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ON THE AUTOMORPHISM GROUP OF STRONGLY PSEUDOCONVEX DOMAINS IN ALMOST COMPLEX MANIFOLDS

by Jisoo BYUN, Hervé GAUSSIER & Kang-Hyurk LEE (*)

Abstract. — In contrast with the integrable case there exist infinitely many non-integrable homogeneous almost complex manifolds which are strongly pseudoconvex at each boundary point. All such manifolds are equivalent to the Siegel half space endowed with some linear almost complex structure.

We prove that there is no relatively compact strongly pseudoconvex representation of these manifolds. Finally we study the upper semi-continuity of the automorphism group of some hyperbolic strongly pseudoconvex almost complex manifolds under deformation of the structure.

Résumé. — Contrairement au cas intégrable, il existe une infinité de variétés presque complexes homogènes, non intégrables, strictement pseudoconvexes en tout point de leur bord. De telles variétés sont équivalentes au demi-espace de Siegel muni d'une structure presque complexe linéaire.

Nous démontrons qu'il n'existe pas de représentation relativement compacte, strictement pseudoconvexe, de ces variétés. Enfin, nous étudions la semi-continuité du groupe des automorphismes de certaines variétés presque complexes hyperboliques, strictement pseudoconvexes, par déformation de la structure.

1. Introduction

The main purpose of the present paper is a structural description of automorphisms groups in almost complex manifolds. The study of pseudoholomorphic maps started with the work of A.Nijenhuis-W.Woolf [22] in which the authors proved the local existence of pseudoholomorphic curves for Hölderian almost complex structures and their stability under small deformations of the structure. Generically no nontrivial map f between two

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almost complex manifolds (M', J') and (M, J) satisfies the holomorphicity condition :

$$df \circ J' = J \circ df \tag{1.1}$$

since this is an overdetermined system. The local existence of pseudoholomorphic curves in [22] relies on the crucial fact that System (1.1), considered for $(M', J') = (\Delta, J_{st})$ (the unit disc in $\mathbb{C}$ endowed with the standard complex structure), is a small deformation of the classical elliptic Cauchy-Riemann equation $\bar{\partial} f = 0$. The precise regularity of a pseudoholomorphic curve (f is of class $\mathcal{C}^{k+1,\alpha}$ whenever J is of class $\mathcal{C}^{k,\alpha}$) also follows from the elliptic theory; S.Ivashkovich-J.P.Rosay [14] gave a transparent presentation of these facts.

In complex manifolds the unit ball is, up to biholomorphic equivalence, the only strongly pseudoconvex domain with noncompact automorphism group. This result known as the Wong-Rosay theorem is purely local, since one may localize the assumptions near a boundary accumulation point [27, 24, 23, 6]. Quite surprisingly strongly pseudoconvex homogeneous almost complex manifolds appeared as limits under a scaling process in [7] (some of their main properties were exhibited in [8, 19, 20]). It follows from [20] that these almost complex manifolds form a family of non equivalent models, in contrast with the complex setting. The first result of the paper is to explain this pathology by proving that there is no representation of a non-integrable model almost complex manifold as a relatively compact strongly pseudoconvex domain; model manifolds may be viewed as almost complex manifolds with a singularity at infinity. In particular Theorem 1.1 recovers the classical Wong-Rosay theorem for relatively compact domains in almost complex manifolds. In real dimension four this was proved in [8].

Theorem 1.1. — *Let D be a relatively compact domain in an almost complex manifold (M, J). If D is strongly J-pseudoconvex and if (D, J) is not equivalent to $(\mathbb{B}^n, J_{st})$, then the group* $\text{Aut}(D, J)$ *is compact.*

As usual two almost complex manifolds (M', J') and (M, J) are equivalent if there exists a biholomorphism f between (M', J') and (M, J), i.e. a diffeomorphism $f : M' \to M$ satisfying Condition (1.1). Here $\text{Aut}(D, J)$ denotes the set of automorphisms of (D, J), J_{st} denotes the standard complex structure on the Euclidean complex space $\mathbb{C}^n$ and $\mathbb{B}^n$ denotes the unit ball in $\mathbb{C}^n$.

The second result deals with the "upper semi-continuity" of automorphism groups in almost complex manifolds with boundary. R.Greene-

S.G.Krantz proved in [9, 10] that if D is a strongly pseudoconvex domain in $\mathbb{C}^n$, then the automorphism group of any smooth deformation of D is Lie isomorphic to a subgroup of the automorphism group of D. Independently, R.Hamilton proved in [12] that every smooth integrable deformation of a prescribed complex structure on a strongly pseudoconvex domain can be realized as a smooth deformation of the domain. The result in [9] may therefore be considered as a stability phenomenon under a deformation of the structure. Theorem 1.2 is a partial generalization of this result to almost complex manifolds :

Theorem 1.2. — *Let D be a $\mathcal{C}^\infty$ smooth relatively compact domain in a $\mathcal{C}^\infty$ almost complex manifold (M, J). Assume that (D, J) is hyperbolic, strongly pseudoconvex and not biholomorphic to $(\mathbb{B}^n, J_{st})$. Then for every $\mathcal{C}^\infty$ almost complex structure J' defined in a neighborhood of $\bar{D}$ and sufficiently close to J on $\bar{D}$ in the $\mathcal{C}^\infty$ convergence topology,* $\mathrm{Aut}(D, J')$ *is Lie isomorphic to a subgroup of* $\mathrm{Aut}(D, J)$.

The statement of Theorem 1.2 deserves some comments.

(a) The "$\mathcal{C}^\infty$ convergence topology" consists of the uniform convergence of all the partial derivatives on $\bar{D}$.

(b) The set of relatively compact domains may be endowed with the Hausdorff distance, measuring the distance between the boundaries of two given domains. Theorem 1.2 is no more true for this distance, even in the complex setting. Indeed, one can create any finite group of order less than or equal to the complex dimension of the manifold as the automorphism group of sufficiently small deformations of a given domain, see [3]. However, the dimension of the automorphism group is an upper semi-continuous function for this distance, see [4].

(c) Under the assumptions of Theorem 1.2, (D, J') is hyperbolic for a small deformation of J, as shown by B.Kruglikov-M.Overholt in [17]. In particular the automorphism group $\mathrm{Aut}(D, J')$ is a real Lie group by standard arguments.

(d) The study of the automorphism group consists of two cases. The first case deals with a domain D with compact automorphism group, considered in Theorem 1.2. The second case deals with a domain D with noncompact automorphism group. In view of Theorem 1.1 and of the Fefferman extension theorem (see [8]), $(\bar{D}, J)$ is equivalent to $(\bar{\mathbb{B}}^n, J_{st})$. The situation may be reduced to considering a smooth deformation J' of the standard structure J_{st} on $\bar{\mathbb{B}}^n$. Generically $\mathrm{Aut}(\mathbb{B}^n, J')$ is reduced to the identity. If $\mathrm{Aut}(\mathbb{B}^n, J')$ is noncompact, then Theorem 1.1 means that $(\mathbb{B}^n, J')$ is also equivalent to $(\mathbb{B}^n, J_{st})$ so that $\mathrm{Aut}(\mathbb{B}^n, J')$ is Lie isomorphic to $\mathrm{Aut}(\mathbb{B}^n, J_{st})$. In case $\mathrm{Aut}(\mathbb{B}^n, J')$ is compact, the question concerns the classification of compact (pseudo)holomorphic groups acting on the sphere. This will be treated in a forecoming paper.

The third result establishes the upper semi-continuity of the isotropy group $\mathrm{Aut}_q(D, J) := \{f \in \mathrm{Aut}(D, J) : f(q) = q\}$ of $q \in D$:

THEOREM 1.3. — *Let D be a $\mathcal{C}^\infty$ smooth relatively compact domain in a $\mathcal{C}^\infty$ almost complex manifold (M, J). Assume that (D, J) is hyperbolic, strongly pseudoconvex. Then for every point $q \in D$ and for every $\mathcal{C}^\infty$ almost complex structure J' defined in a neighborhood of $\bar{D}$ and sufficiently close to J on $\bar{D}$ in the $\mathcal{C}^\infty$ convergence topology, the isotropy group $\mathrm{Aut}_q(D, J')$ is Lie isomorphic to a subgroup of $\mathrm{Aut}_q(D, J)$.*

The paper is organized as follows. Section 2 is devoted to the stability of (complete) hyperbolicity under deformation of the structure. This is related to a question by S.Kobayashi concerning the hyperbolicity in infinite dimensional fiber bundles. Section 3 gathers some results concerning exotic almost complex manifolds called *model domains*. The crucial point is the non existence of a smooth relatively compact realization of these manifolds (Theorem 3.7). Theorem 1.1 is a consequence of Theorem 3.7 and of previous results (see [8, 19]). Section 4 is devoted to the proofs of Theorem 1.2 and Theorem 1.3, obtained as consequences of the results presented in the preceding Sections.

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2. Stability of the complete (Kobayashi) hyperbolicity

The flexibility of almost complex structures fits to many problems involving deformation. However the game plays in proving the stability of some geometric objects after the deformation; this was an essential part in [11] where M.Gromov proved the persistence of pseudoholomorphic curves after deformation of an almost complex structure in a symplectic manifold. One natural question when working with holomorphic curves in almost complex manifolds concerns the stability of (Kobayashi) hyperbolicity and complete hyperbolicity. Indeed, thanks to the local existence of pseudoholomorphic discs, one may define the *Kobayashi-Royden pseudonorm* $K_{(M,J)}$ in (M,J) for a Hölderian structure J :

Definition 2.1. — *For every $p \in M$ and for every $v \in T_pM$ set :*

$$K_{(M,J)}(p,v) = \inf\Big\{\alpha > 0 \ / \ \exists f : (\Delta, J_{st}) \to (M,J),\ f(0) = p,\\ df(0)(\partial/\partial x) = v/\alpha\Big\}.$$

The upper semi-continuity of $K_{(M,J)}$, proved by H.L.Royden [25] in complex manifolds, relies on the persistence of holomorphic discs under perturbation of the parameters p and v. This stability result is proved in the almost complex setting by B.Kruglikov [16] for smooth $\mathcal{C}^\infty$ structures and by S.Ivashkovich-J.P.Rosay [14] for $\mathcal{C}^{1,\alpha}$ structures. Finally, the upper semi-continuity fails for Hölderian structures; S.Ivashkovich-S.Pinchuk-J.P.Rosay [13] gave an example of a disc that cannot be deformed. By analogy with complex manifolds, the Kobayashi pseudodistance may be defined by integration of the Kobayashi-Royden pseudonorm :

Definition 2.2. —

(i) *For every $x, y \in M$ the* Kobayashi pseudodistance *between x and y is given by*

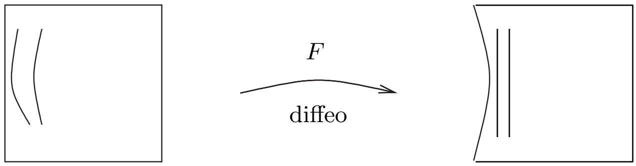
$$d_{(M,J)}(x,y) = \inf\left\{\int_0^1 K_{(M,J)}(\gamma(t),\gamma'(t))dt\right\},$$

where the infimum is taken over all $\mathcal{C}^1$ paths γ joining x and y.

(ii) *(M,J) is* (Kobayashi) hyperbolic *if $d_{(M,J)}$ is a distance (this will induce the usual topology on M)*

(iii) *(M,J) is* complete hyperbolic *if the metric space $(M, d_{(M,J)})$ is complete.*

It turns out that these notions may not be stable under almost complex deformations of the structure. For instance, in the Euclidean complex space $\mathbb{C}^n$, the unit polydisc $\Delta^n = \Delta \times \cdots \times \Delta$ is complete hyperbolic since the Kobayashi metric is the infimum of the Poincaré metric on each unit disc. However let D be a small pseudoconcave deformation of the polydisc, given by some diffeomorphism F in a neighborhood of Δ^n. Then (D, J_{st}) is no more complete hyperbolic since this is not taut (see Figure 2.1). The



$(\Delta^n, F^*(J_{st}))$ (D, J_{st})

Figure 2.1.

(almost) complex structure $F^*(J_{st})$, pullback of J_{st} on Δ^n, is an arbitrary small deformation of J_{st} but $(\Delta^n, F^*(J_{st}))$ is not complete hyperbolic.

The example relies on the non pseudoconvexity of the considered deformation. A more natural question consists in studying such a stability among pseudoconvex deformations. For instance, in the complex Euclidean space $\mathbb{C}^n$, every smooth deformation of a strongly pseudoconvex domain is complete hyperbolic. In the almost complex setting, the situation is different since there exist non hyperbolic strongly pseudoconvex domains. More precisely, according to [14] (Theorem 1), a strongly pseudoconvex domain is either complete hyperbolic or contains a non constant complex line. Moreover such a line is necessarily relatively compact in D, this last situation appearing in complex projective spaces after the blow-up of a point.

Definition 2.3. —

(i) *A $\mathcal{C}^2$ real valued function u on M is* strongly J-plurisubharmonic *on M if $\mathcal{L}^J(u)(p)(v)$ is positive for every $p \in M$, $v \in T_pM \setminus \{0\}$. Here $\mathcal{L}^J(u)$ is the* Levi form *defined by*

$$\mathcal{L}^J(u)(p)(v) = -d(J^*du)(v, Jv)\ .$$

(ii) *A smooth $\mathcal{C}^2$ domain D in M is* strongly J-pseudoconvex *if for every point $p \in \partial D$ there is a neighborhood $p \in U \subset M$ and a smooth $\mathcal{C}^2$ function ρ defined and strongly J-plurisubharmonic on U such that $d\rho \neq 0$ on U and $D \cap U = \{\rho < 0\}$.*

The stability of the Kobayashi hyperbolicity of compact manifolds, under the deformation of the almost complex structure, was achieved by B.Kruglikov-M.Overholt in [17] for smooth $\mathcal{C}^\infty$ structures. This can be carried out for $\mathcal{C}^{1,\alpha}$ structures using a renormalization lemma, first used by R.Brody in [2] (see also [21, 28]), and whose essence goes back to the work of E.Landau [18] . Moreover it gives a positive answer to a question studied by S.Kobayashi (see [15]). Let M be a compact manifold and let $\mathcal{J}$ be the set of almost complex structures on M. Consider the infinite dimensional fiber bundle $M \times \mathcal{J}$. Given a hyperbolic fiber $M \times \{J_0\} \subset M \times \mathcal{J}$ there exists a neighborhood (for the $\mathcal{C}^{1,\alpha}$ topology) of the fiber consisting of hyperbolic fibers. The proof of this stability result may also be combined with the results in [7] to prove that a complete hyperbolic strongly pseudoconvex domain in an almost complex manifold remains complete hyperbolic after a small $\mathcal{C}^{1,\alpha}$ almost complex deformation of the structure on the closure of the domain.

An interesting question concerns the stability of hyperbolicity under different convergences. In particular :

> *Does there exist a sequence $(J_\nu)_\nu$ of (almost) complex structures defined in a fixed neighborhood of the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, that converge to J_{st} in some $\mathcal{C}^k$ sense on compat subsets of $\mathbb{B}^n$, such that $(\mathbb{B}^n, J_\nu)$ is not hyperbolic?*

We thank the referee for mentioning this and for the following improvement of a previous construction. This gives a partial answer to the question.

Example 2.4. — There is a sequence of complex structures $(J_\nu)_\nu$ converging to the standard structure J_{st}, in the $\mathcal{C}^1$ convergence on compact subsets of the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, such that $(\mathbb{B}^n, J_\nu)$ is not hyperbolic.

One may construct J_ν as follows. Let (r_ν) be a sequence of real numbers such that $0 < r_\nu < 1$ and $\lim_{\nu\to\infty} r_\nu = 1$. For every ν consider a diffeomorphism Φ_ν from $\mathbb{B}^n$ to $\mathbb{C}^n$, equal to identity on the ball centered at the origin with radius r_ν. Set $J_\nu := \Phi_\nu^*(J_{st}) = (d\Phi_\nu)^{-1} \circ J_{st} \circ d\Phi$. Then for every integer ν, the almost complex manifold $(\mathbb{B}^n, J_\nu)$ is not hyperbolic and $(J_\nu)_\nu$ converges to J_{st} for the compact open topology on $\mathbb{B}^n$.

3. Model almost complex manifolds

This section is devoted to the study of some almost complex structures, introduced in [7] and studied in [8, 19, 20].

The scaling process, mainly introduced in complex analysis by S.Pinchuk ([23]), reflects the local geometry of a given domain. This emphasizes for instance the osculation of a strongly pseudoconvex domain by spheres. The most striking fact in almost complex manifolds is the convergence of the associated dilated almost complex structures to "model structures", owing particular properties. To present them we first consider a smooth $\mathcal{C}^{1,\alpha}$ almost complex structure J, defined on a smooth $2n$ real dimensional manifold, as a $\mathcal{C}^{1,\alpha}$ almost complex deformation of the standard structure J_{st} on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ (see [7]). Hence we assume that J is defined on $\mathbb{B}^n$ and satisfies $J(0) = J_{st}$. Denoting by $z = (z_1, \dots, z_n)$ the standard coordinates of $\mathbb{C}^n$, we have the following matricial expansion of J :

$$J(z) = \begin{pmatrix} J_{st}^{(n-1)} + \mathcal{O}(\|z\|) & \mathcal{O}(\|z\|) \\ \sum\limits_{j=1}^{n} \left(C_j z_j + \overline{C}_j \bar{z}_j\right) + \mathcal{O}(\|z\|^2) & J_{st}^{(1)} + \mathcal{O}(\|z\|) \end{pmatrix}$$

where $J_{st}^{(1)}$ and $J_{st}^{(n-1)}$ are matricial representations of the standard complex structures of $\mathbb{C}^1$ and $\mathbb{C}^{n-1}$ respectively, $C_j z_j$ denotes the multiplication of each component of a $2 \times (2n-2)$ complex matrix C_j by z_j, and $\mathcal{O}(\|z\|)$, $\mathcal{O}(\|z\|^2)$ denote matrices defined in the appropriate Euclidean complex vector space. Now consider the non-isotropic dilation Λ_τ $(\tau > 0)$ defined on $\mathbb{C}^n$ by :

$$\Lambda_\tau(z) = \left(\frac{'z}{\sqrt{\tau}}, \frac{z_n}{\tau}\right) , \tag{3.1}$$

where $('z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Then as $\tau \to 0$, the push-forward almost complex structure $J_\tau := (\Lambda_\tau)_*(J) = d\Lambda_\tau \circ J \circ d(\Lambda_\tau)^{-1}$ converges to J_C uniformly on any compact subset of $\mathbb{C}^n$, where J_C denotes the almost complex structure globally defined on $\mathbb{C}^n$ by their matricial representation :

$$J_C = \begin{pmatrix} J_{st}^{(n-1)} & 0 \\ \sum\limits_{j=1}^{n-1} \left(C_j z_j + \overline{C}_j \bar{z}_j\right) & J_{st}^{(1)} \end{pmatrix} . \tag{3.2}$$

One can indeed easily check that $J_C^2 = -I$. For instance, differentiating the identity $J^2 = -I$ in the z_j-direction at 0, one gets the equality $J_{st}^{(1)} \cdot C_j + C_j \cdot J_{st}^{(n-1)} = 0$ for $j = 1, \dots, n-1$. This is equivalent to the condition $J_C^2 = -I$.

For a $(n-1) \times (n-1)$ skew-symmetric complex matrix $\mathsf{B} = (b_{jk})$, define the *model structure* J_{B} on $\mathbb{C}^n$ by its complexification $\tilde{J}_{\mathsf{B}}$ satisfying on $\mathbb{C}T(\mathbb{C}^n)$:

$$\tilde{J}_{\mathsf{B}}\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j} + \sum_{k=1}^{n-1} b_{jk} z_k \frac{\partial}{\partial \bar{z}_n} , \quad \tilde{J}_{\mathsf{B}}\left(\frac{\partial}{\partial \bar{z}_j}\right) = -i\frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{n-1} \bar{b}_{jk} \bar{z}_k \frac{\partial}{\partial z_n} ,$$

for $j = 1, \dots, n-1$ and

$$\tilde{J}_{\mathsf{B}}\left(\frac{\partial}{\partial z_n}\right) = i\frac{\partial}{\partial z_n} , \quad \tilde{J}_{\mathsf{B}}\left(\frac{\partial}{\partial \bar{z}_n}\right) = -i\frac{\partial}{\partial \bar{z}_n} .$$

Then for every dilated structure J_C given by the identity (3.2), there is a biholomorphism from $(\mathbb{C}^n, J_C)$ to $(\mathbb{C}^n, J_{\mathsf{B}})$ for some model structure J_{B} (see Proposition 3.4 in [20]).

Definition 3.1. — *Let* $\mathbb{H}$ *be the Siegel half space*

$$\mathbb{H} := \left\{ z = ('z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Re}\,(z_n) + \|'z\|^2 < 0 \right\} .$$

The pair $(\mathbb{H}, J_{\mathsf{B}})$ *is called a* model domain.

The main properties of a model domain $(\mathbb{H}, J_{\mathsf{B}})$, essentially studied in [19, 20], are summarized in the following proposition :

Proposition 3.2. —

(i) *A model structure* J_{B} *is integrable if and only if* $\mathsf{B} = 0$,
(ii) *The model almost complex manifold* $(\mathbb{H}, J_{\mathsf{B}})$ *is hyperbolic and strongly pseudoconvex.*

The classical Wong-Rosay Theorem (see [27, 24, 23, 6]) states that in a complex manifold of complex dimension n, a domain D is biholomorphic to $\mathbb{B}^n$ if there is an automorphism orbit accumulating at a strongly pseudoconvex boundary point.

This is no more valid in almost complex manifolds. Indeed the dilations Λ_τ as in (3.1) belong to $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$ for every B and for every positive constant τ, and accumulate at the origin when $\tau \to \infty$. The origin is a point of strong pseudoconvexity by Statement (ii) of Proposition 3.2 and J_{B} is generically not integrable by Statement (i) of Proposition 3.2. Model domains are, up to biholomorphism, the only almost complex manifolds presenting such a pathology :

Theorem 3.3 ([8, 19]). — *Let* (M, J) *be an almost complex manifold. If a domain* D *in* M *admits an automorphism orbit accumulating at a strongly* J*-pseudoconvex boundary point, then* (D, J) *is biholomorphic to a model domain* $(\mathbb{H}, J_{\mathsf{B}})$ *for some* B.

3.1. Automorphism groups of model domains

Here is a brief description of the automorphisms of model domains. A complete description is given in [20].

Let $(\mathbb{H}, J_{\mathsf{B}})$ be a model domain. For any points $\zeta = ('\zeta, \zeta_n), \xi = ('\xi, \xi_n) \in \mathbb{C}^n$, define a binary operation $*_{\mathsf{B}}$ by :

$$\zeta *_{\mathsf{B}} \xi = ('\zeta + '\xi,\ \zeta_n + \xi_n - 2\,\langle '\xi, '\zeta\rangle_{\mathbb{C}} + i\,\mathrm{Re}\ \mathsf{B}('\xi, '\zeta))\ ,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is the standard hermitian inner product of $\mathbb{C}^{n-1}$, $\omega(\cdot,\cdot) = Im\ \langle\cdot,\cdot\rangle_{\mathbb{C}}$ is the standard symplectic form and $\mathsf{B}('\xi, '\zeta) = \sum_{j,k=1}^{n-1} b_{jk}\xi_j\zeta_k$ is a skew-symmetric bilinear form of $\mathbb{C}^{n-1}$. Then the boundary $\partial\mathbb{H}$ is closed under this operation so that $H_{\mathsf{B}} = (\partial\mathbb{H}, *_{\mathsf{B}})$ is a Lie group. Note that H_0 is the usual Heisenberg group.

One can check that for each $\zeta \in \partial\mathbb{H}$, the map $\Psi_{\mathsf{B}}^{\zeta}(z) := \zeta *_{\mathsf{B}} z$ belongs to $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$. Since $\Psi_{\mathsf{B}}^{\zeta} \circ \Psi_{\mathsf{B}}^{\xi}(z) = \Psi_{\mathsf{B}}^{\zeta *_{\mathsf{B}} \xi}(z)$, the group H_{B} can be identified as a subgroup of $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$.

Theorem 3.4. — *The automorphism group of every model domain admits the following decomposition :*

$$\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}}) = \mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}, J_{\mathsf{B}}) \circ \mathcal{D} \circ H_{\mathsf{B}}$$

where $\mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}, J_{\mathsf{B}})$ *is the isotropy group of* $-\mathbf{1} = (0,\dots,0,-1)$ *and* $\mathcal{D} = \{\Lambda_\tau : \tau > 0\}$. *If* $\mathsf{B} \neq 0$, *then*

$$\mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}, J_{\mathsf{B}}) = \{\Phi_A(z) = (A('z), z_n) : A^t\mathsf{B}A = \mathsf{B} \text{ and } A \in \mathrm{U}(n-1)\}\ ,$$

where $A('z)$ *is a complex linear transformation of* $\mathbb{C}^{n-1}$ *generated by* A.

The homogeneity of $(\mathbb{H}, J_{\mathsf{B}})$ is a consequence of the transitivity of the action of $\mathcal{D} \circ H_{\mathsf{B}}$. One difference between the integrable model $(\mathbb{H}, J_{st})$ and a non-integrable model $(\mathbb{H}, J_{\mathsf{B}})$, that is for $\mathsf{B} \neq 0$, concerns their isotropy group. The isotropy group of $(\mathbb{H}, J_{st})$ is isomorphic to the unitary group $\mathrm{U}(n)$ whereas the isotropy group of $(\mathbb{H}, J_{\mathsf{B}})$ is a subgroup of the group $\mathrm{U}(n-1)$ for $\mathsf{B} \neq 0$. The following corollary will play a crucial rôle in the proof of Theorem 1.1.

Corollary 3.5. — *If* $\mathsf{B} \neq 0$, *then every automorphism of* $(\mathbb{H}, J_{\mathsf{B}})$ *extends smoothly to* $\partial\mathbb{H}$ *and preserves* $\partial\mathbb{H}$.

When B tends to 0, the structure J_{B} converges to J_{st} in local $\mathcal{C}^\infty$ sense. The Lie group structure of $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$ with respect to $\mathrm{Aut}(\mathbb{H}, J_{st})$ is particularly interesting when studying the upper semi-continuity of the automorphism groups. One can easily see that the isotropy group $\mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}, J_{\mathsf{B}})$ is a Lie subgroup of $\mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}, J_{st}) \simeq \mathrm{U}(n)$. And the transitive subgroup $\mathcal{D} \circ H_{\mathsf{B}}$ of $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$ is also Lie isomorphic to that of $\mathrm{Aut}(\mathbb{H}, J_{st})$ by the following proposition. But it is not yet clear whether $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$ is Lie group isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{H}, J_{st})$ or not.

Proposition 3.6. —

(i) *The skew-symmetric bilinear form*

$$\omega_{\mathsf{B}}('\xi, '\zeta) = -2\omega('\xi, '\zeta) + \mathrm{Re}\ \mathsf{B}('\xi, '\zeta)$$

is non-degenerate, i.e. a real symplectic form on the vector space $\mathbb{C}^{n-1}$.

(ii) *If* B *and* $\tilde{\mathsf{B}}$ *are two complex skew-symmetric matrices, then* H_{B} *and* $H_{\tilde{\mathsf{B}}}$ *are Lie isomorphic.*

Proof. — Since Statement (i) is a direct computation let us focus on the proof of Statement (ii). For a real transformation A of $\mathbb{C}^{n-1}$, define the map $h_A : \partial\mathbb{H} \to \partial\mathbb{H}$ by

$$h_A(\zeta) = \left(A('\zeta), -\|A('\zeta)\|^2 + iIm\,\zeta_n\right) .$$

Given two complex skew-symmetric matrices B and $\tilde{\mathsf{B}}$, there is a real linear transformation A such that $\omega_{\mathsf{B}}('\xi, '\zeta) = \omega_{\tilde{\mathsf{B}}}(A('\xi), A('\zeta))$. Then h_A is a Lie group isomorphism from H_{B} to $H_{\tilde{\mathsf{B}}}$. Indeed :

$$\begin{aligned}
&h_A(\zeta) *_{\tilde{\mathsf{B}}} h_A(\xi)\\
&= \left(A('\zeta), -\|A('\zeta)\|^2 + iIm\,\zeta_n\right) *_{\tilde{\mathsf{B}}} \left(A('\xi), -\|A('\xi)\|^2 + iIm\,\xi_n\right)\\
&= \Big(A('\zeta) + A('\xi), -\|A('\zeta) + A('\xi)\|^2\\
&\qquad\qquad\qquad + i\left(Im\,\zeta_n + Im\,\xi_n + \omega_{\tilde{\mathsf{B}}}(A('\xi), A('\zeta))\right)\Big)\\
&= \left(A('\zeta + '\xi), -\|A('\zeta + '\xi)\|^2 + i\left(Im\,\zeta_n + Im\,\xi_n + \omega_{\mathsf{B}}('\xi, '\zeta)\right)\right)\\
&= h_A(\zeta *_{\mathsf{B}} \xi) .
\end{aligned}$$

So H_{B} and $H_{\tilde{\mathsf{B}}}$ are Lie isomorphic. □

3.2. Relatively compact representation of model domains

Non-integrable model almost complex manifolds may be viewed as degenerate in the following sense. The Cayley transform $('z, z_n) \mapsto (2'z/(z_n - 1), (z_n + 1)/(z_n - 1))$ transforms $\overline{\mathbb{H}} \cup \{\infty\}$ biholomorphically onto $\bar{\mathbb{B}}^n$. This is a particularity of the standard complex structure. One can indeed prove that there is no relatively compact strongly pseudoconvex realization of $(\mathbb{H}, J_{\mathsf{B}})$ unless J_{B} is integrable :

Theorem 3.7. — *Let* $(\mathbb{H}, J_{\mathsf{B}})$ *be a model almost complex manifold and let* D *be a relatively compact, strongly pseudoconvex domain in an almost complex manifold* (M, J). *There is no biholomorphism from* (D, J) *to* $(\mathbb{H}, J_{\mathsf{B}})$ *unless* J_{B} *is integrable.*

The idea of the proof is the following. Suppose that the model structure J_{B} is non-integrable and that there exists a biholomorphism $F : (\mathbb{H}, J_{\mathsf{B}}) \to (D, J)$. Corollary 3.5 implies that there is no automorphism $\Phi \in \mathrm{Aut}(D, J)$ such that :

$$\Phi(p) \in F(\partial\mathbb{H}) \quad \text{for some } p \in \partial D \setminus F(\partial\mathbb{H}). \tag{3.3}$$

Using the homogeneity of $\mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}})$ we construct an automorphism orbit in $\mathbb{H}$ whose image by F accumulates at a point $p \in \partial D \setminus F(\partial\mathbb{H})$. By a scaling procedure centered at p, this orbit generates a biholomorphism $S : (D, J) \to (\mathbb{H}, J_{\mathsf{B}'})$ satisfying $S(p) = 0$. Moreover there exists an automorphism $\Psi \in \mathrm{Aut}(\mathbb{H}, J_{\mathsf{B}'})$ such that $\Psi(0)$ belongs to $\partial\mathbb{H} \cap S(F(\partial\mathbb{H}))$. Then the automorphism $S^{-1} \circ \Psi \circ S$ of (D, J) satisfies (3.3). This makes a contradiction. See Figure 3.1.

Proof. — Let D be a relatively compact domain, strongly pseudoconvex in an almost complex manifold (M, J) and let F be a biholomorphism from a model domain $(\mathbb{H}, J_{\mathsf{B}})$ to (D, J). By the Fefferman theorem proved in [8], F has a smooth extension on $\overline{\mathbb{H}}$. Let us consider the restriction $F_{|\mathbf{H}}$ of F to the half plane $\mathbf{H} = \{('z, z_n) \in \mathbb{H} : 'z = 0\}$. Since $\partial\mathbf{H} \subset \partial\mathbb{H}$, $F_{|\mathbf{H}}$ smoothly extends on $\partial\mathbf{H}$. In order to consider the extension of $F_{|\mathbf{H}}$ at infinity along $\mathbf{H}$, we need the following lemma (whose proof is postponed to the end of this section).

Lemma 3.8. — *There is a point $p \in \partial D$ such that*

$$\lim_{\substack{Z \in \mathbf{H} \\ \|Z\| \to \infty}} F(Z) = p\,.$$

Consider now the J_{B}-holomorphic disc $u : \Delta \to \mathbb{H}$ defined by

$$u(\zeta) = \left(0, \dots, 0, \frac{\zeta - 1}{\zeta + 1}\right).$$

Since $u(\partial\Delta \setminus \{-1\}) = \partial\mathbf{H}$, the J-holomorphic disc $f = F \circ u$ has a smooth extension to $\overline{\Delta} \setminus \{-1\}$. Moreover by Lemma 3.8, f extends continuously on $\overline{\Delta}$ and $f(-1) = p$. Hence f has a smooth extension on $\overline{\Delta}$. By the strong J-pseudoconvexity of ∂D, the set $f(\Delta) = F(\mathbf{H})$ meets ∂D transversally at p. Therefore there exists a coordinate system $z : U \ni p \to \mathbb{C}^n$ such that

(1) $z(p) = 0$ and $z_* J(0) = J_{st}$,
(2) the hyperplane $\{\mathrm{Re}\, z_n = 0\}$ is tangent to $z(\partial D \cap U)$ at 0,
(3) $z \circ f(\zeta) = (0, \dots, 0, \frac{\zeta+1}{\zeta-1})$ for ζ sufficiently close to -1.

Consider now the automorphism $\Phi := F \circ \Lambda_{2^{-1}} \circ F^{-1}$ of (D, J) where $\Lambda_{2^{-1}}$ is the dilation Λ_τ defined in (3.1), with $\tau = 2^{-1}$. The iterated sequence

$(\Phi^k := \Phi \circ \Phi^{k-1} = F \circ \Lambda_{2^{-k}} \circ F^{-1})_k$ generates an automorphism orbit $(p_k)_k$ that accumulates at p since :

$$p_k := \Phi^k(F(0,\dots,0,-1)) = F(0,\dots 0,-2^k)\ .$$

Notice that $p_k = f((1-2^k)/(1+2^k))$ and $\Phi^j(p_k) = p_{j+k}$ for all integers j,k. By the construction of z it follows that

$$z(p_k) = z \circ f\left(\frac{1-2^k}{1+2^k}\right) = (0,\dots,0,-2^{-k})$$

and $\operatorname{dist}(z(p_k), z(\partial D \cap U)) = \operatorname{dist}(z(p_k), 0) = 2^{-k}$ for large k. Here dist denotes the Euclidean distance. Let $(S^j := \Lambda_{2^{-j}} \circ z \circ \Phi^j)_j$ be a sequence of dilations. According to the scaling method ([7]) there is a subsequence of $(S^j)_j$ that converges to a biholomorphism $S : (D,J) \to (\mathbb{H}, J_{\mathsf{B}'})$ for some B'. Moreover by the definitions of S^j and of p_k :

(3.4)
$$S^j(p_k) = \Lambda_{2^{-j}} \circ z(p_{k+j}) = \Lambda_{2^{-j}}(0,\dots,0,-2^{-k-j}) = (0,\dots,0,-2^{-k})$$

for every sufficiently large j and k.

Passing to the limit when $j \to \infty$ in Equation (3.4), we obtain that $S(p_k) = (0,\dots,0,-2^{-k})$ for every k. Since $(p_k)_k$ converges to p, the continuous extension of S maps p to 0. By Theorem 3.4 there is $\Psi \in \operatorname{Aut}(\mathbb{H}, J_{\mathsf{B}'})$ such that $\Psi(0)$ is a point in $\partial\mathbb{H} \cap S(F(\partial\mathbf{H}))$. Therefore :

$$S^{-1} \circ \Psi \circ S \in \operatorname{Aut}(D,J) \quad \text{and} \quad S^{-1} \circ \Psi \circ S(p) \in F(\partial \mathbf{H})$$

(see Figure 3.1). Since $F(\infty) = p$ by Lemma 3.8, one get $F^{-1} \circ S^{-1} \circ \Psi \circ S \circ F \in \operatorname{Aut}(\mathbb{H}, J_{\mathsf{B}'})$ and $(F^{-1} \circ S^{-1} \circ \Psi \circ S \circ F)(\infty) \in \partial\mathbf{H}$. Therefore $\mathsf{B} = 0$ by Corollary 3.5. This proves Theorem 3.7. ☐

The Wong-Rosay Theorem (Theorem 1.1) is a corollary of Theorem 3.7.

Proof of Theorem 1.1. Assume by contradiction that $\operatorname{Aut}(D,J)$ is noncompact. It follows from Theorem 3.3 that there is a biholomorphism between (D,J) and $(\mathbb{H}, J_{\mathsf{B}})$. According to Theorem 3.7, B vanishes identically, meaning that $J_{\mathsf{B}} = J_{st}$. ☐

We conclude this section with the proof of Lemma 3.8.

Proof of Lemma 3.8. Let $Z = ('0, z_n)$ be a point of $\mathbf{H}$ and $V = ('0, v_n)$ be a tangent vector to $\mathbf{H}$ at Z. Since $\mathbf{H}$ is contained in $\mathbb{H}$, the decreasing property of the Kobayashi-Royden pseudonorm implies that

$$K_{(\mathbb{H}, J_{\mathsf{B}})}(Z,V) \leqslant K_{(\mathbf{H}, J_{\mathsf{B}})}(Z,V)\ .$$

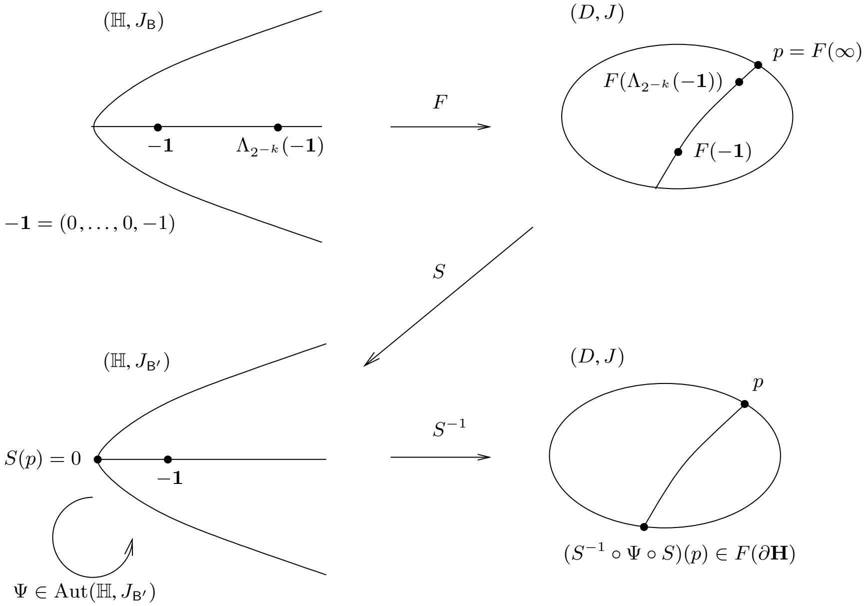


Figure 3.1.

Since $J_{\mathsf{B}} = J_{st}$ on $\mathbf{H}$, there is a positive constant c such that

$$K_{(\mathbb{H}, J_{\mathsf{B}})}(Z, V) \leqslant c \frac{|v_n|}{|\operatorname{Re}(z_n)|}\,, \tag{3.5}$$

for every $Z = ('0, z_n) \in \mathbf{H}$ and every $V = ('0, v_n) \in \mathbb{C}^n$.

Consider now the subharmonic function φ defined on the half plane $H_0 = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\} \subset \mathbb{C}$ by

$$\varphi(\lambda) = \left| \frac{\lambda + 1}{\lambda - 1} \right|^2 - 1\,.$$

The function φ satisfies the following inequalities on H_0 :

$$|\varphi(\lambda)| \leqslant \frac{4}{|\lambda|} \tag{3.6}$$

and, if $|\lambda| \geqslant 1$ and $-1 \leqslant \operatorname{Re}(\lambda) < 0$:

$$|\varphi(\lambda)| \leqslant 4 \frac{|\operatorname{Re}(\lambda)|^{1/2}}{|\lambda|^{3/2}}\,. \tag{3.7}$$

Let dist denote a Riemannian distance on M. According to [8] there is a positive constant c' such that the Kobayashi-Royden pseudonorm satisfies

$$K_{(D,J)}(F(Z), dF(Z)(V)) \geqslant c' \frac{\|dF(Z)(V)\|}{\operatorname{dist}(F(Z), \partial D)^{1/2}}\,. \tag{3.8}$$

The decreasing property of the Kobayashi-Royden pseudonorm under holomorphic maps implies the following inequalities :

$$\begin{aligned}\frac{\|dF(Z)(V)\|}{\operatorname{dist}(F(Z),\partial D)^{1/2}} &\leqslant \frac{1}{c'}K_{(D,J)}(F(Z),dF(Z)(V))\\ &= \frac{1}{c'}K_{(\mathbb{H},J_{\mathrm{B}})}(Z,V)\\ &\leqslant \frac{c}{c'}\frac{\|V\|}{|\operatorname{Re}(z_n)|}\ ,\end{aligned}$$

the last inequality being given by inequality (3.5). Hence :

$$\|dF(Z)(V)\| \leqslant \frac{c}{c'}\ \frac{\operatorname{dist}(F(Z),\partial D)^{1/2}}{|\operatorname{Re}(z_n)|}\|V\|\ . \tag{3.9}$$

Let $\Gamma := F(\mathbf{H})$. This is a one-dimensional proper analytic set in D. By the Hopf Lemma, this is transversal to ∂D. More precisely, there exist two positive constants C_1 and C_2 such that

$$C_1\operatorname{dist}(F(Z),\partial D) \leqslant \operatorname{dist}(F(Z),\partial\Gamma) \leqslant C_2\operatorname{dist}(F(Z),\partial D)\ . \tag{3.10}$$

It follows from the inequalities (3.9) and (3.10) :

$$\|dF(Z)(V)\| \leqslant \frac{c}{c'\sqrt{C_1}}\frac{\operatorname{dist}(F(Z),\partial\Gamma)^{1/2}}{|\operatorname{Re}(z_n)|}\|V\|\ . \tag{3.11}$$

Finally, the function $\varphi\circ F^{-1}_{|\Gamma}$ being subharmonic and negative on Γ, it follows from the Hopf Lemma that there is a positive constant C_3 such that $\operatorname{dist}(F(Z),\partial\Gamma) \leqslant C_3|(\varphi\circ F^{-1})(F(Z))|$. So from inequalities (3.11) and (3.6) one has :

$$\|dF(Z)(V)\| \leqslant \frac{2c\sqrt{C_3}}{c'\sqrt{C_1}}\ \frac{|v_n|}{|\operatorname{Re}(z_n)||z_n|^{1/2}}\ .$$

It follows now from a classical integration argument (see [1] p.145) that $\lim_{\|Z\|\to\infty} F(Z) = p$ on the set $\{Z\in\mathbf{H} : \operatorname{Re}(z_n)\leqslant -1\}$.

Let now $Z\in\mathbf{H}$ be such that $-1<\operatorname{Re}(z_n)<0$ and let γ be the path defined on $[0,1]$ by $\gamma(t)=(1-t)Z+tW$ for $W := ('0,-1+iIm(z_n))$. It follows from inequalities (3.11) and (3.7) :

$$\|dF(\gamma(t))(\gamma'(t))\| \leqslant \frac{2c\sqrt{C_3}}{c'\sqrt{C_1}}\frac{|\gamma_n'(t)|}{|\operatorname{Re}(\gamma_n(t))|^{3/4}|\gamma_n(t)|^{3/4}}\quad \text{for all } t\in[0,1].$$

In particular we have :

$$\begin{aligned}\|F(W)-F(Z)\| &= \int_0^1 \|dF(\gamma(t))(\gamma'(t))\|dt\\ &\leqslant \frac{2c\sqrt{C_3}}{c'\sqrt{C_1}}\int_0^1 \frac{|\gamma_n'(t)|}{|\operatorname{Re}(\gamma_n(t))|^{3/4}|\gamma_n(t)|^{3/4}}\\ &\leqslant \left(\frac{2c\sqrt{C_3}}{c'\sqrt{C_1}}\int_0^1 \frac{dt}{t^{3/4}}\right)\frac{1}{|z_n|^{3/4}}\,.\end{aligned}$$

This last quantity converges to zero when $|z_n|$ goes to infinity, with the condition $-1 < \operatorname{Re}(z_n) < 0$. □

4. Proofs of Theorem 1.2 and Theorem 1.3

Let D be a $\mathcal{C}^\infty$ smooth relatively compact domain in a $\mathcal{C}^\infty$ almost complex manifold (M, J). Assume that (D, J) is hyperbolic and strongly pseudoconvex. Consider a sequence $(J_\nu)_\nu$ of almost complex structures defined on the closure $\bar{D}$ of D and converging to J in the $\mathcal{C}^\infty$ convergence on $\bar{D}$. Theorem 7 of [17] implies that (D, J_ν) is also hyperbolic. In particular $\operatorname{Aut}(D, J_\nu)$ is a Lie group for every sufficiently large ν. Let G_ν be a compact subgroup of $\operatorname{Aut}(D, J_\nu)$. Suppose that the sequence $(G_\nu)_\nu$ satisfies the following condition :

$$\bigcup_{\nu>>1}\ \bigcup_{g^\nu\in G_\nu} g^\nu(q) \subset\subset D \quad \text{for any } q \in D. \tag{4.1}$$

Then one has :

Lemma 4.1. — *Fix $g^\nu \in G_\nu$. The family $(g^\nu)_\nu$ is a normal family and every cluster point g^∞ of $(g^\nu)_\nu$ belongs to* $\operatorname{Aut}(D, J)$.

Proof. — Let K be a relatively compact open subset in D. Let $p \in K$ and $v \in \mathbb{C}^n$ with $\|v\| = 1$. For every sufficiently large ν there exists a J_ν-holomorphic disc φ^ν contained in D such that $\varphi^\nu(0) = p$ and $d\varphi^\nu(0)(\partial/\partial x) = \alpha_\nu v$, where $|\alpha_\nu| \geqslant c > 0$, uniformly with respect to ν and v. Moreover it follows from the assumption on $(G_\nu)_\nu$ and from the estimates of the Kobayashi-Royden pseudonorm given by Theorem 1 of [7] that there exists a compact subset K' of D such that $\cup_{\nu>>1}(g^\nu \circ \varphi^\nu)(\Delta) =: K'_\nu \subset K' \subset\subset D$. It follows then from [26], Proposition 2.3.6 Part (i), that the derivatives of the composition $g^\nu \circ \varphi^\nu$ are bounded from above, uniformly with respect to sufficiently large ν. Hence according to the Ascoli-Arzela

Theorem we may extract from $(g^\nu)_\nu$ a subsequence (still denoted $(g^\nu)_\nu$) that converges, uniformly with its first derivatives, to a (J, J)-holomorphic map g_K^∞ from D to $\bar{D}$. Moreover, it follows from Theorem 2.2.1 in [26] that g_K^∞ is of class $\mathcal{C}^\infty$ on K and that $(g^\nu)_\nu$ converges to g_K^∞ in the $\mathcal{C}^\infty$ topology on K. Considering a compact exhaustion of D, we finally construct a (J, J)-holomorphic map g^∞ from D to $\bar{D}$, setting $(g^\infty)_{|K} = g_K^\infty$. By the strong pseudoconvexity of (D, J) the set $g^\infty(D)$ is contained in D. The same argument applied to $(g^\nu)^{-1}$ implies that g^∞ belongs to $\mathrm{Aut}(D, J)$. □

Theorem 1.2 and Theorem 1.3 are consequences of Theorem 1.1, of Lemma 4.1 and of the following :

Theorem 4.2 (R.Greene-S.G.Krantz [10]). — *Let M be a paracompact manifold and let $\{G_\nu\}$, $G_\nu \subset \mathrm{Diff}(M)$, be a sequence of compact groups converging $\mathcal{C}^\infty$ to a compact group G_0 of $\mathrm{Diff}(M)$. Then the group G_ν is Lie isomorphic to a subgroup of G_0 for sufficiently large ν.*

Proof of Theorem 1.2. Assume that (D, J) is not biholomorphic to $(\mathbb{B}^n, J_{st})$, or equivalently that $\mathrm{Aut}(D, J)$ is compact by Theorem 1.1.

Claim. — *For sufficiently large ν the automorphism group $G_\nu := \mathrm{Aut}(D, J_\nu)$ satisfies (4.1). In particular G_ν is compact for sufficiently large ν.*

Proof of the Claim. Suppose by contradiction that Condition (4.1) is not satisfied. Then there is a point $q \in D$ and a sequence $(g^\nu)_\nu$ of elements of G_ν such that $\lim_{\nu\to\infty} g^\nu(q) = p \in \partial D$ (we keep the same notation G_ν for subsequences). Since the sequence $(J_\nu)_\nu$ converges to J on $\bar{D}$ the scaling process (see [8, 19]) allows to construct a biholomorphism F between (D, J) and a model domain $(\mathbb{H}, J_{\mathsf{B}})$. This contradicts the compactness of $\mathrm{Aut}(D, J)$ by Theorem 3.4. The compactness of G_ν for large ν is a direct consequence of Condition (4.1). □

According to Lemma 4.1 the sequence of compact groups $(G_\nu)_\nu$ converges to the compact group $G_0 := \mathrm{Aut}(D, J)$. Hence the proof of Theorem 1.2 is complete by applying Theorem 4.2. □

Proof of Theorem 1.3. Let q be a point in D. Let J' be an almost complex structure such that D is strongly J'-pseudoconvex and (D, J') is hyperbolic. Then the isotropy group $\mathrm{Aut}_q(D, J')$ is a closed subgroup of the Lie group $\mathrm{Aut}(D, J')$. If $\mathrm{Aut}(D, J')$ is noncompact, then (D, J') is biholomorphic to $(\mathbb{B}^n, J_{st})$ by Theorem 1.1 and $\mathrm{Aut}_q(D, J')$ is isomorphic to the compact group $\mathrm{Aut}_0(\mathbb{B}^n, J_{st}) \simeq \mathrm{U}(n)$. If $\mathrm{Aut}(D, J')$ is compact, the closed subgroup

$\mathrm{Aut}_q(D, J')$ is also compact. Therefore one gets that $G_0 = \mathrm{Aut}_q(D, J)$ and $G_\nu = \mathrm{Aut}_q(D, J_\nu)$ are compact for sufficiently large ν.

In order to apply Lemma 4.1, we shall verify Condition (4.1). Let $p \in D$. By the stability of the Kobayashi distance (see Lemma 2.4 in [5]) there exists $R > 0$ such that $d_{(D,J_\nu)}(p, q) < R$ for sufficiently large ν. Hence for any $g^\nu \in \mathrm{Aut}_q(D, J_\nu)$, $d_{(D,J_\nu)}(g^\nu(p), q) = d_{(D,J_\nu)}(p, q) < R$ and the point $g^\nu(p)$ belongs to the Kobayashi ball $B^K_{(D,J_\nu)}(q, R)$ of (D, J_ν), centered at q with radius R. It follows from the estimates of the Kobayashi-Royden pseudonorm in [7] that the set $\bigcup_{\nu>>1} B^K_{(D,J_\nu)}(q, R)$ is relatively compact in D. Therefore the family $(G_\nu)_\nu$ satisfies (4.1).

Since by definition every element g^ν of G_ν satisfies $g^\nu(q) = q$, every cluster point g^∞ of $(g^\nu)_\nu$ belongs to $\mathrm{Aut}(D, J)$ (see Lemma 4.1) and satisfies $g^\infty(q) = q$. This means that g^∞ belongs to G_0. We may apply Theorem 4.2 to conclude that $\mathrm{Aut}_q(D, J_\nu)$ is Lie isomorphic to a subgroup of $\mathrm{Aut}_q(D, J)$. □

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