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RIESZ TRANSFORMS ON CONNECTED SUMS

by Gilles CARRON

Abstract. — Assume that M_0 is a complete Riemannian manifold with Ricci curvature bounded from below and that M_0 satisfies a Sobolev inequality of dimension $\nu > 3$. Let M be a complete Riemannian manifold isometric at infinity to M_0 and let $p \in (\nu/(\nu-1), \nu)$. The boundedness of the Riesz transform of $L^p(M_0)$ then implies the boundedness of the Riesz transform of $L^p(M)$

Résumé. — Soit M_0 une variété riemannienne complète à courbure de Ricci bornée inférieurement et qui vérifie l'inégalité Sobolev de dimension $\nu > 3$. Si M est une variété riemannienne complète isométrique à M_0 en dehors d'un compact et si $p \in (\nu/(\nu-1), \nu)$ alors lorsque la transformée de Riesz est bornée sur $L^p(M_0)$ elle est également bornée sur $L^p(M)$.

1. Introduction

Let (M, g) be a complete Riemannian manifold with infinite volume, we denote by $\Delta = \Delta^g$ its Laplace operator, it has an unique self-adjoint extension on $L^2(M, d\,\mathrm{vol}_g)$ which is also denoted by Δ. The Green formula and the spectral theorem show that for any $\varphi \in C_0^\infty(M)$:

$$\|d\varphi\|_{L^2}^2 = \langle \Delta\varphi, \varphi \rangle = \|\Delta^{1/2}\varphi\|_{L^2}^2;$$

hence the Riesz transform $T := d\Delta^{-1/2}$ extends to a bounded operator

$$T : L^2(M) \to L^2(M; T^*M).$$

On the Euclidean space, it is well known that the Riesz transform has also a bounded extension $L^p(M) \to L^p(M; TM)$ for any $p \in]1, \infty[$. However, this is not a general feature of the Riesz transform on complete Riemannian manifolds, as a matter of fact, on the connected sum of two copies of the Euclidean space $\mathbb{R}^n$, the Riesz transform is not bounded on L^p for any

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$p \in [n,\infty[\cap]2,\infty[$ ([7, 5]). It is of interest to figure out the range of p for which T extends to a bounded map $L^p(M) \to L^p(M;T^*M)$. The main result of [5] answered to this question for manifolds with Euclidean ends :

THEOREM 1.1. — *Let M be a complete Riemannian manifold of dimension $n \geqslant 3$ which is the union of a compact part and a finite number of Euclidean ends. Then the Riesz transform is bounded from $L^p(M)$ to $L^p(M;T^*M)$ for $1 < p < n$, and is unbounded on L^p for all other values of p if the number of ends is at least two.*

The proof of this result used an asymptotic expansion of the Schwarz kernel of the resolvent $(\Delta + k^2)^{-1}$ near $k \to 0$. In [5] using L^p cohomology, we also find a criterion which insures that the Riesz transform is unbounded on L^p :

THEOREM 1.2. — *Assume that (M,g) is a complete Riemannian manifold with Ricci curvature bounded from below such that for some $\nu > 2$ and $C > 0$, (M,g) satisfies the Sobolev inequality*

$$\forall \varphi \in C_0^\infty(M), \|\varphi\|_{L^{\frac{2\nu}{\nu-2}}} \leqslant C\|d\varphi\|_{L^2}$$

and

$$\forall x \in M, \forall r > 1,\ \operatorname{vol} B(x,r) \leqslant Cr^\nu. \tag{1.1}$$

If M has at least two ends, then the Riesz transform is not bounded on L^p for any $p \geqslant \nu$.

Let (N,g_0) be a simply connected nilpotent Lie group of dimension $n > 2$ (endowed with a left invariant metric). According to [1] we know that the Riesz transform on (N,g_0) is bounded on L^p for every $p \in]1,\infty[$. Let ν be the homogeneous dimension of N; for instance we can set

$$\nu = \lim_{R\to\infty} \frac{\log \operatorname{vol} B(o,R)}{\log R},$$

$o \in N$ being a fixed point. Let (M,g) be a manifold isometric at infinity to $k \geqslant 1$ copies of (N,g_0). That is to say there are compact sets $K \subset M$ and $K_0 \subset N$ such that $(M \setminus K, g)$ is isometric to k copies of $(N \setminus K_0, g_0)$. According to [7] we know that on (M,g) the Riesz transform is bounded on L^p for $p \in]1,2]$. And the theorem 1.2 says that the Riesz transform is not bounded on L^p when $p \geqslant \nu$. In [5], we make the following conjecture : *show that the Riesz transform on (M,g) is bounded on L^p for $p \in]1,\nu[$.* The main result of this paper gives a positive answer to this conjecture ; in fact we obtain a more general result concerning the boundedness of Riesz transform for connected sums, under some mild geometrical conditions :

THEOREM 1.3. — *Let (M_0, g_0) be a complete Riemannian manifold, we assume that the Ricci curvature of (M_0, g_0) is bounded from below and that for some $\nu > 3$ and $C > 0$, (M_0, g_0) satisfies the Sobolev inequality*

$$\forall \varphi \in C_0^\infty(M_0), \ \|\varphi\|_{L^{\frac{2\nu}{\nu-2}}} \leqslant C\|d\varphi\|_{L^2}.$$

Let $p \in]\nu/(\nu-1), \nu[$, if on (M_0, g_0) the Riesz transform is bounded on L^p then the Riesz transform is also bounded on L^p for any manifold M isometric at infinity to several copies of (M_0, g_0).

Moreover under a uniform upper growth control of the volume of geodesic balls (such as (1.1)), the result of [7] implies that under the assumption of the theorem 1.3, the Riesz transform is bounded on M for any $p \in]1, 2]$; hence the restriction of $p > \nu/(\nu-1)$ is not really a serious one. Our method is here less elaborate than the one of [5], its gives a more general result but it is less sharp ; there are two restrictions : the first one is the dimension restriction $\nu > 3$ which is unsatisfactory, and the second concerns the limitation $p < \nu$ which is perhaps also unsatisfactory when M has only one end. However there are recent results of T. Coulhon and N. Dungey in this direction [6].

There is now a long list of complete Riemannian manifolds (M_0, g_0) satisfying our hypothesis and on which the Riesz transform is known to be bounded on L^p for every $p \in]1, \infty[$. For instance Cartan-Hadamard manifolds with a spectral gap [17], non-compact symmetric spaces [2] and Lie groups of polynomial growth [1], manifolds with nonnegative Ricci curvature and maximal volume growth [3] (see the discussion at the end of the proof of theorem 1.3 about the case of manifolds with nonnegative Ricci curvature and non maximal volume growth). Also H.-Q. Li [16] proved that the Riesz transform on n-dimensional cones with compact basis is bounded on L^p for $p < p_0$, where

$$p_0 = \begin{cases} n\left(\frac{n}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1}\right)^{-1}, & \lambda_1 < n-1 \\ +\infty, & \lambda_1 \geqslant n-1, \end{cases}$$

where λ_1 is the smallest nonzero eigenvalue of the Laplacian on the basis. Note that $p_0 > n$. Our proof also applies to a manifold isometric at infinity to several copies of cones, hence our theorem 1.3 also gives a partial answer to the open problem 8.1 of [5] :

COROLLARY 1.4. — *If (M, g) is a smooth Riemannian $n-$manifold of dimension $n \geqslant 4$ with conic ends, then the Riesz transform is bounded on L^p for any $p \in]1, n[$.*

Our manifold (M_0, g_0) is not assumed to be connected, for instance the theorem 1.3 implies that on the connected sum of a hyperbolic space and a euclidean space of dimension $n > 3$, the Riesz transform is bounded on L^p, for $p \in]n/(n-1), n[$.

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2. Analytic preliminaries

2.1. A Sobolev inequality

Proposition 2.1. — *Let (M, g) be a complete Riemannian manifold with Ricci curvature bounded from below then for any $p \in [1, \infty[$, there is a constant C such that for all $\varphi \in C_0^\infty(M)$*

$$\|df\|_{L^p} \leqslant C\big[\|\Delta f\|_{L^p} + \|f\|_{L^p}\big].$$

Remark 2.2.

i) In [8], T. Coulhon and X. Duong have shown that for every complete Riemannian manifolds and any $p \in]1, 2]$, there is a constant C such that

$$\forall f \in C_0^\infty(M), \|df\|_{L^p}^2 \leqslant C\|\Delta f\|_{L^p}\|f\|_{L^p}.$$

When $p \in]1, 2]$, this is clearly a stronger result.

ii) When the injectivity radius is assumed moreover to be positive, this result is due to B. Davies (see corollary 10 in [10]) ; in this setting, another proof along the idea of [14] can be given.

Proof. — According to (theorem 4.1 in [3]) we know that if (M, g) is a complete manifold with Ricci curvature bounded from below then for any $p \in]1, \infty[$ there is a constant C such that

$$\forall f \in C_0^\infty(M),\ \|df\|_{L^p} \leqslant C\left[\left\|\Delta^{1/2} f\right\|_{L^p} + \|f\|_{L^p}\right].$$

Then an interpolation argument (see for instance proposition 5.5 in [15]) implies that

$$\left\|\Delta^{1/2} f\right\|_{L^p}^2 \leqslant \|\Delta f\|_{L^p} \|f\|_{L^p},$$

the proposition is now straightforward. □

2.2. Some estimates on the Poisson operator

Lemma 2.3. — *Let (M,g) be a complete Riemannian manifold which for some $\nu > 2$ and $C > 0$ satisfies the Sobolev inequality :*

$$\forall \varphi \in C_0^\infty(M), \|\varphi\|_{L^{\frac{2\nu}{\nu-2}}} \leqslant C\|d\varphi\|_{L^2}$$

then the Schwarz kernel $P_\sigma(x,y)$ of the Poisson operator $e^{-\sigma\sqrt{\Delta}}$ satisfies

$$P_\sigma(x,y) \leqslant \frac{C\sigma}{(\sigma^2 + d(x,y)^2)^{\frac{\nu+1}{2}}}.$$

Moreover if $1 \leqslant r \leqslant p \leqslant +\infty$ then

$$\left\| e^{-\sigma\sqrt{\Delta}} \right\|_{L^r \to L^p} \leqslant \frac{C}{\sigma^{\nu\left(\frac{1}{r}-\frac{1}{p}\right)}}.$$

We know that the heat operator $e^{-t\Delta}$ and the Poisson operator are related through the subordination identity :

$$e^{-\sigma\sqrt{\Delta}} = \frac{\sigma}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{\sigma^2}{4t}} e^{-t\Delta} \frac{dt}{t^{3/2}}.$$

Hence these properties follow directly from the corresponding ones for the heat operator $e^{-t\Delta}$ and its Schwarz kernel $H_t(x,y)$:

$$H_t(x,y) \leqslant \frac{C}{t^{\nu/2}} e^{-\frac{d(x,y)^2}{5t}} \tag{2.1}$$

and if $1 \leqslant r \leqslant p \leqslant +\infty$ then

$$\left\| e^{-t\Delta} \right\|_{L^r \to L^p} \leqslant \frac{C}{t^{\frac{\nu}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}},$$

which are consequences of the Sobolev inequality [18, 9].

We will also need an estimate for the derivative of the Poisson kernel :

Lemma 2.4. — *Under the assumptions of lemma (2.3), let $\Omega \subset M$ be an open subset and K be a compact set in the interior of $M \setminus \Omega$ then*

$$\left\| e^{-\sigma\sqrt{\Delta}} \right\|_{L^p(\Omega) \to L^\infty(K)} \leqslant \frac{C}{(1+\sigma)^{\nu/p}},$$

$$\left\| \nabla e^{-\sigma\sqrt{\Delta}} \right\|_{L^p(\Omega) \to L^\infty(K)} \leqslant \frac{C}{(1+\sigma)^{\nu/p}}.$$

Proof. — The first estimate is only a consequence of the lemma 2.3 because by assumption there is a constant $\varepsilon > 0$ such that

$$(x,y) \in K \times \Omega \Rightarrow d(x,y) \geqslant \varepsilon. \tag{2.2}$$

To prove the second inequality, we will again only show the corresponding estimate for the heat operator. First, according to the local Harnack inequality (see V.4.2 in [9]), there is a constant C such that for any $x \in K$, $t \in]0,1]$ and $y \in M$:

$$|\nabla_x H_t(x,y)| \leqslant \frac{C}{\sqrt{t}} H_{2t}(x,y). \tag{2.3}$$

But hence by (2.2) and (2.1), we get : for all $(x,y) \in K \times \Omega$ then

$$H_{2t}(x,y) \leqslant \frac{c}{t^{\nu/2}} e^{-\frac{\varepsilon^2}{10t}}.$$

It follows easily that there is a certain constant C such that

$$\forall t \in]0,1] : \left\|\nabla e^{-t\Delta}\right\|_{L^p(\Omega)\to L^\infty(K)} \leqslant C.$$

Now assume that $t > 1$:

$$\left\|\nabla e^{-t\Delta}\right\|_{L^p(\Omega)\to L^\infty(K)} \leqslant \left\|\nabla e^{-\frac{1}{2}\Delta}\right\|_{L^\infty(M)\to L^\infty(K)} \left\|e^{-(t-\frac{1}{2})\Delta}\right\|_{L^p(\Omega)\to L^\infty(M)}.$$

But we have

$$\left\|e^{-(t-\frac{1}{2})\Delta}\right\|_{L^p(\Omega)\to L^\infty(M)} \leqslant \frac{C}{(t-1/2)^{\nu/2p}}.$$

But with 2.3, we obtain :

$$\begin{aligned}\left\|\nabla e^{-\frac{1}{2}\Delta}\right\|_{L^\infty(M)\to L^\infty(K)} &\leqslant \sup_{x\in K} \int_M |\nabla_x H_{1/2}(x,y)|dy \\ &\leqslant C \sup_{x\in K} \int_M H_1(x,y)dy \leqslant C.\end{aligned}$$

Hence for all $t > 0$, we obtain

$$\left\|\nabla e^{-t\Delta}\right\|_{L^p(\Omega)\to L^\infty(K)} \leqslant \frac{C}{(1+t)^{\frac{\nu}{2p}}}$$

and the second estimate follows from the subordination identity. □

3. Proof of the main theorem

Let (M_0, g_0) be a complete Riemannian manifold, we assume that the Ricci curvature of (M_0, g_0) is bounded from below and that for some $\nu > 3$ and $C > 0$, that (M, g) satisfies the Sobolev inequality

$$\forall \varphi \in C_0^\infty(M_0),\ \|\varphi\|_{L^{\frac{2\nu}{\nu-2}}} \leqslant C\|d\varphi\|_{L^2}.$$

We assume that on (M_0, g_0) the Riesz transform is bounded on L^p for some $p \in]\nu/(\nu-1), \nu[$. And we consider M a complete Riemannian manifold such that outside compact sets $K \subset M$ and $K_0 \subset M_0$, $M \setminus K$ is isometric to $M_0 \setminus K_0$, the case where $M \setminus K$ is isometric to several copies of $M_0 \setminus K_0$ can be done similarly by considering the disjoint union of k copies of M_0. We are going to prove that on M the Riesz transform is also bounded on L^p. The first step is to build a good parametrix for the Poisson operator on M. The first problem is that the operator $\sqrt{\Delta}$ is not a differential operator, we circumvent these difficulties by working on $\mathbb{R}_+ \times M$. As a matter of fact, the Poisson operator solves the Dirichlet problem :

$$\begin{cases} \left(-\frac{\partial^2}{\partial\sigma^2} + \Delta\right) u(\sigma, x) = 0 \quad \text{on }]0,\infty[\times M \\ u(0,x) = u(x) \\ \lim_{\sigma\to\infty} u(\sigma,.) = 0. \end{cases} \tag{3.1}$$

The construction of the parametrix will be standard, the non locality nature of the operator $\sqrt{\Delta}$ implies that we can not use the Duhamel formula, instead we used the Green operator. The idea is to find $E_\sigma(u)$ an approximate solution for (3.1) and then to use the fact that, if G is the Green operator of the operator $-\frac{\partial^2}{\partial\sigma^2} + \Delta$ for the Dirichlet boundary condition, then

$$e^{-\sigma\sqrt{\Delta}}u = E_\sigma(u) - G\left(-\frac{\partial^2}{\partial\sigma^2} + \Delta\right) E_\sigma(u).$$

3.1. The parametrix construction

Let $\tilde{K}$ be another compact set in M containing K in its interior. We identify

$$\Omega = M \setminus K = M_0 \setminus K_0.$$

Let ρ_0, ρ_1 a smooth partition of unity such that

$$\operatorname{supp}\rho_0 \subset \Omega \quad \text{and} \quad \operatorname{supp}\rho_1 \subset \tilde{K},$$

let also φ_0, φ_1 be smooth functions, such that

$$\operatorname{supp}\varphi_0 \subset \Omega \quad \text{and} \quad \operatorname{supp}\varphi_1 \subset \tilde{K}$$

Moreover we require that $\varphi_i = 1$ on a neighborhood of the support of ρ_i so that we have :

$$\varphi_i\rho_i = \rho_i.$$

Let Δ_1 be the realization of the Laplace operator on $\tilde{K}$ for the Dirichlet boundary condition and let Δ_0 be the Laplace operator on M_0. Let $e^{-\sigma\sqrt{\Delta_i}}$ their associated Poisson operator then we define for $u \in L^p(M)$:

$$E_\sigma(u) = \sum_{i=0}^{1} \varphi_i(e^{-\sigma\sqrt{\Delta_i}}\rho_i u),$$

where we think of $\rho_0 u$ as a function on $\Omega \subset M_0$ and of $\varphi_0(e^{-\sigma\sqrt{\Delta_0}}\rho_0 u)$ as a function on $\Omega \subset M$.

We can easily compute :

$$\left(-\frac{\partial^2}{\partial\sigma^2} + \Delta\right) E_\sigma(u) = \sum_{i=0}^{1}[\Delta, \varphi_i](e^{-\sigma\sqrt{\Delta_i}}\rho_i u) = f(\sigma, x) = \sum_{i=0}^{1} f_i(\sigma, x),$$

where

$$f_i(\sigma, x) = [\Delta, \varphi_i](e^{-\sigma\sqrt{\Delta_i}}\rho_i u)(x) \tag{3.2}$$
$$= \Delta\varphi_i(x)(e^{-\sigma\sqrt{\Delta_i}}\rho_i u)(x) - 2\left\langle d\varphi_i(x), d(e^{-\sigma\sqrt{\Delta_i}}\rho_i u)(x)\right\rangle.$$

From lemma 2.4 and the fact that the support of $d\varphi_0$ and ρ_0 are disjoint, we easily get that for all $\sigma \geqslant 0$:

$$\|f_0(\sigma)\|_{L^1} + \|f_0(\sigma)\|_{L^p} \leqslant \frac{C}{(1+\sigma)^{\nu/p}}\|\rho_0 u\|_{L^p}. \tag{3.3}$$

Let us explain why this estimate also holds for f_1. Note that the operator

$$\mathcal{S}(\sigma) = [\Delta, \varphi_1]e^{-\sigma\sqrt{\Delta_1}}\rho_1$$

is an operator with smooth Schwarz kernel and compact support, moreover because the corresponding estimate of the lemma (2.4) also holds for $\sigma \in [0, 1]$ on a compact manifold, the Schwarz kernel of $S(\sigma)$ is uniformly bounded when $\sigma \to 0$. Hence there is a constant C such that

$$\forall\sigma \in [0, 1], \|\mathcal{S}(\sigma)u\|_{L^\infty} \leqslant C\|\rho_1 u\|_{L^p}.$$

Now the operator Δ_1 has a spectral gap on L^p (its L^p spectrum is also its L^2 spectrum), hence there is a constant C such that for all $\sigma \geqslant 0$ then

$$\|e^{-\sigma\sqrt{\Delta_1}}\|_{L^p \to L^p} \leqslant Ce^{-\sigma/C}.$$

Hence for $\sigma \geqslant 1$:

$$\|\mathcal{S}(\sigma)u\|_{L^\infty} \leqslant \|[\Delta, \varphi_1]e^{\frac{1}{2}\sqrt{\Delta_1}}\|_{L^p \to L^\infty}\|e^{-(\sigma-1/2)\sqrt{\Delta_1}}\rho_1 u\|_{L^p}$$
$$\leqslant Ce^{-\sigma/C}\|\rho_1 u\|_{L^p}.$$

The result follows by noticing that the f_i's have compact support in $\tilde{K} \setminus K$. Eventually we obtain the estimate :

Lemma 3.1. — *When $u \in L^p(M)$ and we define an operator S_σ by $S_\sigma u = f = f_0 + f_1$ where f_0, f_1 are defined by (3.2) then*

$$\forall\sigma \geqslant 0,\ \|S_\sigma(u)\|_{L^1} + \|S_\sigma(u)\|_{L^p} \leqslant \frac{C}{(1+\sigma)^{\nu/p}}\|u\|_{L^p}.$$

3.2. The Riesz transform on M

We introduce now G, the Green operator of the operator $\left(-\frac{\partial^2}{\partial\sigma^2}+\Delta\right)$ on $\mathbb{R}_+\times M$ for the Dirichlet boundary condition. Its Schwarz kernel is given by

$$G(\sigma,s,x,y)=\int_0^\infty\left[\frac{e^{-\frac{(\sigma-s)^2}{4t}}-e^{-\frac{(\sigma+s)^2}{4t}}}{\sqrt{4\pi t}}\right]H_t(x,y)dt$$

where H_t is the heat kernel on M and

$$\frac{e^{-\frac{(\sigma-s)^2}{4t}}-e^{-\frac{(\sigma+s)^2}{4t}}}{\sqrt{4\pi t}}$$

the heat kernel on the half-line $\mathbb{R}_+$ for the Dirichlet boundary condition. We have

$$e^{-\sigma\sqrt{\Delta}}u=E_\sigma(u)-G(S_\sigma(u)).$$

Hence

$$\Delta^{-1/2}u=\int_0^\infty e^{-\sigma\sqrt{\Delta}}ud\sigma=\sum_{i=0}^{1}\varphi_i\Delta_i^{-1/2}\rho_iu$$
$$-\int_{\mathbb{R}_+^2\times M}G(\sigma,s,x,y)f(s,y)d\sigma dsdy.$$

Let

$$g(x)=\int_{\mathbb{R}_+^2\times M}G(\sigma,s,x,y)f(s,y)d\sigma dsdy$$

then we have

$$\Delta^{-1/2}u=\sum_{i=0}^{1}\varphi_i\Delta_i^{-1/2}\rho_iu-g. \tag{3.4}$$

But

$$\int_0^\infty G(\sigma,s,x,y)d\sigma=\frac{1}{\sqrt{4\pi}}\int_0^\infty\left[\int_{-s}^{s}e^{-\frac{v^2}{4t}}dv\right]H_t(x,y)\frac{dt}{\sqrt{t}}$$
$$=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-r^2}\left[\int_0^{\frac{s^2}{4r^2}}H_t(x,y)dt\right]dr.$$

It follows from the above computation that

$$g(x)=\frac{2}{\sqrt{\pi}}\int_{\mathbb{R}_+^2}e^{-r^2}\left[\int_0^{\frac{s^2}{4r^2}}(e^{-t\Delta}f(s))(x)dt\right]drds.$$

The following lemma is now the last crucial estimate :

LEMMA 3.2. — *There is a constant C such that*

$$\|\Delta g\|_{L^p} + \|g\|_{L^p} \leqslant C\|u\|_{L^p}.$$

Proof. — Recall that according to [4], (M,g) itself satisfies the same Sobolev inequality :

$$\forall \varphi \in C_0^\infty(M),\ \|\varphi\|_{L^{\frac{2\nu}{\nu-2}}} \leqslant C\|d\varphi\|_{L^2}.$$

Hence the heat operator satisfies the following mapping properties : for $1 \leqslant q \leqslant p \leqslant +\infty$ we have

$$\left\|e^{-t\Delta}\right\|_{L^q\to L^p} \leqslant \frac{C}{t^{\frac{\nu}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}}.$$

As a consequence, for all $t \in [0,1]$, then

$$\|(e^{-t\Delta} f(s))\|_{L^p} \leqslant \|f(s))\|_{L^p} \leqslant \frac{C}{(1+s)^{\nu/p}}\,\|u\|_{L^p}$$

and if $t > 1$, then

$$\|(e^{-t\Delta} f(s))\|_{L^p} \leqslant \left\|e^{-t\Delta}\right\|_{L^1\to L^p} \|f(s))\|_{L^1} \leqslant \frac{1}{t^{\frac{\nu}{2}(1-\frac{1}{p})}}\frac{C}{(1+s)^{\nu/p}}\,\|u\|_{L^p}.$$

Hence

$$\begin{aligned}\|g\|_{L^p} &\leqslant \frac{2}{\sqrt{\pi}}\int_{\mathbb{R}_+^2} e^{-r^2}\left[\int_0^{\frac{s^2}{4r^2}} \|(e^{-t\Delta} f(s))\|_{L^p} dt\right] dsdr \\ &\leqslant \frac{2}{\sqrt{\pi}}\left(\int_{\mathbb{R}_+^2} e^{-r^2}\left[\int_0^{\frac{s^2}{4r^2}} \frac{C}{\max\left(1, t^{\frac{\nu}{2}(1-\frac{1}{p})}\right)}\frac{1}{(1+s)^{\nu/p}} dt\right] dsdr\right)\|u\|_{L^p}.\end{aligned}$$

But because $p < \nu$, we have

$$\begin{aligned}&\int_{\{2r\sqrt{t}\leqslant s\}} e^{-r^2}\frac{1}{\max\left(1, t^{\frac{\nu}{2}(1-\frac{1}{p})}\right)}\frac{1}{(1+s)^{\nu/p}} dsdtdr \\ &= \frac{\nu}{\nu-p}\int_{\mathbb{R}_+^2} e^{-r^2}\frac{1}{\max\left(1, t^{\frac{\nu}{2}(1-\frac{1}{p})}\right)}\frac{1}{(1+2r\sqrt{t})^{\nu/p-1}} dtdr\end{aligned}$$

and this integral is finite exactly when $p > \nu/(\nu-1)$ and $\nu > 3$.

It remains to estimate $\|\Delta g\|_{L^p}$, which is easier because

$$\begin{aligned}\Delta g &= \frac{2}{\sqrt{\pi}}\int_{\mathbb{R}^2_+} e^{-r^2}\left[\int_0^{\frac{s^2}{4r^2}} \Delta(e^{-t\Delta}f(s))dt\right]drds\\ &= -\frac{2}{\sqrt{\pi}}\int_{\mathbb{R}^2_+} e^{-r^2}\left[\int_0^{\frac{s^2}{4r^2}} \frac{d}{dt}(e^{-t\Delta}f(s))dt\right]drds\\ &= \frac{2}{\sqrt{\pi}}\int_{\mathbb{R}^2_+} e^{-r^2}\left[f(s)-(e^{-\frac{s^2}{4r^2}\Delta}f(s))\right]drds.\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta g\|_{L^p} &\leqslant \frac{4}{\sqrt{\pi}}\int_{\mathbb{R}^2_+} e^{-r^2}\|f(s)\|_{L^p}drds\\ &\leqslant \frac{4}{\sqrt{\pi}}\left(\int_{\mathbb{R}^2_+} e^{-r^2}\frac{C}{(1+s)^{\nu/p}}drds\right)\|u\|_{L^p}.\end{aligned}$$

□

Now we can finish the proof of the main theorem : let T_i be the Riesz transform associated with the operator Δ_i. With the formula (3.4), we obtain

$$d\Delta^{-1/2}u = \sum_{i=0}^{1}\varphi_i T_i \rho_i u + \sum_{i=0}^{1} d\varphi_i(\Delta_i^{-1/2}\rho_i u) - dg.$$

By hypothesis, T_0 is bounded on L^p. Moreover since $\varphi_1 T_1 \rho_1$ is a pseudo differential operator of order 0 with compact support it is also bounded on L^p. The operator $d\varphi_1(\Delta_i^{-1/2}\rho_1 u)$ has a smooth kernel with compact support, hence it is bounded on L^p. Moreover, the Sobolev inequality

$$\forall \varphi \in \mathbb{C}_0^\infty(M_0),\ \|\varphi\|_{L^{\frac{2\nu}{\nu-2}}} \leqslant C\|d\varphi\|_{L^2}.$$

also implies the following mapping properties of the $\Delta_0^{-1/2}$ ([18]) :

$$\left\|\Delta_0^{-1/2}\right\|_{L^p\to L^{\frac{p\nu}{\nu-p}}} \leqslant C.$$

Hence

$$\begin{aligned}\left\|d\varphi_0(\Delta_0^{-1/2}\rho_0 u)\right\|_{L^p} \leqslant C\|\Delta_0^{-1/2}\rho_0 u\|_{L^p(\tilde{K}\setminus K)} &\leqslant C'\|\Delta_0^{-1/2}\rho_0 u)\|_{L^{\frac{p\nu}{\nu-p}}(\tilde{K})}\\ &\leqslant C\|\rho_0 u\|_{L^p}.\end{aligned}$$

Moreover the lemmas (3.2) and (2.1) imply that

$$\|dg\|_{L^p} \leqslant C\|u\|_{L^p}.$$

All these estimates yield the fact that the Riesz transform is bounded on L^p.

3.3. A comment on manifolds with non negative Ricci curvature

The proof of theorem 1.3 is fairly general, we can easily make a list of the properties which makes it runs ; let (M_i, g_i) $i = 1, ..., b$ be complete Riemannian manifolds and let (M, g) be isometric at infinity to the disjoint union $M_1 \cup ... \cup M_b$. That is to say there are compact sets $K \subset M$, $K_i \subset M_i$ such that $M \setminus K$ is isometric to $(M_1 \setminus K_1) \cup ... \cup (M_b \setminus K_b)$. Let $\tilde{K} \subset \widehat{K}$ such that $\tilde{K}$ (resp. $\widehat{K}$) contains K in its interior (resp. $\tilde{K}$). And let $\widehat{K}_i, \tilde{K}_i \subset M_i$ such that :

$$M \setminus \tilde{K} \simeq (M_1 \setminus \tilde{K}_1) \cup ... \cup (M_b \setminus \tilde{K}_b),\ M \setminus \widehat{K} \simeq (M_1 \setminus \widehat{K}_1) \cup ... \cup (M_b \setminus \widehat{K}_b),$$

let Δ_i be the Laplace operator on M_i. We assume that on each M_i, the Ricci curvature is bounded from below such that on each M_i and M, we get the estimate induced by the Sobolev inequality (2.1). Assume that for some functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+^*$ we have the estimate :

$$\left\| e^{-\sigma\sqrt{\Delta_i}} \right\|_{L^p(M_i \setminus \widehat{K}_i) \to L^\infty(\tilde{K}_i)} + \left\| \nabla e^{-\sigma\sqrt{\Delta_i}} \right\|_{L^p(M_i \setminus \widehat{K}_i) \to L^\infty(\tilde{K}_i)} \leqslant \frac{1}{f(\sigma)},$$

and that on the manifold M :

$$\left\| e^{-t\Delta} \right\|_{L^1(\widehat{K}) \to L^p(M)} \leqslant \frac{1}{g(t)}.$$

with

$$\int_0^\infty \frac{ds}{f(s)} < \infty \tag{3.5}$$

$$\int_{\mathbb{R}_+^2} e^{-u^2} \min\left(1, \frac{1}{g(t)}\right) \left[\int_{2u\sqrt{t}}^\infty \frac{ds}{f(s)}\right] du dt < \infty. \tag{3.6}$$

Then if for all i, the Riesz transform $T_i := d\Delta_i^{-1/2}$ is bounded on L^p, then on M, the Riesz transform is also bounded on L^p.

A natural and well study class of manifolds satisfying such estimates are manifolds satisfying the so called relative Faber-Krahn inequality : for some $\alpha > 0$ and $c > 0$, we have :

$$\forall B(x,R), \forall \Omega \subset B(x,R),\ \lambda_1(\Omega) \geqslant \frac{c}{R^2} \left(\frac{\operatorname{vol} \Omega}{\operatorname{vol} B(x,R)} \right)^{-\alpha}$$

where

$$\lambda_1(\Omega) = \inf_{f \in C_0^\infty(\Omega)} \frac{\int_\Omega |df|^2}{\int_\Omega f^2}$$

is the first eigenvalue of the Laplace operator on Ω for the Dirichlet boundary condition. According to A. Grigor'yan [11] this inequality is equivalent to the conjunction of the doubling property : uniformly in x and $R > 0$ we have

$$\frac{\operatorname{vol} B(x,2R)}{\operatorname{vol} B(x,R)} \leqslant C$$

and of the upper bound on the heat operator

$$H_t(x,y) \leqslant \frac{C}{\operatorname{vol} B(x,\sqrt{t})} e^{-\frac{d(x,y)^2}{5t}}.$$

Manifolds with non negative Ricci curvature are examples of manifolds satisfying this relative Faber-Krahn inequalities.

Assume that each M_i satisfies this relative Faber-Krahn inequality and if we assume that for $i = 1, ..., b$, there is a point $o_i \in K_i$ and all $R \geqslant 1$

$$\operatorname{vol} B(o_i, R) := V_i(R) \geqslant CR^\nu$$

then we get easily from the subordination identity :

$$\left\| e^{-\sigma\sqrt{\Delta_i}} \right\|_{L^p(M_i \setminus \widehat{K}_i) \to L^\infty(\tilde{K}_i)} + \left\| \nabla e^{-\sigma\sqrt{\Delta_i}} \right\|_{L^p(M_i \setminus \widehat{K}_i) \to L^\infty(\tilde{K}_i)} \leqslant \frac{1}{(1+\sigma)^{\nu/p}}.$$

Now the problem comes from the fact that we don't know how to obtain a relative Faber-Krahn inequality on M from the one we assume on the M_i's. However, recently in (page 877 of [13]), A. Grigor'yan and L. Saloff-Coste have announced the following very useful result (see also [12]) : when the M_i's satisfy the relative Faber-Krahn inequality then

$$\forall B(x,R) \subset M, \forall \Omega \subset B(x,R), \quad \lambda_1(\Omega) \geqslant \frac{c}{R^2} \left(\frac{\operatorname{vol} \Omega}{\mu(x,R)} \right)^{-\alpha},$$

where

$$\mu(x,R) = \begin{cases} \operatorname{vol} B(x,R) & \text{if } B(x,R) \subset M \setminus K \\ \inf_i V_i(R) & \text{else.} \end{cases}$$

Hence from our volume growth estimate, we will obtain (see [11]) when $t \geqslant 1$:

$$\left\| e^{-t\Delta} \right\|_{L^1(\widehat{K}) \to L^p(M)} \leqslant \frac{C}{t^{\frac{\nu}{2}\frac{p-1}{p}}}.$$

With this result of A. Grigor'yan and L. Saloff-Coste and with the result of D. Bakry [3], we will obtain :

Proposition 3.3. — *Let $(M_1, g_1), ..., (M, g_b)$ be complete Riemannian manifolds with non negative Ricci curvature. Assume that on all M_i's we have the volume growth lower bound :*

$$\operatorname{vol} B(o_i, R) \geqslant CR^\nu.$$

Then assume that $\nu > 3$ and $p \in]\nu/(\nu-1), \nu[$ then on any manifold isometric at infinity to the disjoint union of the M_i's, the Riesz transform is bounded on L^p.

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