ANNALES DE L'INSTITUT FOURIER

A. K. MOOKHOPADHYAYA

# On restricted measurability

*Annales de l'institut Fourier*, tome 16, nº 2 (1966), p. 159-166

<http://www.numdam.org/item?id=AIF_1966__16_2_159_0>

Ann. Inst. Fourier, Grenoble
**16,** 2 (1966), 159-166.

# ON RESTRICTED MEASURABILITY

**by A. K. MOOKHOPADHYAYA**

## 1. Introduction and Definitions.

The purpose of the present paper is to study some properties of the restricted measurability [5] and to show that a Radon measure similar to that of [4] can be constructed with the help of the notion of the restricted measurability. Before we go into details, we write out, for the sake of completeness, a few definitions and notations some of which are borrowed from the above papers and the standard texts such as Halmos [1] and Kelley [2].

1.1. Definition. — *$\mu$ is a measure (Carathéodory) on $X$ if $\mu$ is a function on the family of all subsets of $X$ to $0 \leqslant t \leqslant \infty$ such that*

(i) $$\mu(0) = 0$$

(ii) $$0 \leqslant \mu A \leqslant \sum_{n=1}^{\infty} \mu B_n, \qquad \textit{whenever} \qquad A \subset \bigcup_{n=1}^{\infty} B_n \subset X.$$

1.2. Definition. — *$A \subset X$ is $\mu$-measurable if for every $T \subset X$*

$$\mu T = \mu(T \cap A) + \mu(T \sim A)$$

*where $\mu$ is a measure on $X$.*

1.3. Definition. — *A partition is a finite or infinite disjoint sequence $\{E_i\}$ of sets such that $\bigcup_i E_i = X$.*

1.4. Definition. — *A partition* $\{E_i\}$ *is called a* $\mu$*-partition if*

$$\mu A = \sum_{i=1}^{\infty} \mu(A \cap E_i)$$

*for every* A *in* X *and where* $\mu$ *is a measure on* X.

1.5. Definition. — *If* $\mu$ *is a measure on* X, *then a partition* $\{E_i\}$ *is called a* $\mu$*-partition* F *if for every* E *of* F

$$\mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap E_i) \qquad \text{whenever} \qquad T \subset X.$$

1.6. Definition. — *If* $\{E_i\}$ *and* $\{F_j\}$ *are partitions, then* $\{E_i\}$ *is called a subpartition of* $\{F_j\}$ *if each* $E_i$ *is contained in some* $F_j$.

1.7. Definition. — *A set* E *is a* $\mu$-set *if the partition* $\{E, E'\}$ *is a* $\mu$*-partition.*

1.8. Definition. — *A set* D *is a* $\mu$*-set* F *if the partition* $\{D, D'\}$ *is a* $\mu$*-partition* F.

1.9. Definition. — *A is* $\mu$*-measurable* F *if* $\mu$ *is a measure and, for each member* E *of* F

$$\mu(TE) = \mu(TE \cap A) + \mu(TE \sim A)$$

*whenever* $T \subset X$.

1.10. Definition. — F *is* $\mu$*-convenient if* $\mu$ *is a measure,* F *is hereditary, and corresponding to each* T *of finite* $\mu$ *measure there exists such a sequence* C *that* $\mu\left(T \sim \bigcup_{j=0}^{\infty} C_j\right) = 0$ *and for each integer* $n$, $C_n \subset C_{n+1} \in F$ *and* $C_n$ *is a* $\mu$*-set* F.

1.11. Definition. — *Sect* $(\mu, B)$ *is the function* $f$ *on the subsets of* X *such that* $f(\alpha) = \mu(\alpha B)$ *for* $\alpha \subset X$.

1.12. Definition. — *If* $\rho$ *metrizes* X, *then*

$$\text{dist}\,(A, B) = \inf\,\{\rho(x, y);\, x \in A,\, y \in B\}.$$

1.13. Definition. — *If* X *is a topological space, then* $\mu$ *is a Radon measure on* X *if* $\mu$ *is a measure and*

(i) *open sets are* $\mu$*-measurable*

(ii) *if* C *is compact, then* $\mu C < \infty$

(iii) *if* $\alpha$ *is open, then* $\mu\alpha = \sup\{\mu C;\ C \text{ compact},\ C \subset \alpha\}$

(iv) *if* $A \subset X$, *then* $\mu A = \inf\{\mu\alpha,\ \alpha \text{ open},\ A \subset \alpha\}$.

1.14. Definition. — $(D, <)$ *is a directed set if* $D \neq 0$, D *is partially ordered by* $<$ *such that for any* $i, j \in D$, *there exists* $k \in D$ *with* $i < k$, $j < k$.

*Let* X *be a regular topological space;* $\mathcal{B}$ *be a base for the topology;* $(D, <)$ *be a directed set and for each* $i \in D$, $\mu_i$ *be a Radon measure on* X.

*For each* $\alpha \in \mathcal{B}$, *let*

$$g(\alpha, E) = \frac{\mathcal{L}t}{i \in D} \operatorname{Sect}(\mu_i, E)\alpha \quad \textit{where } E \textit{ is a member of } F.$$

*Let* $\varphi(A, E) = \inf\left\{\sum_{\alpha \in H} g(\alpha, E);\ H \textit{ countable},\ H \subset \mathcal{B},\ A \subset \bigcup_{\alpha \in H} \alpha\right\}$
*and* $\varphi^*(A, E) = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E)$ *where* $A \subset X$.

*Then* $\varphi$ *is a measure on* X *generated by* g *and* $\mathcal{B}$ [3].

## 2. Theorems and Corollaries.

2.1. Theorem. — *Product of two* $\mu$*-partitions* F *is a* $\mu$*-partition* F.

*Proof.* — Let $\{E_i\}$ and $\{F_i\}$ be two $\mu$-partitions F, then for every E of F

$$\mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap E_i) \qquad \text{and} \qquad \mu(TE) = \sum_{i=1}^{\infty} \mu(TE \cap F_i)$$

whenever $T \subset X$.

Since

$$\begin{aligned} \sum_{i,j} \mu(TE \cap E_i \cap F_j) &= \sum_j \left\{\sum_i \mu(TEF_j \cap E_i)\right\} \\ &= \sum_j \mu(TE \cap F_j) \\ &= \mu(TE), \end{aligned}$$

the proof is complete.

2.2. Theorem. — *If a subpartition* $\{F_i\}$ *of a partition* $\{E_i\}$ *is a* $\mu$*-partition* F, *then* $\{E_i\}$ *is a* $\mu$*-partition* F.

*Proof.* — For $T \subset X$ and any member E of F, we have

$$\sum_i \mu(TE \cap E_i) = \sum_i \mu[TE \cap \{\bigcup_j E_{ji}\}]$$

where $\bigcup_j E_{ji} = E_i$ and $E_{ji}$ is a member of $\{F_i\}$

$$\leqslant \sum_i \sum_j \mu(TE \cap E_{ji}) = \mu(TE),$$

since $\{F_i\}$ is a $\mu$-partition F. The reverse inequality is, however, clear. This proves the theorem.

2.3. Theorem. — *A partition* $\{E_i\}$ *is a* $\mu$*-partition* F *if each* $E_i$ *is a* $\mu$*-set* F.

*Proof.* — Suppose that each $E_i$ is a $\mu$-set F. Then for E in F and $T \subset X$, we have

$$\begin{aligned}\mu(TE) &= \mu(TE \cap E_1) + \mu(TE \cap E_1') \\ &= \mu(TE \cap E_1) + \mu(TE \cap \{E_2 \cup E_3 \cup \cdots\}).\end{aligned}$$

And

$$\begin{aligned}\mu(TE \cap \{E_2 \cup E_3 \cup \cdots\}) \\ &= \mu(TE \cap \{E_2 \cup E_3 \cup \cdots\} \cap E_2) \\ &+ \mu(TE \cap \{E_2 \cup E_3 \cup \cdots\} \cap E_2') \\ &= \mu(TE \cap E_2) + \mu(TE \cap \{E_3 \cup E_4 \cup \cdots\}).\end{aligned}$$

So,

$$\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \mu(TE \cap \{E_3 \cup E_4 \cup \cdots\}).$$

Proceeding in this way, we ultimately obtain

$$\mu(TE) = \mu(TE \cap E_1) + \mu(TE \cap E_2) + \cdots = \sum_i \mu(TE \cap E_i).$$

This proves that $\{E_i\}$ is a $\mu$-partition F.

Conversely, suppose that $\{E_i\}$ is a $\mu$-partition F. Then for every E of F and $T \subset X$, we have $\mu(TE) = \sum_i \mu(TE \cap E_i)$.

Replacing T by $T \cap \{E_2 \cup E_3 \cup \cdots\}$, we obtain

$$\begin{aligned}\mu(TE \cap \{E_2 \cup E_3 \cup \cdots\}) &= \sum_i \mu(TE \cap \{E_2 \cup E_3 \cup \cdots\} \cap E_i) \\ &= \mu(TE \cap E_2) + \mu(TE \cap E_3) + \cdots\end{aligned}$$

So,

$$\begin{aligned}\mu(\mathrm{TE}) &= \mu(\mathrm{TE} \cap \mathrm{E}_1) + \mu(\mathrm{TE} \cap \{\mathrm{E}_2 \cup \mathrm{E}_3 \cup \cdots\}) \\ &= \mu(\mathrm{TE} \cap \mathrm{E}_1) + \mu(\mathrm{TE} \cap \mathrm{E}_1').\end{aligned}$$

This shows that $E_1$ is a $\mu$-set F. Similarly, it can be shown that each $E_i$, $i = 2,3,\ldots$ is a $\mu$-set F.

Corollary. — *If* F *is $\mu$-convenient, then any $\mu$-partition* F *is a $\mu$-partition.*

*Proof.* — Let the partition $\{E_i\}$ be a $\mu$-partition F, then each $E_i$ is a $\mu$-set F. Since F is $\mu$-convenient, by Theorem 3.4 [5], a $\mu$-set F is a $\mu$-set. So, each $E_i$ is a $\mu$-set and consequently the partition $\{E_i\}$ is a $\mu$-partition {p. 48 [1]}.
In the following two theorems, we shall suppose that $\rho$ metrizes X.

2.4. Theorem. — *If* F *is hereditary and* $\mu(A \cup B) = \mu A + \mu B$ *whenever* A *and* B *are such members of* F *that* $d(A, B) > 0$, *then each open set is a $\mu$-set* F.

*Proof.* — This theorem is due to Trevor J. McMinn [5].

2.5. Theorem. — *If* F *is $\mu$-convenient and every open set is a $\mu$-set* F, *then $\mu$ is a metric outer measure.*

*Proof.* — It follows from Theorem 3.4 [5] that each open set is a $\mu$-set. Let A and B be two sets with $d(A, B) > 0$.
Let $\alpha$ be an open set such that $A \subset \alpha$ and $\alpha \cap B = 0$. Then

$$\begin{aligned}\mu(A \cup B) &= \mu(\{A \cup B\} \cap \alpha) + \mu(\{A \cup B\} \sim \alpha) \\ &= \mu A + \mu B.\end{aligned}$$

In the following theorems, we shall suppose that X is a regular topological space and $\mathcal{B}$ be a base for the topology.

2.6. Theorem. — *If* A *and* B *are disjoint, closed compact sets, then*

$$\varphi(A \cup B, E) = \varphi(A, E) + \varphi(B, E) \qquad \textit{for each} \qquad E \textit{ of } F.$$

*Proof.* — Let $\alpha$ and $\beta$ be two open sets such that $A \subset \alpha$, $B \subset \beta$ and $\alpha \cap \beta = 0$. This is possible, since X is regular. If $\varepsilon > 0$ is arbitrary, there exists a sequence $\{\nu_n\}$ of open sets such that

$$A \cup B \subset \bigcup_n \nu_n \qquad \text{and} \qquad \sum_n g(\nu_n, E) \leqslant \varphi(A \cup B, E) + \varepsilon.$$

Let $\nu'_n = \nu_n \cap \alpha$ and $\nu''_n = \nu_n \cap \beta$, then $\nu'_n$, $\nu''_n$ are open and

$$A \subset \bigcup_n \nu'_n, \qquad B \subset \bigcup_n \nu''_n.$$

So,

$$\begin{aligned}
\varphi(A, E) + \varphi(B, E) &\leqslant \sum_n \{g(\nu'_n, E) + g(\nu''_n, E)\} \\
&= \sum_n \{g(\nu_n \cap \alpha, E) + g(\nu_n \cap \beta, E)\} \\
&= \sum_n \left\{ \frac{\mathfrak{L}t}{i \in D} \operatorname{Sect}(\mu_i, E)(\nu_n \cap \alpha) \right. \\
&\quad \left. + \frac{\mathfrak{L}t}{i \in D} \operatorname{Sect}(\mu_i, E)(\nu_n \cap \beta) \right\} \\
&\leqslant \sum_n \left\{ \frac{\mathfrak{L}t}{i \in D} \operatorname{Sect}(\mu_i, E)\nu_n \right\} \\
&= \sum_n g(\nu_n, E) \\
&\leqslant \varphi(A \cup B, E) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\varphi(A, E) + \varphi(B, E) \leqslant \varphi(A \cup B, E).$$

The reverse inequality is clear, because $\varphi$ is a Carathéodory measure. This proves the theorem.

2.7. Theorem. — *For each* E *of* F, $\varphi^*$ *is a Radon measure on* X.

*Proof.* — (i) If $\alpha$ is any open set, by definition

$$\varphi^*(\alpha, E) = \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E) \leqslant \varphi(\alpha, E).$$

So, for any $A \subset X$, we have

$$\begin{aligned}
\varphi^*(A, E) &= \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E) = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \varphi^*(\alpha, E) \\
&\leqslant \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \varphi(\alpha, E) = \varphi(A, E).
\end{aligned}$$

If C is compact and $\alpha$ is open, $C \subset \alpha$, we have

$$\varphi(C, E) \leqslant \varphi^*(\alpha, E), \qquad \text{so} \qquad \varphi(C, E) \leqslant \varphi^*(C, E).$$

Therefore, if C is compact, $\varphi(C, E) = \varphi^*(C, E)$. But, it is clear that for any compact C, $\varphi(C, E) < \infty$ and hence $\varphi^*(C, E) < \infty$.

(ii) Let $\alpha$ be an open set, $T \subset X$ and $\varepsilon > 0$ arbitrary. Since for any $A \subset X$, $\varphi^*(A, E) = \inf\limits_{\substack{\nu \text{ open} \\ A \subset \nu}} \varphi^*(\nu, E)$, there exists open set $T'$, $T \subset T'$ and $\varphi^*(T', E) < \varphi^*(T, E) + \varepsilon$.

Also, $\varphi^*(\alpha, E) = \sup\limits_{\substack{C \text{ compact} \\ C \subset \alpha}} \varphi(C, E)$.

Therefore, since X is regular, there exists a closed compact set $C_1 \subset T' \cap \alpha$ such that $\varphi^*(T' \cap \alpha, E) \leqslant \varphi(C_1, E) + \varepsilon$. Similarly, there exists a closed compact set $C_2 \subset T' \sim C_1$ such that $\varphi^*(T' \sim C_1, E) \leqslant \varphi(C_2, E) + \varepsilon$.

So,

$$\begin{aligned}
\varphi^*(T \cap \alpha, E) + \varphi^*(T \sim \alpha, E)& \\
&\leqslant \varphi^*(T' \cap \alpha, E) + \varphi^*(T' \sim C_1, E) \\
&\leqslant \varphi(C_1, E) + \varphi(C_2, E) + 2\varepsilon \\
&= \varphi^*(C_1 \cup C_2, E) + 2\varepsilon, \text{ by Theorem 2.6} \\
&\leqslant \varphi^*(T', E) + 2\varepsilon \\
&\leqslant \varphi^*(T, E) + 3\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that $\alpha$ is $\varphi^*$-measurable.

The other properties are evident. This proves the theorem.

2.8. Theorem. — *If* A *and* B *are sets of which any one of them is open and* $A \cap B = 0$, *then*

$$\varphi^*(A \cup B, E) = \varphi^*(A, E) + \varphi^*(B, E) \qquad \textit{for each} \quad E \ \textit{of} \ F.$$

*Proof.* — Let A be open and so it is $\varphi^*$-measurable. Hence

$$\begin{aligned}
\varphi^*(A \cup B, E) &= \varphi^*\{(A \cup B) \cap A, E\} + \varphi^*\{(A \cup B) \sim A, E\} \\
&= \varphi^*(A, E) + \varphi^*(B, E).
\end{aligned}$$

2.9. Theorem. — *If* X *is a metric space, then* $\varphi^*$ *is a metric outer measure.*

*Proof.* — This is clear.

In conclusion, I offer my best thanks to Dr. B. K. Lahiri of Calcutta University for his helpful guidance in the preparation of this paper.

## BIBLIOGRAPHY

[1] P. R. Halmos, Measure Theory (1950).
[2] J. -L. Kelley, General Topology (1955).
[3] M. E. Munroe, Measure and Integration (1952).
[4] M. Sion, A characterization of weak convergence, *Pacific Journal of Mathematics* (1964), vol. 14, n° 3, 1059.
[5] Trevor J. McMinn, Restricted Measurability, *Bull. Amer. Math. Soc.* (1948), vol. 54, July-Dec., 1105.

Manuscrit reçu le 24 septembre 1965.

Asim K. Mookhopadhyaya,
Department of Mathematics,
Charuchandra College,
22 Lake Road,
Calcutta-29 (Inde)