant q_1 \leqslant q$ such that F_{q_1} contains at most one of the x_j. □

We can now prove Lemma 3.7.

Proof of Lemma 3.7 from Lemma 4.3. — Suppose $E \in \mathcal{H}$. If we set

$$E_1 = E_2 = \cdots = E_q = E$$

then, automatically, E_1, E_2, $\ldots E_q$ are closed subsets of $\mathbb{T}$ such that

$$m(E_1 + E_2 + \cdots + E_q) \geqslant 1/2$$

and so, by Lemma 4.3 (supplemented by the observation of Lemma 4.4), we can find F_1, F_2, $\ldots F_q$ closed subsets of $\mathbb{T}$ with the following properties.

(i) $m(F_1 + F_2 + \cdots + F_q) \geqslant 1/2$.

(ii) The Hausdorff distance $d_{\mathcal{F}}(E_j, F_j) < \epsilon/q$ for $1 \leqslant j \leqslant q$.

(iii) If $x_j \in \bigcup_{s=1}^{q} F_s$ for $1 \leqslant j \leqslant 2q-1$ and $|x_j - x_k| \geqslant \delta$ for $1 \leqslant j < k \leqslant 2q-1$, we have

$$\sum_{j=1}^{2q-1} m_j x_j \neq 0.$$

If we now set $F = \bigcup_{s=1}^{q} F_s$, then simple estimates (not the best possible) give

$$d_{\mathcal{F}}(E, F) \leqslant \sum_{s=1}^{q} d_{\mathcal{F}}(E, F_s) = \sum_{s=1}^{q} d_{\mathcal{F}}(E_s, F_s) < \epsilon$$

and the remaining conclusions can be read off.

□

The following observation may be worth making.

Lemma 4.5. — *If E is as in Theorem 2.6 then the set*

$$\left\{ \sum_{j=1}^{q-1} m_j x_j \;:\; m_j \in \mathbb{Z},\, x_j \in E \right\}$$

has Lebesgue measure zero.

Proof. — If not, we can find an $\mathbf{m} \in \mathbb{Z}^{q-1}$ such that

$$F = \left\{ \sum_{j=1}^{q-1} m_j x_j \ : \ x_j \in E \right\}$$

has positive Lebesgue measure. But then $F + F$ contains a non-trivial interval which is impossible. □

5. Proof of the main theorem from the main lemma

Although we have simply demanded that $\psi(r) \to \infty$ as $r \to \infty$ in Theorem 2.4, we can demand rather better behaviour.

LEMMA 5.1. — *If $\tilde{\psi} : \mathbb{N} \to \mathbb{R}$ is a sequence of strictly positive numbers such that $\tilde{\psi}(r) \to \infty$ as $r \to \infty$ and $\delta > 0$ we can find an increasing sequence of strictly positive numbers $\psi(r) \to \infty$ such that*

(i) $\min(\tilde{\psi}(r), \delta) \geqslant \psi(r)$ *for all* $r \in \mathbb{N}$,

(ii) $2\psi(n) \geqslant \psi(r) \geqslant \psi(n)$ *for all* $2n \geqslant r \geqslant n \geqslant 1$.

Proof. — Immediate. □

Throughout the rest of this paper q is a fixed integer with $q \geqslant 1$ and $\psi : \mathbb{N} \to \mathbb{R}$ is a fixed sequence of strictly positive numbers obeying the conditions of Lemma 5.1 with

$$\delta = \big(\max_{r \geqslant 1} |r|^{-1/(2q)} \big(\log(1 + |r|) \big)^{1/2} \big)^{-1}.$$

We write

$$\phi(r) = |r|^{-1/(2q)} \big(\log(1 + |r|) \big)^{1/2} \psi(|r|).$$

Observe that $0 < \phi(r) \leqslant 1$ for all $r \geqslant 1$ and there is a constant $K \geqslant 1$ such that

$$K\phi(n) \geqslant \phi(r) \geqslant K^{-1}\phi(n)$$

for all $2n \geqslant r \geqslant n \geqslant 1$.

Once again we use a Baire category argument but our metric space is a little more complicated than the Hausdorff metric space $(\mathcal{F}, D_{\mathcal{F}})$.

DEFINITION 5.2. — *We take $\mathcal{G}$ to be the set of ordered pairs (E, μ) where E is a non-empty closed set, and μ is a probability measure such that*

(i) $E \supseteq$ *supp* μ.

(ii) $|\hat{\mu}(r)|\phi(|r|)^{-1} \to 0$ *as* $r \to \infty$.

If (E,μ), $(F,\tau) \in \mathcal{G}$ *we define*

$$d_{\mathcal{G}}\big((E,\mu),(F,\tau)\big) = d_{\mathcal{F}}(E,F) + \sup_{r\in\mathbb{Z}} |\hat{\mu}(r) - \hat{\tau}(r)|\phi(|r|)^{-1}.$$

It is easy to check that $(\mathcal{G}, d_{\mathcal{G}})$ is a complete metric space. Since $(\mathbb{T}, m) \in \mathcal{G}$, the space is non-empty.

Theorem 2.4 thus follows from its Baire category version.

THEOREM 5.3. — *The set* $\mathcal{E}$ *of* (E,μ) *such that there exist distinct points*

$$x_1,\, x_2,\, \dots,\, x_{q+1} \in E,$$

and integers m_j, *not all zero such that*

$$\sum_{j=1}^{q+1} m_j x_j = 0$$

is of first category in $(\mathcal{G}, d_{\mathcal{G}})$.

It may be worth remarking that we shall use Baire category, not because it gives an apparently more general theorem, but because it makes the book keeping aspects of the proof rather easier. It should also be said that, even if the arguments of this section appear complicated, they are not deep.

In order to attack Theorem 5.3, we introduce some temporary definitions reflecting the conditions of Lemma 2.5. If $\mathbf{m} \in \mathbb{Z}^{q+1} \setminus \{\mathbf{0}\}$ we write

$$N_0(\mathbf{m}) = 12\left(q + 1 + \sum_{j=1}^{q+1} |m_j|\right).$$

If $N \geqslant 24(q+1)$ we write $\mathcal{J}(N)$ for the collection of ordered $(q+1)$tuples

$$\mathbf{I} = (I_1, I_2, \dots, I_{q+1})$$

where $I_j = [(n_j + \frac{1}{2})/N, (n_j - \frac{1}{2})/N]$, with n_j an integer and

$$\left|\frac{n_j}{N} - \frac{n_k}{N}\right| \geqslant \frac{6}{N} \text{ for } 1 \leqslant j < k \leqslant q+1.$$

If $\mathbf{m} \in \mathbb{Z}^q \setminus \{\mathbf{0}\}$, $N \geqslant N_0(\mathbf{m})$ and $\mathbf{I} \in \mathcal{J}(N)$ we write $\mathcal{E}(\mathbf{m}, N, \mathbf{I})$ for the set of (E,μ) with the property that if $x_j \in E \cap I_j$ then

$$\sum_{j=1}^{q} m_j x_j \neq 0.$$

By the definition of first category, it suffices to prove the following simpler result.

Lemma 5.4. — *If* $\mathbf{m} \in \mathbb{Z}^{q+1} \setminus \{\mathbf{0}\}$, $N \geqslant N_0(\mathbf{m})$ *and* $\mathbf{I} \in \mathcal{J}(N)$, *then* $\mathcal{E}(\mathbf{m}, N, \mathbf{I})$ *is open and dense in* $(\mathcal{G}, d_{\mathcal{G}})$.

Proof of Theorem 5.3 from Lemma 5.4. — It suffices to show that

$$\mathcal{E} = \bigcap_{\mathbf{m}\in\mathbb{Z}^{q+1}\setminus\{\mathbf{0}\}} \bigcap_{N\geqslant N_0(\mathbf{m})} \bigcap_{\mathbf{I}\in\mathcal{J}(N)} \mathcal{E}(\mathbf{m}, N, \mathbf{I}),$$

and, since

$$\mathcal{E} \subseteq \mathcal{E}(\mathbf{m}, N, \mathbf{I}),$$

we need only show that

$$\mathcal{E} \supseteq \bigcap_{\mathbf{m}\in\mathbb{Z}^{q+1}\setminus\{\mathbf{0}\}} \bigcap_{N\geqslant N_0(\mathbf{m})} \bigcap_{\mathbf{I}\in\mathcal{J}(N)} \mathcal{E}(\mathbf{m}, N, \mathbf{I}).$$

To this end, let

$$(E, \mu) \in \bigcap_{\mathbf{m}\in\mathbb{Z}^{q+1}\setminus\{\mathbf{0}\}} \bigcap_{N\geqslant N_0(\mathbf{m})} \bigcap_{\mathbf{I}\in\mathcal{J}(N)} \mathcal{E}(\mathbf{m}, N, \mathbf{I}).$$

Suppose $\tilde{\mathbf{m}} \in \mathbb{Z}^q \setminus \{\mathbf{0}\}$ and $x_1, x_2, \dots x_{q+1}$ are distinct points in E. If we choose $\tilde{N} \geqslant N_0(\tilde{\mathbf{m}})$ with

$$\tilde{N} \geqslant 48\big(1 + \max_{1\leqslant i,j\leqslant q+1} |x_i - x_j|^{-1}\big),$$

then we can find $\tilde{\mathbf{I}} \in \mathcal{J}(\tilde{N})$ such that $x_j \in \tilde{I}_j$. Since

$$(E, \mu) \in \mathcal{E}(\tilde{\mathbf{m}}, \tilde{N}, \tilde{\mathbf{I}})$$

we have

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

Thus $(E, \mu) \in \mathcal{E}$ and we are done. □

We split the proof of Lemma 5.4 into two parts, the easy Lemma 5.5 and the harder Lemma 5.6 (this depends on Lemma 2.5 which we still have to prove).

Lemma 5.5. — *If* $\mathbf{m} \in \mathbb{Z}^{q+1} \setminus \{\mathbf{0}\}$, $N \geqslant N_0(\mathbf{m})$ *and* $\mathbf{I} \in \mathcal{J}(N)$, *then* $\mathcal{E}(\mathbf{m}, N, \mathbf{I})$ *is open in* $(\mathcal{G}, d_{\mathcal{G}})$.

Proof. — Imitate the proof of Lemma 3.6. □

Thus the proof of Lemma 5.4 reduces to the proof of the next lemma.

Lemma 5.6. — *If* $\mathbf{m} \in \mathbb{Z}^{q+1} \setminus \{\mathbf{0}\}$, $N \geqslant N_0(\mathbf{m})$ *and* $\mathbf{I} \in \mathcal{J}(N)$, *then* $\mathcal{E}(\mathbf{m}, N, \mathbf{I})$ *is dense in* $(\mathcal{G}, d_{\mathcal{G}})$.

The proof of Lemma 5.6 from Lemma 5.6 will occupy the rest of this section. The next lemma merely serves to establish notation.

Lemma 5.7. — *Let $K : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with the following properties.*

(i)′ $K(x) \geqslant 0$ *for all* $x \in \mathbb{R}$.
(ii)′ $\int_{\mathbb{R}} K(x)\,dx = 1$.
(iii)′ $K(x) = 0$ *for* $|x| \geqslant 1/4$.

If M is a positive integer and we define $K_M : \mathbb{T} \to \mathbb{R}$ by

$$K_M(t) = \begin{cases} MK(Mt) & \text{if } |t| \leqslant 1/(4M), \\ 0 & \text{otherwise,} \end{cases}$$

then K_M is an infinitely differentiable function having the following properties.

(i) $K_M(t) \geqslant 0$ *for all* $t \in \mathbb{T}$.
(ii) $\int_{\mathbb{T}} K_M(t)\,dt = 1$.
(iii) $K_M(t) = 0$ *for* $|t| \geqslant 1/(4M)$.
(iv) $|\hat{K}_M(r)| \leqslant 1$ *for all* r.
(v) *There exists a constant A, independent of M, such that $|\hat{K}_M(r)| \leqslant A(M/r)^2$ for all $r \neq 0$.*

Proof. — This is entirely straightforward. □

Lemma 5.8. — *Given $(E, \mu) \in \mathcal{E}$ and $\epsilon > 0$, we can find $(F, \tau) \in \mathcal{E}$ such that $d\tau = gdm$ with g infinitely differentiable.*

Proof. — Observe that $(K_M * \mu, E + [-M^{-1}, M^{-1}]) \in \mathcal{E}$ and

$$d\big((K_M * \mu, E + [-M^{-1}, M^{-1}]), (E, \mu)\big) \to 0$$

as $M \to \infty$. □

Proof of Lemma 5.6 from Lemma 2.5. — In view of Lemma 5.8, it is sufficient to show that, given $(E, \mu) \in \mathcal{E}$ such that $d\mu = gdm$ with g infinitely differentiable and $\epsilon > 0$, we can find $(F, \tau) \in \mathcal{E}(\mathbf{m}, N, \mathbf{I})$ with $d\big((E, \mu), (F, \tau)\big) < \epsilon$.

Note that, since g is infinitely differentiable there exists a constant A such that

$$|\hat{g}(r)| \leqslant A|r|^{-3}$$

for all $r \neq 0$ and, since $(E, \mu) \in \mathcal{E}$, there exists a constant B such that

$$|\hat{g}(r)| \leqslant B\phi(|r|)$$

for all $r \neq 0$.

To this end, observe that given $\eta > 0$ (to be fixed later), Lemma 2.5 tells us that, since $\eta\psi(|r|) \to \infty$ as $r \to \infty$, we can find an infinitely differentiable function f with the following properties.

(i) $f(t) \geqslant 0$ for all $t \in \mathbb{T}$.
(ii) $\hat{f}(0) = 1$.
(iii) $|\hat{f}(r)| \leqslant \eta |r|^{-1/(2q)}\big(\log(1+|r|)\big)^{1/2}\psi(|r|) = \eta\phi(|r|)$ for all $r \neq 0$.
(iv) If $x_j \in \operatorname{supp} f \cap I_j$ then

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(v) If $x \in \mathbb{T}$ we can find a $y \in \operatorname{supp} f$ with $|x-y| < \eta$.

Note also that, since g is infinitely differentiable, there exists a constant A such that

$$|\hat{g}(r)| \leqslant A|r|^{-3}$$

for all $r \neq 0$. Finally we observe that $r^{-2}\phi(r) \to 0$ as $r \to \infty$ so there exists a $C > 0$ such that

$$C\phi(r) > r^{-2}$$

for all $r \geqslant 1$.

Set $h(t) = g(t)f(t)$ and choose some $1 > \delta > 0$ (to be fixed later). We seek to estimate $\hat{h}(r)$. If $r \neq 0$

$$\begin{aligned} |\hat{h}(r) - \hat{g}(r)| &= \left|\sum_{m\neq 0} \hat{f}(r-m)\hat{g}(m)\right| \leqslant \sum_{m\neq 0} |\hat{f}(r-m)||\hat{g}(m)| \\ &= \sum_{0\neq |m|\leqslant |r|/2} |\hat{f}(r-m)||\hat{g}(m)| + \sum_{|m|>|r|/2} |\hat{f}(r-m)||\hat{g}(m)| \end{aligned}$$

Using the remarks about the behaviour of ϕ at the beginning of this section, we have

$$|\hat{f}(r-m)| \leqslant \eta\phi(|r-m|) \leqslant \eta K\phi(r)$$

whenever $|m| \leqslant |r|/2$ and

$$|\hat{f}(r-m)| \leqslant \eta\phi(|r-m|) \leqslant \eta$$

whenever $|r-m| \neq 0$. Thus

$$\begin{aligned} |\hat{h}(r) - \hat{g}(r)| &\leqslant KA\eta\phi(|r|) \sum_{0\neq |m|\leqslant |r|/2} |m|^{-3} + A\eta \sum_{|m|>|r|/2} |m|^{-3} \\ &\leqslant KA\eta \sum_{m\neq 0} |m|^{-3} + A\eta \sum_{|m|>|r|/2} |m|^{-3} \\ &\leqslant \eta(10KA\phi(|r|) + 10Ar^{-2}) \leqslant 10A\eta(K+C)\phi(|r|) \\ &\leqslant \delta\phi(|r|) \end{aligned}$$

for all $r \neq 0$ provided only that we choose η small enough.

A similar but simpler argument shows that

$$|\hat{h}(0) - \hat{g}(0)| \leqslant \delta$$

provided only that we choose η small enough. If we now set $H(t) = |\hat{h}(0)|^{-1}h(t)$ then, since

$$|\hat{H}(r) - \hat{g}(r)| \leqslant |\hat{h}(r) - \hat{g}(r)| + |1 - \hat{h}(0)^{-1}|(|\hat{g}(r)| + |\hat{h}(r) - \hat{g}(r)|),$$

it follows that, provided we pick δ small enough,

$$\sup_r \phi(r)^{-1}|\hat{H}(r) - \hat{g}(r)| < \epsilon/2.$$

Taking $\tau = H dm$ and $F = E \cap \operatorname{supp} f$ we see that $(F, \tau) \in \mathcal{E}$ by construction. Provided η is small enough, condition (v) implies that $d_{\mathcal{H}}(E, F) < \epsilon/2$ and so, using the conclusion of the previous paragraph, $d\big((E, \mu), (F, \tau)\big) < \epsilon$. Condition (iv) shows that $(F, \tau) \in \mathcal{E}(\mathbf{m}, N, \mathbf{I})$ so we are done. □

6. Preparations for the main lemma

Before we start the start the proof of Lemma 2.5 in earnest we need to do some cleaning up.

Lemma 6.1. — *If y_1, y_2, $\dots y_m$ are distinct points of $\mathbb{T}$, $\epsilon > 0$ and ϕ is as specified at the beginning of section 5, we can find an infinitely differentiable function f with the following properties.*

(i) $f(t) \geqslant 0$ *for all* $t \in \mathbb{T}$.
(ii) $\hat{f}(0) = 1$.
(iii) $|\hat{f}(r)| \leqslant \phi(|r|)$ *for all* $r \neq 0$.
(iv) If $y_k \notin$ *supp* f *for* $1 \leqslant k \leqslant m$.
(v) If $x \in \mathbb{T}$ *we can find a* $y \in$ *supp* f *with* $|x - y| < \epsilon$.

Proof. — (The reader may prefer to supply her own proof.) Choose K in Lemma 5.7 In such a way that

$$K(x) = \|K\|_\infty \text{ for } |x| \leqslant 1/16$$

and set

$$L_M(t) = \|K\|_\infty^{-1} M^{-1} K_M(t).$$

If we set

$$g(t) = 1 - \sum_{j=1}^{m} L_M(t - y_k)$$

then

$$|\hat{g}(r)| \leqslant \begin{cases} \frac{m\|K\|_\infty}{M} & \text{for } |r| \leqslant M \\ \frac{mA\|K\|_\infty}{M}\left(\frac{M}{r}\right)^2 & \text{for } |r| \geqslant M \end{cases}$$

If $\eta > 0$ then, provided only that M is large enough, we have

$$\frac{m(A+1)\|K\|_\infty}{M} \leqslant \eta M^{-3/4}$$

and so

$$|\hat{g}(r)| \leqslant \eta |r|^{-3/4}$$

for all $r \neq 0$. If we now set $f = \|g\|_1^{-1} g$ then, provided that M is large enough, all the conditions of the lemma follow. □

Lemma 6.1 gives a proof of Lemma 2.5 in the particular case when all the m_j except one are zero. Lemma 2.5 is also trivial in the case when

$$0 \notin \sum_{j=1}^{q} m_j I_j$$

since we can then take $f = 1$. Thus we need only prove the following version of Lemma 2.5

Lemma 6.2. — *Let ϕ be as specified at the beginning of section 5 and let $\epsilon > 0$. Suppose $\mathbf{m} \in \mathbb{Z}^{q+1}$, m_1, $m_2 \neq 1$ and N is a positive integer such that*

$$N \geqslant 12\left(q + 1 + \sum_{j=1}^{N} |m_j|\right).$$

Suppose further that we are given $I_j = [(n_j - \frac{1}{2})/N, (n_j + \frac{1}{2})/N]$, with n_j an integer $[1 \leqslant j \leqslant q+1]$, such that

$$\left|\frac{n_j}{N} - \frac{n_k}{N}\right| \geqslant \frac{6}{N} \text{ for } 1 \leqslant j < k \leqslant q+1$$

and

$$0 \in \sum_{j=1}^{q} m_j I_j,$$

Then we can find an infinitely differentiable function f with the following properties.

(i) *$f(t) \geqslant 0$ for all $t \in \mathbb{T}$.*
(ii) *$\hat{f}(0) = 1$.*
(iii) *$|\hat{f}(r)| \leqslant \phi(|r|)$ for all $r \neq 0$.*

(iv) If $x_j \in$ supp $f \cap I_j$ then

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(v) If $x \in \mathbb{T}$ we can find a $y \in$ supp f with $|x-y| < \epsilon$.

Lemma 6.2 follows in turn from the following result.

Lemma 6.3. — *Suppose the hypotheses of Lemma 6.2 hold and in addition we are given infinitely differentiable g_j $[1 \leqslant j \leqslant q+1]$ such that*

(i)′ $1 \geqslant g_j(t) \geqslant 0$ for all $t \in \mathbb{T}$.
(ii)′ $g_j(t) = 0$ if $t \notin [(n_j - \frac{1}{2} - \epsilon)/N, (n_j + \frac{1}{2} + \epsilon)/N]$.
(iii)′ $g_j(t) = 1$ if $t \in [(n_j - \frac{1}{2})/N, (n_j + \frac{1}{2})/N]$.

Then we can find infinitely differentiable functions f_j with the following properties.

(i)″ $f_j(t) \geqslant 0$ for all $t \in \mathbb{T}$.
(ii)″ $\hat{f}_j(0) = \hat{g}_j(0)$.
(iii)″ $|\hat{f}_j(r) - \hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$ for all $r \neq 0$.
(iv)″ If $x_j \in$ supp f_j then

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(v) If $x \in I_j$ we can find a $y \in$ supp f_j with $|x-y| < \epsilon$.

Proof of Lemma 6.2 from Lemma 6.3. — Choose g_j satisfying conditions (i)′, (ii)′ and (iii)′ and set $g = 1 - \sum_{j=1}^{q+1} g_j$. If we choose f_j satisfying the conclusions of Lemma 6.3 and set $f = g + \sum_{j=1}^{q+1} f_j$, then f satisfies the conclusions of Lemma 6.2. □

We can deduce Lemma 6.3 from a result on sums of point masses. Here and elsewhere we write $|E|$ for the number of elements in a finite set E.

Lemma 6.4. — *Suppose the hypotheses of Lemma 6.3 hold. Then we can find N_0, N_1, $B \geqslant 1$ and $\gamma > 0$ with the following properties. If $n \geqslant N_1$ we can find finite sets of points E_j $[1 \leqslant j \leqslant q+1]$ such that writing*

$$\mu_j = |E_j|^{-1} \|g_j\|_1 \sum_{x \in E_j} \delta_x$$

the following conditions hold.

(1) $|\hat{\mu}_j(r) - \hat{g}_j(r)| \leqslant 2^{-1}(q+1)^{-1}\phi(|r|)$ for all $|r| \leqslant N_0$.
(2) $|\hat{\mu}_j(r)| + |\hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$ for all $N_0 \leqslant |r| \leqslant n^{2(q+1)}$.

(3) If $x_j \in E_j + [-4\gamma n^{-q}, 4\gamma n^{-q}]$ then

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(4) If $x \in I_j$ we can find a $y \in E_j$ with $|x - y| < \epsilon$.

Proof of Lemma 6.3 from Lemma 6.4. — let $M = M(n)$ be the integer satisfying

$$(\gamma n^{-q})^{-1} + 1 \geqslant M > (\gamma n^{-q})^{-1}$$

and set $f_j = \mu_j * K_M$ where K_M is defined as in Lemma 5.7. Automatically the f_j satisfy the conclusions of Lemma 6.3. with the possible exception of (iii)$''$. We now show that (iii)$''$ holds, provided only that n is large enough.

First observe that, provided only that n is large enough, standard results on approximate identities tell us that

$$|\hat{f}_j(r) - \hat{\mu}_j(r)| \leqslant 2^{-1}(q+1)^{-1}\phi(|r|)$$

and so, using (1),

$$|\hat{g}_j(r) - \hat{f}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$$

for all $|r| \leqslant N_0$ provided only that n is large enough. Next we note that

$$|\hat{f}_j(r)| = |\hat{\mu}_j(r)||\hat{K}_M(r)| \leqslant |\hat{\mu}_j(r)|$$

so, using (2),

$$|\hat{f}_j(r)| + |\hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$$

whence

$$|\hat{\mu}_j(r) - \hat{f}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$$

for all $N_0 \leqslant |r| \leqslant n^{2(q+1)}$.

Note that, since g_j is infinitely differentiable, there exists a constant C such that

$$|\hat{g}_j(r)| \leqslant C|r|^{-1}$$

for all $r \neq 0$. Thus, provided only that n is large enough,

$$|\hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)/2$$

for all $n^{2q}+1 \leqslant |r|$. Using the equality $|\hat{f}_j(r)| = |\hat{\mu}_j(r)||\hat{K}_M(r)|$, we observe that

$$|\hat{f}_j(r)| \leqslant A(M/r)^2 \leqslant 2A\gamma^{-2}(n^q/r)^2 \leqslant 2A\gamma^{-2}|r|^{-1}$$

for $n^{2q} + 1 \leqslant |r|$. Thus, provided only that n is large enough,

$$|\hat{f}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)/2$$

and so

$$|\hat{f}_j(r) - \hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$$

for all $n^{2q} + 1 \leqslant |r|$ and so we are done. □

7. Completion of the proof of the main lemma

In this final section we obtain Lemma 6.4 by means of a probabilistic construction. All parts of following theorem are well known (see for example [1]) but it may be helpful to recall the proofs.

THEOREM 7.1. — *(i) If Y is a real valued random variable with $|Y| \leqslant 1$ and $\mathbb{E}Y = 0$ then*

$$\mathbb{E}e^{\lambda Y} \leqslant e^{\lambda^2}.$$

(ii) If $Y_1, Y_2 \dots$ are independent real valued random variable with $|Y_k| \leqslant 1$ and $\mathbb{E}Y_k = 0$ then

$$\Pr\left(\sum_{k=1}^{n} Y_k \geqslant y\right) \leqslant e^{-y^2/4n}.$$

(iii) If $Z_1, Z_2 \dots$ are independent complex valued random variable with $|Z_k| \leqslant 1$ and $\mathbb{E}Z_k = 0$ then

$$\Pr\left(\left|\sum_{k=1}^{n} Z_k\right| \geqslant y\right) \leqslant 4e^{-y^2/4n}$$

(iv) Suppose $U_1, U_2 \dots$ are independent identically distributed random variables taking values on $\mathbb{T}$. If

$$\Pr\big(U_1 \in [a,b)\big) = \mu\big([a,b)\big)$$

for some probability measure μ then

$$\Pr\left(\left|n^{-1}\sum_{j=1}^{n} e^{irU_k} - \hat{\mu}(r)\right| \geqslant y\right) \leqslant 4e^{-y^2/(16n)}.$$

(v) Suppose $0 \leqslant \alpha \leqslant 1$ and $W_1, W_2, \dots$ are independent complex valued random variables with $|W_k| \leqslant 1$ and $|\mathbb{E}W_k| \leqslant \alpha$ is as in (iv). Then

$$\Pr\left(\left|n^{-1}\sum_{j=1}^{n} W_j\right| \geqslant \alpha + y\right) \leqslant 4e^{-y^2/(16n)}.$$

Proof. —

(i) The result is immediate if $|\lambda| \geqslant 1$. If $|\lambda| \leqslant 1$,

$$\mathbb{E}e^{\lambda Y} = \sum_{r=0}^{\infty} \mathbb{E}Y^r \frac{\lambda^r}{r!} = 1 + \sum_{r=2}^{\infty} \mathbb{E}Y^r \frac{\lambda^r}{r!} \leqslant = 1 + \sum_{r=2}^{\infty} \frac{|\lambda|^r}{r!} \leqslant \sum_{r=0}^{\infty} \frac{|\lambda|^{2r}}{r!} = e^{\lambda^2}.$$

(ii) Observe that the random variables $e^{\lambda Y_k}$ are independent so

$$\mathbb{E}e^{\lambda \sum_{j=1}^{n} Y_k} = \mathbb{E}\prod_{j=1}^{n} e^{\lambda Y_k} = \prod_{j=1}^{n} \mathbb{E}e^{\lambda Y_k} \leqslant e^{n\lambda^2}.$$

Thus by a Tchebychev estimate

$$\Pr\left(\sum_{k=1}^{n} Y_k \geqslant y\right) \leqslant e^{-\lambda y}\mathbb{E}e^{\lambda \sum_{j=1}^{n} Y_k} = e^{n\lambda^2 - \lambda y}$$

and setting $\lambda = y/2n$ we have the desired result.

(iii) Apply part (ii) to $\Re Z_k$, $-\Re Z_k$, $\Im Z_k$ and $-\Im Z_k$.

(iv) Observe that

$$\mathbb{E}e^{irU_k} = \hat{\mu}(r),$$

so applying part (iii) with $Z_k = \big(e^{irU_k} - \hat{\mu}(r)\big)/2$ gives the required result.

(v) Observe that

$$\begin{aligned}\Pr\left(\left|n^{-1}\sum_{k=1}^{n} W_k\right| \geqslant |\hat{\mu}(r)| + y\right) &\leqslant \Pr\left(\left|n^{-1}\sum_{k=1}^{n} W_k\right| \geqslant |\mathbb{E}W_1| + y\right) \\ &\leqslant \Pr\left(\left|n^{-1}\sum_{k=1}^{n} (W_k - \mathbb{E}W_k)\right| \geqslant y\right) \\ &\leqslant 4e^{-y^2/(16n)}\end{aligned}$$

as in part (iv).

□

We now state our probabilistic version of Lemma 6.4.

Lemma 7.2. — *Suppose the hypotheses of Lemma 6.3 hold. Set* $M = \sum_{j=1}^{q+1} |m_j|$. *Then we can find* N_0, N_1, $B \geqslant 1$ *and* $\gamma > 0$ *with the following properties such that whenever* $n \geqslant N_1$ *the following is true.*

Suppose X_{jk} *are independent random variables taking values on* $\mathbb{T}$ $[1 \leqslant j \leqslant q+1,\ 1 \leqslant k \leqslant n]$ *such that* X_{jk} *has probability density* $\|g_j\|_1^{-1} g_j$. *If* $2 \leqslant j \leqslant q+1$ *take*

$$E_j = \{X_{jk} \,:\, 1 \leqslant k \leqslant n\}$$

set

$$\tilde{E}_1 = \{X_{jk} \,:\, 1 \leqslant k \leqslant n\}$$

and

$$E_1 = \left\{ x \in E_1 \,:\, 0 \notin [-4M\gamma n^{-q}, 4M\gamma n^{-q}] + m_1 x + \sum_{j=2}^{q+1} m_j E_j \right\}.$$

If we take

$$\mu_j = |E_j|^{-1} \|g_j\|_1 \sum_{x \in E_j} \delta_x$$

then, with probability at least $1/2$*, following conditions hold.*

(1) $|\hat{\mu}_j(r) - \hat{g}_j(r)| \leqslant 2^{-1}(q+1)^{-1}\phi(|r|)$ *for all* $|r| \leqslant N_0$*.*

(2) $|\hat{\mu}_j(r)| + |\hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$ *for all* $N_0 \leqslant |r| \leqslant n^{2(q+1)}$*.*

(3) If $x_j \in E_j + [-4\gamma n^{-q}, 4\gamma n^{-q}]$*, then*

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(4) If $x \in I_j$ *we can find a* $y \in E_j$ *with* $|x - y| < \epsilon$*.*

Since any event which has strictly positive probability must have an instance Lemma 7.2 follows from Lemma 6.4.

Most of Lemma 7.2 is easy to prove.

Lemma 7.3. — *Suppose the hypotheses of Lemma 6.3 hold. Set* $M = \sum_{j=1}^{q+1} |m_j|$*. Then we can find* N_0'*,* N_1' *and* $B' \geqslant 1$ *such that whenever* $n \geqslant N_1'$ *and* $B \geqslant B'$ *the following is true.*

Suppose X_{jk} *are independent random variables taking values on* $\mathbb{T}$ $[1 \leqslant j \leqslant q+1,\ 1 \leqslant k \leqslant n]$ *such that* X_{jk} *has probability density* $\|g_j\|_1^{-1} g_j$ *and suppose* $\gamma > 0$*. If* $2 \leqslant j \leqslant q+1$ *take*

$$E_j = \{X_{jk} \,:\, 1 \leqslant k \leqslant n\}$$

set

$$\tilde{E}_1 = \{X_{jk} \,:\, 1 \leqslant k \leqslant n\}$$

and

$$E_1 = \left\{ x \in E_1 \,:\, 0 \notin [-4M\gamma n^{-q}, 4M\gamma n^{-q}] + m_1 x + \sum_{j=2}^{q+1} m_j E_j \right\}.$$

If we take

$$\mu_j = |E_j|^{-1} \|g_j\|_1 \sum_{x \in E_j} \delta_x$$

then, with probability at least $3/4$*, the following conditions hold.*

(1)′ $|\hat{\mu}_j(r)-\hat{g}_j(r)| \leqslant 2^{-1}(q+1)^{-1}\phi(|r|)$ *for all* $|r| \leqslant N_0'$ *and* $2 \leqslant j \leqslant q+1$.

(2)′ $|\hat{\mu}_j(r)| + |\hat{g}_j(r)| \leqslant (q+1)^{-1}\phi(|r|)$ *for all* $N_0 \leqslant |r| \leqslant n^{2(q+1)}$ *and* $2 \leqslant j \leqslant q+1$.

(3) *If* $x_j \in E_j + [-4\gamma n^{-q}, 4\gamma n^{-q}]$, *then*

$$\sum_{j=1}^{q+1} m_j x_j \neq 0.$$

(4)′ *If* $2 \leqslant j \leqslant q+1$ *and* $x \in I_j$ *we can find a* $y \in E_j$ *with* $|x-y| < \epsilon$.

Proof. — Observe that (3) is always true by virtue of the definition of E_1. The weak law of large numbers tells us that, provided only that n is large enough, condition (4)′ will hold with probability at least 7/8.

Since g_j is once continuously differentiable we can find a C_j such that

$$|\hat{g}_j(r)| \leqslant C_j |r|^{-1}$$

for all $|r| > 0$ and so we can find an N_0' such that

$$|\hat{g}_j(r)| \leqslant 4^{-1}(q+1)^{-1}\phi(|r|)$$

for all $|r| \leqslant N_0'$ and $2 \leqslant j \leqslant q+1$.

By Theorem 7.1

$$\begin{aligned}\Pr\big(|\hat{\mu}_j(r) - \hat{g}_j(r)| &\geqslant Bn^{-1/2}(\log n)^{1/2}\big) \\ &= \Pr\left(\left|n^{-1}\sum_{k=1}^{n} e^{irX_k} - \hat{\mu}(r)\right| \geqslant Bn^{-1/2}(\log n)^{1/2}\right) \\ &\leqslant 4e^{-B\log n/(16n)}.\end{aligned}$$

Thus, if we choose $B \geqslant 64(q+1)$, we have

$$\Pr\big(|\hat{\mu}_j(r) - \hat{g}_j(r)| \geqslant Bn^{-1/2}(\log n)^{1/2}\big) \leqslant 4n^{-4(q+1)}$$

for all r and all $2 \leqslant j \leqslant q+1$. Thus provided only that n is large enough,

$$\Pr\big(|\hat{\mu}_j(r) - \hat{g}_j(r)| \geqslant Bn^{-1/2}(\log n)^{1/2}\big) \leqslant 4n^{-4(q+1)}$$

will hold with probability at least 7/8.

Using the results of the two previous paragraphs we see that conditions (1)′ and (2)′ will both hold (with probability at least 7/8) provided only that n is large enough. The result follows. □

We now prove the harder part of Lemma 7.2.

Lemma 7.4. — *Suppose the hypotheses of Lemma 6.3 hold. Set* $M = \sum_{j=1}^{q+1} |m_j|$. *Then we can find* N_0'', N_1'', $B'' \geqslant 1$ *and* $\gamma > 0$ *such that whenever* $n \geqslant N_1''$ *and* $B \geqslant B_1$ *the following is true.*

Suppose X_{jk} are independent random variables taking values on $\mathbb{T}$ $[1 \leqslant j \leqslant q+1,\ 1 \leqslant k \leqslant n]$ such that X_{jk} has probability density $\|g_j\|_1^{-1} g_j$. If $2 \leqslant j \leqslant q+1$ take

$$E_j = \{X_{jk} \,:\, 1 \leqslant k \leqslant n\}$$

set

$$\tilde{E}_1 = \{X_{jk} \,:\, 1 \leqslant k \leqslant n\}$$

and

$$E_1 = \left\{ x \in \tilde{E}_1 \,:\, 0 \notin [-4M\gamma n^{-q}, 4M\gamma n^{-q}] + m_1 x + \sum_{j=2}^{q+1} m_j E_j \right\}.$$

If we take

$$\mu_1 = |E_1|^{-1} \|g_j\|_1 \sum_{x \in E_1} \delta_x$$

then, with probability at least $3/4$, following conditions hold.

(1)″ $|\hat{\mu}_1(r) - \hat{g}_1(r)| \leqslant 2^{-1}(q+1)^{-1}\phi(|r|)$ for all $|r| \leqslant N_0''$.

(2)″ $|\hat{\mu}_1(r)| + |\hat{g}_1(r)| \leqslant (q+1)^{-1}\phi(|r|)$ for all $N_0'' \leqslant |r| \leqslant n^{2(q+1)}$.

(4)″ If $x \in I_1$ we can find a $y \in E_1$ with $|x-y| < \epsilon$.

Proof. — Let

$$\begin{aligned} E_* &= E_1 \setminus \tilde{E}_1 \\ &= \left\{ x \in \tilde{E}_1 \,:\, 0 \in [-4M\gamma n^{-q}, 4M\gamma n^{-q}] + m_1 x + \sum_{j=2}^{q+1} m_j E_j \right\} \end{aligned}$$

and

$$\tau = n^{-1} \|g\|_1 \sum_{x \in E_*} \delta_x.$$

The main step of the proof involves finding an upper bound for $\hat{\tau}(r)$ which holds with high probability independent of the choice of γ.

First observe that, if we set $W_k = e^{irX_{1k}}$ when $X_{1k} \in E_*$ and $W_k = 0$ otherwise, then

$$\hat{\tau}(r) = \|g\|_1 \sum_{k=1}^{n} Z_k$$

the W_k satisfy the conditions of Theorem 7.1 (v).

Since X_{2k} has density function $g_2/\|g_2\|_1$ it follows that, to first order in δt

$$\Pr(m_2 X_{2k} \in [t, t+\delta t]) = G(t)\delta t$$

where G is differentiable density function with first and second derivatives bounded by some K_1 depending only on m_2 and g_2. Thus

$$\Pr(m_2X_{2k} \in [t, t+\delta t] \,:\, \text{for some } 1 \leqslant k \leqslant n)\} = nG(t)\delta t$$

to first order in δt and

$$\Pr\big((m_2E_2 + m_3E_3 + \cdots + m_{q+1}E_{q+1}) \cap [t, t+\delta t] \neq \emptyset\big) = nG * H(t)\delta t$$

for some H. We observe that $G * H$ is differentiable density function with first and second derivatives bounded by K_1. It follows that, if t is fixed

$$\Pr\big((t+m_2E_2+m_3E_3+\cdots+m_{q+1}E_{q+1})\cap[-4M\gamma n^{-q}, 4M\gamma n^{-q}] \neq \emptyset\big) = F_\gamma(t)$$

where F_γ has continuous first and second derivatives bounded by $\|F\|_\gamma^{-1}K_1$. Thus the density function G_γ of X_{11} given that

$$(m_1X_{11} + m_2E_2 + m_3E_3 + \cdots + m_{q+1}E_{q+1}) \cap [-4M\gamma n^{-q}, 4M\gamma n^{-q}] \neq \emptyset$$

has a continuous derivative bounded by K_2, where K_2 is independent both of γ and n.

We now have

$$|\mathbb{E}W_k| = \Pr(W_k \neq 0)\big|\mathbb{E}(W_k|W_k \neq 0)\big| = |\hat{G}_\gamma(r)| \leqslant \frac{K_2}{|r|}$$

for $r \neq 0$. Using Theorem 7.1 (v), we see that that, if take $B \geqslant 64(q+1)$ then provided we take n large enough, there is a probability at least 31/32 that

$$|\hat{\tau}(r)| \leqslant K_2|r|^{-1} + Bn^{-1/2}(\log n)^{1/2} \tag{1}$$

for all $1 \leqslant |r| \leqslant n^{2(q+1)}$.

Next we observe that

$$Pr(X_{1k} \in E_*) \leqslant n^q \times (8M\gamma n^{-q}) = 8M\gamma$$

so the expected number of points in E^* is no greater than $8M\gamma$. Since

$$y \Pr(Y \geqslant y) \leqslant \mathbb{E}Y,$$

it follows that given $\eta > 0$ (to be fixed later) we can choose γ so small that with probability at least 31/32 E^* contains at most ηn points and so

$$\|\tau\| \leqslant \eta\|g\|_1. \tag{2}$$

If we set

$$\mu = n^{-1}\|g\|_1 \sum_{x\in E_1} \delta_x,$$

the argument of Lemma 7.3 shows that, provided that n is large enough, then with probability at least 31/32,

$$|\hat{\mu}(r) - \hat{g}_1(r)| \leqslant Bn^{-1/2}(\log n)^{1/2} \tag{3}$$

for all $|r| \leqslant n^{2(q+1)}$. Since g_1 is continuously differentiable there exists a C such that $|\hat{g}_1(r)| \leqslant C|r|^{-1}$ for $r \neq 0$.

For the moment we suppose simply that $\eta \leqslant 1/2$. Since

$$\mu_1 = (\|g\|_1 - \|\tau\|)^{-1}(\mu - \tau)$$

it follows that, if (**1**), (**2**) and (**3**) hold

$$\begin{aligned}|\hat{\mu}_1(r)| + |\hat{g}_1(r)| &\leqslant 2(|\hat{\mu}(r)| + |\hat{\tau}(r)) + |\hat{g}_1(r)| \\ &\leqslant 2(|\hat{\mu}(r) - \hat{g}_1(r)| + |\hat{\tau}(r)) + 3|\hat{g}_1(r)| \\ &\leqslant 4Bn^{-1/2}(\log n)^{1/2} + \frac{2K_2 + C}{|r|}\end{aligned}$$

for all $|r| \leqslant n^{2(q+1)}$. Thus we can find N_0'' independent of η (provided $\eta < 1/2$) such that, if (**1**), (**2**) and (**3**), hold

$$|\hat{\mu}_1(r)| + |\hat{g}_1(r)| \leqslant 4^{-1}(q+1)^{-1}\phi(|r|)$$

for all $|r| \geqslant N_0''$.

Once N_0'' is fixed, we see that, provided only that η (and so γ) is taken sufficiently small, we will have

$$|\hat{\mu}_1(r) - \hat{\mu}(r)| \leqslant Bn^{-1/2}(\log n)^{1/2}$$

for all $|r| \leqslant N_0'$ and so

$$|\hat{\mu}_1(r) - \hat{g}_1(r)| \leqslant 2^{-1}(q+1)^{-1}\phi(|r|)$$

for all $|r| \leqslant N_0''$ whenever (**2**) (**3**) hold and n is sufficiently large.

Once γ is fixed, the weak law of large numbers tells us that, provided only that n is large enough, condition $(4)''$ will hold with probability at least 31/32. Thus, provided only that n is large enough (**1**), (**2**), (**3**) and $(4)''$ will hold simultaneously with probability at least 7/8 and imply the conclusions of the lemma. □

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