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THE NASH PROBLEM OF ARCS AND THE RATIONAL DOUBLE POINTS D_n

by Camille PLÉNAT

Abstract. — This paper deals with the Nash problem, which consists in comparing the number of families of arcs on a singular germ of surface U with the number of essential components of the exceptional divisor in the minimal resolution of this singularity. We prove their equality in the case of the rational double points D_n ($n \geqslant 4$).

Résumé. — Dans cet article, on étudie le problème des arcs de Nash, qui consiste à comparer le nombre de composantes irréductibles de l'espace des arcs passant par une singularité isolée de surface normale avec les courbes exceptionnelles apparaissant dans la résolution minimale de cette singularité. On montre que les deux nombres sont égaux dans le cas des points doubles rationnels D_n.

1. Introduction

In this paper, $\Bbbk$ is an algebraically closed field of characteristic 0.
Let $(S, 0)$ be a normal surface singularity over $\Bbbk$ and $\pi : (X, E) \longrightarrow (S, 0)$ be the *minimal resolution* of $(S, 0)$, where X is a smooth surface and $E = \pi^{-1}(0)$ is the exceptional set. Let $E = \bigcup_{i\in\Delta} E_i$ be the decomposition of E into its irreducible components, that we will call *exceptional divisors.*
In order to study such a resolution, J. Nash (around 1968, published as [14]) looked at the space H of *arcs* passing through the singular locus 0. Recall that an arc is a formal parametrized curve, i.e. a $\Bbbk$-morphism from the local ring $\mathcal{O}_{S,0}$ to the formal series ring $\Bbbk[[t]]$.
Nash had shown that H is the union of finitely many *families*, (which turn out to be the irreducible components of H viewed as a scheme endowed with the Zarsiski topology) , and that there exists an injection from the set of

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families of arcs to the set of exceptional divisors of the minimal resolution. The natural question of surjectivity then arose [14].

Later on, M. Lejeune (in [11]) proposed the following decomposition of the space H : let N_i be the set of arcs lifting transversally to E_i but not intersecting any other exceptional divisor E_j. M.Lejeune showed that $H = \bigcup_{i\in\Delta} \overline{N_i}$ and the set $\overline{N_i}$ is an irreducible algebraic subset of the space of arcs; therefore the families of arcs are among the $\overline{N_i}$'s. Moreover, notice that there are as many $\overline{N_i}$ as divisors E_i.Then the Nash problem reduces to showing that the $\overline{N_i}$ are the irreducible components, i.e.to proving $card(\Delta)(card(\Delta)-1)$ non-inclusions :

Problem 1.1. — *Is it true that $\overline{N_i} \not\subset \overline{N_j}$ for all $i \neq j$?*

This question has found some positive answers : for singularities A_n by Nash, for minimal surface singularities by A. Reguera [18] (other proofs in J. Fernandez-Sanchez [5] or C. Plénat [15]), for sandwiched singularities by M. Lejeune and A. Reguera(cf.[12]), for toric vareties by S. Ishii and J. Kollar ([8] using the previous work of C. Bouvier and G. Gonzalez-Sprinberg [2] and [3]), and for a family of non rational surface singularities by P. Popescu-Pampu and C. Plénat ([17]).

In [8], S. Ishii and J. Kollar also gave a counter example in dimension greater than or equal to 4.

The singularity D_n is the first "natural" singularity for which the answer was unknown till now. We present in this paper a proof of the following theorem:

Theorem 1.2. — *The Nash problem has an affirmative answer for rational double points D_n ($n \geqslant 4$).*

Notation. — We give a detailed proof for the D_{2n}, the proof for D_{2n+1} will follow easily.

By [15], corollary 3.5, we have the following corollary :

Corollary 1.3. — *Let $(S,0)$ be a normal surface singularity whose graph is the same as the graph of D_n (but with different weights). Then the problem also has an affirmative answer for $(S,0)$.*

The proof of the theorem is divided into two steps. For the first step we use the following valuative criterion (for a proof see [15]; it is a generalisation of a result of A. Reguera [18]):

Proposition 1.4. — *Let $(S,0)$ be a normal surface singularity. If there exists an element f in $\mathcal{O}_{S,0}$ such that $ord_{E_i} f < ord_{E_j} f$ then $\overline{N_i} \not\subset \overline{N_j}$.*

This condition allows us to prove more than half the non-inclusions (cf Problem 1.1).
The second step consists in proving the remaining non-inclusions. For it, we use the algebraic machinery developed in section 4. The "geometric" idea is the following:
Let E_i and E_j be two divisors such that

$$ord_{E_i} f \leqslant ord_{E_j} f \ for\ all\ f \in \mathcal{O}_{(S,0)} \ (1)$$

In other words, $\overline{N_i} \not\subset \overline{N_j}$ by the valuative criterion (proposition 1.4).
By contradiction, suppose that we have $\overline{N_j} \subset \overline{N_i}$. Let ϕ_j be a general arc in N_j. Then there exists a sequence of arcs $(\phi_i)_n$ in N_i converging to ϕ_j. The arcs on D_n (embedded in $\Bbbk^3 = spec\ \Bbbk[x,y,z]$) are described by three formal power series

$$\begin{cases} x(t) = \sum a_k t^k \\ y(t) = \sum b_k t^k \\ z(t) = \sum c_k t^k \end{cases}$$

whose coefficients are subjected to algebraic constraints; for a general arc of N_k, the coefficients $a_{ord_{E_k}(x)}, b_{ord_{E_k}(y)}, c_{ord_{E_k}(z)}$ are the first non-zero coefficients and must be nonzero. Convergence here means that the coefficients of $(\phi_i)_n$ converge to the respective coefficients of ϕ_j, and the algebraic constraints are satisfied at each step. The inequality (1), if strict, implies that the coefficients $a_{ord_{E_i}(x),n}, b_{ord_{E_i}(y),n}, c_{ord_{E_i}(z),n}$ converge to 0. In order to obtain the contradiction, we show that the constraints imply the vanishing of at least one of the coefficients $a_{ord_{E_j}(x)}, b_{ord_{E_j}(y)}, c_{ord_{E_j}(z)}$ of the limit ϕ_j.
In order to deal with the fact that the scheme H is non noetherian, we use the following description of H :

Definition 1.5. — *An i-jet is a $\Bbbk$-morphism $\mathcal{O}_{S,0} \to \frac{\Bbbk[[t]]}{t^{i+1}}$.*

The schemes $H(i)$ are of finite type. With the natural maps (called truncation map) $\rho_i : H \to H(i)$ and $\rho_{ij} : H(i) \to H(j)$ (for $j < i$) they form a projective system whose limit is H. Easily one can see that if there exists j such that $\rho_j(N_\alpha) \not\subset \overline{\rho_j(N_\beta)}$ then $\overline{N_\alpha} \not\subset \overline{N_\beta}$. It is then enough to work in a "good" $H(j)$.
The paper is organized as follows : in section 2 we first recall one description of the singularity D_n we will use and by using the valuative criterion we develop the first step of the proof. In section 3, we reformulate the "geometric" idea described above as an algebraic problem. In section 4, we partially describe the spaces $H(k)$ for a general k and for quasi-homogeneous hypersurface singularities. The two last sections are devoted to the proof of second step.

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2. Step one

2.1. The rational double points D_{2n}

Let $(S,0)$ be the rational double point D_{2n}. Embedded in $\Bbbk^3$, take as equation the following one : $f(x,y,z)=z^2-y^2x-x^{2n-1}=0$. Its dual graph of resolution is :

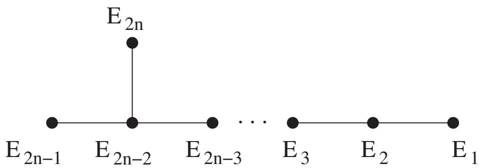


Figure 2.1. Dual graph (singularity of type D_{2n})

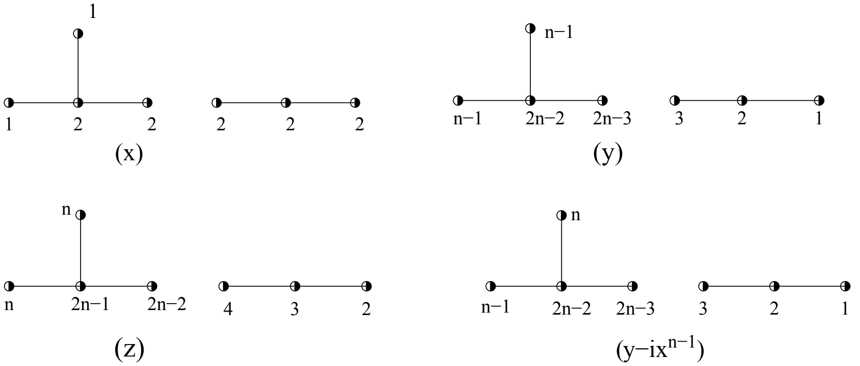
Let $\overline{N_1},...\overline{N_{2n}}$ be the irreducible subsets of H associated to the exceptional divisors $E_1,...,E_{2n}$.
As D_{2n} is embedded in $\Bbbk^3$, the arcs are described by three formal power series :

$$x(t)=a_1t+a_2t^2+...$$
$$y(t)=b_1t+b_2t^2+...$$
$$z(t)=c_1t+c_2t^2+...$$

with $f(x(t),y(t),z(t))=0$. Let f_m be the coefficients of t^m in $f\big(x(t),y(t),z(t)\big)=0$ for all m, and let I be the ideal they generate.

2.2. Step One

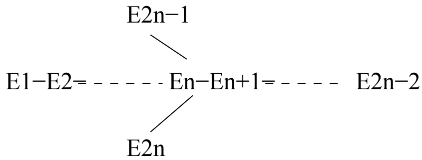
For this first step, we use the criterion (cf. proposition 1.4) with the functions x, y, z, $y+ix^n$ and $y-ix^n$ pulled back to X, whose order of vanishing at each E_i is written on the graphs below :



(We compute the orders of vanishing of each function by pulling back it to the resolution, which is done by a sequence of blowing-ups. We let the details of the computation to the reader.)
The criterion allows us to define a partial order on the families of arcs :

Definition 2.1. — *(Partial order)*
We say that $E_i \leqslant E_j$ if and only if for any element $f \in \mathcal{O}_{S,0}$ we have $ord_{E_i}(f) \leqslant ord_{E_j}(f)$.
We say that $E_i < E_j$ if one of the inequality above is strict.

This partial order can be translated by the following scheme :



(The relation $E_i - E_j$ means $E_i < E_j$; cf.[15]):

Definition 2.2. — *We define $E_1 - E_2 - ... - E_{2n-2}$ to be the "principal branch" and $E_{2n-1} - E_n - ... - E_{2n-2}$ and $E_{2n} - E_n - ... - E_{2n-2}$ to be the non-principal ones.*

Remark. — A general arc ϕ in N_k is described by three formal power series :

$$x(t) = a_{ord_{E_k}(x)} t^{ord_{E_k}(x)} + a_{ord_{E_k}(x)+1} t^{ord_{E_k}(x)+1} + ...$$
$$y(t) = b_{ord_{E_k}(y)} t^{ord_{E_k}(y)} + b_{ord_{E_k}(y)+1} t^{ord_{E_k}(y)+1} + ...$$
$$z(t) = c_{ord_{E_k}(z)} t^{ord_{E_k}(z)} + c_{ord_{E_k}(z)+1} t^{ord_{E_k}(z)+1} + ...$$

with $f(x(t), y(t), z(t)) = 0$.

3. Second step: algebraic reformulation

We can read from the above that the remaining non-inclusions to be shown are :

- $\overline{N_{2n-1-k}} \not\subset \overline{N_{2n-1-l}}$ for $1 \leqslant k < l \leqslant 2n-2$
- $\overline{N_{2n-1-k}} \not\subset \overline{N_{2n-1}}, \overline{N_{2n}}$ for $1 \leqslant k \leqslant n$

As one can notice, two different series of difficulties appear : the first, that we call "principal branch" is to show that $\overline{N_{2n-1-k}} \not\subset \overline{N_{2n-1-l}}$ for $1 \leqslant k < l \leqslant 2n-2$; the second are "the non-principal branches" (they are of two types but these are totally symmetric).

To solve the two series of non inclusions, we will use the same idea, described below.

First, in order to deal with finite dimensional varieties, we truncate the arcs at order $4n-2$.

Let $\overline{N_\alpha}$ and $\overline{N_\beta}$ be two families such that $\overline{N_\alpha} \not\subset \overline{N_\beta}$. By the previous section, we have $ord_{E_\alpha} f \leqslant ord_{E_\beta} f$ for all f in the local ring $\mathcal{O}_{S,o}$.

Notation. —

- Let $N_\alpha(4n-2) = \rho_{4n-2}(N_\alpha)$ and $N_\beta(4n-2) = \rho_{4n-2}(N_\beta)$.
- Let P_α and P_β be prime ideals such that $V(P_\alpha) = \overline{N_\alpha(4n-2)}$ and $V(P_\beta) = \overline{N_\beta(4n-2)}$.
- Let $L_l = \{a_1, ..., a_{ord_{E_l}(x)-1}, b_1, ..., b_{ord_{E_l}(y)-1}, c_1, ..., c_{ord_{E_l}(z)-1}\}$.
- Let
$$\begin{aligned} I_l = {} & I \cap \Bbbk[a_1, ..., c_{4n-2}] \\ & \cap (a_1, ..., a_{ord_{E_l}(x)-1}, b_1, ..., b_{ord_{E_l}(y)-1}, c_1, ..., c_{ord_{E_l}(z)-1}) \end{aligned}$$
  be the first equations verified by an arc in N_l (for $l = \alpha$ or β).
- Let $J = P_\alpha \cap P_\beta$.

Suppose that $P_\alpha \subset P_\beta$. In order to have a contradiction, we find elements in P_α not contained in P_β by the following way:

We have L_l and I_l in P_l for each l. Unfortunately, we also have L_α and I_α in P_β by hypothesis. Thus those "simple" elements will not lead us to the contradiction. But $a_{ord_{E_l}(x)}, b_{ord_{E_l}(y)}, c_{ord_{E_l}(z)}$ are not in P_α. It implies that $(I_\alpha : d^\infty) = \bigcup (I_\alpha : d^r)$, for d equal to one of the three elements $a_{ord_{E_l}(x)}$, $b_{ord_{E_l}(y)}$, $c_{ord_{E_l}(z)}$ ($(I_\alpha : d^r)$ is the saturation of I_α by d^r), is in the prime ideal P_α.

But the computation of the saturation is not obvious; so to make computation easier, we do it in an extension S of $\frac{\Bbbk[a_1,\dots,c_{4n-2}]}{P_\alpha \cap P_\beta}$. More precisely, we choose a prime ideal Q_α in S over $\frac{P_\alpha}{J}$ and get the following commutative diagram :

$$\begin{array}{ccccc} \frac{P_\alpha}{J} & \subset & \frac{P_\beta}{J} & \subset & \frac{\Bbbk[a_1,\dots,c_{4n-2}]}{J} \\ \cap & & \cap & & \downarrow \\ Q_\alpha & \subset & Q_\beta & \subset & S \end{array}$$

Then we show that for any prime ideal Q_β over $\frac{P_\beta}{J}$ in the above diagram, one cannot have $(I_\alpha : d^\infty)S \subset Q_\beta$, which gives the desired contradiction.

Notation. — Our computation of the saturation (c.f sections 5.2 and 6.2) is the algebraic analogue of sequences of arcs (cf introduction).

We now solve the two series of non-inclusions following the same plan: first we describe the images of N_α and N_β in $H(4n-2)$ and their associated ideal. Then we construct the extension to find non trivial elements of P_α. Finally, we show that those elements cannot live in P_β. But let us start with some algebra.

4. General study of the k-jet scheme H(k)

In this section, after giving general lemmas from commutative algebra, we will use them to study the space of jets passing through the singularity of a normal quasi-homogeneous hypersurface.

4.1. The principal lemma

LEMMA 4.1. — *Let* $R = \Bbbk[y_1, \dots y_n, x_{21}, \dots, x_{2m}, \dots, x_{k1}, \dots, x_{km}]$.
Let $f_1, \dots, f_k$ *be a sequence of elements in the following form :*

$$
\begin{aligned}
f_1 = f_1(y_1, ..., y_n) = g_1 ... g_s \\
f_2 = a_1 x_{21} + ... + a_m x_{2m} + h_2(y_1, ..., y_n) \\
f_3 = a_1 x_{31} + ... + a_m x_{3m} + h_3(y_1, ..., y_n, x_{21}, ..., x_{2m}) \\
\vdots \\
f_k = a_1 x_{k1} + ... + a_m x_{km} + h_k(y_1, ..., y_n, x_{21}, ..., x_{(k-1)m})
\end{aligned}
$$

with $g_1, ..., g_s$ distinct irreducible polynomials and $a_1, ..., a_m \in \Bbbk[y_1, ..., y_n]$. For fixed j ($1 \leqslant j \leqslant s$), let $S_j = \{a_{j_1}, ..., a_{j_{l(j)}}\} \subset \{a_1, ..., a_m\}$ be the set of a_l such that $a_l \notin (g_j)$.
Let us denote $I = (f_1, ..., f_k)$.
Suppose $S_j \neq \emptyset$.
Then there exists a unique minimal prime ideal $\mathcal{P}_j$ of I such that $g_j \in \mathcal{P}_j$ and $a_l \notin \mathcal{P}_j$ for all $a_l \in S_j$.
Let $\mathcal{Q}$ be a minimal prime ideal of I different from $\mathcal{P}_1, ..., \mathcal{P}_s$; then $(a_1, ..., a_m) \subset \mathcal{Q}$.
Let g_i and g_j be two irreducible factors of f_1. Then $\mathcal{P}_i \neq \mathcal{P}_j$. And finally, we have $ht(\mathcal{P}_j) = k$.

Definition 4.2. — *We call the prime ideal $\mathcal{P}_j$ of the lemma the distinguished ideal of I, associated to g_j.*

Proof. — Let $j \in \{1, ..., s\}$ and $a_l \in S_j$. Take x to be

$$(x_{21}, ..., \hat{x}_{2l}, ..., x_{2m}, ..., \hat{x}_{kl}, ..., x_{km})$$

and $y = (y_1, ..., y_n)$. One has

$$\frac{R_{a_l}}{(I)_{a_l}} \simeq \frac{\Bbbk[y_1, ..., y_n, x_{21}, ..., x_{2l}, ..., x_{2m}, ..., x_{rl}, ..., x_{km}]_{a_l}}{(f_1, ..., f_k)_{a_l}} \simeq \frac{\Bbbk[x, y]_{a_l}}{(f_1)_{a_l}}. \tag{4.1}$$

The decomposition into irreducible factors of f_1 in $\Bbbk[x, y]$ is $f_1 = g_1 ... g_s$; then the minimal prime ideals of (f_1) in $\Bbbk[x, y]_{a_l}$ have the form (g_q), where $a_l \notin (g_q)$. In particular, (g_j) is the unique minimal prime ideal $\mathcal{P}'_j$ of (f_1) containing g_j. By (4.1), one has a unique minimal prime ideal $\mathcal{P}''_j$ of (I_{a_l}) containing g_j. Let $\mathcal{P}_j$ be the inverse image of $\mathcal{P}''_j$ in R, under the bijection between the prime ideals of R_{a_l} and those of R not containing a_l . But $(\mathcal{P}_j)_{a_l} = (g_j, f_2, ..., f_k)_{a_l}$ and the sequence $(g_j, f_2, ..., f_k)$ is regular in R_{a_l}, while the length of this sequence is k; then the height of $\mathcal{P}_j$ is k.
Let $a_q \in S_j$, $a_q \neq a_l$. Let $\tilde{\mathcal{P}}_j$ be the unique minimal prime ideal of I such

that $g_j \in \tilde{\mathcal{P}}_j$ and $a_q \notin \tilde{\mathcal{P}}_j$. Then $(\mathcal{P}_j)_{a_l a_q} = (g_j, f_2, ..., f_r)_{a_l a_q} = (\tilde{\mathcal{P}}_j)_{a_l a_q}$, so $\mathcal{P}_j = \tilde{\mathcal{P}}_j$. The ideal $\mathcal{P}_j$ satisfies the conclusion of the lemma.

Let $\mathcal{Q} \notin \{\mathcal{P}_1, ..., \mathcal{P}_s\}$ be a prime ideal of I. We want to show that $(a_1, ..., a_m) \subset \mathcal{Q}$. We reason by contradiction : let us suppose that there exist $l \in \{1, ..., m\}$ such that $a_l \notin \mathcal{Q}$. The image of $\mathcal{Q}_{a_l}$ by (4.1) is a minimal prime ideal of (f_1); thus it has the form (g_j), where $a_l \notin (f_1)$. Then $\mathcal{Q} = \mathcal{P}_j$, a contradiction.

It remains to prove that distinguished ideals of I are distinct one from the other. Let $\mathcal{P}_i$ and $\mathcal{P}_j$ be minimal distinguished prime ideals of I associated to g_i and g_j respectively. If $S_i \cap S_j = \emptyset$, then $\mathcal{P}_i \neq \mathcal{P}_j$. Let $a_l \in S_i \cap S_j$. The image of $(\mathcal{P}_i)_{a_l}$ and $(\mathcal{P}_j)_{a_l}$ by (4.1) are respectively (g_i) and (g_j), thus $\mathcal{P}_i \neq \mathcal{P}_j$. □

4.2. Application to the space of k-jets of a quasi-homogeneous hyper-surface singularity

Let $f(x,y,z) = \sum c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma = 0$ be the equation of a normal quasi-homogeneous hypersurface embedded in $\mathbb{k}^3$ with singularity at 0. Any k-jet $\phi(t)$ passing through the singularity can be written as three polynomials of degree k, $\phi(t) = (x(t), y(t), z(t)) = (a_1 t + ... a_k t^k, b_1 t + ... b_k t^k, c_1 t + ... c_k t^k)$, with $a_0 = 0 = b_0 = c_0$ (because the singularity is at 0). Let $f_1 = 0, .., f_k = 0$ be the equations of the k-jet scheme $H(k)$ $(k > 0)$ (namely f_i is the coefficient of t^i in $f(x(t), y(t) z(t))$). These coefficients are polynomials in variables a_l, b_m, c_n, where l, n, m are positive integers.
Let K be the subset of the k-jet scheme defined in $H(k)$ by the ideal

$$(a_1, ..., a_{l-1}, b_1, ..., b_{m-1}, c_1, ... c_{n-1}).$$

Suppose there exists an integer r such that

$$f_r \notin (a_1, ..., a_{l-1}, b_1, ..., b_{m-1}, c_1, ... c_{n-1}).$$

Take the smallest such r. Then K is defined by the ideal

$$(a_1, ..., a_{l-1}, b_1, ..., b_{m-1}, c_1, ... c_{n-1}, f_r, ..., f_k).$$

Then $f_r = \sum_{l\alpha + m\beta + n\gamma = r} c_{\alpha\beta\gamma} {a_l}^\alpha {b_m}^\beta {c_n}^\gamma$. The polynomial f_r being quasi-homogeneous of degree r, one can write :

$$f_r = \frac{l}{r} a_l \frac{\partial f_r}{\partial a_l} + \frac{m}{r} b_m \frac{\partial f_r}{\partial b_m} + \frac{n}{r} c_n \frac{\partial f_r}{\partial c_n}$$

Let us look at the monomial $x^\alpha y^\beta z^\gamma$. (The coefficient does not play any role).
Let us write

$$\phi(t) = (a_l t^l + \sum_{j \geqslant l+1} a_j t^j, b_m t^m + \sum_{p \geqslant m+1} b_p t^p, c_n t^n + \sum_{q \geqslant n+1} c_q t^q).$$

Then we have :

$$(a_l t^l + \sum_{j \geqslant l+1} a_j t^j)^\alpha = a_l^\alpha t^{l\alpha} + \alpha a_l^{\alpha-1} t^{l(\alpha-1)} \sum_{j \geqslant l+1} a_j t^j + A(t)$$
$$(b_m t^m + \sum_{p \geqslant m+1} b_p t^p)^\beta = b_m^\beta t^{m\beta} + \beta b_m^{\beta-1} t^{m(\beta-1)} \sum_{p \geqslant m+1} b_p t^p + B(t)$$
$$(c_n t^n + \sum_{q \geqslant n+1} c_q t^q)^\gamma = c_n^\gamma t^{n\gamma} + \gamma c_n^{\gamma-1} t^{n(\gamma-1)} \sum_{q \geqslant n+1} c_q t^q + C(t)$$

(with $degA_1 \geqslant l+1$, $degB_1 \geqslant m+1$ and $degC_1 \geqslant n+1$, A_1, B_1, C_1 being the monomial of lowest degree of A, B, C respectively).

Therefore

$$\begin{aligned} x^\alpha y^\beta z^\gamma = a_l^\alpha b_m^\beta c_n^\gamma t^r &+ \alpha a_l^{\alpha-1} b_m^\beta c_n^\gamma t^{l\alpha+m\beta+n\gamma-l} \sum_{j \geqslant l+1} a_j t^j \\ &+ \beta a_l^\alpha c_n^\gamma b_m^{\beta-1} t^{l\alpha+m\beta+n\gamma-m} \sum_{p \geqslant m+1} b_p t^p \\ &+ \gamma a_l^\alpha b_m^\beta c_n^{\gamma-1} t^{l\alpha+m\beta+n\gamma-n} \sum_{q \geqslant n+1} c_q t^q + R(t) \end{aligned}$$

with $degR_1 \geqslant r+1$, where R_1 is the monomial of lowest degree in R. The coefficient of t^{r+i} is then

$$\begin{aligned} &\alpha a_l^{\alpha-1} b_m^\beta c_n^\gamma a_{l+i} + \beta a_l^\alpha c_n^\gamma b_m^{\beta-1} b_{m+i} + \gamma a_l^\alpha b_m^\beta c_n^{\gamma-1} c_{n+i} \\ &\qquad + S(a_l, ..., a_{l+i-1}, b_m, ..., b_{m+i-1}, c_n, ..., c_{n+i-1}) \end{aligned}$$

We recognize the three partial derivatives of $a_l^\alpha b_m^\beta c_n^\gamma$. This holds for each monomial, thus we have

$$\begin{aligned} f_{r+i} = \left(\frac{\partial f_r}{\partial a_l}\right) a_{l+i} + \left(\frac{\partial f_r}{\partial b_m}\right) b_{m+i} + \left(\frac{\partial f_r}{\partial c_n}\right) c_{n+i} \\ + S_{r+i}(a_l, ..., a_{l+i-1}, b_m, ..., b_{m+i-1}, c_n, ..., c_{n+i-1}). \end{aligned}$$

These equations satisfy the hypothesis of Lemma 5. If f_r is irreducible, then there exists a unique distinguished ideal $\mathcal{P}$, the one which corresponds to the closure of the set $G = \{a_1 = ... = a_{l-1} = 0 = b_1 = ... = b_{m-1} = c_1 = ... = c_{n-1} = f_1 = .. = f_k\} \cap \{\frac{\partial f_r}{\partial a_l} \neq 0\} \cap \{\frac{\partial f_r}{\partial b_m} \neq 0\} \cap \{\frac{\partial f_r}{\partial c_n} \neq 0\}$.

Proposition 4.3. — *Let us suppose that $f_r = g_1...g_s$, the factors g_j being irreducible and different from x, y, z,and suppose f_r reduced.*
Then $I = (f_r, ..., f_k)$ has exactly s distinguished ideals.

Proof. — By Lemma 5, it suffices to show that each (g_j) does not contain one of the partial derivatives of f_r. Let g_j be one of the irreducible factor of f_r. Then

$$\begin{aligned} f_r &= g_j h \\ \frac{\partial f_r}{\partial a_l} &= g_j \frac{\partial h}{\partial a_l} + h \frac{\partial g_j}{\partial a_l} \\ \frac{\partial f_r}{\partial b_m} &= g_j \frac{\partial h}{\partial b_m} + h \frac{\partial g_j}{\partial b_m} \\ \frac{\partial f_r}{\partial c_n} &= g_j \frac{\partial h}{\partial c_n} + h \frac{\partial g_j}{\partial c_n}. \end{aligned}$$

Suppose all three partial derivatives of f_r are in (g_j). Then

$$\left(h\frac{\partial g_j}{\partial a_l}, h\frac{\partial g_j}{\partial b_m}, h\frac{\partial g_j}{\partial c_n} \right) \subset (g_j).$$

But $h \notin (g_j)$ and (g_j) is prime, so $\left(\frac{\partial g_j}{\partial a_l}, \frac{\partial g_j}{\partial b_m}, \frac{\partial g_j}{\partial c_n} \right) \subset (g_j)$, which is false since f_r is reduced.
Finally, if $f_r = g_1...g_s$, $(f_r, ..., f_k)$ has exactly s distinct distinguished ideals, then each of them associated to a factor g_i. □

5. The principal branch

In this section, we first study $\overline{N_\alpha(4n-2)}$ and $\overline{N_\beta(4n-2)}$ in $H(4n-2)$ for E_α and E_β in the principal branch. Then we construct concrete elements and the extension they live in, which will give us the desired contradiction (cf. introduction). Finally we solve the non-inclusions of the principal branch.

Notation. — Let $\mu_k(g) = ord_{E_k} g$ for $g \in \mathcal{O}_{(S,0)}$.
Let $i = 4n-2$, and $R = \Bbbk[a_1, ..., a_{4n-2}, b_1, ..., b_{4n-2}, c_1, ..., c_{4n-2}]$.

5.1. The images of the families of arcs in H(4n-2)

Let $\overline{N}_{2n-k-1}$ and $\overline{N}_{2n-l-1}$ be two families of arcs so that $1 \leqslant l < k \leqslant 2n-2$ (thus $\overline{N}_{2n-k-1} \not\subset \overline{N}_{2n-l-1}$, by the valuative criterion (proposition 1.4)). Our aim is to show that $\overline{N}_{2n-l-1} \not\subset \overline{N}_{2n-k-1}$.

Consider the $(4n-2)$-jet scheme $H(4n-2)$ (here one has $\mu_{2n-k-1}(z) = 2n-k$ and $\mu_{2n-k-1}(y) = 2n-k-1$). Let $K_{2n-k-1}(4n-2) = \{a_1 = b_1 = \ldots = b_{2n-k-2} = c_1 = \ldots = c_{2n-k-1} = 0\} \cap H(4n-2)$ be the sub-space of $H(4n-2)$ defined by the ideal $I_{2n-k-1}(4n-2)$ which is generated by the following equations :

$$\begin{aligned}
f_{(2n-k-1,4n-2k)} &= c_{2n-k}^2 - a_2 b_{2n-k-1}^2 \\
f_{(2n-k-1,4n-2k+1)} &= 2c_{2n-k}c_{2n-k+1} - a_3 b_{2n-k-1}^2 - 2a_2 b_{2n-k-1} b_{2n-k} \\
f_{(2n-k-1,4n-2k+2)} &= c_{2n-k+1}^2 + 2c_{2n-k}c_{2n-k+2} \\
&\quad - g_{4n-2k+2}(A_{4n-2k+2}, B_{4n-2k+2}) \\
f_{(2n-k-1,4n-2k+3)} &= 2c_{2n-k+1}c_{2n-k+2} + 2c_{2n-k}c_{2n-k+3} \\
&\quad - g_{4n-2k+3}(A_{4n-2k+3}, B_{4n-2k+3}) \\
&\vdots \\
f_{(2n-k-1,4n+l-k-2)} &= 2c_{2n-k}c_{2n+l-2} + \ldots \\
&\quad + g_{4n+l-k-2}(A_{4n+l-k-2}, B_{4n+l-k-2}) \\
&\vdots \\
f_{(2n-k-1,4n-2)} &= 2c_{2n-k}c_{2n+k-2} + \ldots + g_{4n-2}(A_{4n-2}, B_{4n-2})
\end{aligned}$$

in $R = \Bbbk[a_1, \ldots, a_{4n-2}, b_1 \ldots, b_{4n-2}, c_1, \ldots, c_{4n-2}]$. (where $A_{4n-2k+j} = \{a_2, \ldots, a_{j+1}\}$ and $B_{4n-2k+j} = \{b_{2n-k-1}, \ldots, b_{2n-k+j-1}\}$, the g_i are polynomials in variables A_i and B_i, and $f_{(2n-k-1,i)}$ are the coefficients of t^i in $f(x(t), y(t), z(t)) = 0$ modulo the ideal $(a_1, b_1, \ldots b_{2n-k-2}, c_1, \ldots, c_{2n-k-1})$).

Image of $\overline{N}_{2n-k-1}$.

We have $a_1 = b_1 = \ldots = b_{2n-k-2} = c_1 = \ldots = c_{2n-k-1} = 0$, thus the image of $\overline{N}_{2n-k-1}$ is in $K_{2n-k-1}(4n-2)$.
Let $\mathcal{Q}_{2n-k-1}(4n-2)$ be the defining ideal of $\overline{N_{2n-k-1}(4n-2)}$.
The ideal $\mathcal{Q}_{2n-k-1}(4n-2)$ contains all equations defining N_{2n-k-1} in H whose variables are in R, that is to say the ideal

$$\begin{aligned}
I_{2n-k-1}(4n-2) = (a_1, b_1, , \ldots, b_{\mu_{2n-k-1}(y)-1}, c_1, \ldots, c_{\mu_{2n-k-1}k(z)-1}, \\
f_{(2n-k-1,4n-2k)}, \ldots, f_{(2n-k-1,6n-k-2)})
\end{aligned}$$

Moreover, $c_{\mu_{2n-k-1}(z)}$ and $b_{\mu_{2n-k-1}(y)}$ are not in $\mathcal{Q}_{2n-k-1}(4n-2)$. Thus $\mathcal{Q}_{2n-k-1}(4n-2)$ contains the distinguished prime ideal of $I_{2n-k-1}(4n-2)$, called $\mathcal{P}_{2n-k-1}(4n-2)$.

Let $\phi_i \in V(\mathcal{P}_{2n-k-1}(4n-2)) - \{c_{2n-k} \neq 0\}$; we can lift ϕ_i to an arc in $N_{2n-k-1}(4n-2)$: we freely choose $b_{2n+l-3+r-1}$ and $a_{k+l+r-1}$ and we set

$$c_{2n+l-2+r} = \frac{1}{2c_{2n-k}}(f_{2n+k} - 2c_{2n-k}c_{2n+l+r})$$

Then $\phi_i \in N_{2n-k-1}(4n-2)$ and $\mathcal{Q}_{2n-k-1}(4n-2) = \mathcal{P}_{2n-k-1}(4n-2)$.

Finally, as $b_{2n-k-1}, c_{2n-k} \notin \mathcal{P}_{2n-k-1}(4n-2)$, we get that $\mathcal{P}_{2n-k-1}(4n-2)$ contains $(I_{2n-k-1}(4n-2) : {c_{2n-k}}^{\infty})$ and $(I_{2n-k-1}(4n-2) : {b_{2n-k-1}}^{\infty})$.

Image of $\overline{N}_{2n-l-1}$.
Similarly, one has that $\overline{N}_{2n-l-1}(4n-2)$ has its generic point on at least one of the irreducible components of $K_{2n-k-1}(4n-2)$, but is not equal in general to the whole component. Let $\mathcal{Q}_{2n-l-1}(4n-2)$ be its defining ideal. The ideal $\mathcal{Q}_{2n-l-1}(4n-2)$ contains all equations defining N_{2n-l-1} in H whose variables live in R_{4n-2}, i.e. the ideal

$$I_{2n-l-1}(4n-2) = (a_1, b_1, , ..., b_{\mu_{2n-l-1}(y)-1}, c_1, ..., c_{\mu_{2n-l-1}(z)-1}, \\ f_{(2n-l-1,4n-2l)}, ..., f_{(2n-l-1,6n-l-2)})$$

Moreover, $c_{\mu_{2n-l-1}(z)}$ and $b_{\mu_{2n-l-1}(y)}$ are not in $\mathcal{Q}_{2n-l-1}(4n-2)$. Thus $\mathcal{Q}_{2n-l-1}(4n-2)$ contains the distinguished prime ideal of $I_{2n-l-1}(4n-2)$, called $\mathcal{P}_{2n-l-1}(4n-2)$.
Let $\phi_i \in V(\mathcal{P}_{2n-l-1}(4n-2)) - \{c_{2n-l} \neq 0\}$; we can lift ϕ_i to an arc in $N_{2n-l-1}(4n-2)$ by elimination.
In conclusion, $\mathcal{Q}_{2n-l-1}(4n-2) = \mathcal{P}_{2n-l-1}(4n-2)$.

In order to show that $\overline{N}_{2n-l-1}(4n-2) \not\subset \overline{N}_{2n-k-1}(4n-2)$, we have to find non trivial elements in $\mathcal{P}_{2n-k-1}(4n-2)$ (they will be in $(I_{2n-k-1}(4n-2) : {c_{2n-k}}^{\infty})$ and $(I_{2n-k-1}(4n-2) : {b_{2n-k-1}}^{\infty})$), not in $\mathcal{P}_{2n-l-1}(4n-2)$.

5.2. Looking for non trivial elements : The ideal $\mathcal{P}_{2n-v-1}(4n-2)$ for $1 \leqslant v \leqslant 2n-2$

First of all, notice that the elements c_{2n-v} and b_{2n-v-1} are not in $\mathcal{P}_{2n-v-1}(4n-2)$, but they are in the other minimal prime ideals of $I_{2n-v-1}(4n-2)$. We deduce that $\mathcal{P}_{2n-v-1}(i)$ contains $(I_{2n-v-1}(4n-2) : c^{\infty}_{2n-v}) = \bigcup(I_{2n-v-1}(4n-2) : c^{r}_{2n-v})$ and the ideal $(I_{2n-v-1}(4n-2) : b^{\infty}_{2n-v-1}) = \bigcup(I_{2n-v-1}(4n-2) : b^{r}_{2n-v-1})$ One can construct elements of $\mathcal{P}_{2n-v-1}(4n-2)$ in the following way:
We work in $R = \Bbbk[a_1, ..., a_{4n-2}, b_1, ..., b_{4n-2}, c_1, ..., c_{4n-2}]$. Note that $a_2 \neq 0$ on the generic points of all the families of the principal branch; we can then

consider a_2 as a unit.
Let $a_2 = \alpha^2$ and look at the algebraic extension

$$\frac{R}{\mathcal{P}_{2n-k-1}(4n-2) \cap \mathcal{P}_{2n-l-1}(4n-2)} \to \frac{R}{I_{\mathcal{P}_{2n-k-1}(4n-2) \cap \mathcal{P}_{2n-l-1}(4n-2)}}[\alpha]_{a_2} = S.$$

In this extension, one can rewrite the equation of the singularity as follows:

$$z^2 - xy^2 - x^{2n-1} = (z - \sqrt{x}y)(z + \sqrt{x}y) - x^{2n-1} = 0.$$

In fact we are looking at the families of the principal branch, that is to say families with $a_1 = 0$ and $a_2 \neq 0$, so :

$$x(t) = a_2 t^2 + a_3 t^3 + ... = \alpha^2 t^2 + a_3 t^3 .... \ \in S[[t]]$$

and

$$\sqrt{x} = \alpha t + \frac{a_3}{2\alpha} t^2 + ... \in S[[t]] = \alpha t + \alpha_2 t^2 + \alpha_3 t^3 ...$$

Let $g^{(1)}_{(2n-v-1,j)}(\alpha)$be the coefficient of t^j in $(z - \sqrt{x}y)$ and $g^{(2)}_{(2n-v-1,j)}(\alpha)$ be the coefficient of t^j in $(z + \sqrt{x}y)$. The elements $g^{(1)}_{(2n-v-1,j)}(\alpha)$ and $g^{(2)}_{(2n-v-1,j)}(\alpha)$ are conjugate to each other under the involution $\alpha \to -\alpha$.

We have :

$$\begin{aligned} f_{(2n-v-1,4n-2v)} &= c^2_{2n-v} - a_2 b^2_{2n-v-1} \\ &= (c_{2n-v} - \alpha b_{2n-v-1})(c_{2n-v} + \alpha b_{2n-v-1}). \end{aligned}$$

Consider now the prime ideal $\mathcal{P}$ over $\mathcal{P}_{2n-v-1}(4n-2)$ in $\frac{R}{I_{2n-v-1}(4n-2)}[\alpha]$ such that $c_{2n-v} - \alpha b_{2n-v-1} = 0$; then $c_{2n-v} + \alpha b_{2n-v-1} = 2c_{2n-v}$.

We compute $z^2 - xy^2 - x^{2n-1} = (z - \sqrt{x}y)(z + \sqrt{x}y) - x^{2n-1}$ for this family in $\frac{R}{I_{2n-v-1}(4n-2)}[\alpha]$:

$$\begin{aligned}
f_{(2n-v-1,4n-2v)} &= (c_{2n-v} - \alpha b_{2n-v-1})2c_{2n-v} = 2c_{2n-v} g^{(1)}_{(2n-v-1,2n-v)}(\alpha) \in \mathcal{P}\\
f_{(2n-v-1,4n-2v+1)} &= 2c_{2n-v} g^{(1)}_{(2n-v-1,2n-v+1)}(\alpha)\\
&\quad + g^{(1)}_{(2n-v-1,2n-v)}(\alpha) g^{(2)}_{(2n-v-1,2n-v+1)}(\alpha) \in \mathcal{P}\\
&\vdots\\
f_{(2n-v-1,4n-3)} &= 2c_{2n-v} g^{(1)}_{(2n-v-1,2n+v-3)}(\alpha)\\
&\quad + g^{(1)}_{(2n-v-1,2n-v)}(\alpha) g^{(2)}_{(2n-v-1,2n+v-3)}(\alpha) + \dots\\
&\quad \dots + g^{(1)}_{(2n-v-1,2n+v-4)}(\alpha) g^{(2)}_{(2n-v-1,2n-v+1)}(\alpha) \in \mathcal{P}\\
f_{(2n-v-1,4n-2)} &= 2c_{2n-v} g^{(1)}_{(2n-v-1,2n+v-2)}(\alpha)\\
&\quad + g^{(1)}_{(2n-v-1,2n-v)}(\alpha) g^{(2)}_{(2n-v-1,2n+v-2)}(\alpha) + \dots\\
&\quad \dots + g^{(1)}_{(2n-v-1,2n+v-3)}(\alpha) g^{(2)}_{(2n-v-1,2n-v+1)}(\alpha) - a_2^{2n-1} \in \mathcal{P}
\end{aligned}$$

and thus

$$\begin{aligned}
2c_{2n-v} g^{(1)}_{(2n-v-1,2n-v)} &\in \mathcal{P}\\
2c_{2n-v} g^{(1)}_{(2n-v-1,2n-v+1)}(\alpha) &\in \mathcal{P}\\
&\vdots\\
2c_{2n-v} g^{(1)}_{(2n-v-1,2n+v-3)}(\alpha) &\in \mathcal{P}\\
2c_{2n-v} g^{(1)}_{(2n-v-1,2n+v-2)}(\alpha) - a_2^{2n-1} &\in \mathcal{P}.
\end{aligned}$$

As $2c_{2n-v} \notin \mathcal{P}$, **one has** $g^{(1)}_{(2n-v-1,j)} \in \mathcal{P}$ **for** $2n - v \leqslant j \leqslant 2n + v - 3$.
We can solve the non-inclusions of the principal branch.

5.3. Resolution of the principal branch

Consider $K_{2n-k-1}(4n-2)$; let $\overline{N_{2n-k-1}}$ and $\overline{N_{2n-l-1}}$ be two families such that $l < k$ (then $\overline{N}_{2n-k-1} \not\subset \overline{N}_{2n-l-1}$ from the scheme of partial order). We show that $\overline{N}_{2n-l-1}(4n-2) \not\subset \overline{N}_{2n-k-1}(4n-2)$, then we will have $\overline{N}_{2n-l-1} \not\subset \overline{N}_{2n-k-1}$.

Suppose that $\mathcal{P}_{2n-k-1}(4n-2) \subset \mathcal{P}_{2n-l-1}(4n-2)$.
Let

$$J = \mathcal{P}_{2n-k-1}(4n-2) \cap \mathcal{P}_{2n-l-1}(4n-2) = \mathcal{P}_{2n-k-1}(4n-2)$$

and consider the algebraic extension $S=\frac{R}{J}[\alpha]_{a_2}$ of $\frac{R}{J}$. Let $\mathcal{P}$ be the prime ideal over $\mathcal{P}_{2n-k-1}(i)$ in $\frac{R'_i}{J}[\alpha]_{a_2}$ such that $c_{2n-k}-\alpha b_{2n-k-1}=0$; then $c_{2n-k}+\alpha b_{2n-k-1}=2c_{2n-k}$. Let $\mathcal{Q}$ be a prime ideal over $\mathcal{P}_{2n-l-1}(4n-2)$ such that

$$\begin{array}{ccccc}
\mathcal{P}_{2n-k-1}(4n-2) & \subset & \mathcal{P}_{2n-l-1}(4n-2) & \subset & \frac{R}{J} \\
\cap & & \cap & & \downarrow \\
\mathcal{P} & \subset & \mathcal{Q} & \subset & S
\end{array}$$

If $\mathcal{P}_{2n-k-1}(4n-2)\subset\mathcal{P}_{2n-l-1}(4n-2)$, then

$$g^{(1)}_{(2n-k-1,2n+l-2)}=g^{(1)}_{(2n-l-1,2n+l-2)}\in\mathcal{Q}$$

because $l<k$ and thus, as $2c_{2n-l}g^{(1)}_{(2n-l-1,2n+l-2)}(\alpha)-a_2^{2n-1}\in\mathcal{Q}$, we have $a_2^{2n-1}\in\mathcal{Q}$ which gives the desired contradiction.

6. The two non-principal branches

The two branches $E_{2n-1}-E_n-...-E_{2n-2}$ and $E_{2n}-E_n-...-E_{2n-2}$ are symmetric, thus we can restrict ourselves to $E_{2n}-E_n-...-E_{2n-2}$. The only non-inclusions left to be proved are $\overline{N_l}\not\subset\overline{N_{2n}}$ for all l such that $n\leqslant l\leqslant 2n-2$.

Let $i_l=2l+1$.

6.1. The images of the families in $\mathbf{H(i}_l\mathbf{)}$

Let $K(i_l)=\{b_1=...=b_{n-2}=c_1=...=c_{n-1}=0\}\cap H(i_l)$. It is the subspace of $H(i_l)$ whose defining ideal $I(i_l)$ in $R_{i_l}=\Bbbk[a_1,...,a_{i_l},b_1...,b_{i_l},c_1,...,c_{i_l}]$ is generated by the following equations :

$$\begin{aligned}
f_{2n-1}&=ia_1^{n-1}-b_{n-1}\\
f_{2n}&=c_n^2-2a_1b_{n-1}b_n-a_2b_{n-1}^2-(2n-1)a_1^{2n-2}a_2\\
f_{2n+1}&=2c_nc_{n+1}-g_{2n+1}(A_{2n+1},B_{2n+1})-a_1^{2n-3}h_{2n+1}(A_{2n+1})\\
&\vdots\\
f_{2l}&=c_l^2+...+2c_nc_{2l-n}-g_{2l}(A_{2l},B_{2l})-a_1^{4n-2l-2}h_{2l}(A_{2l})\\
f_{2l+1}&=2c_lc_{l+1}+...+2c_nc_{2l-n+1}-g_{2l+1}(A_{2l+1},B_{2l+1})\\
&\qquad -a_1^{4n-2l-3}h_{2l+1}(A_{2l+1})
\end{aligned}$$

(where $A_r = \{a_1, ..., a_{r+2-2n}\}$ et $B_r = \{b_n, ..., b_{r+1-n}\}$, the g_i being certain polynomials in variables A_i and B_i ; the h_i being polynomials in A_i and the f_{2n+i} are the coefficients of t^{2n+i} in $f(x(t), y(t), z(t)) = 0$ modulo the ideal $(b_1, ...b_{n-2}, c_1, ..., c_{n-1})$).

Remark. — Alternatively one could work in $H(4n-2)$ for all l.

We have to find the ideals defining the closure of the sets $N_{2n}(i_l) = \rho_{i_l}(N_{2n})$ and $N_l(i_l) = \rho_{i_l}(N_l)$.
Let $\mathcal{Q}_{2n}(i_l)$ be the defining ideal of $N_{2n}(i_l)$ and $\mathcal{Q}_l(i_l)$ be the defining ideal of $\overline{N_l(i_l)}$. By the same argument as for the families of the principal branches, we have $\mathcal{Q}_{2n}(i_l) = \mathcal{P}_{2n}(i_l)$ where $\mathcal{P}_{2n}(i_l)$ is the distinguished prime ideal of $I_{2n}(i_l)$ and

$$I_{2n}(i_l) = (a_1, b_1, , ..., b_{\mu_{2n}(y)-1}, c_1, ..., c_{\mu_{2n}(z)-1}, f_{(2n,2n-1)}, ..., f_{(2n,2l+n+1)}).$$

We also have that $\mathcal{Q}_l(i_l)$ is the distinguished minimal prime ideal of

$$(a_1, b_1, , ..., b_{l-1}, c_1, ..., c_l, f_{(l,2l)}, ..., f_{(l,3l+1)}).$$

Moreover we have, as $a_1, b_{n-1}, ..., b_{l-1}, c_n, ..., c_l \notin \mathcal{P}_{2n}(i_l)$, that $\mathcal{P}_{2n}(i_l)$ contains the ideals $(I(i_l) : {a_1}^{\infty})$, $(I(i_l) : {c_{r+1}}^{\infty})$ and $(I(i_l) : {b_r}^{\infty})$ where $r \in \{n-1, ..., l-1\}$.

In the same way as for the principal branch, we want to construct elements of $\mathcal{Q}_{2n}(i_l)$, by studying the ideal $(I(i_l) : {a_1}^{\infty})$. The extension we find is not the same as for the principal branch, we need an extension where we are allowed to divide by a_1.

6.2. Looking for non trivial elements

Study of the ideal $\mathcal{Q}_{2n}(i_l)$.
In what follows, we fix an l such that $n-1 \leqslant l \leqslant m-1$ (we want to show that $\overline{N_l(i_l)} \not\subset \overline{N_{2n}(i_l)}$)
In this section, we show that each equation f_j for $2n-1 \leqslant j \leqslant 2l$ is in the integral closure of $(a_1^d)R$ (for some $d \in \mathbb{N}$ depending on j). For each b_r and c_r, we find the greatest $d \in \mathbb{N}$ such that b_r -or c_r- are in the integral closure of $(a_1^d)R$ ($d \in \mathbb{N}$).
Recall the valuative characterization of the integral closure of an ideal (cf. [4] and [13], theorem 38):

Definition 6.1. — *Let $\mathcal{R}$ be a normal noetherian domain, $b \in \mathcal{R}$ and I an ideal of $\mathcal{R}$.*

The element b is in the integral closure of I if and only if for every positive valuation μ over $\mathcal{R}$ of rank one, there exists an element x of I such that $\mu(x) \leqslant \mu(b)$.

This characterization motivates the following definition :

DEFINITION 6.2. — *Let p and q be two integers.*
We say that $a^{\frac{p}{q}}$ divides b in a normal ring R (or equivalently that b is in the integral closure of $(a^{\frac{p}{q}})$) if a^p divides b^q in R.

Notation. —
- in what follows, we will denote "a divides b" by "a/b".
- take $J(i_l) = \mathcal{Q}_l(i_l) \cap \mathcal{Q}_{2n}(i_l)$

Suppose $\overline{N}_l(i_l) \subset \overline{N}_{2n}(i_l)$; then $J(i_l) = \mathcal{Q}_{2n}(i_l)$. Let $\tilde{R}(i_l) = \overline{\frac{R(i_l)}{J(i_l)}}$ be the normalization of $\frac{R(i_l)}{J(i_l))}$; it is a normal domain.
The system generated by the equations $(f_{2n-1}, ..., f_{2l})$ in $\tilde{R}(i_l)$ is:

$$f_{2n-1} = ia_1^{n-1} - b_{n-1}$$

(6.1)

$$f_{2m+1} = -a_1 b_m^2 + \sum_{r=n}^{m} C_r^{2m+1} c_r c_{2m-r+1} - \sum_{r=2}^{2m+3-2n} a_r \left(\sum_{u+v+r=2m+1} B_{uv}^{2m+1} b_u b_v \right) - a_1 \left(\sum_{r=n-1}^{m-1} B_r^{2m+1} b_r b_{2m-r} \right) + a_1^{4n-2m-3} g_{2m+1}(A)$$

(6.2)

$$f_{2m+2} = c_{m+1}^2 + \sum_{r=n}^{m} C_r^{2m+2} c_r c_{2m+2-r} - \sum_{r=2}^{2m+4-2n} a_r \left(\sum_{u+v+r=2m+2} B_{uv}^{2m+2} b_u b_v \right) - a_1 \left(\sum_{r=n-1}^{m} B_r^{2m+2} b_{r-1} b_{2m+2-r} \right) + a_1^{4n-2m-4} g_{2m+2}(A)$$

for $n-1 \leqslant m \leqslant l-1$ (where C_i^j and B_i^j are constants, $A \in \Bbbk[a_1, ..., a_{2l-2n+3}]$ and the polynomials g_s are not divisible by a_1).
This system is a system of $2(l-n)+2$ equations with $2(l-n)+3$ unknowns $a_1, b_{n-1} ..., b_{l-1}, c_n, ..., c_l$. We want to find positive rational numbers $\beta_{n-1}, ..., \beta_{l-1}, \gamma_n, ..., \gamma_l \in \mathbb{Q}$ so that $a_1^{\beta_k}$ divides b_k and $a_1^{\gamma_r}$ divides c_r in $\tilde{R}(i_l)$.

DEFINITION 6.3. — *Let $\beta_k = sup\{\alpha \in \mathbb{Q} : a_1^{\alpha}/b_k$ in $\tilde{R}(i_l)\}$ and $\gamma_{k+1} = sup\{\alpha \in \mathbb{Q} : a_1^{\alpha}/c_{k+1}$ in $\tilde{R}(i_l)\}$ for $n-1 \leqslant k \leqslant l-1$*

Remark. — A priori, β_k is in $\mathbb{R}\cup\{\infty\}$ and so is γ_k. Below, we will calculate lower bounds for β_k and γ_k which will be rational numbers.

We prove the following proposition:

Proposition 6.4. — *For all k and r such that $n-1 \leqslant k \leqslant l-1 < 2n-3$ and $n \leqslant r \leqslant l < 2n-2$, one has $\beta_k > 1$ and $\gamma_r > 1$. For $k = l-1 = 2n-3$, one has $\beta_k \geqslant 1$ and $\gamma_{k+1} \geqslant 1$.*

Remark. — $\beta_{n-1} = n-1$ by $f_{2n-1} = 0$.

Proof. — We define the sequences $(\beta_k)_s$ and $(\gamma_{k+1})_s$ recursively in k. These sequences $(\beta_k)_s$ and $(\gamma_{k+1})_s$ will be increasing, converging, with $(\beta_k)_s \leqslant \beta_k$, $(\gamma_{k+1})_s \leqslant \gamma_{k+1}$ and with limit greater than or equal to 1.

We will use the following trivial lemma :

Lemma 6.5. — *Let $f = g - h$ be elements of S. If a_1^α divides h and a_1^α divides f, then a_1^α divides g.*

Construction of the sequences.
For $k = n-1$, consider :

$$f_{2n-1} = ia_1^{n-1} - b_{n-1}$$
$$f_{2n} = c_n^2 - 2a_1b_{n-1}b_n - a_2b_{n-1}^2 - (2n-1)a_1^{2n-2}a_2$$

We already have $\beta_{n-1} = n-1$. Set $(\beta_{n-1})_s = n-1$ for all s. Moreover, we have $a_1^n/f_{2n}(=0)$ and $a_1^n/2a_1b_{n-1}b_n - a_2b_{n-1}^2 - (2n-1)a_1^{2n-2}a_2$, thus a_1^n/c_n^2, i.e. $a_1^{\frac{n}{2}}/c_n$: set $(\gamma_n)_s = \frac{n}{2}$. (for $n > 2$, we get $\gamma_n > 1$; for $n = 2$, i.e. the case D_4, $k = l-1$, $\gamma_n \geqslant 1$).
Let $l > k \geqslant n-1$. Suppose we have already constructed for all $n-1 \leqslant m \leqslant k-1$ increasing sequences $(\beta_m)_s$ and $(\gamma_{m+1})_s$ which converge to a limit strictly greater than 1. There exists a positive integer S such that $(\beta_m)_S > 1$ and $(\gamma_{m+1})_S > 1$ for all $n-1 \leqslant m \leqslant k-1$.
Rewrite the equations :

$$f_{2m+1} = \sum_{w\mu+v\nu+u\lambda=2m+1} C_{\mu\nu\lambda}^{2m+1} a_w^\mu b_v^\nu c_u^\lambda$$
$$f_{2m+2} = \sum_{w\mu+v\nu+u\lambda=2m+2} C_{\mu\nu\lambda}^{2m+2} a_w^\mu b_v^\nu c_u^\lambda$$

where $C_{\mu\nu\lambda}^i$ are constants.
Define $(\beta_k)_S = \min\Big\{\frac{\beta-1}{2} \,:\, \beta = \mu(\alpha_w)_S + \nu(\beta_v)_S + \lambda(\gamma_u)_S / C_{\mu\nu\lambda}^{2k+1} \neq$

0 *and* $C^{2m+1}_{\mu\nu\lambda} \neq C^{2m+1}_{120}\Big\}$, then $(\gamma_{k+1})_S = \min\Big\{\frac{\gamma}{2} : \gamma = \mu(\alpha_w)_S + \nu(\beta_v)_S + \lambda(\gamma_u)_S / C^{2k+2}_{\mu\nu\lambda} \neq 0$ *and* $C^{2m+1}_{\mu\nu\lambda} \neq C^{2m+1}_{00k+1}\Big\}$, with $(\alpha_w)_s = 0$ if $w \neq 1$, $(\alpha_w)_s = 1$ if not.
We can thus define :

Definition 6.6. — *Recursively in s, we define* $(\beta_m)_s = \min\Big\{\frac{\beta-1}{2} : \beta = \mu(\alpha_w)_{s_w} + \nu(\beta_v)_{s_v} + \lambda(\gamma_u)_{s_u} / C^{2m+1}_{\mu\nu\lambda} \neq 0 \textit{ et } C^{2m+1}_{\mu\nu\lambda} \neq C^{2m+1}_{120}\Big\}$ *and* $(\gamma_{m+1})_s = \min\Big\{\frac{\gamma}{2} : \gamma = \mu(\alpha_w)_{s_w} + \nu(\beta_v)_{s_v} + \lambda(\gamma_u)_{s_u} / C^{2m+2}_{\mu\nu\lambda} \neq 0 \textit{ et } C^{2m+1}_{\mu\nu\lambda} \neq C^{2m+1}_{00k+1}\Big\}$
with $s_x = s - 1$ *if* $x > m$, $s_x = s$ *if not.*
For all $s \leqslant S$, *we pose* $(\beta_k)_s = 0$ *and* $(\gamma_{k+1})_s = 0$

Lemma 8 shows that $a_1^{(\beta_m)_s}$ divides b_m for all s.
The sequences are increasing by construction, thus the limits of $(\beta_m)_s$ and $(\gamma_{m+1})_s$ for all $n-1 \leqslant m \leqslant k-1$ are strictly greater than 1 by construction. It remains to show that the limits for the sequences $(\beta_k)_s$ and $(\gamma_{k+1})_s$ are greater than or equal to 1.
For notational convenience, we set for $m > k$, $(\beta_m)_s = 0$ and $(\gamma_{m+1})_s = 0$ (even for $m > l$) and $(\beta_{n-1})_s = n - 1$.
We write the equations in the following form:
For m such that $n \leqslant m \leqslant k$,

$$f_{2n-1} = ia_1^{n-1} - b_{n-1}$$

$$f_{2n} = c_n^2 - 2a_1 b_{n-1} b_n - a_2 b_{n-1}^2 - (2n-1) a_1^{2n-2} a_2 \tag{6.3}$$

(6.4)
$$f_{2m+1} = -a_1 b_m^2 + \sum_{r=n}^{m} C_r^{2m+1} c_r c_{2m-r+1} - \sum_{r=2}^{2m+3-2n} a_r \left(\sum_{u+v+r=2m+1} B_{uv}^{2m+1} b_u b_v \right) - a_1 \left(\sum_{r=n-1}^{m-1} B_r^{2m+1} b_r b_{2m-r} \right) + a_1^{4n-2m-3} g_{2m+1}(A)$$

(6.5)
$$f_{2m+2} = c_{m+1}^2 + \sum_{r=n}^{m} C_r^{2m+2} c_r c_{2m+2-r} - \sum_{r=2}^{2m+4-2n} a_r \left(\sum_{u+v+r=2m+2} B_{uv}^{2m+2} b_u b_v \right) - a_1 \left(\sum_{r=n-1}^{m} B_r^{2m+2} b_r b_{2m+1-r} \right) + a_1^{4n-2m-4} g_{2m+2}(A)$$

for $n-1 \leqslant k \leqslant l-1$ (where C_i^j and B_i^j are constants, $A \in \Bbbk[a_1, ..., a_{2l-2n+3}]$ and the polynomials g_s are not divisible by a_1).

Then, by definition, we have the following properties:

$$by\ (6.3)\ :\ (\gamma_n)_s = min\left\{(\beta_{n-1})_s, \frac{(\beta_{n-1})_s + (\beta_n)_{s-1} + 1}{2}, \frac{2n-2}{2}\right\};$$

(6.6)

$$by\ (6.4)\ :$$

$$(\beta_m)_s = \min\left\{\frac{(\gamma_m)_s + (\gamma_{m+1})_{s-1} - 1}{2}, ..., \frac{(\gamma_n)_s + (\gamma_{2m-n+1})_{s-1} - 1}{2},\right.$$
$$\frac{(\beta_u)_{s_u} + (\beta_v)_{s_v} - 1}{2}\ with\ u+v+r = 2m+1\ ,\ r \geqslant 2,$$
$$s_w = s-1\ if\ w > m,\ \ s_w = s,\ if\ not,\ with\ w = u, v$$
$$\left.\frac{(\beta_{m-1})_s + (\beta_{m+1})_{s-1}}{2}, ..., \frac{(\beta_{n-1})_s + (\beta_{2m-n+1})_{s-1}}{2}, \frac{4n-2m-4}{2}\right\};$$

(6.7)

$$by\ (6.5)\ :$$

$$(\gamma_{m+1})_s = \min\left\{\frac{(\gamma_m)_s + (\gamma_{m+2})_{s-1}}{2}, ..., \frac{(\gamma_n)_s + (\gamma_{2m+2-n})_{s-1}}{2}\right.$$
$$\frac{(\beta_u)_{s_u} + (\beta_v)_{s_v}}{2}\ with\ u+v+r = 2m+2r \geqslant 2,\ s_w = s-1$$
$$if\ w > m+1,\ s_w = s\ if\ not,\ with\ w = u, v$$
$$\left.\frac{1 + (\beta_m)_s + (\beta_{m+1})_{s-1}}{2}, ..., \frac{1 + (\beta_{n-1})_s + (\beta_{2m+2-n})_{s-1}}{2}, \frac{4n-2m-4}{2}\right\}$$

The sequences are bounded above by $\frac{4n-2m-4}{2}$, so they converge. Let $\tilde{\gamma}_{m+1} = \lim_s (\gamma_{m+1})_s$, $\tilde{\beta_m} = \lim_s (\beta_m)_s$. One has: $\gamma_{m+1} \geqslant \tilde{\gamma}_{m+1}$, $\beta_m \geqslant \tilde{\beta_m}$. We compute the minimum of (6.6) and (6.7) for $s > S$.

a. Equations (6.6) give :

$$(6.8)\quad (\beta_m)_s \leqslant \frac{(\beta_{m-1})_s + (\beta_m)_{s-1} - 1}{2} \leqslant \frac{(\beta_{m-1})_s + (\beta_m)_s - 1}{2}$$

thus $(\beta_m)_s \leqslant (\beta_{m-1})_s - 1$, information we inject in (6.7), thus we get :

$$(6.9)\qquad (\gamma_m)_s \leqslant \frac{(\beta_{m-1})_s + (\beta_m)_{s-1} + 1}{2} \leqslant (\beta_{m-1})_s.$$

Lemma 6.7. — *We have :*

(6.10)

$$(\beta_m)_s = \min\left\{\frac{(\gamma_m)_s + (\gamma_{m+1})_{s-1} - 1}{2}, ..., \frac{(\gamma_n)_s + (\gamma_{2m-n+1})_{s-1} - 1}{2},\right.$$
$$\left.\frac{(\beta_{m-1})_s + (\beta_{m+1})_{s-1}}{2}, ..., \frac{(\beta_{n-1})_s + (\beta_{2m-n+1})_{s-1}}{2}, \frac{4n-2m-4}{2}\right\};$$

Proof. — We use the inequalities (6.8) and (6.9) with the fact that the sequences are increasing.

Let $2 \leqslant r' < r \leqslant 2m+3-2n$, (u',v') and (u,v) such that $u'+v'+r' = 2m+1 = u+v+r$ and $u=u'$; then $v'>v$ and :

$$\frac{(\beta_u)_{s_u} + (\beta_v)_{s_v} - 1}{2} \geqslant \frac{(\beta_u)_{s_u} + (\beta_{v'})_{s_v} - 1}{2}.$$

This allows us to eliminate the terms $\frac{(\beta_u)_{s_u}+(\beta_v)_{s_v}-1}{2}$ for $2m+1 = u+v+r$ and $r>2$.

It remains to eliminate terms for $r=2$. One has $u+v = 2m-1$; we can suppose that $n-1 \leqslant u \leqslant m-1$ and $2m-n \geqslant v \geqslant m$ (so as not to consider the same monomial twice). Thus we get thanks to inequalities (6.8) and (6.9):

$$\frac{(\beta_{n-1})_s + (\beta_{2m-n})_{s-1} - 1}{2} \geqslant \frac{(\gamma_n)_s + (\gamma_{2m-n+1})_{s-1} - 1}{2}$$

$$\vdots$$

$$\frac{(\beta_{m-1})_s + (\beta_m)_{s-1} - 1}{2} \geqslant \frac{(\gamma_m)_s + (\gamma_{m+1})_{s-1} - 1}{2}$$

□

Lemma 6.8. — *We have (for $n \leqslant m \leqslant k$):*

(6.11)

$$(\gamma_m)_s = \min\left\{\frac{(\gamma_{m-1})_s + (\gamma_{m+1})_{s-1}}{2}, \ldots, \frac{(\gamma_n)_s + (\gamma_{2m-n})_{s-1}}{2},\right.$$
$$\left.\frac{1+(\beta_{m-1})_s + (\beta_m)_{s-1}}{2}, \ldots, \frac{1+(\beta_{n-1})_s + (\beta_{2m-n})_{s-1}}{2}, \frac{4n-2m-4}{2}\right\}.$$

Proof. — As before we have to eliminate terms for $r=2$:

we have $u+v = 2m-2$; we can suppose $n-1 \leqslant u \leqslant m-1$ and $2m-n \geqslant v \geqslant m-1$ (not to consider the same monomial twice). Thus we get:

$$(\beta_{m-1})_s + (\beta_{m+1})_{s-1} \geqslant (\gamma_m)_s + (\gamma_{m+2})_{s-1}$$

$$\vdots$$

$$(\beta_{n-1})_s + (\beta_{2m-n+1})_{s-1} \geqslant (\gamma_n)_s + (\gamma_{2m-n+2})_{s-1}$$

We have to eliminate $(\beta_m)_s$.

For all $n-1 \leqslant m \leqslant k-1$ we have $(\beta_m)_s \geqslant (\beta_{m+1})_s + 1$, thus

$$(\beta_m)_s \geqslant \frac{(\beta_m)_s + (\beta_{m+1})_{s-1} + 1}{2}.$$

□

LEMMA 6.9. — *For $m = k+1$ we obtain :*

(6.12)

$$(\gamma_{k+1})_s = \min\left\{\frac{(\gamma_k)_s + (\gamma_{k+2})_{s-1}}{2}, ..., \frac{(\gamma_n)_s + (\gamma_{2k+2-n})_{s-1}}{2}, (\beta_k)_s,\right.$$
$$\left.\frac{1+(\beta_k)_s + (\beta_{k+1})_{s-1}}{2}, ..., \frac{1+(\beta_{n-1})_s + (\beta_{2k+1-n})_{s-1}}{2}, \frac{4n-2k-4}{2}\right\}.$$

Proof. — Same proof as before, except that we cannot eliminate the term $(\beta_k)_s$. □

b. Show by induction on l :

$$\frac{(\gamma_{m-l})_s + (\gamma_{m+l})_{s-1}}{2} \leqslant \frac{(\gamma_{m-l-1})_s + (\gamma_{m+l+1})_{s-1}}{2}.$$

For $l = 1$:

$$\begin{aligned}\frac{(\gamma_{m-1})_s + (\gamma_{m+1})_{s-1}}{2} &\leqslant \frac{(\gamma_{m-2})_s + (\gamma_m)_{s-1} + (\gamma_m)_{s-1} + (\gamma_{m+2})_{s-2}}{4}\\ &\leqslant \frac{(\gamma_{m-2})_s + (\gamma_{m+2})_{s-1}}{4} + \frac{(\gamma_m)_s}{2}\\ &\leqslant \frac{(\gamma_{m-2})_s + (\gamma_{m+2})_{s-1}}{4} + \frac{(\gamma_{m-1})_s + (\gamma_{m+1})_{s-1}}{4}\end{aligned}$$

So :

$$\frac{(\gamma_{m-1})_s + (\gamma_{m+1})_{s-1}}{4} \leqslant \frac{(\gamma_{m-2})_s + (\gamma_{m+2})_{s-1}}{4}.$$

Let $l \geqslant 1$.

$$\begin{aligned}&\frac{(\gamma_{m-l-1})_s + (\gamma_{m+l+1})_{s-1}}{2}\\ &\qquad\leqslant \frac{(\gamma_{m-l-2})_s + (\gamma_{m+l})_{s-1} + (\gamma_{m+l})_{s-1} + (\gamma_{m+l+2})_{s-2}}{4}\\ &\qquad\quad \frac{(\gamma_{m-l-2})_s + (\gamma_{m+l+2})_{s-1}}{4} + \frac{(\gamma_{m+l})_s}{2}\\ &\qquad\quad \frac{(\gamma_{m-l-2})_s + (\gamma_{m+l+2})_{s-1}}{4} + \frac{(\gamma_{m-l})_s + (\gamma_{m+l})_{s-1}}{4}.\end{aligned}$$

The result follows.
In the same way, one can show that:

$$\frac{(\beta_{m-l})_s+(\beta_{m+l})_{s-1}}{2}\leqslant\frac{(\beta_{m-l-1})_s+(\beta_{m+l+1})_{s-1}}{2}$$
$$\frac{(\gamma_{m-l})_s+(\gamma_{m+l+1})_{s-1}}{2}\leqslant\frac{(\gamma_{m-l-1})_s+(\gamma_{m+l+2})_{s-1}}{2}$$
$$\frac{(\beta_{m-l})_s+(\beta_{m+l+1})_{s-1}}{2}\leqslant\frac{(\beta_{m-l-1})_s+(\beta_{m+l+2})_{s-1}}{2}.$$

c. Moreover, we have that:

$$\begin{aligned}\frac{(\gamma_m)_s+(\gamma_{m+1})_{s-1}-1}{2}&\leqslant\frac{\frac{(\beta_{m-1})_s+(\beta_m)_{s-1}-1}{2}+\frac{(\beta_m)_{s-1}+(\beta_{m+1})_{s-2}-1}{2}+1}{2}\\&\leqslant\frac{(\beta_{m-1})_s+(\beta_{m+1})_{s-1}}{4}+\frac{(\beta_m)_s}{2}\\&\leqslant\frac{(\beta_{m-1})_s+(\beta_{m+1})_{s-1}}{4}+\frac{(\gamma_m)_s+(\gamma_{m+1})_{s-1}-1}{4}\end{aligned}$$

and

$$\begin{aligned}\frac{(\beta_{m-1})_s+(\beta_m)_{s-1}+1}{2}&\leqslant\frac{\frac{(\gamma_{m-1})_s+(\gamma_m)_{s-1}+1}{2}+\frac{(\gamma_m)_{s-1}+(\gamma_{m+1})_{s-2}+1}{2}-1}{2}\\&\leqslant\frac{(\gamma_{m-1})_s+(\gamma_{m+1})_{s-1}}{4}+\frac{(\gamma_m)_s}{2}\\&\leqslant\frac{(\gamma_{m-1})_s+(\gamma_{m+1})_{s-1}}{4}+\frac{(\beta_{m-1})_s+(\beta_m)_{s-1}+1}{4}.\end{aligned}$$

d. We also have for $n\leqslant m\leqslant k$:

$$\begin{aligned}\frac{(\beta_{m-1})_s+(\beta_m)_{s-1}+1}{2}&\leqslant\frac{\frac{4n-2m+2-4}{2}+\frac{4n-2m-4}{2}+1}{2}\\&\leqslant\frac{4n-2m-2}{2}\end{aligned}$$

and

$$\begin{aligned}\frac{(\gamma_m)_s+(\gamma_{m+1})_{s-1}-1}{2}&\leqslant\frac{\frac{4n-2m-2}{2}+\frac{4n-2m-4}{2}-1}{2}\\&\leqslant\frac{4n-2m-4}{2}.\end{aligned}\tag{6.13}$$

Finally : for $m=k+1$, we have :

$$\begin{aligned}(\gamma_{k+1})_s&=\min\left\{\frac{(\beta_k)_s+1}{2},(\beta_k)_s,\frac{4n-2k-4}{2}\right\}\\&=\min\left\{\frac{(\beta_k)_s+1}{2},(\beta_k)_s\right\}\end{aligned}$$

and for $n \leqslant m \leqslant k$

$$(\gamma_m)_s = \frac{(\beta_{m-1})_s + (\beta_m)_{s-1} + 1}{2}$$
$$(\beta_m)_s = \frac{(\gamma_m)_s + (\gamma_{m+1})_{s-1} - 1}{2}.$$

Lemma 6.10. — *We have $\tilde{\beta}_k \geqslant 1$*

Proof. — Suppose that $\tilde{\beta}_k < 1$ (then $(\gamma_{k+1})_s = (\beta_k)_s$).
Taking the limits as s goes to infinity, we obtain the following system of equations :

$$\tilde{\gamma}_m = \frac{\tilde{\beta}_{m-1} + \tilde{\beta}_m + 1}{2}$$
$$\tilde{\beta}_m = \frac{\tilde{\gamma}_m + \tilde{\gamma}_{m+1} - 1}{2}$$
$$\tilde{\gamma}_{k+1} = \tilde{\beta}_k.$$

Solving the system, we obtain :

$$\tilde{\beta}_k = \frac{2n-2-k}{2k-2n+3} + \frac{(2k-2n+2)\tilde{\beta}_k}{2k-2n+3}.$$

Then :

$$\tilde{\beta}_k = 2n-2-k > 1 \; for \; k < 2n-3$$
$$\tilde{\beta}_k = 2n-2-k = 1 \; for \; k = 2n-3.$$

Contradiction. □

Then for all $n-1 \leqslant m \leqslant k$ and $n \leqslant r \leqslant k+1$, one has $\beta_m \geqslant 1$ and $\gamma_r \geqslant 1$.
Passing to the limit as s goes to infinity, obtain the following system :

$$\tilde{\gamma}_m = \frac{\tilde{\beta}_{m-1} + \tilde{\beta}_m + 1}{2}$$
$$\tilde{\beta}_m = \frac{\tilde{\gamma}_m + \tilde{\gamma}_{m+1} - 1}{2}$$
$$\tilde{\gamma}_{k+1} = \frac{\tilde{\beta}_k + 1}{2}.$$

We find :

$$\tilde{\beta}_k = \frac{n-1}{k-n+2};$$
$$\tilde{\beta}_m = (k-m+1)\tilde{\beta}_k$$
$$\tilde{\gamma}_m = \tilde{\gamma}_{k+1} + (k-m+1)\tilde{\beta}_k = \frac{2k-2m+3}{2}\tilde{\beta}_k + \frac{1}{2}.$$

□

Thus $\beta_k > 1$ and $\gamma_{k+1} > 1$ for all k and l such that $k \leqslant l-1 < 2n-3$ and for $k = l-1 = 2n-3$, one has $\beta_k \geqslant 1$ and $\gamma_{k+1} \geqslant 1$.
Therefore we have

- for $l-1 < 2n-3$, for all $n-1 \leqslant k \leqslant l-1$: $\beta_k > 1$ and $\gamma_{k+1} > 1$.
- For $l-1 = 2n-3$: $\beta_{l-1} \geqslant 1$, $\gamma_l \geqslant 1$ and for all $n-1 \leqslant k < l-1$ one has $\beta_k > 1$ and $\gamma_{k+1} > 1$.

Let us fix l such that $n-1 \leqslant l \leqslant 2n-2$.
Let $n-1 \leqslant m \leqslant l-1$. We have shown that for all pairs of integers (p,q) such that $\frac{p}{q} < \tilde{\beta_m} = \frac{p_m}{q_m}$, for all positive valuations μ of rank 1, we have :

$$\mu(a_1^p) \leqslant \mu(b_m^q)$$

in $\overline{\frac{R'(i_l)}{J(i_l)}}$.
Thus we have :

$$\mu(a_1^{p_m}) \leqslant \mu(b_m^{q_m}).$$

for all valuations μ of rank one 1.
Indeed, if not, there exists a valuation ν such that

$$\nu(a_1^{p_m}) > \nu(b_m^{q_m})$$

Then there exist two positive integers p and q such that $\frac{p}{q} > 1$ and $\nu(a_1^{p_m}) > \frac{p}{q}\nu(b_m^{q_m})$, i.e. $\nu(a_1^{p_m q}) > \nu(b_m^{q_m p})$. Contradiction.
We can define the following finite algebraic extension

$$\frac{R(i_l)}{J(i_l))} \to \overline{\frac{R(i_l)}{J(i_l))}} = S.$$

Note that the fractions $\frac{b_{n-1}}{{a_1}^{n-1}}, \frac{b_n}{{a_1}^{\tilde{\beta}_n}}, ..., \frac{b_{l-1}}{{a_1}^{\tilde{\beta}_{l-1}}}, \frac{c_n}{{a_1}^{\tilde{\gamma}_n}}, ..., \frac{c_l}{{a_1}^{\tilde{\gamma}_l}}$ are in S.

6.3. Proof for the non-principal branches

We look at the branch $E_{2n} - E_n - ... - E_{2n-2}$ (the proof for the other one being symmetrical)
The notations are the same as in section 6.2
Truncate at the order $i_l = 2l+1$.

Proposition 6.11. — *For all $n \leqslant l \leqslant 2n-2$, one has $\overline{N}_l(i_l) \not\subset \overline{N}_{2n}(i_l)$.*

Proof. — Consider the extension :$\frac{R_{i_l}}{J(i_l)} \to S$. Let $\mathcal{P} \subset S$ be the prime ideal over $\mathcal{P}_{2n}(i_l)$ and $\mathcal{Q}$ be the prime ideal over $\mathcal{P}_l(i_l)$. (We suppose $\mathcal{P} \subset \mathcal{Q}$).

For $n \leqslant l < 2n-2$, consider $\frac{f_{2l+1}}{a_1}$ in S. We have $\frac{f_{2l+1}}{a_1} \in \mathcal{P}$ (because $f_{2l+1} \in \mathcal{P}_{2n}(i_l)$, $a_1 \frac{f_{2l+1}}{a_1} = f_{2l+1}$ and $a_1 \notin \mathcal{P}_{2n}(i_l)$). But $\frac{f_{2l+1}}{a_1} - {b_l}^2 \in \mathcal{Q}$ and $b_l \notin \mathcal{Q}$. Contradiction.

Suppose $l = 2n-2$. We still suppose that $\mathcal{P}_{2n}(i_{2n-2}) \subset \mathcal{Q}_{2n-2}(i_{2n-2})$. Let $J = \mathcal{P}_{2n}(i_{2n-2}) \cap \mathcal{Q}_{2n-2}(i_{2n-2})$. Let $c'_r = \frac{c_r}{a_1^{2n-r-1}}$ and $b'_{r_1} = \frac{b_{r-1}}{a_1^{2n-r-2}}$ for $n \leqslant r \leqslant 2n-2$. Let S be the birational extension obtained by adding the elements c'_r and b'_{r-1} $(n \leqslant r \leqslant 2n-2)$ (It is, as we have just shown, contained in the normalization of the ring $\frac{R(i_{2n-2})}{J}$).

Let $\mathcal{P}$ and $\mathcal{Q}$ be prime ideals over $\mathcal{P}_{2n}(i_{2n-2})$ and $\mathcal{Q}_{2n-2}(i_{2n-2})$ respectively in the extension S.

Let $h'_m = \frac{f_{2n,m}}{a_1^{4n-2-m}}$ for $2n-1 \leqslant m \leqslant 4n-3$ and $h_m = h'_m$ modulo the ideal $\mathcal{Q}$ (to obtain h_m one sets the coefficients $a_1, b_{n-1}, ...b_{2n-3}, c_n, ...c_{2n-2}$ be zero in (h'_m)).

By construction, the equations h_m for $2n-1 \leqslant m \leqslant 4n-3$ live in

$$A = \Bbbk[a_2, b'_n, ..., b'_{2n-3}, b_{2n-2}, c'_n, ..., c'_{2n-2}, c_{2n-1}]$$

(we have replaced b'_{n-1} by i and we are not considering anymore the equation $f_{2n,2n-1} = 0$).

We also have $f_{2n-2,4n-2} \in A$. There exists a natural homomorphism ϕ from A to $\frac{S}{\mathcal{Q}}$.

Let P be the ideal in $\frac{S}{\mathcal{Q}}$ generated by all relations satisfied by b'_r, c'_r and a_2, c_{2n-1}, b_{2n-2}; it contains in particular the ideal generated by $(h_m)_{m=2n,...,4n-3}$ and $f_{2n,4n-2}$.

Let $P" = ker(\phi)$ be the inverse image of P in A. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & \frac{R}{J} \arrow[r] \arrow[d] & S \arrow[d] \\
 & \frac{R}{\mathcal{Q}_{2n-2}(i_{2n-2})} \arrow[r] & \frac{S}{\mathcal{Q}} \\
\Bbbk[a_2, b_{2n-2}] \arrow[r, "f_1"] & \frac{\Bbbk[a_2, b_{2n-2}, c_{2n-1}]}{(f_{2n,4n-2})} \arrow[r, "f_2"] \arrow[u] & \frac{A}{P"} \arrow[u]
\end{tikzcd}$$

Proposition 6.12. — *We have that $\frac{A}{P"}$ is finite over $k[a_2, b_{2n-2}]$.*

Proof. — We already have that $\frac{\Bbbk[a_2, b_{2n-2}, c_{2n-1}]}{(f_{2n,4n-2})}$ is finite over $\Bbbk[a_2, b_{2n-2}]$. The finiteness of the second arrow f_2 comes from the following :

Consider the equations

$$(f'_{2n-2,m}) = (f_{2n-2,m}) mod(a_3, ..., a_{4n-3}, b_{2n-1}, ..., b_{4n-3}, c_{2n}, ..., c_{4n-3})$$

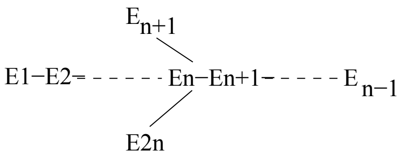
for $2n \leqslant m \leqslant 4n-3$ in $\mathbb{k}[a_1, a_2, b_1, ..., b_{2n-2}, c_1, ..., c_{2n-1}]$; these equations allow us to show that $\frac{c_k}{a_1^{\gamma_k}}$ and $\frac{b_{k-1}}{a_1^{\beta_{k-1}}}$ are in the normalization of $\frac{\mathbb{k}[a_1,a_2,b_1,...,b_{2n-21},c_1,...,c_{2n-1}]}{(f'_{2n-2,m})_{m=2n,...,4n-3}}$ and that $\gamma_k = 2n-k-1$ and $\beta_k = 2n-k-2$ in the same way as we show that S is an algebraic extension of $\frac{R}{J}$ (cf. 3.2.2) . Denote by S' the algebraic extension of $\frac{\mathbb{k}[a_1,a_2,b_1,...,b_{2n-21},c_1,...,c_{2n-1}]}{(f'_{2n-2,m})_{m=2n,...,4n-3}}$ obtained by adding to it the elements $c'_r = \frac{c_r}{a_1^{2n-r-1}}$ and $b'_{r-1} = \frac{b_{r-1}}{a_1^{2n-r-2}}$ for $n \leqslant r \leqslant 2n-2$.
We get the following commutative diagram:

$$\begin{tikzcd}
\frac{\mathbb{k}[a_1,a_2,b_1,...,b_{2n-2},c_1,...,c_{2n-1}]}{((f'_{2n-2,m})_{m=2n,...,4n-3},f_{2n,4n-2})} \arrow[r] \arrow[d] & S' \arrow[d] \\
\frac{\mathbb{k}[a_2,b_{2n-2},c_{2n-1}]}{(f_{2n,4n-2})} \arrow[r, "f_2"] & \frac{A}{P"}
\end{tikzcd}$$

Then the first horizontal arrow is finite by definition and then so is f_2. ☐
Thus, there exists an ideal $\mathcal{M}$ over $(a_2, b_{2n-2}-1)$ in $\frac{A}{P"}$. This maximal ideal does not contain h_{4n-3} by definition, which implies that P" does not contain h_{4n-3} either. This is false by definition of P". ☐

7. Comments

The proof developed above works for the singularities D_{2n}. For the singularities D_{2n+1}, the proof is almost the same : first we use the valuative criterion with the functions x, y, z , $z+ix^n$ and $z-ix^n$. It gives the following scheme:



(where E_{2n} and E_{2n+1} are the two symmetric exceptional curves).
As for D_{2n} it remains to solve two series of non-inclusions (the principal branch and the two symmetrical branches). The resolution of the principal branch works exactly as for D_{2n}, because $a_2 \neq 0$ for all the families of this branch. The resolution for the non-principal branches is slightly different:

the first equation of the arcs corresponding to (for example) E_{2n+1} is $c_n - ia_1^n = 0$. Thus we still construct an extension where it is allowed "to divide by a_1", but the roles played by c_r and b_m are exchanged.

The method seems to work for the three rational points left.

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