ANNALES

DE

L'INSTITUT FOURIER

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Tome 57, n° 7 (2007), p. 2183-2195.

<http://aif.cedram.org/item?id=AIF_2007__57_7_2183_0>

Ann. Inst. Fourier, Grenoble
57, 7 (2007) 2183-2195

RIEMANN SUMS OVER POLYTOPES

by Victor GUILLEMIN & Shlomo STERNBERG

Abstract. — It is well-known that the N-th Riemann sum of a compactly supported function on the real line converges to the Riemann integral at a much faster rate than the standard $O(1/N)$ rate of convergence if the sum is over the lattice, Z/N. In this paper we prove an n-dimensional version of this result for Riemann sums over polytopes.

Résumé. — Il est bien connu que l'intégrale de Riemann d'une fonction d'une variable est beaucoup mieux approximée par la N-ième somme de Riemann si la somme est effectuée sur le réseau Z/N. Dans cet article nous démontrons un résultat similaire en plusieurs variables pour des sommes de Riemann sur des polytopes.

1. Introduction

Given a $\mathcal{C}^\infty$ function, f, on the interval $[0,1]$ let $R_N(f)$ be the Riemann sum

$$\frac{1}{N}\sum_{i=1}^{N} f(t_i), \qquad \frac{i}{N} \leqslant t_i < \frac{i+1}{N}. \tag{1.1}$$

In freshman calculus one learns that

$$R_N(f) = \int_0^1 f(x)\,dx + O\left(\frac{1}{N}\right). \tag{1.2}$$

What is not perhaps as well known is that if one chooses the t_i's judiciously, *i.e.*, lets $t_i = \frac{i}{N}$ the $O\left(\frac{1}{N}\right)$ in (1.2) can be replaced by a much better error term, an asymptotic series:

$$\frac{1}{2N}\left(f(1)-f(0)\right)+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}\left(f^{\langle 2k-1\rangle}(1)-f^{\langle 2k-1\rangle}(0)\right)N^{-2k} \tag{1.3}$$

Keywords: Riemann sums, Euler-Maclaurin formula for polytopes, Ehrhart's theorem.
Math. classification: 52B20.

in which the B_k's are the Bernoulli numbers. In particular if f is periodic of period 1 the $O\left(\frac{1}{N}\right)$ in (1.2) is actually an $O(N^{-\infty})$. (For an expository account of this "Euler-Maclaurin formula for Riemann sums" see [6].)

In this article we will prove an n-dimensional version of this result in which the interval $[0,1]$ gets replaced by a convex polytope. We will give a precise formulation of our result in §4; however, roughly speaking, it asserts that if Δ is a simple convex polytope whose vertices lie on the lattice, $\mathbb{Z}^n$, and if f is in $\mathcal{C}^\infty(\Delta)$ the difference

$$\int_\Delta f(x)\,dx - \frac{1}{N^n}\sum_{k\in N\Delta\cap\mathbb{Z}^n} f\left(\frac{k}{N}\right) \tag{1.4}$$

can be expanded in an asymptotic series in N^{-1} in which the coefficients are explicitly computable by recipes resembling (1.3). Our formula bears a formal resemblance to the generalized Euler-Maclaurin formulas of [9], [7], [2], [4], [1] et al., however in these so-called "exact" Euler-Maclaurin formulas the functions involved are polynomials, not as in the case here, arbitrary $\mathcal{C}^\infty$ functions. Somewhat closer in spirit to our result is the Euler-Maclaurin formula with remainder of [8] and the Ehrhart theorem for symbols of [5]. (Our result also yields an Ehrhart theorem for symbols, and its relation to the theorem in [5] will be discussed in §4.)

A word about the organization of this paper. In §2 we will review the proof of the Riemann sum version of Euler-Maclaurin, for the interval, $(-\infty,0]$ and in §3 show how to extend this result to regions in $\mathbb{R}^n$ which are defined by systems of k linearly-independent inequalities

$$\langle u_i, x\rangle \leqslant c_i, \quad u_i\in\mathbb{Z}^n, \quad c_i\in\mathbb{Z}. \tag{1.5}$$

(We will call such regions *k-wedges*.)

In §4 we will derive from this result a Euler-Maclaurin formula for Riemann sums over polytopes and in §5 show that our result has an equivalent formulation as an Ehrhart theorem for symbols.

We would like to thank Dan Stroock and Hans Duistermaat for helpful discussions concerning the material in Section 2.

2. Euler-Maclaurin for the interval $(-\infty,0]$

Let $\tau(s)$ be the Todd function

$$\frac{s}{1-e^{-s}} = 1 + \frac{s}{2} + \sum(-1)^{n-1}B_n\frac{s^{2n}}{(2n)!}. \tag{2.1}$$

In this section we will show that for Schwartz functions, $f \in S(\mathbb{R})$ the difference

$$\frac{1}{N}\sum_{k=0}^{\infty} f\left(-\frac{k}{N}\right) - \int_{-\infty}^{0} f(x)\,dx \tag{2.2}$$

has an asymptotic expansion:

$$\frac{f(0)}{2N} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!} f^{(2n-1)}(0)N^{-2n}. \tag{2.3}$$

In view of (2.1) this formula can be written more succinctly in the form

$$\frac{1}{N}\sum_{k=0}^{\infty} f\left(-\frac{k}{N}\right) \sim \left(\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h} f(x)\,dx\right)(h=0) \tag{2.4}$$

and it is this version of it which we will prove.

We first of all observe that if $f(x) = e^{\lambda x}$, $\lambda > 0$, then

$$\int_{-\infty}^{h} f(x)\,dx = \frac{1}{\lambda}e^{\lambda h}.$$

So for $N > 2\pi\lambda$ we may apply the infinite order constant coefficient operator $\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)$ to this expression:

$$\begin{aligned}\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h} f(x)\,dx &= \tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\frac{e^{\lambda h}}{\lambda} = \tau\left(\frac{\lambda}{N}\right)\frac{e^{\lambda h}}{\lambda}\\ &= \frac{1}{N}\frac{\lambda}{1-e^{-\lambda/N}}\frac{e^{\lambda h}}{\lambda} = \frac{1}{N}\left(\sum_{k=0}^{\infty} e^{-\frac{k}{N}\lambda}\right)e^{\lambda h},\end{aligned}$$

all series being convergent. We conclude that

$$\frac{1}{N}\sum_{k=0}^{\infty} e^{-\frac{k}{N}\lambda} = \left(\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h} e^{\lambda x}\,dx\right)(h=0). \tag{2.5}$$

More generally differentiating this identity n times with respect to λ we obtain

$$\frac{1}{N}\sum_{k=0}^{\infty}\left(-\frac{k}{N}\right)^{n} e^{-\frac{k}{N}\lambda} = \left(\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h} x^n e^{\lambda x}\,dx\right)(h=0) \tag{2.6}$$

verifying (2.4) for the function $x^n e^{\lambda x}$ and hence for the functions of the form $p(x)e^{\lambda x}$ where p is a polynomial. Now let f be a Schwartz function and p a polynomial having the property that $f(x) - p(x)e^{\lambda x}$ vanishes to order $n+2$ at $x=0$. Let

$$g(x) = \begin{cases} 0\,, & x \geqslant 0 \\ f(x) - p(x)e^{\lambda k}\,, & x < 0. \end{cases} \tag{2.7}$$

Then

$$\|g^{(i)}(x)\|_1 < \infty \text{ for } i \leqslant n+2 \tag{2.8}$$

and by the Poisson summation formula

$$\sum_{-\infty<k<\infty} g\left(-\frac{k}{N}\right) = N \sum_{-\infty<k<\infty} \hat{g}(Nk). \tag{2.9}$$

However, by (2.8)

$$|\hat{g}(Nk)| \leqslant \text{Const.}\, N^{-n}k^{-2} \tag{2.10}$$

for $k \neq 0$, and

$$\hat{g}(0) = \int_{-\infty}^{0} g(x)\,dx. \tag{2.11}$$

Hence

$$\frac{1}{N}\sum_{k=0}^{\infty} g\left(-\frac{k}{N}\right) = \int_{-\infty}^{0} g(x)\,dx + O(N^{-n}).$$

This shows that (2.4) is true for g modulo $O(N^{-n})$ and hence is true for f modulo $O(N^{-\infty})$. $\square$

In §3 we will also need a version of the theorem above for "twisted" Riemann sums. Let $\omega \neq 1$ be a q^{th} root of unity and let

$$\tau_\omega(s) = \frac{s}{1-\omega e^{-s}} = \frac{s}{1-\omega} + \sum_{i>1} b_i^\omega s^i. \tag{2.12}$$

For $f \in S(\mathbb{R})$ we will show that the twisted Riemann sum

$$\frac{1}{N}\sum_{k=0}^{\infty} \omega^k f\left(-\frac{k}{N}\right) \tag{2.13}$$

is asymptotic to the series

$$\frac{1}{1-\omega}\frac{f(0)}{N} + \sum_{i>1} b_i^\omega f^{(i)}(0)N^{-i}. \tag{2.14}$$

As above we can rewrite this in the more succinct form

$$\frac{1}{N}\sum_{k=0}^{\infty} \omega^k f\left(-\frac{k}{N}\right) \sim \left(\tau_\omega\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h} f(x)\,dx\right)(h=0) \tag{2.15}$$

and we will prove this by essentially the same proof as before: If $f = e^{\lambda x}$ the expression in parentheses is

$$\tau_\omega\left(\frac{\lambda}{N}\right)\frac{e^{\lambda h}}{\lambda} = \frac{1}{N}\left(\frac{\lambda}{1-\omega e^{-\lambda/N}}\right)\frac{e^{\lambda h}}{\lambda} = \frac{1}{N}\left(\sum_{k=0}^{-\infty}\omega^k e^{-k\lambda/N}\right)e^{\lambda h}, \tag{2.16}$$

and by setting $h = 0$ we see that (2.15) is valid for $f = e^{\lambda x}$; and by differentiating both sides of (2.16) by $\left(\frac{d}{d\lambda}\right)^n$ that it's valid for $x^n e^{\lambda x}$ and hence for $p(x)e^{\lambda x}$ where $p(x)$ is a polynomial. Thus, as above, we're reduced to showing that for the function g defined by (2.7):

$$\frac{1}{N}\sum_{-\infty<k<\infty}\omega^k g\left(\frac{k}{N}\right) = O(N^{-n}). \tag{2.17}$$

For $r = 0, 1, \ldots, q-1$, let $g_r(x) = g(qx + \frac{r}{N})$. Then

$$\frac{1}{N}\sum_{-\infty<k<\infty}\omega^k g\left(\frac{k}{N}\right) = \frac{1}{N}\sum_{r=0}^{q-1}\omega^r\left(\sum_{-\infty<k<\infty} g_r\left(\frac{k}{N}\right)\right). \tag{2.18}$$

Since

$$\hat{g}_r(Nk) = \frac{1}{q}e^{i\frac{rk}{q}}\hat{g}\left(\frac{Nk}{q}\right)$$

the Poisson summation formula yields, as before, the estimate

$$\sum_{r=0}^{q-1}\omega^r\int_{-\infty}^{\infty} g_r(x)\,dx + O(N^{-n}), \tag{2.19}$$

for the right hand side of (2.18). However,

$$\int_{-\infty}^{\infty} g_r(x)\,dx = \int_{-\infty}^{\infty} g_0(x)\,dx$$

and $\sum_{r=0}^{q-1}\omega^r = 0$ so the first summand in (2.19) is zero. □

We will conclude this discussion of one dimensional Euler-Maclaurin formulas by describing analogues of (2.4) and (2.15) in which the sum over $-\infty < k < 0$ gets replaced by a sum over $-\infty < k < \infty$. For simplicity assume that $f \in \mathcal{C}_0^\infty(\mathbb{R})$. We claim:

$$\frac{1}{N}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{N}\right) = \int_{-\infty}^{\infty} f(x)\,dx + O(N^{-\infty}) \tag{2.20}$$

and, for ω a q^{th} root of unity, $\omega \neq 1$,

$$\sum_{k=-\infty}^{\infty}\omega^k f\left(\frac{k}{N}\right) = O(N^{-\infty}). \tag{2.21}$$

To prove (2.20) we first observe that for c a large positive integer, the left and right hand sides of (2.20) are unchanged if one substitutes the function, $f(x+c)$, for f, so without loss of generality we can assume that f is supported on the interval, $x < 0$, in which case (2.3) is of order $O(N^{-\infty})$ and (2.20) is a consequence of (2.4). Similarly if we replace $f(x)$ by $f(x+cq)$, with c a large positive integer, the left and right hand sides of (2.21) are unchanged; so we can assume that f is supported on the interval $x < 0$, and (2.21) is a consequence of (2.15).

3. Euler-Maclaurin for wedges

Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$, $(\mathbb{Z}^n)^*$ its dual lattice in $(\mathbb{R}^n)^*$ and $\langle u, x\rangle$ the usual pairing of vectors, $x \in \mathbb{R}^n$, and $u \in (\mathbb{R}^n)^*$. Given m linearly independent vectors, $u_i \in (\mathbb{R}^n)^*$ we will call the subset of $\mathbb{R}^n$ defined by the inequalities

$$\langle u_i, x\rangle \leqslant c_i \qquad i = 1, \dots, m \tag{3.1}$$

an *integer m-wedge* if the c_i's are integers and the u_i's primitive lattice vectors in $(\mathbb{Z}^n)^*$. Let W be the set (3.1) and U the subspace of $(\mathbb{R}^n)^*$ spanned by the u_i's. We will call W a *regular* integer m-wedge if $u_1, \dots, u_m$ is a lattice basis of the lattice $U \cap (\mathbb{Z}^n)^*$ *i.e.*, if

$$U \cap (\mathbb{Z}^n)^* = \operatorname{span}_{\mathbb{Z}}\{u_1, \dots, u_m\}. \tag{3.2}$$

We will need below the following criterion for regularity.

Lemma 3.1. — *If (3.2) holds, $u_i, \dots, u_m$ can be extended to a lattice basis, $u_1, \dots, u_m$ of $(\mathbb{Z}^n)^*$.*

Proof. — Let $u_{m+1}, \dots, u_n$ be vectors in $(\mathbb{Z}^n)^*$ whose projections onto the quotient of $(\mathbb{Z}^n)^*$by $U \cap (\mathbb{Z})^*$ are a lattice basis of this quotient. ☐

For an integer m-wedge satisfying (3.2) the n-dimensional generalization of Euler-Maclaurin is relatively straightforward.

Theorem 3.2. — *Let $h \in \mathbb{R}^m$ and let W_h be the subset of $\mathbb{R}^n$ defined by the inequalities*

$$\langle u_i, x\rangle \leqslant c_i + h_i, \quad i = 1, \dots, m. \tag{3.3}$$

Then, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\frac{1}{N^n} \sum_{k \in \mathbb{Z}^n \cap NW} f\left(\frac{k}{N}\right) \sim \left(\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right) \int_{W_h} f(x)\, dx\right)(h = 0) \tag{3.4}$$

where $\tau(s_1, \dots, s_m) = \prod_{i=1}^m \tau(s_i)$.

Proof. — By Lemma 3.1 we can incorporate $u_1, \dots, u_m$ in a lattice basis $u_1, \dots, u_n$ of $(\mathbb{Z}^n)^*$. Let $\alpha_1, \dots, \alpha_n$ be the dual basis of $\mathbb{Z}^n$ and let $v = \sum_{i=1}^m c_i\alpha_i$. Then via the map

$$x \in \mathbb{R}^n \to \sum x_i\alpha_i + v \tag{3.5}$$

one is reduced to proving the theorem for the standard m-wedge: $x_1 \leqslant 0, \dots, x_m \leqslant 0$, *i.e.*, showing that the sum

$$\frac{1}{N^n}\sum f\left(\frac{k_1}{N},\cdots,\frac{k_n}{N}\right) \tag{3.6}$$

summed over all $(k_1,\dots,k_n)\in\mathbb{Z}^n$, with $k_i \leqslant 0$ for $i \leqslant m$, is equal to the expression

$$\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h_1}\cdots\int_{-\infty}^{h_m}dx_1\dots dx_m\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(x)\,dx_{m+1}\cdots dx_n, \tag{3.7}$$

evaluated at $h=0$, modulo $O(N^{-\infty})$. Moreover, since the subalgebra of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ generated by the products

$$f(x)=f_1(x_1)\dots f_n(x_n),\quad f_i\in\mathcal{C}_0^\infty(\mathbb{R}),$$

is dense in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ it suffices to prove the theorem for functions of this form, and hence it suffices to prove the theorem for $n=1$ and $m=0$ or 1. However, these two cases were dealt with in §2. (See (2.4) and (2.20).) ☐

We will next describe how (3.4) has to be modified if the condition (3.2) isn't satisfied. As above let $u_{m+1},\dots,u_n$ be vectors in $(\mathbb{Z}^n)^*$ whose projections onto the quotient of $(\mathbb{Z}^n)^*$ by $U\cap(\mathbb{Z}^n)^*$ are a lattice basis of this quotient lattice. The vectors, $u_1,\dots,u_n$ are now no longer a lattice basis of $(\mathbb{Z}^n)^*$ but they span a sublattice

$$\mathbb{A}^*=\mathrm{span}_{\mathbb{Z}}\{u_1,\dots,u_n\} \tag{3.8}$$

of $(\mathbb{Z}^n)^*$ of rank n, so the quotient

$$\Gamma=(\mathbb{Z}^n)^*/\mathbb{A}^* \tag{3.9}$$

is a finite group. Let $\alpha_1,\dots,\alpha_n$ be the basis vectors of $\mathbb{R}^n$ dual to $u_1,\dots,u_n$. Since $\mathbb{A}^*$ is a sublattice of $(\mathbb{Z}^n)^*$ the dual lattice,

$$\mathbb{A}=\mathrm{span}_{\mathbb{Z}}\{\alpha_1,\dots,\alpha_n\}, \tag{3.10}$$

contains $\mathbb{Z}^n$ as a sublattice. Moreover, each element, $x\in\mathbb{A}$, defines a character of the group, Γ, via the pairing

$$\gamma\in\Gamma\to e^{2\pi i\langle\gamma,x\rangle} \tag{3.11}$$

and this character is trivial if and only if x is in $\mathbb{Z}^n$. Recall now that if Γ is a finite group and γ a character of this group then

$$\sum_h\gamma(gh)=\gamma(g)\sum_h\gamma(h),$$

hence if $\gamma(g) \neq 1$ the sum above is zero. Thus we have

$$\frac{1}{|\Gamma|}\sum e^{2\pi i\langle\gamma,x\rangle} = \begin{cases} 1 \text{ if } x\in\mathbb{Z}^n \\ 0 \text{ if } x\notin\mathbb{Z}^n. \end{cases} \tag{3.12}$$

For each $\gamma \in \Gamma$ let

$$\tau_\gamma(s_1,\dots,s_m) = \tau_{\omega_1}(s_1)\cdots\tau_{\omega_m}(s_m) \tag{3.13}$$

where $\tau_\gamma(s)$ is defined by (2.12) and where $\omega_k = e^{2\pi i\langle\gamma,\alpha_k\rangle}$. We will generalize Theorem 3.2 by showing that for integer m-wedges which don't satisfy condition (3.2) one has

Theorem 3.3. — *For $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$*

$$\begin{aligned}\frac{1}{N^n}&\sum_{k\in NW\cap\mathbb{Z}^n} f\left(\frac{k}{N}\right) \\ &= \left(\sum_{\gamma\in\Gamma}\tau_\gamma\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{W_h} f(x)\,dx\right)(h=0) \mod O(N^{-\infty}).\end{aligned} \tag{3.14}$$

Proof. — By (3.11) the sum on the left coincides with the sum

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1}{N^n}\sum_{x\in\mathbb{A}\cap NW} e^{2\pi i\langle\gamma,x\rangle} f\left(\frac{x}{N}\right) \tag{3.15}$$

so it suffices to show that the γ-$^{\text{th}}$ summand in (3.14) is equal to the γ-$^{\text{th}}$ summand in (3.15). Via the map (3.5) the γ-$^{\text{th}}$ summand in (3.15) becomes

$$\frac{1}{N^n|\Gamma|}\sum_{k_1\leqslant 0,\dots,k_m\leqslant 0}\omega_1^{k_1}\dots\omega_m^{k_m}\left(\sum_{k_{m+1},\dots,k_n} g\left(\frac{k}{N}\right)\right) \tag{3.16}$$

where $g(x_1,\dots,x_n) = f(v + x_1\alpha_1 + \dots + x_n\alpha_n)$, and the γ-$^{\text{th}}$ summand in (3.14) becomes

$$\begin{aligned}\frac{1}{|\Gamma|}\tau_\gamma\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{-\infty}^{h_1}\cdots\int_{-\infty}^{h_m} dx_1\dots dx_m\int_{-\infty}^{\infty}& \\ \cdots\int_{-\infty}^{\infty} g(x)dx_{m+1}\cdots dx_k&\end{aligned} \tag{3.17}$$

evaluated at $h = 0$. (The reason for the factor, $1/|\Gamma|$, is that this is the Jacobian determinant of the mapping (3.5).) To prove that (3.16) and (3.17) are equal mod $O(N^{-\infty})$ it suffices as above to prove this for functions of the form $g = g_1(x)\dots g_n(x_n)$ with $g_i \in \mathcal{C}_0^\infty(\mathbb{R})$ and hence to show, for $i \leqslant m$

$$\frac{1}{N}\sum_{k_i=0}^{-\infty}\omega^{k_i} g_i\left(\frac{k_i}{N}\right) \sim \left(\tau_{\omega_i}\left(\frac{1}{N}\frac{\partial}{\partial h_i}\right)\int_{-\infty}^{h_i} g_i(x_i)\,dx_i\right)(h_i=0) \tag{3.18}$$

and, for $i > m$

$$\frac{1}{N}\sum_{-\infty}^{\infty} g_i\left(\frac{k_i}{N}\right) = \int_{-\infty}^{\infty} g_i(x_i)\,dx_i + O(N^{-\infty}), \tag{3.19}$$

and these follow from the identities (2.15) and (2.20). □

4. Riemann sums over polytopes

Let $\Delta \subseteq \mathbb{R}^n$ be an n-dimensional polytope whose vertices lie on the lattice $\mathbb{Z}^n$. Δ is said to be a simple polytope if each codimension k face is the intersection of exactly k facets. (It suffices to assume that the vertices of Δ, *i.e.*, the codimension n faces, have this property or, alternatively, that there are exactly n edges of Δ meeting at each vertex.) If the number of facets is d then Δ can be defined by a set of d inequalities

$$\langle u_i, x\rangle \leqslant c_i \tag{4.1}$$

where c_i is an integer and $u_i \in (\mathbb{Z}^n)^*$ is a primitive lattice vector which is perpendicular to the i^{th} facet and points “ outward” from Δ. By the simplicity assumption each codimension k face of Δ is the intersection of k facets lying in the hyperplanes

$$\langle u_i, x\rangle = c_i, \quad i \in F \tag{4.2}$$

where F is a k element subset of $\{1, \dots, d\}$. Let W_F be the k-wedge

$$\langle u_i, x\rangle \leqslant c_i \quad i \in F. \tag{4.3}$$

We will say that Δ is regular if each of these k-wedges is regular. (As above it suffices to assume this for the zero faces, *i.e.*, the vertices of Δ, or alternatively to assume that for every vertex, v, the edges of Δ which intersect at v lie on rays

$$v + t\alpha_i, \quad 0 \leqslant t < \infty$$

where $\alpha_1, \dots, \alpha_n$ is a lattice basis of $\mathbb{Z}^n$.)

For regular simple lattice polytopes one has the following Euler-Maclaurin formula.

Theorem 4.1. — *Let Δ_h be the polytope*

$$\langle u_i, x\rangle \leqslant c_i + h_i \quad i = 1, \dots, d. \tag{4.4}$$

Then, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$

$$\frac{1}{N^n}\sum_{k\in\mathbb{Z}^n\cap N\Delta} f\left(\frac{k}{N}\right) \sim \left(\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{\Delta_h} f(x)\,dx\right)(h=0) \tag{4.5}$$

where $\tau(s_1,\dots,s_d) = \tau(s_1)\dots\tau(s_d)$.

Proof. — For each face, F, of Δ let $\mathcal{O}_F$ be the open subset of Δ consisting of all faces of Δ which contain F in their closure. The $\mathcal{O}_F$'s are an open cover of Δ and by choosing a partition of unity subordinate to this cover, we can assume that $\operatorname{supp} f$ is contained in a small neighborhood of the set (4.2) and doesn't intersect the hyperplanes, $\langle u_i, x\rangle = c_i$, $i \notin F$. Then for $i \notin F$

$$\tau\left(\frac{1}{N}\frac{\partial}{\partial h_i}\right)\int_{\Delta_h} f\,dx = \int_{\Delta_h} f\,dx + \frac{1}{2N}\frac{\partial}{\partial h_i}\int_{\Delta_h} f(x)\,dx + \cdots .$$

However, by (2.4) all the terms on the right except the first are integrals of derivatives of f over the hyperplane $\langle u_i, x\rangle = c_i + h_i$, and hence for h_i small are zero. Thus the left hand side of (4.5) becomes

$$\left(\prod_{i\in F}\tau\left(\frac{1}{N}\frac{\partial}{\partial h_i}\right)\int_{(W_F)_h} f(x)\,dx\right)(h=0)$$

and the theorem above reduces to Theorem 3.2. □

If Δ is simple but not regular, one gets a slightly more complicated result. To the codimension k-face of Δ defined by (4.2) attach the subspace

$$U_F = \operatorname{span}_{\mathbb{R}}\{u_i, \quad i \in F\}$$

of $(\mathbb{R}^n)^*$, the sublattice

$$\mathbb{Z}_F = \operatorname{span}_{\mathbb{Z}}\{u_i, \quad i \in F\}$$

and the finite group

$$\Gamma_F = U_F \cap (\mathbb{Z}^n)^*/\mathbb{Z}_F.$$

This group coincides with the "torsion group" (3.9) of the wedge W_F. Moreover, if E is a subset of F, U_E is contained in U_F and $\mathbb{Z}_E$ in $\mathbb{Z}_F$, so Γ_E is contained in Γ_F. Let $\Gamma_F^\sharp$ be the set of points in Γ_F which are *not* contained in Γ_E for some proper subset, E of F.

Theorem 4.2. — *For* $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *the sum*

$$\frac{1}{N^n}\sum_{k\in\mathbb{Z}^n\cap N\Delta} f\left(\frac{k}{N}\right) \tag{4.6}$$

is equal $\mod O(N^{-\infty})$ *to*

$$\left(\sum_F\sum_{\gamma\in\Gamma_F^\sharp}\tau_\gamma\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{\Delta_h} f(x)\,dx\right)(h=0). \tag{4.7}$$

Proof. — As above it suffices to prove this for $\operatorname{supp} f$ contained in a small neighborhood of the set (4.2), and not intersecting the hyperplanes, $\langle u_i, x\rangle = c_i$, $i \notin F$. Then as above, the only contribution to the sum (4.7) is

$$\left(\sum_{\gamma\in\Gamma_F} \tau_\gamma \left(\frac{1}{N}\frac{\partial}{\partial h}\right) \int_{(W_F)_h} f(x)\,dx\right)(h=0)$$

and Theorem 4.2 reduces to Theorem 3.3. □

5. An Ehrhart theorem for symbols

A function, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a *polyhomogeneous symbol* of degree d if, for large values of x, it admits an asymptotic expansion

$$f(x) \sim \sum_{j=d}^{-\infty} f_j(x) \tag{5.1}$$

whose summands are homogeneous functions $f_j \in \mathcal{C}^\infty(\mathbb{R}^n - \{0\})$ of degree j. Let f be such a function and let Δ be a simple lattice polytope in $\mathbb{R}^n$ containing the origin in its interior. The Ehrhart function of the pair, f, Δ, is defined to be the function

$$E(f,\Delta,N) = \sum_{k\in N\Delta\cap\mathbb{Z}^n} f(k)\,, \quad N \in \mathbb{Z}_+\,.$$

In [5] it was shown that

$$E(f,\Delta,N) - \int_{N\Delta} f dx$$

had an asymptotic expansion

$$\sum_{j=n+d}^{-\infty} c_j N^j + c \tag{5.2}$$

for N large.

The main result of this section is a variant of this result. As above let Δ be a simple lattice polytope in $\mathbb{R}^n$ and let C_Δ be the polyhedral cone consisting of all points, $(x_1,\dots,x_n,x_{n+1})$, in $\mathbb{R}^{n+1}$ with $x_{n+1} > 0$ and $(x_1,\dots,x_n)/x_{n+1} \in \Delta$. Then, for $N \in \mathbb{Z}_+$, $N\Delta$ is just the slice of C_Δ by the hyperplane, $x_{n+1} = N$. We will prove that if $f \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ is a polyhomogeneous symbol of degree d the sum

$$\sum_{k\in N\Delta\cap\mathbb{Z}^n} f(k) \tag{5.3}$$

admits an asymptotic expression of the form (5.2).

Remarks.

1. (1) This result, albeit very close in spirit to the theorem in [5] cited above, doesn't, as far as we can see, seem to be a trivial consequence of it.
2. (2) This result has a number of applications to spectral theory on toric varieties which we'll explore in future publications.
3. (3) As a corollary of this result one gets another variant of the Ehrhart function theorem: Let

$$\Delta^\sharp := \{(x_1, \ldots, x_{n+1}) \in C_\Delta, \quad x_{n+1} \leqslant 1\}$$

be the pyramid over Δ with vertex 0. This $(n+1)$-dimensional polytope is *not* in general simple. However a version of theorem described at the beginning of this section is still true, namely

$$E(f, \Delta^\sharp, N) \sim \sum_{i=d+n+1}^{-\infty} c_i^\sharp + c^\sharp \log N.$$

as one can see by summing the differences

$$E(f, \Delta^\sharp, N) - E(f, \Delta^\sharp, N-1)$$

and noting that each difference is exactly (5.3). By combining this result with the Danilov "desingularization trick" [3] one can extend the Ehrhart theorem to a much larger class of convex lattice polytopes. We will discuss the details elsewhere.

Proof. — As above let

$$f \sim \sum_{i=d}^{-\infty} f_i$$

where $f_i(x_1, \ldots, x_{n+1})$ is a homogeneous function of degree i. Then on the cone, C_Δ:

$$f_i(x_1, \ldots, x_{n+1}) = x_{n+1}^i \, f_i\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\right)$$

so if we set $\tilde{f}_i(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_n, 1)$ the sum (5.3) is equal to the sum

$$N^i \sum_{k \in N\Delta \cap \mathbb{Z}^n} \tilde{f}_i\left(\frac{k}{N}\right). \tag{5.4}$$

which is N^{i+n} times the Riemann sum

$$\frac{1}{N^n}\sum_{k\in N\Delta\cap\mathbb{Z}^n}\tilde{f}_i\left(\frac{k}{N}\right). \tag{5.5}$$

Thus, by Theorem 4.2, each of these summands admits an asymptotic expansion:

$$\sum_{k=n+i}^{-\infty} c_{i,k}N^k$$

and hence so does the sum (5.3). □

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Manuscrit reçu le 2 octobre 2006,
accepté le 9 novembre 2006.

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