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RAABE'S FORMULA FOR p-ADIC GAMMA AND ZETA FUNCTIONS

by Henri COHEN & Eduardo FRIEDMAN (*)

Abstract. — The classical Raabe formula computes a definite integral of the logarithm of Euler's Γ-function. We compute p-adic integrals of the p-adic $\log\Gamma$-functions, both Diamond's and Morita's, and show that each of these functions is uniquely characterized by its difference equation and p-adic Raabe formula. We also prove a Raabe-type formula for p-adic Hurwitz zeta functions.

Résumé. — La formule de Raabe classique donne la valeur de l'intégrale de la fonction log gamma d'Euler sur un intervalle de longueur 1. Nous calculons des intégrales p-adiques analogues pour les fonctions log gamma p-adiques de Diamond et de Morita, et nous montrons que chacune de ces fonctions est caractérisée de manière unique par son équation fonctionnelle et sa formule de Raabe p-adique. Nous démontrons aussi une formule de type Raabe pour les fonctions zêta de Hurwitz p-adiques.

1. Introduction

Some 30 years ago Diamond [2] defined a p-adic analogue $\operatorname{Log}\Gamma_{\mathrm{D}}(x)$ of Euler's classical function $\log\Gamma(x)$. It takes values in the completion $\mathbb{C}_p$ of the algebraic closure of the p-adic field $\mathbb{Q}_p$ and is defined for $x\in\mathbb{C}_p-\mathbb{Z}_p$ [8, §60], where $\mathbb{Z}_p\subset\mathbb{Q}_p$ denotes the ring of p-adic integers. We recall the definition of $\operatorname{Log}\Gamma_{\mathrm{D}}(x)$ in (1.18) below, but the most important feature of $\operatorname{Log}\Gamma_{\mathrm{D}}(x)$ is its difference equation

$$\operatorname{Log}\Gamma_{\mathrm{D}}(x+1)-\operatorname{Log}\Gamma_{\mathrm{D}}(x)=\log_p x,\quad (x\in\mathbb{C}_p-\mathbb{Z}_p), \tag{1.1}$$

where $\log_p$ on the right denotes the Iwasawa p-adic logarithm (so $\log_p p=0$).

Diamond's $\operatorname{Log}\Gamma_{\mathrm{D}}(x)$ is not the logarithm of any function, but our notation is meant to recall the analogy with the classical $\log\Gamma$ function and

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with Morita's [4] [8, §35] alternative p-adic analogue $\operatorname{Log}\Gamma_{\mathrm{M}}(x)$, which we define in (1.14) below. Morita's function has for domain the p-adic integers $\mathbb{Z}_p$, satisfies the modified difference equation

$$\operatorname{Log}\Gamma_{\mathrm{M}}(x+1) - \operatorname{Log}\Gamma_{\mathrm{M}}(x) = \begin{cases} \log_p x & \text{if } x \in \mathbb{Z}_p^*, \\ 0 & \text{if } x \in p\mathbb{Z}_p, \end{cases} \tag{1.2}$$

and is actually the Iwasawa logarithm of Morita's $\Gamma_p(x)$ [7, §7.1] [8, §58].

Morita's $\operatorname{Log}\Gamma_{\mathrm{M}}$, being continuous and having domain $\mathbb{Z}_p$, is uniquely determined by $\operatorname{Log}\Gamma_{\mathrm{M}}(1) = 0$ and by its difference equation (1.2). Diamond's $\operatorname{Log}\Gamma_{\mathrm{D}}$, on the other hand, is far from being characterized by its difference equation (1.1), as there are many non-constant continuous periodic functions on $\mathbb{C}_p - \mathbb{Z}_p$.[(1)]

In this paper we compute a Volkenborn integral of $\operatorname{Log}\Gamma_{\mathrm{D}}$ in terms of its derivative $(\operatorname{Log}\Gamma_{\mathrm{D}})'$, and show that this integral formula and the difference equation characterize $\operatorname{Log}\Gamma_{\mathrm{D}}$.

THEOREM. — *Diamond's* $\operatorname{Log}\Gamma_{\mathrm{D}}$ *satisfies*

$$\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{D}}(x+t)\,dt = (x-1)(\operatorname{Log}\Gamma_{\mathrm{D}})'(x) - x + \frac{1}{2}, \quad (x \in \mathbb{C}_p - \mathbb{Z}_p). \tag{1.3}$$

It is the unique strictly differentiable function $f : \mathbb{C}_p - \mathbb{Z}_p \to \mathbb{C}_p$ *satisfying the difference equation*

$$f(x+1) - f(x) = \log_p x \tag{1.4}$$

and the Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (x-1)f'(x) - x + \frac{1}{2}. \tag{1.5}$$

We prove slightly more. Namely, the uniqueness statement above also holds if we replace $\mathbb{C}_p - \mathbb{Z}_p$ by $\mathbb{Q}_p - \mathbb{Z}_p$, or by any subset $D \subset \mathbb{C}_p - \mathbb{Z}_p$ such that $x \in D$ and $t \in \mathbb{Z}_p$ imply $x + t \in D$.

Recall that the Volkenborn integral of a function $g\colon \mathbb{Z}_p \to \mathbb{C}_p$ is defined by

$$\int_{\mathbb{Z}_p} g(t)\,dt := \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n-1} g(j), \tag{1.6}$$

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(1) For instance, $h(x+1) = h(x)$ for all $x \in \mathbb{C}_p$ if we take $h(x) := 1$ for $|x|_p \leqslant 1$, $h(x) := \|x\|_p$ for $|x|_p > 1$. Here $|x|_p$ denotes the usual absolute value on $\mathbb{C}_p$, and $\|x\|_p$ is the embedding of this value into $\mathbb{C}_p$ obtained by choosing an injective homomorphism of $p^{\mathbb{Q}}$ into $\mathbb{C}_p^*$ [7, p. 147].

and that this limits exists if g is strictly differentiable on $\mathbb{Z}_p$ [7, p. 264], [8, p. 167]. A function $g : X \to \mathbb{C}_p$ is strictly differentiable on a subset $X \subset \mathbb{C}_p$ (assumed not to have isolated points) if at all points $a \in X$

$$\lim_{(x,y)\to(a,a)} \frac{g(x) - g(y)}{x - y} \tag{1.7}$$

exists, the limit being restricted to $x, y \in X,\ x \neq y$ [7, p. 221].

There are several characterizations of the classical function $\log \Gamma(x)$, the most famous one being the 1922 Bohr-Mollerup theorem [1, p. 35] stating that $\log\big(\Gamma(x)\big)$ is the unique convex function on $(0, \infty)$ satisfying $F(1) = 0$ and

$$F(x + 1) - F(x) = \log(x). \tag{1.8}$$

In the p-adic domain the above theorem is the first characterization of $\mathrm{Log}\,\Gamma_{\mathrm{D}}$ known to the authors.

The uniqueness part of the theorem is actually very easy, as p-adic functions satisfying both a difference equation and a linear integro-differential equation are highly restricted.

Proposition. — *Let $D \subset \mathbb{C}_p$ be any subset of $\mathbb{C}_p$ such that $x \in D$ and $t \in \mathbb{Z}_p$ imply $x + t \in D$, let $f : D \to \mathbb{C}_p$ and $g : D \to \mathbb{C}_p$ be strictly differentiable and n times differentiable on D for some $n \geqslant 1$. Suppose finally that $F = f$ and $F = g$ are solutions of the difference equation*

$$F(x + 1) - F(x) = b_0(x), \quad (x \in D) \tag{1.9}$$

and of the n-th order Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} F(x + t)\, dt = b_1(x) + a_0(x)F(x) + \sum_{j=1}^{n} a_j(x)F^{(j)}(x), \quad (x \in D) \tag{1.10}$$

for some arbitrary functions $a_j : D \to \mathbb{C}_p$ and $b_k : D \to \mathbb{C}_p$ $(0 \leqslant j \leqslant n,$ $0 \leqslant k \leqslant 1)$. Then $f(x) = g(x)$ for all $x \in D$ such that $a_0(x) \neq 1$.

We prove the theorem and proposition above in §2.

As a corollary of the proposition we see that the Volkenborn integral $\int_{\mathbb{Z}_p} f(x + t)\, dt$ together with the difference $f(x + 1) - f(x)$ uniquely characterize f.

Corollary. — *Suppose $D \subset \mathbb{C}_p$ is as above and that $w : D \to \mathbb{C}_p$ and $v : D \to \mathbb{C}_p$ are arbitrary p-adic functions on D. Then there is at most one strictly differentiable function $f : D \to \mathbb{C}_p$ satisfying both*

$$f(x+1) - f(x) = w(x) \quad \text{and} \quad \int_{\mathbb{Z}_p} f(x+t)\, dt = v(x), \quad (x \in D). \tag{1.11}$$

More precisely, an "integration by parts" (see Lemma 2.1) shows that

$$f(x) = v(x) + \int_{\mathbb{Z}_p} (t+1)w(x+t)dt, \quad (x \in D). \tag{1.12}$$

We note that $f(x+1) - f(x) = w(x)$ implies that $w(x)$ is strictly differentiable on D, so that the above Volkenborn integral is defined.

We turn next to Morita's p-adic Γ-function $\Gamma_{\mathrm{M}} : \mathbb{Z}_p \to \mathbb{Z}_p^*$ (usually denoted Γ_p). For positive integers n [4] [7, pp. 368, 385] [8, §35] it is defined as

$$\Gamma_{\mathrm{M}}(n) = (-1)^n \prod_{\substack{1 \leqslant j < n \\ p \nmid j}} j.$$

Being a continuous function on $\mathbb{Z}_p$, Γ_{M} is characterized by $\Gamma_{\mathrm{M}}(1) = -1$ and

$$\frac{\Gamma_{\mathrm{M}}(x+1)}{\Gamma_{\mathrm{M}}(x)} = \begin{cases} -x & \text{if } x \in \mathbb{Z}_p^*, \\ -1 & \text{if } x \in p\mathbb{Z}_p. \end{cases} \tag{1.13}$$

In §2 we prove a p-adic Raabe formula for the Iwasawa logarithm $\operatorname{Log}\Gamma_{\mathrm{M}}$ of Γ_{M}

$$\operatorname{Log}\Gamma_{\mathrm{M}}(x) := \log_p \Gamma_{\mathrm{M}}(x). \tag{1.14}$$

Namely,

$$\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{M}}(x+t)\, dt = (x-1)(\operatorname{Log}\Gamma_{\mathrm{M}})'(x) - x + \left\lceil \frac{x}{p} \right\rceil, \tag{1.15}$$

where $\lceil \frac{x}{p} \rceil$ is the p-adic limit of the usual integer ceiling function $\lceil \frac{x_n}{p} \rceil$ as $x_n \to x$ through $x_n \in \mathbb{Z}$. Actually, $\operatorname{Log}\Gamma_{\mathrm{M}}(x)$ is given by a power series convergent on the open disk $\{x \in \mathbb{C}_p \big| \; |x|_p < 1\}$ [7, p. 376], but we have not investigated how the above formula might extend beyond $x \in \mathbb{Z}_p$.

Taking $x = 1$ in (1.15) we find

$$\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{M}}(1+t)\, dt = 0. \tag{1.16}$$

It is easy to see that the difference equation (1.2) and the Raabe formula (1.15) uniquely characterize $\operatorname{Log}\Gamma_{\mathrm{M}}$ among all continuous functions on $\mathbb{Z}_p$. In fact, the single integral (1.16) suffices to fix the constant left undetermined by the difference equation.

The p-adic integrals $\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{D}}(x+t)\, dt$ and $\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{M}}(x+t)\, dt$ are analogues of Raabe's 1843 formula [6, p. 89]

$$\int_0^1 \log\left(\frac{\Gamma(x+t)}{\sqrt{2\pi}}\right) dt = x\log x - x, \quad (x \geqslant 0). \tag{1.17}$$

Although Raabe's formula has largely been ignored, an extension of it to higher Γ-functions [3] was recently used to simplify work of Barnes and Shintani on these functions. The p-adic Raabe formulas proved here give another indication that Raabe-type formulas may be more than a curiosity.

J. Diamond [2] defined his function as the Volkenborn integral

$$\operatorname{Log}\Gamma_{\mathrm{D}}(x) := \int_{\mathbb{Z}_p} \big((x+t)\log_p(x+t) - (x+t)\big)\, dt. \tag{1.18}$$

The appearance above of the p-adic analogue of the $x \log x - x$ in Raabe's formula (1.17) is no coincidence [8, §58]. If a classical (continuously differentiable) function f satisfies a difference equation

$$f(x+1) - f(x) = g(x), \tag{1.19}$$

then its "Raabe" integral

$$G(x) := \int_0^1 f(x+t)\, dt$$

is connected to a p-adic version f_p of f as follows. Notice first that $G'(x) = g(x)$, so G determines the difference equation. If G has a (strictly differentiable) p-adic version G_p, then $g_p(x) := G_p'(x)$ is a p-adic version of $g(x) = G'(x)$. If we set

$$f_p(x) := \int_{\mathbb{Z}_p} G_p(x+t)\, dt, \tag{1.20}$$

then f_p satisfies

$$f_p(x+1) - f_p(x) = g_p(x). \tag{1.21}$$

The function $f_p(x)$ is thus a good canditate for a p-adic version of $f(x)$, for it satisfies the difference equation (1.21), in analogy with (1.19).

Raabe's classical formula (1.17) is related to the integral formula (see §3)

$$\int_0^1 \zeta(s, x+t)\, dt = \frac{x^{1-s}}{s-1}, \quad (\mathrm{Re}(x) > 0,\ s \in \mathbb{C},\ s \neq 1) \tag{1.22}$$

for the Hurwitz zeta function

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad (\mathrm{Re}(x) > 0,\ \mathrm{Re}(s) > 1). \tag{1.23}$$

The Hurwitz zeta function has an analytic s-continuation to $s = 0$ and satisfies Lerch's formula [1, p. 17]

$$\log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) = \frac{\partial \zeta}{\partial s}(0, x). \tag{1.24}$$

One easily deduces Raabe's original formula (1.17) from Lerch's and (1.22). In §3 we prove Raabe-type formulas for the p-adic Hurwitz zeta functions, which like $\operatorname{Log}\Gamma_{\mathrm{D}}$ and $\operatorname{Log}\Gamma_{\mathrm{M}}$ also come in two varieties according to the domain. In §2 and §3 we also apply our p-adic Raabe formula characterizations to give quick proofs of some basic properties of p-adic $\log\Gamma$ and p-adic Hurwitz zeta functions.

2. p-adic Raabe formulas

Throughout this section $D \subset \mathbb{C}_p$ is a an arbitrary subset closed under $x \to x+t$ for $t \in \mathbb{Z}_p$ and $x \in D$. In particular, D could be $\mathbb{C}_p - \mathbb{Z}_p$, $\mathbb{Q}_p - \mathbb{Z}_p$ or $\mathbb{Z}_p$. Suppose $f : D \to \mathbb{C}_p$ is strictly differentiable on D (see (1.7)), so that for fixed $x \in D$ the function $t \to f(x+t)$ is strictly differentiable on $\mathbb{Z}_p$. The Volkenborn integral

$$F(x) := \int_{\mathbb{Z}_p} f(x+t)\,dt, \quad (x \in D) \tag{2.1}$$

is then defined and satisfies [7, p. 265]

$$F(x+1) - F(x) = f'(x). \tag{2.2}$$

We shall need to cite a result allowing us to differentiate F. Call a function $f : D \to \mathbb{C}_p$ twice strictly differentiable on D if it admits a second order expansion

$$f(x) = f(y) + (x-y)\alpha(y) + (x-y)^2\beta(x,y), \quad (x,\, y \in D), \tag{2.3}$$

where α and β are continuous functions from $D \times D$ to $\mathbb{C}_p$ [7, p. 224]. Then F in (2.1) and f' are strictly differentiable and [7, pp. 223, 268]

$$F'(x) := \int_{\mathbb{Z}_p} f'(x+t)\,dt, \quad (x \in D). \tag{2.4}$$

We shall need a kind of integration by parts formula.

Lemma 2.1. — *Suppose $f : D \to \mathbb{C}_p$ is such that $w(x) := f(x+1) - f(x)$ is strictly differentiable on D. Then the Volkenborn integral $\int_{\mathbb{Z}_p} f(x+t)\,dt$ exists for any $x \in D$ and is given by*

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = f(x) - \int_{\mathbb{Z}_p} (t+1)w(x+t)\,dt, \quad (x \in D). \tag{2.5}$$

Proof. — Note that $t \to (t+1)w(x+t)$ is strictly differentiable on $\mathbb{Z}_p$, being a product of two such functions, so the Volkenborn integral on the right-hand side above is defined. Also note the telescoping sum

$$f(x+j) = f(x) + \sum_{k=0}^{j-1} w(x+k), \quad (j \geqslant 0,\ j \in \mathbb{Z}).$$

Hence,

$$\begin{aligned}\frac{1}{p^n}\sum_{j=0}^{p^n-1} f(x+j) &= f(x) + \frac{1}{p^n}\sum_{j=0}^{p^n-1}\sum_{k=0}^{j-1} w(x+k)\\ &= f(x) + \frac{1}{p^n}\sum_{k=0}^{p^n-2}(p^n-1-k)w(x+k)\\ &= f(x) + \frac{1}{p^n}\sum_{k=0}^{p^n-1}(p^n-1-k)w(x+k)\\ &= f(x) + \sum_{k=0}^{p^n-1} w(x+k) - \frac{1}{p^n}\sum_{k=0}^{p^n-1}(k+1)w(x+k).\end{aligned}$$

The lemma follows on taking $\lim_{n\to\infty}$ above, noting that

$$\lim_{n\to\infty}\sum_{k=0}^{p^n-1} w(x+k) = \Big(\lim_{n\to\infty} p^n\Big)\Big(\lim_{n\to\infty}\frac{1}{p^n}\sum_{k=0}^{p^n-1} w(x+k)\Big) = 0,$$

since w is assumed strictly differentiable. □

Formula (1.12) and the corollary in §1 follow from the lemma just proved. The next lemma will lead us to p-adic Raabe formulas.

Lemma 2.2. — *Suppose $G : D \to \mathbb{C}_p$ is twice strictly differentiable (see (2.3)), and for $x \in D$, set $F(x) := \int_{\mathbb{Z}_p} G(x+t)\,dt$. Then*

$$\int_{\mathbb{Z}_p} F(x+t)\,dt = F(x) + (x-1)F'(x) - \int_{\mathbb{Z}_p}(x+t)G'(x+t)\,dt, \quad (x \in D).$$

Proof. — Using the previous lemma and (2.2) we find

$$\begin{aligned}\int_{\mathbb{Z}_p} F(x+t)\,dt &= F(x) - \int_{\mathbb{Z}_p}(t+1)G'(x+t)\,dt\\ &= F(x) + (x-1)\int_{\mathbb{Z}_p} G'(x+t)\,dt - \int_{\mathbb{Z}_p}(x+t)G'(x+t)\,dt,\end{aligned}$$

whence the lemma follows from (2.4). □

Diamond's $\operatorname{Log}\Gamma_{\mathrm{D}}$ is defined by

$$\operatorname{Log}\Gamma_{\mathrm{D}}(x) := \int_{\mathbb{Z}_p} \big((x+t)\log_p(x+t) - (x+t)\big)\,dt, \quad (x \in \mathbb{C}_p - \mathbb{Z}_p), \tag{2.6}$$

where $\log_p$ is the Iwasawa p-adic logarithm [8, §60]. We can now prove a p-adic Raabe formula.

Proposition 2.3.

$$\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{D}}(x+t)\,dt = (x-1)(\operatorname{Log}\Gamma_{\mathrm{D}})'(x) - x + \frac{1}{2}, \quad (x \in \mathbb{C}_p - \mathbb{Z}_p).$$

Proof. — For $x \in \mathbb{C}_p - \{0\}$, let $G(x) := x\log_p(x) - x$. Then one checks that G is twice strictly differentiable on $\mathbb{C}_p - \{0\}$ and that $G'(x) = \log_p(x)$. From Lemma 2.2 and (2.6),

$$\begin{aligned}\int_{\mathbb{Z}_p} &\operatorname{Log}\Gamma_{\mathrm{D}}(x+t)\,dt \\ &= \operatorname{Log}\Gamma_{\mathrm{D}}(x) + (x-1)(\operatorname{Log}\Gamma_{\mathrm{D}})'(x) - \int_{\mathbb{Z}_p}(x+t)\log_p(x+t)\,dt \\ &= (x-1)(\operatorname{Log}\Gamma_{\mathrm{D}})'(x) - \int_{\mathbb{Z}_p}(x+t)\,dt \\ &= (x-1)(\operatorname{Log}\Gamma_{\mathrm{D}})'(x) - x - \int_{\mathbb{Z}_p} t\,dt.\end{aligned}$$

The proposition follows since

$$\int_{\mathbb{Z}_p} t\,dt = \lim_{n\to\infty}\frac{1}{p^n}\sum_{j=0}^{p^n-1} j = \lim_{n\to\infty}\frac{p^n-1}{2} = -\frac{1}{2}.$$

□

We now prove a Raabe formula for Morita's $\operatorname{Log}\Gamma_{\mathrm{M}}$.

Proposition 2.4.

$$\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{M}}(x+t)\,dt = (x-1)(\operatorname{Log}\Gamma_{\mathrm{M}})'(x) - x + \left\lceil\frac{x}{p}\right\rceil, \quad (x \in \mathbb{Z}_p),$$

where $\lceil\frac{x}{p}\rceil$ is the p-adic ceiling function defined after (1.15).

Proof. — We first prove the Volkenborn integral representation [8, p. 177]

$$\operatorname{Log}\Gamma_{\mathrm{M}}(x) = \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(x+t)\big((x+t)\log_p(x+t) - (x+t)\big)\,dt, \quad (x \in \mathbb{Z}_p) \tag{2.7}$$

where $\chi_{\mathbb{Z}_p^*}$ denotes the characteristic function of $\mathbb{Z}_p^*$. Indeed, using (1.14), (1.2) and (2.2) we find that both sides of (2.7) satisfy the same difference equation, hence differ by a constant c for all $x \in \mathbb{Z}_p$. Since $\operatorname{Log}\Gamma_{\mathrm{M}}(0) = \operatorname{Log}\Gamma_{\mathrm{M}}(1) = 0$, taking $x = 0$ we find

$$c = \int_{\mathbb{Z}_p} \chi_{\mathbb{Z}_p^*}(t)(t\log_p(t) - t)\,dt. \tag{2.8}$$

Now, for any odd strictly differentiable function f on $\mathbb{Z}_p$ we have [7, p. 269]

$$\int_{\mathbb{Z}_p} f(t)\,dt = -\frac{f'(0)}{2}.$$

Since the integrand in (2.8) is odd and vanishes identically in a neighborhood of 0, we find $c = 0$.

Having established the integral representation (2.7), the proof of Proposition 2.4 now follows exactly that of Proposition 2.3, except that at the end we must show

$$\int_{\mathbb{Z}_p} (x+t)\cdot\chi_{\mathbb{Z}_p^*}(x+t)\,dt = x - \left\lceil \frac{x}{p} \right\rceil, \quad (x \in \mathbb{Z}_p). \tag{2.9}$$

To prove this it suffices to take x a positive integer (see [7, pp. 230, 265] for a metric on the space of strictly differentiable functions which makes the Volkenborn integral $\int_{\mathbb{Z}_p} f(t)\,dt$ into a continuous function of the integrand f). We calculate

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (x+t)\cdot\chi_{\mathbb{Z}_p^*}(x+t)\,dt \\
&\qquad = \lim_{n\to\infty}\frac{1}{p^n}\sum_{\substack{0\leqslant j<p^n\\ p\nmid(x+j)}}(x+j) \\
&\qquad = \lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leqslant j<p^n}(x+j) - \lim_{n\to\infty}\frac{1}{p^n}\sum_{\substack{0\leqslant j<p^n\\ p\mid(x+j)}}(x+j) \\
&\qquad = x + \lim_{n\to\infty}\frac{1}{p^n}\sum_{0\leqslant j<p^n} j - \lim_{n\to\infty}\frac{1}{p^n}\sum_{\substack{x\leqslant k<x+p^n\\ p\mid k}} k \\
&\qquad = x - \frac{1}{2} - \lim_{n\to\infty}\frac{1}{p^n}\sum_{\lceil \frac{x}{p}\rceil\leqslant \ell<\lceil \frac{x}{p}\rceil+p^{n-1}} p\ell = x - \left\lceil\frac{x}{p}\right\rceil,
\end{aligned}$$

as claimed. □

To complete the proof of the theorem given in §1, it only remains to establish the uniqueness claimed in the proposition in §1. Let $h(x) := f(x) - g(x)$ be the difference of two functions, both satisfying the difference and integro-differential equations (1.9) and (1.10). Thus, $h(x+1)=h(x)$ and

$$\int_{\mathbb{Z}_p} h(x+t)\,dt = a_0(x)h(x) + \sum_{j=1}^{n} a_j(x)h^{(j)}(x) \tag{2.10}$$

for all $x \in D$. Using $h(x+k)=h(x)$ for any non-negative integer k and the definition of the Volkenborn integral, we find $\int_{\mathbb{Z}_p} h(x+t)\,dt = h(x)$. Since $h(x+p^n)=h(x)$ for any integer $n \geqslant 0$, we also find that $h'(x)=0$ for all $x \in D$. Hence all higher derivatives $h^{(j)}(x)$ also vanish. The integro-differential equation (2.10) then simplifies to $h(x)=a_0(x)h(x)$, whence the proposition in §1.

The characterization of $\operatorname{Log}\Gamma_{\mathrm{D}}$ by its difference equation and Raabe formula can be used to give quick proofs of some known properties of p-adic $\log\Gamma$ functions, such as the following [8, pp. 182–183] [7, p. 369].

Proposition 2.5.

$$\operatorname{Log}\Gamma_{\mathrm{D}}(1-x) = -\operatorname{Log}\Gamma_{\mathrm{D}}(x), \quad (x \in \mathbb{C}_p - \mathbb{Z}_p). \tag{2.11}$$

$$\operatorname{Log}\Gamma_{\mathrm{M}}(x) = \sum_{\substack{0\leqslant j\leqslant p^N-1\\ p\nmid(x+j)}} \operatorname{Log}\Gamma_{\mathrm{D}}\Big(\frac{x+j}{p^N}\Big), \qquad (x \in \mathbb{Z}_p,\ N=1,2,3,\cdots). \tag{2.12}$$

$$\operatorname{Log}\Gamma_{\mathrm{M}}(1-x) = -\operatorname{Log}\Gamma_{\mathrm{M}}(x), \quad (x \in \mathbb{Z}_p). \tag{2.13}$$

Proof. — Let $f(x) := -\operatorname{Log}\Gamma_{\mathrm{D}}(1-x)$. A direct calculation using $\log_p(-x)=\log_p x$ shows that f satisfies the same difference equation (1.4) as $\operatorname{Log}\Gamma_{\mathrm{D}}(x)$. Recalling the general property [7, p. 268]

$$\int_{\mathbb{Z}_p} g(t)\,dt = \int_{\mathbb{Z}_p} g(-1-t)\,dt,$$

where $g:\mathbb{Z}_p \to \mathbb{C}_p$ is any strictly differentiable function, we have

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = -\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{D}}(1-x-t)\,dt = -\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{D}}(2-x+t)\,dt.$$

This last integral can be evaluated using the Raabe formula in Proposition 2.3, after which a routine calculation shows that f satisfies the integro-differential equation (1.5). Hence $f=\operatorname{Log}\Gamma_{\mathrm{D}}$, as claimed in (2.11).

We now prove (2.12). Quite generally, given $f:\mathbb{C}_p-\mathbb{Z}_p \to \mathbb{C}_p$ we can define $g:\mathbb{Z}_p \to \mathbb{C}_p$ by

$$g(x) := \sum_{\substack{0\leqslant j\leqslant p^N-1\\ p\nmid(x+j)}} f\Big(\frac{x+j}{p^N}\Big).$$

Then

$$g(x+1)-g(x)=\begin{cases} f\big(1+\dfrac{x}{p^N}\big)-f\big(\dfrac{x}{p^N}\big) & \text{if } x\in\mathbb{Z}_p^*,\\ 0 & \text{if } x\in p\mathbb{Z}_p.\end{cases}$$

Since $\log_p\left(\frac{x}{p^N}\right)=\log_p(x)$, we see that the right-hand side of (2.12) satisfies the same difference equation as $\operatorname{Log}\Gamma_{\mathrm{M}}(x)$. It also vanishes at $x=0$ since

$$\operatorname{Log}\Gamma_{\mathrm{D}}\big(\frac{j}{p^N}\big)=-\operatorname{Log}\Gamma_{\mathrm{D}}\big(\frac{p^N-j}{p^N}\big)\qquad(1\leqslant j\leqslant p^N-1).$$

Hence both sides of (2.12) coincide.

To prove (2.13), take $N=1$ in (2.12) and calculate

$$\begin{aligned}\operatorname{Log}\Gamma_{\mathrm{M}}(1-x)&=\sum_{\substack{0\leqslant k<p\\ p\nmid(1-x+k)}}\operatorname{Log}\Gamma_{\mathrm{D}}\left(\frac{1-x+k}{p}\right)\\ &=\sum_{\substack{0\leqslant j<p\\ p\nmid(1-x+(p-1-j))}}\operatorname{Log}\Gamma_{\mathrm{D}}\left(\frac{1-x+(p-1-j)}{p}\right)\\ &=\sum_{\substack{0\leqslant j<p\\ p\nmid(x+j))}}\operatorname{Log}\Gamma_{\mathrm{D}}\left(1-\frac{x+j}{p}\right)\\ &=-\sum_{\substack{0\leqslant j<p\\ p\nmid(x+j))}}\operatorname{Log}\Gamma_{\mathrm{D}}\left(\frac{x+j}{p}\right)=-\operatorname{Log}\Gamma_{\mathrm{M}}(x).\end{aligned}$$

□

3. Raabe formulas for p-adic Hurwitz zeta functions

Before passing to the p-adic domain, we take a quick look at the classical Hurwitz zeta function

$$\zeta(s,x):=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s},\qquad(\operatorname{Re}(x)>0,\ \operatorname{Re}(s)>1).\tag{3.1}$$

The difference equation

$$\zeta(s,x+1)-\zeta(s,x)=-x^{-s}$$

follows immediately from the series (3.1) defining $\zeta(s,x)$. Differentiating this same series we obtain

$$\frac{\partial\zeta}{\partial x}(s,x)=-s\sum_{n=0}^{\infty}\frac{1}{(n+x)^{s+1}}=-s\zeta(s+1,x).\tag{3.2}$$

Hence

$$\int_0^1 \zeta(s,x+t)\,dt = \frac{\zeta(s-1,x+1)-\zeta(s-1,x)}{1-s} = \frac{x^{1-s}}{s-1}, \tag{3.3}$$

where in the last step we used the difference equation. In this section we give a p-adic version of this Raabe-type formula.

Define the Washington-Hurwitz p-adic zeta function [9] [10, §5.2] by

$$\zeta_{\mathrm{D}}(s,x) = \frac{1}{s-1}\int_{\mathbb{Z}_p} \omega_{\mathrm{D}}(x+t)\langle x+t\rangle^{1-s}\,dt, \quad (x\in\mathbb{Q}_p-\mathbb{Z}_p,\ s\in\mathbb{Z}_p,\ s\neq 1), \tag{3.4}$$

where $\omega_{\mathrm{D}} : \mathbb{Q}_p^* \to \mathbb{Q}_p^*$ is the unique multiplicative extension of the Teichmüller character ω from $\mathbb{Z}_p^*$ to $\mathbb{Q}_p^*$ such that $\omega_{\mathrm{D}}(p) = p$, and $\langle x\rangle := x/\,\omega_{\mathrm{D}}(x)$. Notice that for $x\in\mathbb{Q}_p-\mathbb{Z}_p$ and $t\in\mathbb{Z}_p$,

$$\omega_{\mathrm{D}}(x+t) = \omega_{\mathrm{D}}(x)\,\omega_{\mathrm{D}}(1+x^{-1}t) = \omega_{\mathrm{D}}(x),$$

except for $p=2$ (in which case we need $x\in\mathbb{Q}_2-\frac{1}{2}\mathbb{Z}_2$). Thus the term $\omega_{\mathrm{D}}(x+t)$ in (3.4) could be pulled out of the integral at the expense of making special provisos for $p=2$. Washington's original definition did not have the factor $\omega_{\mathrm{D}}(x+t)$ in (3.4), but we have inserted it to simplify formulas.

From (3.4) and (2.2) it follows that ζ_{D} satisfies the difference equation

$$\zeta_{\mathrm{D}}(s,x+1) - \zeta_{\mathrm{D}}(s,x) = -\langle x\rangle^{-s}. \tag{3.5}$$

This is just as expected from the classical Raabe-type formula (3.3) (*cf.*the general principle in the paragraph containing (1.19)).

We now prove a Raabe formula for ζ_{D}.

Proposition 3.1.

$$\int_{\mathbb{Z}_p} \zeta_{\mathrm{D}}(s,x+t)\,dt = s\,\zeta_{\mathrm{D}}(s,x) + (x-1)\frac{\partial}{\partial x}\,\zeta_{\mathrm{D}}(s,x),$$
$$(x\in\mathbb{Q}_p-\mathbb{Z}_p,\ s\in\mathbb{Z}_p,\ s\neq 1).$$

Proof. — Using Lemma 2.2 with $G(x) = \omega_{\mathrm{D}}(x)\langle x\rangle^{1-s}/(s-1)$ we find

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \zeta_{\mathrm{D}}(s,x+t)\,dt \\
&\quad= \zeta_{\mathrm{D}}(s,x) + (x-1)\frac{\partial}{\partial x}\,\zeta_{\mathrm{D}}(s,x) + \int_{\mathbb{Z}_p}(x+t)\,\langle x+t\rangle^{-s}dt \\
&\quad= \zeta_{\mathrm{D}}(s,x) + (x-1)\frac{\partial}{\partial x}\,\zeta_{\mathrm{D}}(s,x) + \int_{\mathbb{Z}_p}\omega_{\mathrm{D}}(x+t)\,\langle x+t\rangle^{1-s}dt \\
&\quad= \zeta_{\mathrm{D}}(s,x) + (x-1)\frac{\partial}{\partial x}\,\zeta_{\mathrm{D}}(s,x) + (s-1)\,\zeta_{\mathrm{D}}(s,x).
\end{aligned}$$

□

It is easy to see from Proposition 3.1 and the proposition in §1 that $x \to \zeta_{\mathrm{D}}(s,x)$ is the unique strictly differentiable function $f : \mathbb{Q}_p - \mathbb{Z}_p \to \mathbb{C}_p$ satisfying the difference equation

$$f(x+1) - f(x) = -\langle x \rangle^{-s}$$

and the integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = sf(x) + (x-1)f'(x).$$

We now turn to the Morita-Hurwitz p-adic zeta function [5]

$$\zeta_{\mathrm{M}}(s,x) := \frac{1}{s-1}\int_{\mathbb{Z}_p} \omega_{\mathrm{M}}(x+t)\langle x+t\rangle^{1-s}\,dt \quad (x, s \in \mathbb{Z}_p,\ s \neq 1), \tag{3.6}$$

where $\omega_{\mathrm{M}} : \mathbb{Z}_p \to \mathbb{Z}_p$ is the unique multiplicative extension of the Teichmüller character ω from $\mathbb{Z}_p^*$ to $\mathbb{Z}_p$ such that $\omega_{\mathrm{M}}(p) = 0$.[2]

Morita's ζ_{M} satisfies the difference equation

$$\zeta_{\mathrm{M}}(s,x+1) - \zeta_{\mathrm{M}}(s,x) = \begin{cases} -\langle x \rangle^{-s} & \text{if } x \in \mathbb{Z}_p^*, \\ 0 & \text{if } x \in p\mathbb{Z}_p, \end{cases} \tag{3.7}$$

and the Raabe formula

$$\int_{\mathbb{Z}_p} \zeta_{\mathrm{M}}(s,x+t)\,dt = (x-1)\frac{\partial \zeta_{\mathrm{M}}}{\partial x} + s\,\zeta_{\mathrm{M}}(s,x). \tag{3.8}$$

The proofs follow the corresponding ones for Washington's ζ_{D}.

Just as we did for p-adic $\log\Gamma$ functions, the characterization of p-adic Hurwitz zeta functions by their difference equations and Raabe-type formulas allows us to easily prove some of their main properties, which we certainly believe to be known even if we have so far failed to locate them in the literature.

Proposition 3.2.

$$\zeta_{\mathrm{D}}(s,1-x) = -\,\zeta_{\mathrm{D}}(s,x), \quad (x \in \mathbb{Q}_p - \mathbb{Z}_p,\ s \in \mathbb{Z}_p,\ s \neq 1). \tag{3.9}$$

$$\zeta_{\mathrm{M}}(s,x) = \sum_{\substack{0 \leqslant j \leqslant p^N-1 \\ p \nmid (x+j)}} \zeta_{\mathrm{D}}\left(s, \frac{x+j}{p^N}\right), \quad (s, x \in \mathbb{Z}_p,\ s \neq 1,\ N = 1,2,3,\cdots). \tag{3.10}$$

$$\zeta_{\mathrm{M}}(s,1-x) = -\,\zeta_{\mathrm{M}}(s,x), \quad (s, x \in \mathbb{Z}_p,\ s \neq 1). \tag{3.11}$$

---

(2) Actually, the singularity at $s=1$ in (3.6) is only apparent, but for the sake of brevity we shall not deal with $s=1$.

Proof. — Just as in the proof of Proposition 2.5, to prove (3.9) it suffices to show that $f(x) = f_s(x) := -\zeta_{\mathrm{D}}(s, 1-x)$ satisfies

$$f(x+1) - f(x) = -\langle x\rangle^{-s}$$

and

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = sf(x) + (x-1)f'(x). \tag{3.12}$$

To check the difference equation all we need use is $\langle -x\rangle = \langle x\rangle$ and the difference equation for $\zeta_{\mathrm{D}}(s,x)$. The integro-differential equation (3.12) is verified just as in the proof of Proposition 2.5.

The proofs of (3.10) and (3.11) follow exactly along the lines of the proofs of (2.12) and (2.13), using parity in (3.6) to show $\zeta_{\mathrm{M}}(S,0) = 0$ $\square$

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