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Thom polynomials and Schur functions: the singularities $I_{2,2}(-)$

THOM POLYNOMIALS AND SCHUR FUNCTIONS: THE SINGULARITIES $I_{2,2}(-)$

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ABSTRACT. — We give the Thom polynomials for the singularities $I_{2,2}$ associated with maps $(\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$ with parameter $k \geqslant 0$. Our computations combine the characterization of Thom polynomials via the "method of restriction equations" of Rimanyi et al. with the techniques of Schur functions.

RÉSUMÉ. — Nous présentons les polynômes de Thom pour les singularités $I_{2,2}$ associées aux applications $(\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$ de paramètre $k \geqslant 0$. Nos calculs combinent la caractérisation des polynômes de Thom via la « méthode des équations restreintes » de Rimanyi et al. avec les techniques des fonctions de Schur.

1. Introduction

The global behavior of singularities is governed by their *Thom polynomials* (cf. [35], [14], [1], [12], [31]). Knowing the Thom polynomial of a singularity η, denoted $\mathcal{T}^\eta$, one can compute the cohomology class represented by the η-points of a map. We do not attempt here to survey all activities related to computations of Thom polynomials, which are difficult tasks in general.

In the present paper, following a series of papers by Rimanyi et al. [32], [31], [7], [2], we study the Thom polynomials for the singularities $I_{2,2}$ of the maps $(\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$ with parameter $k \geqslant 0$.

The way of obtaining the thought Thom polynomial is through the solution of a system of linear equations, which is fine when we want to find one concrete Thom polynomial, say, for a fixed k. However, if we

want to find the Thom polynomials for a series of singularities, associated with maps $(\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$ with k as a parameter, we have to solve *simultaneously* a countable family of systems of linear equations. As stated by Rimanyi in [31], p. 512 :

"*However, another challenge is to find Thom polynomials containing k as a parameter.*"

We do it here for the restriction equations for the singularities $I_{2,2}$ (any k) with the help of Schur functions. It appears that a use of these functions puts a transparent structure on computations of Thom polynomials. In particular, we get in this way some recursive formulas (cf., e.g., Lemma 14) that are not so easy to find using other bases (e.g. the Chern monomial basis that was used in [31] and in the references quoted there). In fact, various recursions play a prominent role in the present paper – apart from Lemma 14, see Eq. (4.34). Let us note that in a recent paper [6], Feher and Komuves compute Thom polynomials for some second order Thom-Boardman singularities also using Schur functions, and obtain similar recursions for the coefficients.

Another feature of using the Schur function expansions for Thom polynomials is that in all known to us cases, all the coefficients are nonnegative. In fact, we state the following "positivity conjecture":

CONJECTURE. — *The coefficients of the Schur function expansion of a Thom polynomial are nonnegative.*[1]

To be more precise, we use here (the specializations of) *supersymmetric* Schur functions, also called "Schur functions in difference of alphabets" together with their three basic properties: *vanishing, cancellation* and *factorization*, (cf. [34], [3], [17], [27], [29], [19], [9], and [16]). These functions contain resultants among themselves. Their geometric role was illuminated, e.g., in the study of *ideals of polynomials supported on degeneracy loci* of [26], i.e. $\mathcal{P}$-ideals of singularities Σ^i in the terminology of of the present paper (cf. the end of Section 2 and Theorem 11).

The main goal of this paper is to give the Thom polynomials for the singularities $I_{2,2}$ (in Mather's notation) associated with maps $(\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$ with parameter $k \geqslant 0$. We do it via establishing the Schur function expansions for these Thom polynomials. We prove first in Lemma 12 that the partitions appearing nontrivially have not more than three parts. Then,

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[1] Note added in May 2006: this conjecture (formulated also by Feher and Komuves [6]) has been recently proved by Weber and the author in [30].

in Lemma 14, we establish a recursive relation for Thom polynomials associated with the successive values of the parameter k. This reduces our calculation to compute the (sub)sum indexed by partitions with precisely two parts. This is essentially done in Proposition 16 (see also Propositions 17, 18, 19).

Our main result (Theorem 20), combined with Propositions 18 and 19, gives an explicit presentation of the Thom polynomial for the singularities $I_{2,2}$ with parameter $k \geqslant 0$ as a $\mathbb{Z}$-combination of Schur functions. We give *closed algebraic* expressions for the coefficients of these expansions. It turns out that these coefficients are the same as the coefficients of the Schur function expansions of the Segre classes of the second symmetric power of a rank 2 vector bundle, computed in [33], [26], [15], and [28].

Our main result offers a generalization (to any $k \geqslant 0$) of the formulas obtained previously by Porteous [21] and Rimanyi [31] for $k = 0$ and $k = 1$, respectively.

In our calculations, we use extensively the functorial λ-ring approach to symmetric functions developed mainly in Lascoux's book [16].

Main results of the present paper were announced in [22].

The forthcoming author's articles [24] and [23] will be devoted to study the Schur function expansions of Thom polynomials for the Morin's singularities A_i. Inspired by the present paper, [22], [24], and [23], Ozer Ozturk [20] computed the Thom polynomials for A_4 and $k = 2, 3$.

2. Recollections on Thom polynomials

Our main reference for this section is [31]. We start with recalling what we shall mean by a "singularity". Let $k \geqslant 0$ be a fixed integer. By a *singularity* we shall mean an equivalence class of stable germs $(\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$, where $\bullet \in \mathbb{N}$, under the equivalence generated by right-left equivalence (i.e. analytic reparametrizations of the source and target) and suspension (by suspension of a germ κ we mean its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$).

We recall[2] that the *Thom polynomial* $\mathcal{T}^\eta$ of a singularity η is a polynomial in the formal variables $c_1, c_2, \ldots$ that after the substitution

(2.1) $$c_i = c_i(f^*TY - TX) = \left[c(f^*TY)/c(TX) \right]_i,$$

for a general map $f : X \to Y$ between complex analytic manifolds, evaluates the Poincaré dual of $[V^\eta(f)]$, where $V^\eta(f)$ is the cycle carried by the closure

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[2] This statement is usually called the Thom-Damon theorem [35], [4].

of the set

$$(2.2) \qquad \{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\}.$$

By *codimension of a singularity η*, codim(η), we shall mean $\text{codim}_X(V^\eta(f))$ for such an f. The concept of the polynomial $\mathcal{T}^\eta$ comes from Thom's fundamental paper [35]. For a detailed discussion of the *existence* of Thom polynomials, see, e.g., [1]. Thom polynomials associated with group actions were studied by Kazarian in [12], [13].

In fact, the above is the case with singularities without moduli [12], [7] and [2]. The singularities $I_{2,2}$, studied in the present paper, have this property for $k \geqslant 0$. Indeed, the moduli of singularities start at codimension greater than $6k + 8$ (*loc.cit.*) whereas $\text{codim}(I_{2,2}) = 3k + 4$.

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional $\mathbb{C}$-algebras. We shall use the following notation:

▷ A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbb{C}[[x]]/(x^{i+1})$, $i \geqslant 0$;

▷ $I_{2,2}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbb{C}[[x,y]]/(xy, x^2 + y^2)$;

▷ $III_{2,2}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbb{C}[[x,y]]/(xy, x^2, y^2)$, here $k \geqslant 1$.

In the present article, the computations of Thom polynomials shall use the method which stems from a sequence of papers by Rimanyi et al. [32], [31], [7], [2]. We sketch briefly this approach, refering the interested reader for more details to these papers, the main references being the last three mentioned items.

Let $k \geqslant 0$ be a fixed integer, and let $\eta : (\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$ be a singularity with a prototype $\kappa : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$. The *maximal compact subgroup of the right-left symmetry group*

$$(2.3) \quad \text{Aut}\,\kappa = \left\{(\varphi, \psi) \in \text{Diff}(\mathbb{C}^n, 0) \times \text{Diff}(\mathbb{C}^{n+k}, 0) : \psi \circ \kappa \circ \varphi^{-1} = \kappa\right\}$$

of κ will be denoted by G_η. Even if $\text{Aut}\,\kappa$ is much too large to be a finite dimensional Lie group, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way (cf. [10] and [36]). In fact, G_η can be chosen so that the images of its projections to the factors $\text{Diff}(\mathbb{C}^n, 0)$ and $\text{Diff}(\mathbb{C}^{n+k}, 0)$ are linear. Its representations via the projections on the source $\mathbb{C}^n$ and the target $\mathbb{C}^{n+k}$ will be denoted by $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The vector bundles associated with the universal principal G_η-bundle $EG_\eta \to BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be

called E'_η and E_η. The *total Chern class of the singularity* η is defined in $H^*(BG_\eta, \mathbb{Z})$ by

$$(2.4) \qquad\qquad c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}.$$

The *Euler class* of η is defined in $H^{2\,\mathrm{codim}(\eta)}(BG_\eta, \mathbb{Z})$ by

$$(2.5) \qquad\qquad e(\eta) := e(E'_\eta).$$

In the following theorem, we collect information from [31], Theorem 2.4 and [7], Theorem 3.5, needed for the calculations in the present paper.

THEOREM 1. — *Suppose, for a singularity η, that the Euler classes of all singularities of smaller codimension than* $\mathrm{codim}(\eta)$, *are not zero-divisors.*[3] *Then we have:*

(i) *if* $\xi \neq \eta$ *and* $\mathrm{codim}(\xi) \leqslant \mathrm{codim}(\eta)$, *then* $\mathcal{T}^\eta(c(\xi)) = 0$;

(ii) $\mathcal{T}^\eta(c(\eta)) = e(\eta)$.

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

To use this method of determining the Thom polynomials for singularities, one needs their classification, see, e.g., [5].

In Section 4, we shall use these equations to compute Thom polynomials. Sometimes, it is convenient not to work with the whole maximal compact subgroup G_η but with its suitable subgroup; this subgroup should be, however, as "close" to G_η as possible (cf. [31], p. 502). We shall denote this subgroup by the same symbol G_η. We recall the following recipe for computing maximal compact subgroups from [31], pp. 505–507. Let η be a singularity whose prototype is $\kappa : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$. The germ κ is the miniversal unfolding of another germ $\beta : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+k}, 0)$ with $d\beta = 0$. The group G_η is a subgroup of the maximal compact subgroup of the algebraic automorphism group of the local algebra Q_η of η times the unitary group $U(k - d)$, where d is the difference between the minimal number of relations and the number of generators of Q_η. With β well chosen, G_η acts as right-left symmetry group on β with representations μ_1 and μ_2. The representations λ_1 and λ_2 are

$$(2.6) \qquad\qquad \lambda_1 = \mu_1 \oplus \mu_V \quad \text{and} \quad \lambda_2 = \mu_2 \oplus \mu_V,$$

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[3] This is the so-called "Euler condition" (*loc. cit.*) The Euler condition holds true for the singularities $I_{2,2}$ for any $k \geqslant 0$.

where μ_V is the representation of G_η on the unfolding space $V = \mathbb{C}^{n-m}$ given, for $\alpha \in V$ and $(\varphi, \psi) \in G_\eta$, by

$$(2.7) \qquad (\varphi, \psi)\,\alpha = \psi \circ \alpha \circ \varphi^{-1}.$$

For example, for the singularity of type $A_i \colon (\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+k}, 0)$, we have

$$G_{A_i} = U(1) \times U(k)$$

with

$$(2.8) \qquad \mu_1 = \rho_1, \quad \mu_2 = \rho_1^{i+1} \oplus \rho_k, \quad \mu_V = \bigoplus_{j=2}^{i} \rho_1^j \oplus \bigoplus_{j=1}^{i}(\rho_k \otimes \rho_1^{-j}),$$

where ρ_j denotes the standard representation of the unitary group $U(j)$. Hence, we obtain assertion (i) of the following

PROPOSITION 2. — (i) *Let* $\eta = A_i$. *For any* k, *writing* x *and* $y_1,\ldots,y_k$ *for the Chern roots of the universal bundles on* $BU(1)$ *and* $BU(k)$,

$$(2.9) \qquad c(A_i) = \frac{1 + (i+1)x}{1 + x} \prod_{j=1}^{k}(1 + y_j).$$

(ii) *Let* $\eta = I_{2,2}$. *Denote by* H *the extension of* $U(1) \times U(1)$ *by* $\mathbb{Z}/2\mathbb{Z}$ (*"the group generated by multiplication on the coordinates and their exchange"*). *For* $k \geqslant 0$, $G_\eta = H \times U(k)$. *Hence, for the purpose of our computations, we can use* $G_\eta = U(1) \times U(1) \times U(k)$. *Writing* x_1, x_2 *and* $y_1, \ldots, y_k$ *for the Chern roots of the universal bundles on two copies of* $BU(1)$ *and on* $BU(k)$, *we have*

$$(2.10) \qquad c(I_{2,2}) = \frac{(1 + 2x_1)(1 + 2x_2)}{(1 + x_1)(1 + x_2)} \prod_{j=1}^{k}(1 + y_j),$$

$$(2.11) \qquad e(I_{2,2}) = x_1 x_2 (2x_1 - x_2)(2x_2 - x_1)$$

$$\times \prod_{j=1}^{k}(y_j - x_1)(y_j - x_2)(y_j - x_1 - x_2).$$

(iii) *Let* $\eta = III_{2,2}$. *For* $k \geqslant 1$, $G_\eta = U(2) \times U(k-1)$, *and writing* x_1, x_2 *and* $y_1, \ldots, y_{k-1}$ *for the Chern roots of the universal bundles on* $BU(2)$ *and* $BU(k-1)$,

$$(2.12) \qquad c(III_{2,2}) = \frac{(1 + 2x_1)(1 + 2x_2)(1 + x_1 + x_2)}{(1 + x_1)(1 + x_2)} \prod_{j=1}^{k-1}(1 + y_j).$$

(Assertions (ii) and (iii) are obtained, in a standard way, following the instructions of [31], Sect. 4. As for assertion (ii), compare [31, pp. 506–507] whereas assertion (iii) stems from [2, p. 65].)

Let η be a singularity. As it was illuminated in the author's paper [26], in the case of the singularities $\eta = \Sigma^i$, it is natural and useful to consider a certain (homogeneous) ideal in the polynomial ring $R = \mathbb{Z}[c_1, c_2, \ldots]$ whose component of minimal degree is generated by $\mathcal{T}^\eta$. Namely, we denote by $\mathcal{P}^\eta$ the ideal of polynomials in R which – after the substitution (1) – are supported on $V^\eta(f)$, where $f : X \to Y$ is a general map between complex analytic manifolds. (The notion of a "polynomial supported on a subscheme" can be found in [9], Appendix A.) Keeping track of [26], we shall call $\mathcal{P}^\eta$ the $\mathcal{P}$-*ideal of the singularity* η. For example, the $\mathcal{P}$-ideal of the singularity

$$\Sigma^i : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$$

is

$$\mathcal{P}^{\Sigma^i} = \mathcal{P}_{m-i} \,,$$

where on the RHS we have the ideal studied extensively in [26] (cf. also [25] and [27]). We shall use this ideal in the proof of Theorem 11.

In the present paper, it will be more handy to use, instead of k, a "shifted" parameter

(2.13) $$r := k + 1 \,.$$

Sometimes, we shall write $\eta(r)$ for the singularity $\eta : (\mathbb{C}^\bullet, 0) \to (\mathbb{C}^{\bullet+r-1}, 0)$, and denote the Thom polynomial of $\eta(r)$ by $\mathcal{T}_r^\eta$ – to emphasize the dependence of both items on r.

We have

$$c_i(f^*TY - TX) = S_i\big(TX^* - f^*(TY^*)\big) \,,$$

where S_i means the Segre class. We shall follow of the notation on the RHS and use, more generally, Schur functions $S_{(i_1, i_2, \ldots, i_h)}$, indexed by partitions, cf. the next section.

3. Recollections on Schur functions

In this section, we collect needed notions related to symmetric functions. We adopt the functorial point of view of [16] for what concerns symmetric functions. Namely, given a commutative ring, we treat symmetric functions as operators acting on the ring. (Here, these commutative rings are mostly $\mathbb{Z}$-algebras generated by the Chern roots of the vector bundles from Proposition 2.)

DEFINITION 3. — *By an* alphabet $\mathbb{A}$, *we understand a (finite) multi-set of elements in a commutative ring.*

For $m \in \mathbb{N}$, by "an alphabet $\mathbb{A}_m$" we shall mean an alphabet $\mathbb{A} = (a_1, \dots, a_m)$ (of cardinality m); ditto for $\mathbb{B}_n = (b_1, \dots, b_n)$, $\mathbb{Y}_k = (y_1, \dots, y_k)$, and $\mathbb{X}_2 = (x_1, x_2)$.

DEFINITION 4. — *Given two alphabets* $\mathbb{A}$, $\mathbb{B}$, *the* complete functions $S_i(\mathbb{A} - \mathbb{B})$ *are defined by the generating series (with z an extra variable):*

$$(3.1) \qquad \sum S_i(\mathbb{A} - \mathbb{B})z^i = \prod_{b \in \mathbb{B}}(1 - bz) / \prod_{a \in \mathbb{A}}(1 - az).$$

So $S_i(\mathbb{A} - \mathbb{B})$ interpolates between $S_i(\mathbb{A})$ – the complete homogeneous symmetric function of degree i in $\mathbb{A}$ and $S_i(-\mathbb{B})$ – the ith elementary function in $\mathbb{B}$ times $(-1)^i$. The notation $\mathbb{A} - \mathbb{B}$ is compatible with the multiplication of series

$$(3.2) \quad \sum S_i(\mathbb{A} - \mathbb{B})z^i \cdot \sum S_j(\mathbb{A}' - \mathbb{B}')z^j = \sum S_i\big((\mathbb{A} + \mathbb{A}') - (\mathbb{B} + \mathbb{B}')\big)z^i,$$

the sum $\mathbb{A} + \mathbb{A}'$ denoting the union of two alphabets $\mathbb{A}$ and $\mathbb{A}'$.

CONVENTION 5. — We shall often identify an alphabet $\mathbb{A} = \{a_1, \dots, a_m\}$ with the sum $a_1 + \cdots + a_m$ and perform usual algebraic operations on such elements. For example, $\mathbb{A}b$ will denote the alphabet $(a_1 b, \dots, a_m b)$. We will give priority to the algebraic notation over the set-theoretic one. In fact, in the following, we shall use mostly alphabets of variables.

We have
$$(\mathbb{A} + \mathbb{C}) - (\mathbb{B} + \mathbb{C}) = \mathbb{A} - \mathbb{B},$$
and this corresponds to simplification of the common factor for the rational series:

$$(3.3) \qquad \sum S_i\big((\mathbb{A} + \mathbb{C}) - (\mathbb{B} + \mathbb{C})\big)z^i = \sum S_i(\mathbb{A} - \mathbb{B})z^i.$$

DEFINITION 6. — *By a partition $I = (i_1, i_2, \dots, i_h)$ we mean a weakly increasing sequence $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_h$ of natural numbers.*

In the following, we shall identify partitions with their *Young diagrams*, as is customary.

DEFINITION 7. — *Given a partition I and two alphabets $\mathbb{A}$ and $\mathbb{B}$, the Schur function $S_I(\mathbb{A} - \mathbb{B})$ is defined by the determinant*

$$(3.4) \qquad S_I(\mathbb{A} - \mathbb{B}) := \big|S_{i_p + p - q}(\mathbb{A} - \mathbb{B})\big|_{1 \leqslant p, q \leqslant h}.$$

These functions are often called *supersymmetric Schur functions* or *Schur functions in difference of alphabets*. Their properties were studied, among others, in [34], [3], [17], [27], [29], [19], [9], and [16]. From the last item, we borrow a use of increasing "French" partitions and the determinant of the form (3.4) evaluating a Schur function. We shall use the simplified notation $i_1 i_2 \cdots i_h$ or $i_1, i_2, \ldots, i_h$ for a partition $(i_1, i_2, \ldots, i_h)$ (the latter one if $i_h \geqslant 10$). The rectangle partition $(i, i, \ldots, i)$ (h times) will be denoted (i^h).

For example,

$$
S_{33344}(\mathbb{A} - \mathbb{B}) = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix},
$$

where S_i means $S_i(\mathbb{A} - \mathbb{B})$.

We shall now give some properties of Schur functions. The details can be found in the just quoted references. By Eq. (3.3), we get the following *cancellation property*:

$$(3.5) \qquad S_I\big((\mathbb{A} + \mathbb{C}) - (\mathbb{B} + \mathbb{C})\big) = S_I(\mathbb{A} - \mathbb{B}).$$

We record the following property justifying the notational remark from the end of Section 2; for a partition I,

$$(3.6) \qquad S_I(\mathbb{A} - \mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B} - \mathbb{A}) = S_J(\mathbb{B}^* - \mathbb{A}^*),$$

where J is the conjugate partition of I (i.e. the consecutive rows of J are equal to the corresponding columns of I), and $\mathbb{A}^*$ denotes the alphabet $\{-a_1, -a_2, \ldots\}$.

Fix two positive integers m and n. We shall say that a partition $I = (i_1, i_2, \ldots, i_h)$ *is contained in* the (m, n)-hook if either $h \leqslant m$, or $h > m$ and $i_{h-m} \leqslant n$. Pictorially, this means that the Young diagram of I is contained in the "tickened" hook
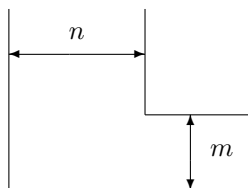
We record the following *vanishing property*. Given alphabets $\mathbb{A}$ and $\mathbb{B}$ of cardinalities m and n, if a partition I is not contained in the (m, n)-hook,

then

(3.7) $$S_I(\mathbb{A} - \mathbb{B}) = 0\,.$$

For example,

$$S_{3569}(\mathbb{A}_2 - \mathbb{B}_4) = S_{3569}(a_1 + a_2 - b_1 - b_2 - b_3 - b_4) = 0$$

because 3569 is not contained in the $(2,4)$-hook. In fact, we have the following result.

THEOREM 8. — *If $\mathbb{A}_m$ and $\mathbb{B}_n$ are alphabets of variables, then the functions $S_I(\mathbb{A}_m - \mathbb{B}_n)$, for I running over partitions contained in the (m,n)-hook, are $\mathbb{Z}$-linearly independent.*

(They form a $\mathbb{Z}$-basis of the abelian group of the so-called "supersymmetric functions".)

In the present paper, by a *symmetric function* we shall mean a $\mathbb{Z}$-linear combination of the operators $S_I(-)$.

DEFINITION 9. — *Given two alphabets $\mathbb{A}, \mathbb{B}$, we define their resultant*

(3.8) $$R(\mathbb{A}, \mathbb{B}) := \prod_{a \in \mathbb{A},\, b \in \mathbb{B}} (a - b)\,.$$

This terminology is justified by the fact that $R(\mathbb{A}, \mathbb{B})$ is the classical resultant of the polynomials $R(x, \mathbb{A})$ and $R(x, \mathbb{B})$. We have

(3.9) $$R(\mathbb{A}_m, \mathbb{B}_n) = S_{(n^m)}(\mathbb{A} - \mathbb{B}) = \sum_I S_I(\mathbb{A}) S_{(n^m)/I}(-\mathbb{B})\,,$$

where the sum is over all partitions $I \subset (n^m)$.

When a partition is contained in the (m,n)-hook and at the same time it contains the rectangle (n^m), then we have the following *factorization property*: for partitions $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_h)$,

(3.10) $$S_{(j_1,\ldots,j_h,i_1+n,\ldots,i_m+n)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A})\ R(\mathbb{A}, \mathbb{B})\ S_J(-\mathbb{B})\,.$$

The following convention stems from Lascoux's paper [18].

CONVENTION 10. — We may need to specialize a letter to 2, but this must not be confused with taking two copies of 1. To allow one, nevertheless, specializing a letter to an (integer, or even complex) number r inside a symmetric function, without introducing intermediate variables, we write $\boxed{r}$ for this specialization. Boxes have to be treated as single variables. For example,

$$S_i(2) = i + 1, \quad \text{but } S_i(\boxed{2}) = 2^i.$$

A similar remark applies to $\mathbb{Z}$-linear combinations of variables. We have:

$\triangleright$ $S_2(\mathbb{X}_2) = x_1^2 + x_1 x_2 + x_2^2$ but $S_2(\boxed{x_1 + x_2}) = x_1^2 + 2x_1 x_2 + x_2^2$,

$\triangleright$ $S_{11}(\mathbb{X}_2) = x_1 x_2$ but $S_{11}(\boxed{x_1 + x_2}) = 0$,

$\triangleright$ $S_2(3x) = 6x^2$ but $S_2(\boxed{3x}) = 9x^2$, etc.

For example,

$$(3.11) \qquad \prod_{j=1}^{k} (x_1 - y_j)(x_2 - y_j)(x_1 + x_2 - y_j) = R(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{Y}_k).$$

This convention will be used in the next section.

We end the present section with the following result which is a consequence of the author's study [26], [25], [27] of the $\mathcal{P}$-ideals of the singularities Σ^i.

THEOREM 11. — *Suppose that a singularity η is of Thom-Boardman type Σ^i. Then all summands in the Schur function expansion of $\mathcal{T}_r^{\eta}$ are indexed by partitions containing*[4] *the rectangle partition $(r + i - 1)^i$.*

Proof. — Since η is of Thom-Boardman type Σ^i, the Thom polynomial $\mathcal{T}_r^{\eta}$ belongs to the $\mathcal{P}$–ideal of the singularity Σ^i with parameter r. We also know by the Thom-Damon theorem (cf. [4]) that $\mathcal{T}_r^{\eta}$ is a $\mathbb{Z}$-linear combination of Schur functions in $TX^* - f^*(TY^*)$. The assertion now follows by combining Theorem 3.4 from [26] with Lemma 2.5 from [25] (see also Claim in the proof of Theorem 5.3(i) in [27]). Indeed, it follows from the former result that any $\mathbb{Z}$-combination of Schur functions indexed by partitions containing $(r + i - 1)^i$ belongs to $\mathcal{P}^{\Sigma_i(r)}$, whereas the latter result implies that no nonzero $\mathbb{Z}$-combination $\sum_I \alpha_I S_I$, where all $I \not\supset (r + i - 1)^i$, belongs to $\mathcal{P}^{\Sigma_i(r)}$. □

4. Thom polynomial for $I_{2,2}(r)$

The codimension of $I_{2,2}(r)$, $r \geqslant 1$, is $3r + 1$. The Thom polynomial for $I_{2,2}(1)$ is $S_{22} = S_{22}(TX^* - f^*(TY^*))$ (cf. [21]). In the following, we shall often omit the arguments of Schur functions.

>From now on, we shall assume that $r \geqslant 2$. The Thom polynomial for $I_{2,2}(2)$ is (cf. [31])

$$S_{133} + 3S_{34}.$$

---

[4] We say that one partition *is contained* in another if this holds for their Young diagrams (cf. [16]).

By virtue of Proposition 2, the equations from Theorem 1 characterizing the Thom polynomial for $I_{2,2}(r)$ are

$$(4.1) \qquad P(-\mathbb{B}_{r-1}) = P\big(x - \boxed{2x} - \mathbb{B}_{r-1}\big) = P\big(x - \boxed{3x} - \mathbb{B}_{r-1}\big) = 0\,,$$

and (using Eq. (3.11))

$$(4.2) \quad P\big(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-1}\big)$$
$$= x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R\big(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-1}\big)\,.$$

Here, without loss of generality, we assume that x, x_1, x_2, and $\mathbb{B}_{r-1}$ are variables. Moreover, $P(-)$ denotes a symmetric function. For the remainder of this paper, we set

$$(4.3) \qquad\qquad \mathbb{D} := \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2}\,.$$

Then, additionally, for variables x_1, x_2 and an alphabet $\mathbb{B}_{r-2}$, we have the vanishing imposed by $III_{2,2}$:

$$(4.4) \qquad\qquad P(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,.$$

Indeed, the singularities $\neq I_{2,2}$ with codimension $\leqslant \operatorname{codim}(I_{2,2})$ are: A_0, A_1, A_2 and $III_{2,2}$.

For $r \geqslant 1$, we set

$$(4.5) \qquad\qquad \mathcal{T}_r := \mathcal{T}_r^{I_{2,2}}\,.$$

Our goal is to give a presentation of $\mathcal{T}_r$ as a $\mathbb{Z}$-linear combination of Schur functions with explicit algebraic expressions of the coefficients:

$$(4.6) \qquad\qquad \mathcal{T}_r = \sum_I \alpha_I S_I.$$

We shall say that a partition I appears nontrivially in Eq. (4.6) if $\alpha_I \neq 0$.

LEMMA 12. — (i) *A partition appearing nontrivially in the Schur function expansion of $\mathcal{T}_r$ contains the partition $(r+1, r+1)$.*

(ii) *A partition appearing nontrivially in the Schur function expansion of $\mathcal{T}_r$ has at most three parts.*

Proof. — (i) Since the singularity $I_{2,2}$ is of Thom-Boardman type Σ^2, this is a particular case of Theorem 11.

(ii) We can assume that $r \geqslant 3$. In addition to information contained in (i), we shall use Eq. (4.4)

$$\mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,.$$

By virtue of (i), we can use factorization property (3.10) to all summands of

$$(4.7) \qquad \mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = \sum_I \alpha_I S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})$$

(we assume that $\alpha_I \neq 0$). We divide each summand of this last polynomial by the resultant

$$R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2}).$$

Suppose that the resulting factor of S_I is

$$(4.8) \qquad S_{p,q}(\mathbb{X}_2)\, S_J(-\mathbb{D} - \mathbb{B}_{r-2}),$$

cf. (3.10). Since $|I| = 3r + 1$, we have

$$(4.9) \qquad |J| \leqslant r - 1.$$

Now, let us assume that I has more than 3 parts, that is, J has 2 or more parts. This assumption (together with the inequality (4.9)) implies that

$$S_J(-\mathbb{B}_{r-2}) \neq 0$$

($\mathbb{B}_{r-2}$ is an alphabet of variables). Expanding (4.8), we get among summands the following one of largest possible degree $|J|$ in $\mathbb{B}_{r-2}$:

$$(4.10) \qquad S_{p,q}(\mathbb{X}_2)\, S_J(-\mathbb{B}_{r-2}) \neq 0.$$

Take in the sum

$$\sum_I \alpha_I S_{p,q}(\mathbb{X}_2)\, S_J(-\mathbb{D} - \mathbb{B}_{r-2})$$

the (sub)sum of all the nonzero summands of the form (4.8) with the largest possible weight of J. Since Schur polynomials are independent, this (sub)sum is nonzero, and moreover, it is $\mathbb{Z}$-linearly independent of other summands both in the sum indexed by partitions I with $\geqslant 3$ parts, and in that indexed by partitions with two parts (this last sum does not depend on $\mathbb{B}_{r-2}$). Hence, there is no $\mathbb{Z}$-linear combination of S_I's involving nontrivially I with more than three parts (and possibly also those with 3 or 2 parts) that satisfies Eq. (4.4). Assertion (ii) has been proved. $\qquad\square$

(For example, S_{1144} cannot appear in the Schur function expansion of $\mathcal{T}_3$ because $S_{1144}(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_1)$ after division by the resultant contains the summand $S_{11}(-\mathbb{B}_1) = S_2(\mathbb{B}_1)$, which does not occur in similar expressions for $S_{55}, S_{46}, S_{244}, S_{145}$.)

COROLLARY 13. — *If S_{i_1,i_2} appears nontrivially in the Schur function expansion of $\mathcal{T}_r$, then $i_1 = r + 1 + p$ and $i_2 = 2r - p$, where $0 \leqslant 2p \leqslant r - 1$.*

The following lemma gives a recursive description of $\mathcal{T}_r$. Denote by Φ the linear endomorphism on the $\mathbb{Z}$-module spanned by Schur functions indexed by partitions of length $\leqslant 3$, that sends a Schur function S_{i_1,i_2,i_3} to S_{i_1+1,i_2+1,i_3+1}. Let $\overline{\mathcal{T}}_r$ denote the sum of those terms in the Schur function expansion of $\mathcal{T}_r$ which are indexed by partitions of length $\leqslant 2$. Note that $\overline{\mathcal{T}}_1 = \mathcal{T}_1 = S_{22}$.

LEMMA 14. — *With this notation, for $r \geqslant 2$, we have the following recursive equation:*

$$(4.11) \qquad \mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\mathcal{T}_{r-1}).$$

Proof. — Write

$$(4.12) \qquad \mathcal{T}_r = \sum_I \alpha_I S_I = \sum_J \alpha_J S_J + \sum_K \alpha_K S_K,$$

where J have two parts and $K = (k_1, k_2, k_3)$ have three parts (we assume that $\alpha_I \neq 0$). We set

$$(4.13) \qquad Q = \sum_K \alpha_K S_{k_1-1,k_2-1,k_3-1},$$

and our goal is to show that $Q = \mathcal{T}_{r-1}$. Since a partition I appearing nontrivially in the Schur function expansion of $\mathcal{T}_r$ must contain the partition $(r+1, r+1)$, any partition $(k_1 - 1, k_2 - 1, k_3 - 1)$ above contains the partition (r, r). Since this last partition is not contained in the $(1, r-1)$-hook, Eqs. (4.1) with r replaced by $r - 1$ and P by Q are automatically fulfilled by virtue of the vanishing property (3.7). Note that Eq. (4.4) is a particular case of Eq. (4.2). Indeed, specializing b_{r-1} to $\boxed{x_1 + x_2}$ in Eq. (4.2), we get Eq. (4.4). Therefore it suffices to show that

$$(4.14) \quad Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) \, R\big(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-2}\big),$$

where $\mathbb{E} = \boxed{2x_1} + \boxed{2x_2}$. We apply to each summand

$$\alpha_K S_{k_1-1,k_2-1,k_3-1}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$$

of $Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$ the factorization property (3.10), and divide it by the resultant

$$R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-2}).$$

Suppose that the resulting factor is

$$(4.15) \qquad \alpha_K S_{a,b}(\mathbb{X}_2) \, S_c(-\mathbb{E} - \mathbb{B}_{r-2}),$$

where $(k_1 - 1, k_2 - 1, k_3 - 1) = (c, r + a, r + b)$.

Performing the same division of

$$x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R\big(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-2}\big),$$

we get $R(\boxed{x_1 + x_2}, \mathbb{B}_{r-2})$. Thus, the desired equation $Q = \mathcal{T}_{r-1}$ is equivalent to

$$(4.16) \qquad \sum_{a+b+c=r-2} \alpha_K S_{a,b}(\mathbb{X}_2) \, S_c(-\mathbb{E} - \mathbb{B}_{r-2}) = R(\boxed{x_1 + x_2}, \mathbb{B}_{r-2}).$$

To prove Eq. (4.16), we use Eqs. (4.2) and (4.12) for $\mathcal{T}_r$

$$\sum_I \alpha_I S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) R(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-1}).$$

Using again the factorization property (this time with respect to the larger rectangle $(r+1, r+1)$) and dividing both sides of the last equation by the resultant

$$R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1}),$$

we get the identity

$$(4.17) \qquad \sum_{p+q+j=r-1} \alpha_I S_{p,q}(\mathbb{X}_2) \, S_j(-\mathbb{E} - \mathbb{B}_{r-1}) = R(\boxed{x_1 + x_2}, \mathbb{B}_{r-1}).$$

Since

$$S_j(-\mathbb{E} - \mathbb{B}_{r-1}) = S_j(-\mathbb{E} - \mathbb{B}_{r-2}) - b_{r-1} S_{j-1}(-\mathbb{E} - \mathbb{B}_{r-2})$$

and

$$R(\boxed{x_1 + x_2}, \mathbb{B}_{r-1}) = (x_1 + x_2 - b_{r-1}) R(\boxed{x_1 + x_2}, \mathbb{B}_{r-2}),$$

taking the coefficients of $(-b_{r-1})$ in both sides of Eq. (4.17), we get the desired Eq. (4.16). The lemma has been proved. $\qquad\square$

(For example, writing $\mathcal{T}_3 = \alpha S_{46} + \beta S_{55} + \gamma S_{244} + \delta S_{145}$, we get that

$$\gamma S_1(-\mathbb{E} - B_1) + \delta S_1(\mathbb{X}_2) = R(\boxed{x_1 + x_2}, B_1),$$

by taking the coeficients of $(-b_2)$ in both sides of

$$\alpha S_2(\mathbb{X}_2) + \beta S_{11}(\mathbb{X}_2) + \gamma S_2(-\mathbb{E} - \mathbb{B}_2) + \delta S_1(-\mathbb{E} - \mathbb{B}_2) S_1(X_2) = R(\boxed{x_1 + x_2}, \mathbb{B}_2).)$$

Iterating Eq. (4.11) gives

COROLLARY 15. — *With the above notation, we have*

$$(4.18) \qquad \mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi(\overline{\mathcal{T}}_{r-1}) + \Phi^2(\overline{\mathcal{T}}_{r-2}) + \cdots + \Phi^{r-1}(\overline{\mathcal{T}}_1).$$

Of course, $\overline{\mathcal{T}}_r$ is uniquely determined by its value on $\mathbb{X}_2$. The following result gives this value.

PROPOSITION 16. — *For any $r \geqslant 1$, we have*

$$(4.19) \qquad \overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1 x_2)^{r+1} \, S_{r-1}(\mathbb{D}).$$

Proof. — We use induction on r. For $r = 1, 2$, the assertion holds true. Suppose that the assertion is true for $\overline{\mathcal{T}}_i$ where $i < r$. Fix a partition $I = (j, r + 1 + p, r + 1 + q)$ appearing nontrivially in the Schur function expansion (4.6) of $\mathcal{T}_r$. Note that j varies from 0 to $r - 1$ because $|I| = 3r + 1$. We obtain, by the factorization property (3.10),

$$S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot S_j(-\mathbb{D} - \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2),$$

where $R = R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})$. Hence, using Eq. (4.18), we see that

$$(4.20) \quad \mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot \left(\sum_{j=0}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) \frac{\overline{\mathcal{T}}_{r-j}(\mathbb{X}_2)}{(x_1 x_2)^{r-j+1}} \right).$$

By the induction assumption, for positive $j \leqslant r - 1$,

$$\overline{\mathcal{T}}_{r-j}(\mathbb{X}_2) = (x_1 x_2)^{r-j+1} \, S_{r-1-j}(\mathbb{D}).$$

Substituting this to (4.20), and using the vanishing (4.4), we obtain

$$(4.21) \quad \sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + \frac{\overline{\mathcal{T}}_r(\mathbb{X}_2)}{(x_1 x_2)^{r+1}} = 0.$$

But we also have, by a formula for addition of alphabets,

$$(4.22) \quad \sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + S_{r-1}(\mathbb{D}) = S_{r-1}(-\mathbb{B}_{r-2}) = 0.$$

Combining Eqs. (4.21) and (4.22) gives

$$\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1 x_2)^{r+1} \, S_{r-1}(\mathbb{D}),$$

that is, the induction assertion. The proof of the proposition is now complete.

$\square$

The Schur function expansion of $S_i(\mathbb{D})$ was described in [26], [15], and [28, App. A3] in the context of the *Segre classes* of the second symmetric power of a rank 2 vector bundle. Indeed, $\mathbb{D}$ is the alphabet of the Chern roots of the second symmetric power of a rank 2 bundle with the Chern roots x_1, x_2.

Denote by $\langle p, q \rangle$ the coefficient of $S_{p,q} := S_{p,q}(\mathbb{X}_2)$ in $S_{p+q}(\mathbb{D})$, where $0 \leqslant p \leqslant q$. A proof of the next proposition, due to Lascoux with the help of *divided differences*, can be found in [28], pp. 163–166. We give here another proof without divided differences.

PROPOSITION 17. — *For $p > 0$, we have*

$$(4.23) \qquad\qquad \langle p, q \rangle = \langle p - 1, q \rangle + \langle p, q - 1 \rangle.$$

Proof. — We have

$$(4.24) \qquad S_i(\mathbb{D}) = \sum_{h=0}^{i} S_h\left(\boxed{2x_1} + \boxed{2x_2}\right) S_{i-h}\left(\boxed{x_1 + x_2}\right)$$

$$= \sum_{h=0}^{i} 2^h S_h \cdot (x_1 + x_2)^{i-h},$$

and (cf., e.g., [19] I.4, Ex.3)

$$(4.25) \qquad (x_1 + x_2)^j = \sum_{a,b \geqslant 0} \binom{a+b}{a} \frac{b-a+1}{b+1} S_{a,b},$$

where $a + b = j$ and $a \leqslant b$. Combining Eqs. (4.24), (4.25) with the Pieri formula (cf., e.g., [16], [19]), we get for $0 \leqslant p \leqslant q$,

$$(4.26) \qquad \langle p, q \rangle = \sum_{h=0}^{p+q} 2^h \sum_{h_1, h_2 \geqslant 0} \binom{p+q-h}{p-h_1} \frac{(q-h_2)-(p-h_1)+1}{q-h_2+1},$$

where $h_1 + h_2 = h$ and $h_1 \leqslant p \leqslant q - h_2$.

We also compute the Schur function expansion of $S_{1,i-1}(\mathbb{D})$. Denote by $[p,q]$ the coefficient of $S_{p,q}$ in $S_{1,p+q-1}(\mathbb{D})$, $0 \leqslant p \leqslant q$. We have the following expansion for $S_{1,i-1}(\mathbb{D})$:

$$\sum_{h=1}^{i} S_{(1,i-1)/(i-h)}\left(\boxed{2x_1} + \boxed{2x_2}\right) S_h\left(\boxed{x_1 + x_2}\right)$$

$$= \sum_{h=1}^{i} 2^h S_h \cdot (x_1 + x_2)^{i-h} + \sum_{h=1}^{i} 2^h S_{1,h-1} \cdot (x_1 + x_2)^{i-h}.$$

We get from both sums in the last line that for $p > 0$ the coefficient $[p,q]$ is equal *twice* the RHS of Eq. (4.26), that is,

$$(4.27) \qquad [p,q] = 2\langle p,q \rangle.$$

We have, by the Pieri formula,

$$(4.28) \qquad S_{i-1}(\mathbb{D}) \cdot S_1(\mathbb{D}) = S_{i-1}(\mathbb{D}) \cdot 3S_1 = S_i(\mathbb{D}) + S_{1,i-1}(\mathbb{D}).$$

This equation implies that $S_{p,q}$ appears in $S_i(\mathbb{D}) + S_{1,i-1}(\mathbb{D})$ with multiplicity $3(\langle p-1, q \rangle + \langle p-1, q \rangle)$ (we use the Pieri formula once again). The desired Eq. (4.23) now follows by virtue of Eq. (4.27). □

We now pass to some "closed" algebraic expressions for the $\langle p,q \rangle$'s. We have

$$(4.29) \qquad \langle 0, q \rangle = S_q\left(\boxed{1} + \boxed{2}\right) = 1 + 2 + \cdots + 2^q = 2^{q+1} - 1.$$

The following result was obtained in [33], [26], and [15].

PROPOSITION 18. — *For $0 \leqslant p \leqslant q$, we have*

$$(4.30) \qquad \langle p, q \rangle = \binom{p+q+1}{p+1} + \binom{p+q+1}{p+2} + \cdots + \binom{p+q+1}{q+1}.$$

We propose now an alternative expression involving powers of 2, which is a natural generalization of the equation $\langle 0, q \rangle = 2^{q+1} - 1$, and which stems directly from Eq. (4.23). Namely, with the convention that $\binom{a}{0} = 1$ for any $a \in \mathbb{Z}$, we have

PROPOSITION 19. — *For $0 \leqslant p \leqslant q$,*

$$(4.31) \qquad \langle p, q \rangle = 2^{p+q+1} - \sum_{s=0}^{p} \left[\binom{p+q-2s-1}{p-s} - \binom{2p-2s-1}{p-s+1} \right] 2^{2s}.$$

Proof. — The proof uses double induction on p and q. We use Eq. (4.23) several times:

$$\langle p, q \rangle = \langle p-1, q \rangle + \langle p, q-1 \rangle$$
$$= \langle p-1, q \rangle + \langle p-1, q-1 \rangle + \langle p, q-2 \rangle$$
$$= \cdots = \langle p-1, q \rangle + \cdots + \langle p-1, 1 \rangle + \langle p, 0 \rangle.$$

We know the values of all summands in the last row by the induction assumption (the last summand being equal to $2^{p+1} - 1$). Using several times Eq. (4.29) as well as a well-known equality:

$$1 + \binom{a+1}{a} + \binom{a+2}{a} + \cdots + \binom{2a-2}{a} = \binom{2a-1}{a+1},$$

we get the desired induction assertion (4.31) for $\langle p, q \rangle$.   □

Using Proposition 16, we shall now give the Schur function expansion of $\overline{\mathcal{T}}_r$. Denote by d_{rj} the coefficient of $S_{r+j,2r+1-j}$ in $\overline{\mathcal{T}}_r$ for $r \geqslant 1$ and $j \geqslant 1$. By virtue of Corollary 13, $d_{rj} \neq 0$ entails $j \leqslant [\frac{1}{2}(r+1)]$ (for example, the only Schur functions that can appear with nonzero coefficients in $\overline{\mathcal{T}}_5$ are $S_{6,10}$, S_{79}, and S_{88}), so that we have

$$(4.32) \qquad \overline{\mathcal{T}}_r = \sum_{j=1}^{[\frac{1}{2}(r+1)]} d_{rj} \, S_{r+j,2r+1-j}.$$

We have the following link between the d_{rj}'s and $\langle p, q \rangle$'s: suppose that $d_{rj} \neq 0$, then we have

$$(4.33) \qquad d_{rj} = \langle j-1, r-j \rangle.$$

We may display the d_{rj}'s with the help of the following "Pascal triangle"-type matrix:

$$
\begin{matrix}
d_{11} & 0 & 0 & 0 & 0 & \ldots \\
d_{21} & 0 & 0 & 0 & 0 & \ldots \\
d_{31} & d_{32} & 0 & 0 & 0 & \ldots \\
d_{41} & d_{42} & 0 & 0 & 0 & \ldots \\
d_{51} & d_{52} & d_{53} & 0 & 0 & \ldots \\
d_{61} & d_{62} & d_{63} & 0 & 0 & \ldots \\
d_{71} & d_{72} & d_{73} & d_{74} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix}
\quad = \quad
\begin{matrix}
1 & 0 & 0 & 0 & 0 & \ldots \\
3 & 0 & 0 & 0 & 0 & \ldots \\
7 & 3 & 0 & 0 & 0 & \ldots \\
15 & 10 & 0 & 0 & 0 & \ldots \\
31 & 25 & 10 & 0 & 0 & \ldots \\
63 & 56 & 35 & 0 & 0 & \ldots \\
127 & 119 & 91 & 35 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix}
$$

By Proposition 17, if $d_{rj} > 0$, then we have

$$(4.34) \qquad d_{rj} = d_{r-1,j-1} + d_{r-1,j} .$$

We have the following values:

$$
\begin{aligned}
\overline{T}_1 &= S_{22}, \\
\overline{T}_2 &= 3S_{34}, \\
\overline{T}_3 &= 7S_{46} + 3S_{55}, \\
\overline{T}_4 &= 15S_{58} + 10S_{67}, \\
\overline{T}_5 &= 31S_{6,10} + 25S_{79} + 10S_{88}, \\
\overline{T}_6 &= 63S_{7,12} + 56S_{8,11} + 35S_{9,10}, \\
\overline{T}_7 &= 127S_{8,14} + 119S_{9,13} + 91S_{10,12} + 35S_{11,11} .
\end{aligned}
$$

Summing up all our considerations, we get the main result of the present paper. It gives the desired Thom polynomial in a parametric form (the parameter being r).

THEOREM 20. — *For $r \geqslant 1$, the Thom polynomial for $I_{2,2}(r)$ is equal to*

$$(4.35) \qquad \sum_{i=0}^{r-1} \sum_{\substack{j \geqslant 1 \\ i+2j \leqslant r+1}} d_{r-i,j}\, S_{i,r+j,2r-i-j+1} ,$$

where the coefficients

$$d_{r-i,j} = \langle j-1, r-i-j \rangle$$

are given by Eq. (4.30) (or (4.31)).

We have the following values of $\mathcal{T}_1, \mathcal{T}_2 = \Phi(\mathcal{T}_1) + \overline{\mathcal{T}}_2, \ldots, \mathcal{T}_7 = \Phi(\mathcal{T}_6) + \overline{\mathcal{T}}_7$:

$$\mathcal{T}_1 = S_{22},$$

$$\mathcal{T}_2 = S_{133} + 3S_{34},$$

$$\mathcal{T}_3 = S_{244} + 3S_{145} + 7S_{46} + 3S_{55},$$

$$\mathcal{T}_4 = S_{355} + 3S_{256} + 7S_{157} + 3S_{166} + 15S_{58} + 10S_{67},$$

$$\mathcal{T}_5 = S_{466} + 3S_{367} + 7S_{268} + 3S_{277} + 15S_{169} + 10S_{178}$$
$$+ 31S_{6,10} + 25S_{79} + 10S_{88},$$

$$\mathcal{T}_6 = S_{577} + 3S_{478} + 7S_{379} + 3S_{388} + 15S_{2,7,10} + 10S_{289}$$
$$+ 31S_{1,7,11} + 25S_{1,8,10} + 10S_{199}$$
$$+ 63S_{7,12} + 56S_{8,11} + 35S_{9,10},$$

$$\mathcal{T}_6 = S_{688} + 3S_{589} + 7S_{4,8,10} + 3S_{499} + 15S_{3,8,11} + 10S_{3,10,10}$$
$$+ 31S_{2,8,12} + 25S_{2,9,11} + 10S_{2,9,10} + 63S_{1,8,13}$$
$$+ 56S_{1,9,12} + 35S_{1,10,11} + 127S_{8,14}$$
$$+ 119S_{9,13} + 91S_{10,12} + 35S_{11,11}.$$

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Note. — After completion of the first version [22] of this paper, we received the preprint [8] containing a Chern monomial expression for the Thom series of $I_{2,2}$ supported by a computer evidence. In July 2006, we have received a letter from Kazarian [11] informing us that he has found another derivation of the Thom polynomial for $I_{2,2}$ based on geometric considerations.

BIBLIOGRAPHY

[1] V. ARNOLD, V. VASILEV, V. GORYUNOV & O. LYASHKO, "Singularities. Local and global theory", in *Enc. Math. Sci.*, vol. 6 (Dynamical Systems VI), Springer, 1993.

[2] G. BERCZI, L. FEHER & R. RIMANYI, "Expressions for resultants coming from the global theory of singularities, Topics in algebraic and noncommutative geometry, (L. McEwan et al. eds.)", in *Contemporary Math.*, vol. 324, Amer. Math. Soc., 2003, p. 63-69.

[3] A. BERELE & A. REGEV, "Hook Young diagrams with applications to combinatorics and to representation theory of Lie superalgebras", *Adv. in Math.* **64** (1987), p. 118-175.

[4] J. DAMON, "Thom polynomials for contact singularities", PhD Thesis, Harvard, 1972.

[5] A. DU PLESSIS & C. T. C. WALL, *The geometry of topological stability*, Oxford Math. Monograph, 1995.

[6] L. FEHER & B. KOMUVES, "On second order Thom-Boardman singularities", *Fund. Math.* **191** (2006), p. 249-264.

[7] L. FEHÉR, L. M. & R. RIMÁNYI, "Calculation of Thom polynomials and other cohomological obstructions for group actions", in *Real and complex singularities*, Contemp. Math., vol. 354, Amer. Math. Soc., 2004, p. 69-93.

[8] L. FEHER & R. RIMÁNYI, "On the structure of Thom polynomials of singularities", Preprint, September 2005.

[9] W. FULTON & P. PRAGACZ, *Schubert varieties and degeneracy loci*, Lecture Notes in Mathematics, vol. 1689, Springer-Verlag, Berlin, 1998, Appendix J by the authors in collaboration with I. Ciocan-Fontanine, xii+148 pages.

[10] K. JÄNICH, "Symmetry properties of singularities of C^∞-functions", *Math. Ann.* **238** (1979), p. 147-156.

[11] M. E. KAZARIAN, Letter to the author, dated July 4, 2006.

[12] ———, "Characteristic classes of singularity theory", in *The Arnold-Gelfand mathematical seminars: Geometry and singularity theory*, 1997, p. 325-340.

[13] ———, "Classifying spaces of singularities and Thom polynomials, New developments in singularity theory", in *NATO Sci. Ser. II Math. Phys. Chem.*, vol. 21, Kluwer Acad. Publ., Dordrecht, 2001, p. 117-134.

[14] S. KLEIMAN, "The enumerative theory of singularities", in *Real and complex singularities, Oslo 1976 (P. Holm ed.)*, 1978, p. 297-396.

[15] D. LAKSOV, A. LASCOUX & A. THORUP, "On Giambelli's theorem for complete correlations", *Acta Math.* **162** (1989), p. 143-199.

[16] A. LASCOUX, "Symmetric functions and combinatorial operators on polynomials", in *CBMS/AMS Lectures Notes*, vol. 99, Providence, 2003.

[17] A. LASCOUX & M.-P. SCHÜTZENBERGER, *Formulaire raisonné de fonctions symétriques*, Université Paris 7, 1985.

[18] A. LASCOUX, "Addition of ±1: application to arithmetic", *Sém. Lothar. Combin.* **52** (2004/05), p. Art. B52a, 9 pp. (electronic).

[19] I. G. MACDONALD, "Symmetric functions and Hall-Littlewood polynomials", in *Oxford Math. Monographs*, Amer. Math. Soc., 1995, 2nd ed.

[20] O. OZTURK, "On Thom polynomials for $A_4(-)$ via Schur functions", Preprint, IMPAN Warszawa 2006 (670), to appear in Serdica Math. J.

[21] I. PORTEOUS, "Simple singularities of maps, Proc. Liverpool Singularities I", in *Springer Lecture Notes in Math.*, vol. 192, Springer, 1971, p. 286-307.

[22] P. PRAGACZ, "Thom polynomials and Schur functions I", math.AG/0509234.

[23] ———, "Thom polynomials and Schur functions: the singularities $A_3(-)$", in preparation.

[24] ———, "Thom polynomials and Schur functions: towards the singularities $A_i(-)$", Preprint MPIM Bonn 2006 (139).

[25] ———, "Note on elimination theory", *Indagationes Math.* **49** (1987), p. 215-221.

[26] ———, "Enumerative geometry of degeneracy loci", *Ann. Sci. École Norm. Sup.* **21** (1988), p. 413-454.

[27] ———, "Algebro-geometric applications of Schur S- and Q-polynomials, Topics in invariant theory, Séminaire d'Algèbre Dubreil-Malliavin 1989-1990 (M.-P. Malliavin ed.)", in *Springer Lecture Notes in Math.*, vol. 1478, Springer, 1991, p. 130-191.

[28] ———, "Symmetric polynomials and divided differences in formulas of intersection theory", in *Parameter spaces (P. Pragacz ed.)*, vol. 36, Banach Center Publications, 1996, p. 125-177.

[29] P. Pragacz & A. Thorup, "On a Jacobi-Trudi identity for supersymmetric polynomials", *Adv. in Math.* **95** (1992), p. 8-17.

[30] P. Pragacz & A. Weber, "Positivity of Schur function expansions of Thom polynomials", Preprint, math.AG/0605308 and MPIM Bonn 2006 (60), to appear in Fund. Math.

[31] R. Rimanyi, "Thom polynomials, symmetries and incidences of singularities", *Inv. Math.* **143** (2001), p. 499-521.

[32] R. Rimanyi & A. Szücs, "Generalized Pontrjagin-Thom construction for maps with singularities", *Topology* **37** (1998), p. 1177-1191.

[33] H. Schubert, "Allgemeine Anzahlfunctionen für Kegelschnitte, Flächen und Raüme zweiten Grades in *n* Dimensionen", *Math. Ann.* **45** (1894), p. 153-206.

[34] J. Stembridge, "A characterization of supersymmetric polynomials", *J. Algebra* **95** (1985), p. 439-444.

[35] R. Thom, "Les singularités des applications différentiables", *Ann. Inst. Fourier* **6** (1955–56), p. 43-87.

[36] C. T. C. Wall, "A second note on symmetry of singularities", *Bull. London Math. Soc.* **12** (1980), p. 347-354.

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