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EXPONENTIAL SUMS WITH COEFFICIENTS 0 OR 1 AND CONCENTRATED L^p NORMS

by B. ANDERSON, J. M. ASH, R. L. JONES, D. G. RIDER & B. SAFFARI (*)

Abstract. — A sum of exponentials of the form $f(x) = \exp(2\pi i N_1 x) + \exp(2\pi i N_2 x) + \cdots + \exp(2\pi i N_m x)$, where the N_k are distinct integers is called an *idempotent trigonometric polynomial* (because the convolution of f with itself is f) or, simply, an *idempotent*. We show that for every $p > 1$, and every set E of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $|E| > 0$, there are idempotents concentrated on E in the L^p sense. More precisely, for each $p > 1$, there is an *explicitly calculated* constant $C_p > 0$ so that for each E with $|E| > 0$ and $\epsilon > 0$ one can find an idempotent f such that the ratio $\left(\int_E |f|^p \big/ \int_{\mathbb{T}} |f|^p\right)^{1/p}$ is greater than $C_p - \epsilon$. This is in fact a lower bound result and, though *not* optimal, it is close to the best that *our* method gives. We also give both heuristic and computational evidence for the still open problem of whether the L^p concentration phenomenon fails to occur when $p = 1$.

Résumé. — Une somme d'exponentielles de la forme $f(x) = \exp(2\pi i N_1 x) + \exp(2\pi i N_2 x) + \cdots + \exp(2\pi i N_m x)$, où les N_k sont des entiers distincts, est appelée un *polynôme trigonométrique idempotent* (car $f * f = f$) ou, simplement, un *idempotent*. Nous prouvons que pour tout réel $p > 1$, et tout $E \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}$ avec $|E| > 0$, il existe des idempotents concentrés sur E au sens de la norme L^p. Plus précisément, pour tout $p > 1$, nous *calculons explicitement* une constante $C_p > 0$ telle que pour tout E avec $|E| > 0$, et tout réel $\epsilon > 0$, on puisse construire un idempotent f tel que le quotient $\left(\int_E |f|^p \big/ \int_{\mathbb{T}} |f|^p\right)^{1/p}$ soit supérieur à $C_p - \epsilon$. Ceci est en fait un théorème de minoration qui, bien que *non* optimal, est proche du meilleur résultat que *notre* méthode puisse fournir. Nous présentons également des considérations heuristiques et aussi numériques concernant le problème (toujours ouvert) de savoir si le phénomène de concentration L^p a lieu ou non pour $p = 1$.

Keywords: Idempotents, idempotent trigonometric polynomials, L^p norms, Dirichlet kernel, concentrating norms, sums of exponentials, L^1 concentration conjecture, weak restricted operators.
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1. Introduction

1.1. Concentrated L^p norms

Let $e(x) := \exp(2\pi i x)$. A sum of exponentials of the form

$$f(x) = \sum_{k=1}^{m} e\left(N_k x\right), \quad (x \in \mathbb{R}),$$

where the N_k are distinct integers is called an *idempotent trigonometric polynomial* (because the convolution of f with itself is f) or, simply, an *idempotent*. In the sequel we adopt the term "idempotent" for brevity, and we denote by $\wp$ the set of all such idempotents:

$$\wp := \left\{ \sum_{n \in S} e(nx) : S \text{ is a finite set of non-negative integers} \right\}.$$

The simplest example of an $f \in \wp$ is (one form of) the *Dirichlet Kernel* of length n, defined by

$$D_n(x) := \sum_{\nu=0}^{n-1} e(\nu x) = \frac{\sin(n\pi x)}{\sin(\pi x)} \cdot e^{i(n-1)\pi x}. \tag{1.1}$$

Consider any function $g \in L^p(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and any set $E \subset \mathbb{T}$ with $|E| > 0$. (Throughout, $|E|$ denotes the Lebesgue measure of E.) If

$$\left(\int_E |g(x)|^p dx \Big/ \int_{\mathbb{T}} |g(x)|^p dx \right)^{1/p} \geqslant \alpha \tag{1.2}$$

(where $1 \leqslant p < \infty$ and $0 < \alpha < 1$), we say that "*at least a proportion α of the L^p norm of g is concentrated on E*" or, equivalently, that "*the function g has L^p concentration $\geqslant \alpha$ on E.*" We will now recall a challenging (and still partially open) problem on idempotents which can be expressed in terms of this notion of L^p concentration.

About 25 years ago it was discovered that for any (*arbitrarily small*) arc J of the torus $\mathbb{T}$ with $|J| > 0$, there always exists an *idempotent f* with *at least* 48% *of its L^2 norm concentrated on J*. The origin of this curious fact occurred about 1977 when one of us (J. M. Ash) was attempting to show that an operator T defined on $L^2(\mathbb{T})$, commuting with translations, and of restricted weak type $(2,2)$ is necessarily a bounded operator on $L^2(\mathbb{T})$. That T is of restricted weak type $(2,2)$ means that there is a constant $C = C(T) > 0$ such that, for every characteristic function χ of a subset of $\mathbb{T}$,

$$\sup_{\alpha > 0} \left(\alpha^2 \operatorname{measure}\{x \in \mathbb{T} : |T\chi(x)| > \alpha\} \right) \leqslant C \|\chi\|_{L^2}^2.$$

Ash [2] was only able to show that this condition was equivalent to there being *some* positive amount of L^2 concentration for idempotents. More explicitly, define the absolute constant C_2^* as the largest real number such that for *every* arc $J \subset \mathbb{T}$ with $|J| > 0$, we have the inequality

$$\sup_{f \in \wp} \left(\int_J |f(x)|^2 dx \Big/ \int_{\mathbb{T}} |f(x)|^2 dx \right)^{1/2} \geqslant C_2^*. \tag{1.3}$$

Thus the issue was whether C_2^* was 0 or positive. Luckily, at just the same time, Michael Cowling [5] proved, by another method, that every commuting with translations weak restricted type $(2,2)$ operator *is* necessarily a bounded operator on L^2. (Actually Cowling [5] proved more. His result allowed the underlying group to be *any* amenable group, not just $\mathbb{T}$.) This proved, of course, that C_2^* was indeed positive, but did not give any effective estimate for it. However, a series of concrete estimates quickly followed. The referee of [2] obtained $C_2^* \geqslant .01$, S. Pichorides [14] obtained $C_2^* \geqslant .14$, H. L. Montgomery [12] and J.-P. Kahane [9] obtained several better lower bounds. (The ideas of H. L. Montgomery were "deterministic" while those of J.-P. Kahane used probabilistic methods from [10].) Finally, in [4], three of us achieved the lower bound

$$\gamma_2 := \max_{x>0} \frac{\sin x}{\sqrt{\pi x}} = .4802\ldots, \tag{1.4}$$

which, in [7], was proved to be best possible. (See [6] for a more detailed exposition of the contents of [7].)

To get a little more feel for what to expect, let ζ be any point of density (also called "Lebesgue point") of a set $E \subset \mathbb{T}$. Then for every $p \in [1, \infty[$ the sequence of functions $\{g_n\}$, where $g_n(x) := D_n(x - \zeta) = \sum_{\nu=0}^{n-1} e(-\nu\zeta) \cdot e(\nu x)$, have L^p concentration tending to 1 as $n \to \infty$. However, the trigonometric polynomials g_n are not idempotents, since the nonzero coefficients are not all equal to 1. Note, however, that all the coefficients do have *modulus* 1. The difficulty of the matters studied in [4] and [7], as well as those of the present paper lies precisely in the fact that the trigonometric polynomials $f \in \wp$ have all their coefficients equal to 0 or 1, which is a very drastic constraint.

At this stage we make an obvious remark: in all the L^2 concentration problems on (small) arcs of $\mathbb{T}$ studied in [4] and [7] and in all their L^p analogues studied in the present paper, it is *equivalent* to work on arcs of $\mathbb{T}$ or on intervals of $[0, 1]$. We usually find it convenient to use "arcs of $\mathbb{T}$" in statements of theorems, but "intervals of $[0, 1]$" in their proofs!

1.2. The L^2 and L^p problems

The results in [4] and [7] were satisfying but, as usual, they led to further questions. The first two were:

a) Can we replace "arc" (or "interval") with "set of positive measure?"
b) Can we replace L^2 with L^p for any $p \geqslant 1$?

For each $p \in [1, \infty[$, define C_p as the largest number such that for every set E, $E \subset \mathbb{T}$ with $|E| > 0$, the inequality

$$\sup_{f \in \wp} \|f\|_{L^p,E} \Big/ \|f\|_{L^p} := \sup_{f \in \wp} \left(\int_E |f|^p dx \Big/ \int_{\mathbb{T}} |f|^p dx \right)^{1/p} \geqslant C_p \tag{1.5}$$

holds. Similarly, define C_p^* as the largest number such that for every arc J, $J \subset \mathbb{T}$, the inequality

$$\sup_{f \in \wp} \|f\|_{L^p,J} / \|f\|_{L^p} := \sup_{f \in \wp} \left(\int_J |f|^p dx \Big/ \int_{\mathbb{T}} |f|^p dx \right)^{1/p} \geqslant C_p^* \tag{1.6}$$

holds. The definitions of C_p and C_p^* are extended to the limit cases $p = \infty$ in the usual way. Obviously,

$$C_p \leqslant C_p^*. \tag{1.7}$$

With regard to question a), although the definitions allow the possibility for C_2 to be smaller than C_2^*, in [4] it is shown that both are equal to the constant γ_2 defined in (1.4). Whatever the value of $p \geqslant 1$, there is no result in this paper which changes when the supremum is taken over all sets of positive measure rather than over all arcs. So we *conjecture* that inequality (1.7) is in fact an equality:

$$C_p = C_p^*, \tag{1.8}$$

although we have no proof of this except for $p = 2$ and $p = \infty$.

Question b) is harder. In [7], the constant γ_2 defined in (1.4) is shown to be a lower bound for every C_p when $p \geqslant 2$. This is not altogether satisfying for two reasons. First, the cases $1 \leqslant p < 2$ are not addressed. Second, since the constant function 1 is in $\wp$, and $\|1\|_{L^\infty,A} / \|1\|_{L^\infty} = 1$ for *any* non-empty set $A \subset \mathbb{T}$, so that $C_\infty = 1$, one might hope to show that $\lim_{p\to\infty} C_p = 1$. In Section 1.3 we state new results for the L^p cases, together with some remaining open problems.

1.3. Statement of the result

This paper is devoted to proving one single theorem, Theorem 1.1 below. It was announced in the Comptes Rendus note [1] (in a weaker form, and presented as two distinct results). The aim of this paper is to supply the proofs of [1], to strengthen the first result thereof, and to unify the results in the form of a single theorem (which is valid for all sets of positive measure and not just for all arcs). Our theorem is stated in terms of the "constants" c_p and c_p^*, defined as follows:

$$c_p := \sup_{0<\omega<1/2} \frac{\sin(\pi\omega)/(\pi\omega)}{2^{1+1/p}\left(\lfloor 1/\omega\rfloor + 1 + \frac{1}{p-1}\left(\frac{3}{8}\right)^p \lfloor 1/\omega\rfloor\right)^{1/p}},$$

where for a real number r,

$$\lceil r\rceil = \text{ceiling of } r = \text{the smallest integer greater than or equal to } r,$$
$$\lfloor r\rfloor = \text{floor of } r = \text{the greatest integer less than or equal to } r;$$

and

$$c_p^* := \left(\frac{2}{\pi^{p+1}}\int_0^\infty \left|\frac{\sin x}{x}\right|^p dx\right)^{1/p} \cdot \max_{0\leqslant\omega\leqslant 1} \frac{\sin(\pi\omega)}{\omega^{1-1/p}}.$$

As p increases from 1 to $+\infty$ (resp. from 2 to $+\infty$), c_p (resp. c_p^*) increases from 0 to .5 (resp. from $\gamma_2 = 0.48\ldots$ to 1). That c_p^* tends to 1 as $p \to \infty$ follows from an easy calculation, which is done in Remark 1.2 for the reader's convenience.

THEOREM 1.1. — *Whenever $1 < p < \infty$, we have the estimate*

$$C_p \geqslant \begin{cases} c_p & \text{if } 1 < p \leqslant 2 \\ c_p^* & \text{if } 2 \leqslant p < \infty \end{cases}.$$

In other words, if $p > 1$ and $\epsilon > 0$ are given, then for each set $E \subset \mathbb{T}$ with $|E| > 0$, there is a finite set of integers $S = S(E, p, \epsilon)$ such that

$$\int_E \Big|\sum_{n\in S} e(nx)\Big|^p dx \Big/ \int_{\mathbb{T}} \Big|\sum_{n\in S} e(nx)\Big|^p dx \geqslant \begin{cases} c_p^p - \epsilon & \text{if } 1 < p \leqslant 2 \\ c_p^{*p} - \epsilon & \text{if } 2 \leqslant p < \infty \end{cases}. \tag{1.9}$$

Furthermore, c_p^* (and a fortiori C_p) tends to 1 as p tends to infinity.

For a slightly larger lower estimate of C_p, see inequality (2.19) at the end of Section 2 (where Part I of Theorem 1.1 is proved).

Remark 1.2. — The estimate in Part II of Theorem 1.1, which is quite good although not optimal, does have two virtues. First, it is sharp when $p = 2$, since

$$\frac{2}{\pi}\int_0^\infty \left|\frac{\sin x}{x}\right|^2 dx = 1.$$

Second, it implies that $\lim_{p\to\infty} C_p = 1$. Indeed

$$\begin{aligned}
\liminf_{p\to\infty} C_p &\geqslant \liminf_{p\to\infty} c_p^* \\
&= \liminf_{p\to\infty}\Big(\max_{0\leqslant\omega\leqslant1}\frac{\sin\pi\omega}{\pi\omega^{1-1/p}}\Big)\lim_{p\to\infty}\Big(\frac{2}{\pi}\Big)^{1/p}\lim_{p\to\infty}\Big(\int_0^\infty\left|\frac{\sin x}{x}\right|^p dx\Big)^{1/p} \\
&\geqslant \lim_{p\to\infty}\frac{\sin\pi(1/p)}{\pi(1/p)^{1-1/p}}\cdot 1\cdot \operatorname{Essup}\left|\frac{\sin x}{x}\right| \\
&= \lim_{p\to\infty}\frac{\sin(\pi/p)}{(\pi/p)}\lim_{p\to\infty}(1/p)^{1/p} = 1.
\end{aligned}$$

To prove Theorem 1.1, it obviously suffices to prove the inequalities $C_p \geqslant c_p$ (for all $p > 1$) and $C_p \geqslant c_p^*$ (for all $p \geqslant 2$). These two inequalities will be proved, respectively, in the next Sections 2 and 3.

As for the *open problems* pertaining to the case $p = 1$ (conjectures of non-concentration in the L^1 sense), we shall state them in Section 4 at the end of the paper.

2. Proof of Theorem 1.1 ; Part I: $C_p \geqslant c_p$ (for all $p > 1$)

2.1. Since the proof is quite technical and computational, before giving the full proof we start by sketching a (heuristic) outline of the beginning of the proof.

Outline of (the beginning of) the proof. — Let q be a large odd positive integer and let m be an integer at least as large. We begin with the special case of $J = [\frac{1}{q} - \frac{1}{mq}, \frac{1}{q} + \frac{1}{mq}]$, where $m \geqslant q$. The idea of the proof is to think of $\mathbb{T}$ (or rather of some suitable interval of length 1, which is more convenient in the proofs) as "partitioned" (except for common endpoints) into q congruent arcs of the form $\left[\frac{2\nu-1}{2q}, \frac{2\nu+1}{2q}\right]$, with centers at $\frac{\nu}{q}, (\nu = 0, 1, \ldots, q-1)$ and common length $\frac{1}{q}$. The idempotent $D_{mq}(qx)$, where $D_n(x)$ is the Dirichlet kernel defined by the relation (1.1) of the introduction, has period $1/q$ and when restricted to $\left[-\frac{1}{2}\frac{1}{q}, \frac{1}{2}\frac{1}{q}\right]$ behaves approximately like the Dirac measure. Now consider the idempotent

$D_{\omega q}(x)$, where $\omega \in]0,1/2[$ is chosen to make ωq an integer. This, when restricted to a small neighborhood of the set $\left\{\frac{0}{q}, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}\right\}$, behaves roughly (as far as its *modulus* is concerned) like the function $1/x$. Thus the idempotent $\varpi(x) = D_{mq}(qx)D_{\omega q}(x)$ satisfies $\int_0^1 |\varpi(x)|^p dx \approx \sum_{j=1}^q 1/j^p$ and $\int_J |\varpi(x)|^p dx \approx 1^{-p}$. Concentration at $1/q$ follows since $\sum_{j=1}^q j^{-p}$ is bounded.

This outline will be made rigorous in the proof below. We will also have to treat the problem of concentration at points which are *not* of the form $1/q$.

2.2. To prove Theorem 1.1 we need the following lemma, which will be used in the proof of Part II as well.

Lemma 2.1. — *Let $D_N(x)$ be the Dirichlet Kernel defined by (1.1), so that $|D_N(x)| = |\sin(\pi N x)/\sin(\pi x)|$*, and let p be greater than 1. Then

$$\int_0^1 |D_N(x)|^p \, dx = \delta_p N^{p-1} + o_p(N^{p-1}), \qquad N \to \infty$$

where

$$\delta_p = \frac{2}{\pi}\int_0^\infty \left|\frac{\sin u}{u}\right|^p du$$

and o_p is the "little o" notation of Landau modified to emphasize the dependence of the associated constant on p.

More precisely,

$$\int_0^1 |D_N(u)|^p du = \delta_p N^{p-1} + R_p(N),$$

where the error term $R_p(N)$ satisfies:

$$R_p(N) = \begin{cases} O_p\left(N^{p-3}\right) & \text{if } \ p > 3 \\ O(\log N) & \text{if } \ p = 3 \\ O_p(1) & \text{if } \ 1 < p < 3 \end{cases}.$$

Proof. — This result is classical, but we give the full proof for the reader's convenience. Since D_N is even, we need only estimate $2\int_0^{1/2} |D_N(x)|^p \, dx$. By the triangle inequality, this differs from

$$A_p(N) := 2\int_0^{1/2} \left|\frac{\sin N\pi u}{\pi u}\right|^p du$$

by at most

$$E_p(N) := 2\int_0^{1/2} |\sin N\pi u|^p \left(\frac{1}{\sin^p \pi u} - \frac{1}{(\pi u)^p}\right) du.$$

Substituting $x := N\pi u$ yields

$$\begin{aligned} A_p(N) &= \frac{2}{\pi N}\int_0^{N\pi/2}\left|\frac{\sin x}{x/N}\right|^p dx \\ &= \left(\frac{2}{\pi}\int_0^{\infty}\left|\frac{\sin x}{x}\right|^p dx\right) N^{p-1} - \frac{2}{\pi}N^{p-1}\int_{N\pi/2}^{\infty}\left|\frac{\sin x}{x}\right|^p dx \\ &= \delta_p N^{p-1} + O_p\left(1\right), \end{aligned}$$

since

$$\int_{N\pi/2}^{\infty}\left|\frac{\sin x}{x}\right|^p dx < \int_{N\pi/2}^{\infty} x^{-p}dx = \frac{\pi^{-p+1}2^{p-1}}{p-1}N^{-(p-1)}.$$

So proving the lemma reduces to proving that

$$E_p(N) = \begin{cases} O_p\left(N^{p-3}\right) & \text{if } \ p>3 \\ O\left(\log N\right) & \text{if } \ p=3 \\ O_p\left(1\right) & \text{if } \ 1<p<3 \end{cases}.$$

For $0 < u \leqslant 1/2$, the inequality

$$\begin{aligned} \frac{1}{\sin^p\left(\pi u\right)} - \frac{1}{\left(\pi u\right)^p} \leqslant \frac{\left(\pi u\right)^p - \sin^p\left(\pi u\right)}{\left(\pi u\right)^p\sin^p\left(\pi u\right)} \\ = O_p\left(1\right)\frac{u^p\left(1-\left(1+O\left(u^2\right)\right)^p\right)}{u^{2p}} = O_p\left(u^{2-p}\right) \end{aligned}$$

immediately leads to the two estimates

$$\left|\sin N\pi u\right|^p\left(\frac{1}{\sin^p \pi u} - \frac{1}{\left(\pi u\right)^p}\right) = \left(N\pi u\right)^p O_p\left(u^{2-p}\right) = O_p\left(N^p u^2\right) \tag{2.1}$$

and

$$\left|\sin N\pi u\right|^p\left(\frac{1}{\sin^p \pi u} - \frac{1}{\left(\pi u\right)^p}\right) = 1\cdot O_p\left(u^{2-p}\right) = O_p\left(u^{2-p}\right). \tag{2.2}$$

If $1 < p < 3$, from estimate (2.2) we have

$$E_p(N) \leqslant \int_0^{1/2} O_p\left(u^{2-p}\right)du = O_p\left(1\right);$$

if $p = 3$, from estimates (2.1) and (2.2) we have

$$\begin{aligned} E_3(N) &\leqslant \int_0^{1/N} O\left(N^3u^2\right)du + \int_{1/N}^{1/2} O\left(u^{2-3}\right)du \\ &\leqslant O\left(N^3N^{-3}\right) + O\left(\log N\right) = O\left(\log N\right); \end{aligned}$$

and if $p > 3$, again from estimates (2.1) and (2.2) we have

$$E_p(N) \leqslant \int_0^{1/N} O_p\left(N^p u^2\right) du + \int_{1/N}^{1/2} O_p\left(u^{2-p}\right) du$$
$$= O_p\left(N^p N^{-3}\right) + O_p\left(N^{-(3-p)}\right) = O_p\left(N^{p-3}\right).$$

Thus the lemma is proved. □

2.3. We now proceed to prove Theorem 1.1 in detail. We find it convenient to split the proof into eight steps. The first three steps deal with concentration at $1/q$ and the remaining five steps with the general case.

First Step. Concentration at $1/q$: lower estimation of numerator.

Let $\omega = \omega(q)$ be a constant in $(0, 1/2)$ such that ωq is an integer. We will estimate

$$\left(\int_J |\varpi(x)|^p dx \Big/ \int_0^1 |\varpi(x)|^p dx\right)^{1/p}, \tag{2.3}$$

where $\varpi(x) := D_{mq}(qx) D_{\omega q}(x)$.

We begin with the numerator, N, of (2.3).

Suppose that $|\frac{1}{q} - x| \leqslant \frac{1}{mq}$, and let $\delta := \frac{1}{q} - x$. Then, since $\sin x$ has a bounded derivative,

$$|D_{\omega q}(x)| = |D_{\omega q}(\frac{1}{q} - \delta)| = \left|\frac{\sin \pi\omega + O(\delta q)}{\sin \frac{\pi}{q} + O(\delta)}\right|$$
$$= \left|\frac{\sin \pi\omega}{\sin \frac{\pi}{q}}\right| + O(1) = \left|\frac{\sin \pi\omega}{\frac{\pi}{q}}\right| + O(1),$$

since $m \geqslant q$. Using this and Minkowski's inequality in the form

$$\|F + G\|_{L^p} \geqslant \|F\|_{L^p} - \|G\|_{L^p},$$

we have

$$N \geqslant \left(\int_J |D_{mq}(qx)|^p \left(\frac{q \sin \pi\omega}{\pi}\right)^p dx\right)^{1/p} - O\left(\left(\int_J |D_{mq}(qx)|^p dx\right)^{1/p}\right). \tag{2.4}$$

Substitute $u = qx$ to get

$$N \geqslant \left\{\left(\frac{\sin \pi\omega}{\pi}\right) q^{1-1/p} - O\left(q^{-1/p}\right)\right\} \left(\int_{-1/m}^{1/m} |D_{mq}(u)|^p du\right)^{1/p}. \tag{2.5}$$

Define $\Delta := \left(\int_0^1 |D_{mq}(u)|^p du\right)^{1/p}$. Since D_{mq} is even, we may write

$$\int_{-1/m}^{1/m} |D_{mq}(u)|^p du = \Delta^p - 2\int_{1/m}^{1/2} |D_{mq}(u)|^p du.$$

Use the estimates $|\sin mq\pi u| \leqslant 1$ and $|\sin \pi u| \geqslant 2u$ to control the last integral, thereby obtaining the estimate

$$\int_{-1/m}^{1/m} |D_{mq}(u)|^p du = \Delta^p - O\left(m^{p-1}\right).$$

By the lemma, $\Delta^p \simeq \delta_p q^{p-1} m^{p-1}$, which together with (2.5) implies

$$N \geqslant \Delta \left\{ \left(\frac{\sin \pi\omega}{\pi}\right) q^{1-1/p} - O\left(q^{-1/p}\right) \right\} \left\{ 1 - O\left(q^{-1+1/p}\right) \right\}, \tag{2.6}$$

or, more simply,

$$N \geqslant \Delta \left(\frac{\sin \pi\omega}{\pi}\right) q^{1-1/p} \left\{1 - o(1)\right\}. \tag{2.7}$$

Second Step. Concentration at $1/q$: upper estimation of denominator.

Passing now to the estimate of the denominator, D, of (2.3), we have

$$D^p = \int_0^1 |D_{mq}(qx)|^p \, |D_{\omega q}(x)|^p \, dx. \tag{2.8}$$

We now estimate this in great detail. We decompose

$$D^p = \sum_{j=0}^{q-1} \int_{\frac{j}{q}}^{\frac{j+1}{q}} |D_{mq}(qx)|^p \, |D_{\omega q}(x)|^p \, dx.$$

Let $x = y + \frac{j}{q}$, $dy = dx$, to get

$$D^p = \sum_{j=0}^{q-1} \int_0^{\frac{1}{q}} |D_{mq}(qy+j)|^p \left| D_{\omega q}\left(y + \frac{j}{q}\right)\right|^p dy.$$

Since $|D_{mq}(qy+j)| = |D_{mq}(qy)|$ for any integer j,

$$D^p = \sum_{j=0}^{q-1} \int_0^{\frac{1}{q}} |D_{mq}(qy)|^p \left| D_{\omega q}\left(y + \frac{j}{q}\right)\right|^p dy.$$

Let $t = qy$ to get

$$D^p = \frac{1}{q}\sum_{j=0}^{q-1} \int_0^1 |D_{mq}(t)|^p \left| D_{\omega q}\left(\frac{t}{q} + \frac{j}{q}\right)\right|^p dt.$$

Interchange sum and integral:

$$D^p = \int_0^1 |D_{mq}(t)|^p \, \frac{1}{q} \sum_{j=0}^{q-1} \left| D_{\omega q}\left(\frac{t}{q} + \frac{j}{q}\right)\right|^p dt. \tag{2.9}$$

Replace the sum by its supremum over all $t \in [0,1]$ which is the same as

$$\sup_{x\in[0,\frac{1}{q})} \sum_{j=0}^{q-1} \left| D_{\omega q}\left(x + \frac{j}{q}\right)\right|^p,$$

so recalling that $\Delta^p = \int_0^1 |D_{mq}(t)|^p \, dt$, we have

$$D^p \leqslant \Delta^p \frac{1}{q} \sup_{x\in[0,\frac{1}{q})} \sum_{j=0}^{q-1} \left| D_{\omega q}\left(x + \frac{j}{q}\right)\right|^p.$$

Since $D_{\omega q}$ is even and has period 1,

$$D^p \leqslant 2\Delta^p \frac{1}{q} \sup_{x\in[0,\frac{1}{q})} \sum_{j=0}^{\frac{q-1}{2}} \left| D_{\omega q}\left(x + \frac{j}{q}\right)\right|^p.$$

Break the sum into two pieces using the standard estimates

$$|D_{\omega q}(x + j/q)| \leqslant \omega q$$

when $j \leqslant \lfloor 1/\omega \rfloor$ and

$$\sup_{x\in[0,\frac{1}{q})} \left| D_{\omega q}(x + \frac{j}{q})\right| \leqslant 1/\sin\left(\frac{\pi j}{q}\right) \leqslant \frac{q}{2j}$$

when $j \in [\lfloor 1/\omega \rfloor + 1, (q-1)/2]$. We have

$$\begin{aligned}
D^p &\leqslant 2\Delta^p \frac{1}{q} \left\{ \sum_{j=0}^{\lfloor \frac{1}{\omega} \rfloor} (\omega q)^p + \sum_{j=\lfloor \frac{1}{\omega} \rfloor + 1}^{\frac{q-1}{2}} \left(\frac{q}{2j}\right)^p \right\} \\
&\leqslant 2\Delta^p \frac{1}{q} \left\{ \left(\left\lfloor \frac{1}{\omega} \right\rfloor + 1 \right) (\omega q)^p + \left(\frac{q}{2}\right)^p \int_{\lfloor \frac{1}{\omega} \rfloor}^{\infty} \frac{dx}{x^p} \right\} \\
&= 2\Delta^p q^{p-1} \omega^p \left\{ \left(\left\lfloor \frac{1}{\omega} \right\rfloor + 1 \right) + \frac{\lfloor \frac{1}{\omega} \rfloor}{p-1} \left(\frac{1}{2}\right)^p \left(\frac{\frac{1}{\omega}}{\lfloor \frac{1}{\omega} \rfloor} \right)^p \right\} \\
&= 2\Delta^p q^{p-1} \omega^p \left\{ \left\lfloor \frac{1}{\omega} \right\rfloor + 1 + \frac{\lfloor \frac{1}{\omega} \rfloor \rho}{p-1} \right\},
\end{aligned}$$

where

$$\rho = \left(\frac{1}{2} \frac{1/\omega}{\lfloor 1/\omega \rfloor} \right)^p.$$

Third Step. Concentration at $1/q$: conclusion.

Now combine this estimate with estimate (2.7) to get

$$\frac{N}{D} \geqslant \frac{\Delta\left(\frac{\sin \pi\omega}{\pi}\right) q^{1-1/p}\left\{1-o(1)\right\}}{\left(2\Delta^p q^{p-1}\omega^p\left\{\left\lfloor\frac{1}{\omega}\right\rfloor+1+\frac{\left\lfloor\frac{1}{\omega}\right\rfloor\rho}{p-1}\right\}\right)^{1/p}}$$

or

$$\frac{N}{D} \geqslant \frac{\left(\frac{\sin \pi\omega}{\pi\omega}\right)\left\{1-o(1)\right\}}{\left(2\left\{\left\lfloor\frac{1}{\omega}\right\rfloor+1+\frac{\left\lfloor\frac{1}{\omega}\right\rfloor\rho}{p-1}\right\}\right)^{1/p}}.$$

The numbers $\omega = \omega(q)$ appearing in the above two steps are, by construction, rational numbers. However it is clear, from their definition, that as q varies these $\omega(q)$ are everywhere dense in $[0,1/2]$. (*Cf.* also eighth step of this proof.) As will be made explicit below, this estimate, when extended to general intervals, is sufficient to prove Theorem 1.1. (Notice that $\rho < (3/8)^p$ since $\omega < 1/2$.)

Fourth Step. General case: heuristic search for the concentrated exponential sum.

From now on, our goal is to extend the above estimate (of the third step) to any interval. This fourth step is purely heuristic.

Given any interval $J \subset]0,1[$, we can find q (and in fact infinitely many such q's) so that for some integer $k \in [1,q[$, $[\frac{k}{q}-\frac{1}{qm}, \frac{k}{q}+\frac{1}{qm}] \subset J$, where $m := q$. (This choice of m is for technical reasons that will become clear in the eighth step of the proof.) It is convenient to pick q *prime,* which implies that k and q are relatively prime. Hence there exists a unique pair (a,b) of integers such that

$$ak - bq = 1, \quad (0 < a < q \quad \text{and} \quad 0 < b < k) \tag{2.10}$$

i.e., (the conjugate class of) a is the multiplicative inverse of k in the finite field $Z_q = GF(q)$. Multiplication by a, when reduced modulo q, defines a bijection α from $\{0,1,\dots,q-1\}$ to itself. Furthermore, $\alpha(k) = 1$. For the sake of this heuristic argument, temporarily suppose that (as previously) ω is chosen so that ωq is an integer. (In fact, in the rigorous argument below, we shall choose ω according to another criterion, so that ωq will *not* be an integer, but instead of ωq we shall use the integer $\lceil \omega q \rceil$. Now the idempotent $D_{\omega q}(ax)$ behaves very much like the idempotent $D_{\omega q}(x)$, except that the former does at k/q what the latter does at $1/q$. Since $D_{mq}(qx)$ is constant and large on the set $\left\{\frac{0}{q}, \frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}\right\}$, and $D_{\omega q}(ax)$ restricted to this set takes on the same set of values as $D_{\omega q}(x)$ did, but has its maximum at k/q (instead of at $1/q$); it seems reasonable that, the idempotent $D_{mq}(qx)D_{\omega q}(ax)$ should work here. The definition of the idempotent $G(x)$ analyzed below was motivated by these considerations.

Fifth Step. General case: the desired concentrated exponential sum.

Let E be a subset of $\mathbb{T}$ of positive measure and let $\epsilon > 0$ be given. First we will find an integer Q, and an $\eta = \eta\,(Q, \epsilon) > 0$; then an interval J of the form $J = \left[\frac{k}{q} - \frac{1}{mq}, \frac{k}{q} + \frac{1}{mq}\right]$ so that $|J \cap E| > (1-\eta)\,|J|$; and finally we will define an idempotent f depending on k and Q which is ϵ-close to being sufficiently concentrated on first J and then E. Actually in what follows we will always take m to be equal to q, but we leave m in the calculations since some increase of the concentration constants may be available by taking other values of m.

First we define η. The function f will have the form

$$f(x) = G(x) \sum_{n=0}^{mQ-1} e\,(nqx)\,.$$

Since G will turn out to be a sum of $\leqslant q$ exponentials (see (2.13), $\|G\|_\infty \leqslant q$; so the decomposition $J = (J \smallsetminus E) \cup (J \cap E)$ allows

$$\int_{J\setminus E} |f|^p\,dx \leqslant \eta\,|J|\,(\|f\|_\infty)^p \leqslant \eta \frac{2}{mq}\,(qmQ)^p = (\eta Q)\,2q^{p-1}\,(mQ)^{p-1}\,.$$

In the sixth step below, we will get

$$\int_J |f|^p\,dx \geqslant \left(\frac{\sin \pi\omega}{\pi}\right)^p q^{p-1}\delta_p\,(mQ)^{p-1} + o\left(q^{p-1}\,(mQ)^{p-1}\right)$$

as long as q is large enough. Pick $\eta\,(Q,\epsilon)$ so small that from this will follow

$$\begin{aligned}(2.11)\qquad \int_{J\cap E} |f|^p\,dx &= \int_J |f|^p\,dx - \int_{J\setminus E} |f|^p\,dx\\ &\geqslant (1-\epsilon)^p \left(\frac{\sin \pi\omega}{\pi}\right)^p q^{p-1}\delta_p\,(mQ)^{p-1}\,.\end{aligned}$$

Now that we know how to choose η, we show how to find $J = J\,(E,\eta)$. We will use $m = q$ while choosing J. Almost every point ξ has the property that there are infinitely many primes q and integers k for which

$$(2.12)\qquad \left|\xi - \frac{k}{q}\right| < \frac{1}{q^2}. \quad \text{(See [8].)}$$

We may assume that E has an irrational point of density ξ for which inequality (2.12) holds for infinitely many primes q. Suppose that the prime

q is so large that $K = \left[\xi - \frac{2}{q^2}, \xi + \frac{2}{q^2}\right]$ satisfies $|K \diagdown E| \leqslant \frac{\eta}{2}|K|$ and such that condition (2.12) holds. With $J = \left[\frac{k}{q} - \frac{1}{q^2}, \frac{k}{q} + \frac{1}{q^2}\right]$, we have

$$|J \diagdown E| \leqslant |K \diagdown E| \leqslant \frac{\eta}{2}|K| = \frac{\eta}{2}\frac{4}{q^2} = \eta\frac{2}{q^2} = \eta|J|,$$

so that $|J \cap E| > (1-\eta)|J|$, as required.

Let a be (uniquely) defined by (2.10), which in turn uniquely defines the bijection from $\{0, 1, \ldots, q-1\}$ into itself (reduction modulo q of multiplication by a). We have $\alpha(r) = ra - sq$ where s is the largest (necessarily non-negative) integer such that $sq \leqslant ra$. This leads us to consider the following sets $E_j, j = 1, 2, \ldots$. For each integer $j \geqslant 0$ let E_j denote the set of those $r \in \{0, 1, \ldots, q-1\}$ such that $\alpha(r) = ra - jq$. *A priori* the E_j are pairwise disjoint, and it is straightforward to check that

$$E_j = \begin{cases} \{r \in \mathbb{N} : t_j \leqslant r < t_{j+1}\} & \text{if } \ 0 \leqslant j < a \\ \emptyset & \text{if } \ j \geqslant a \end{cases},$$

where $\{t_j\}$ is the (*strictly increasing*) finite sequence of integers defined by $t_0 := 0$, $t_j = \lceil jq/a \rceil$ if $0 < j < a$, and $t_j = q$ if $j = a$.

Thus $\{E_j\}_{0 \leqslant j < a}$ is a partition of $\{0, 1, \ldots, q-1\}$, and we have $\alpha(r) = ra - jq$ when $r \in E_j$.

Now pick $\omega \in \left]0, 1/2\right[$ so that ωa is an integer ℓ at our disposal (with the obvious constraint $0 < \ell < a/2$). Instead of the "heuristic" idempotent $D_{\omega q}(x)$ suggested in the Fourth Step above, we now consider the idempotent

$$G(x) := \sum_{r=0}^{t_\ell - 1} e(\alpha(r)x), \tag{2.13}$$

where, in view of the above calculations, $t_\ell = \lceil \ell q/a \rceil = \lceil \omega q \rceil$. To make good use of this $G(x)$, we now need to perform a long calculation:

$$\begin{aligned} G(x) &= \sum_{j=0}^{\ell-1} \sum_{r=t_j}^{t_{j+1}-1} e(\alpha(r)x) = \sum_{j=0}^{\ell-1} \sum_{r=t_j}^{t_{j+1}-1} e((ra - jq)x) \\ &= \sum_{j=0}^{\ell-1} e(-jqx) \sum_{s=0}^{t_{j+1}-t_j-1} e((sa + t_j a)x) \\ &= \sum_{j=0}^{\ell-1} e(-jqx)e(t_j ax) \sum_{s=0}^{t_{j+1}-t_j-1} e(sax) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\ell-1} e((t_j a - jq)x)\Big(\frac{e((t_{j+1}-t_j)ax)-1}{e(ax)-1}\Big)\\
&= \frac{1}{e(ax)-1}\sum_{j=0}^{\ell-1} e(-jqx)\Big(e(t_{j+1}ax)-e(t_j ax)\Big)\\
&= \frac{1}{e(ax)-1}\Big(\sum_{j=1}^{\ell} e(-(j-1)qx)e(t_j ax) - \sum_{j=0}^{\ell-1} e(-jqx)e(t_j ax)\Big)\\
&= \frac{1}{e(ax)-1}\Big(\sum_{j=1}^{\ell} e(-jqx)e(qx)e(t_j ax) - \sum_{j=0}^{\ell-1} e(-jqx)e(t_j ax)\Big)\\
&= \frac{1}{e(ax)-1}\Big\{\sum_{j=1}^{\ell-1} e(-jqx)e(t_j ax)\big(e(qx)-1\big)\\
&\qquad + e(-\ell qx)e(qx)e(t_\ell ax)-1\Big\}\\
&= \frac{e(qx)-1}{e(ax)-1}\Big\{\sum_{j=1}^{\ell-1} e(-jqx)\big(e(t_j ax)-1\big) + \sum_{j=1}^{\ell-1} e(-jqx)\Big\}\\
&\qquad + \frac{1}{e(ax)-1}\big\{e(t_\ell ax)\big(e(-\ell qx)e(qx)-1\big)+e(t_\ell ax)-1\big\}\\
&= \frac{e(qx)-1}{e(ax)-1}\Big\{\sum_{j=1}^{\ell-1} e(-jqx)\big(e(t_j ax)-1\big)\\
&\qquad + e(-qx)\frac{e(-(\ell-1)qx)-1}{e(-qx)-1}\Big\}\\
&\qquad + \frac{1}{e(ax)-1}\big\{e(t_\ell ax)\big(e(-(\ell-1)qx)-1\big)+e(t_\ell ax)-1\big\}\\
&= \frac{e(qx)-1}{e(ax)-1}\Big\{\sum_{j=1}^{\ell-1} e(-jqx)\big(e(t_j ax)-1\big)\Big\}\\
&\qquad + \frac{e(qx)-1}{e(ax)-1}e(-qx)\frac{e(-(\ell-1)qx)-1}{e(-qx)-1}\\
&\qquad + e(t_\ell ax)\Big(\frac{e(-(\ell-1)qx)-1}{e(ax)-1}\Big) + \frac{e(t_\ell ax)-1}{e(ax)-1}\\
&= \frac{e(qx)-1}{e(ax)-1}\Big\{\sum_{j=1}^{\ell-1} e(-jqx)\Big(e(t_j ax)-1\Big)\Big\}\\
&\qquad + \frac{e(-(\ell-1)qx)-1}{e(ax)-1}\Big(-1+e(t_\ell ax)\Big) + \frac{e(t_\ell ax)-1}{e(ax)-1}
\end{aligned}
$$

$$= \frac{e(qx)-1}{e(ax)-1}\left\{\sum_{j=1}^{\ell-1} e(-jqx)\big(e(t_j ax)-1\big)\right\} \\ + \frac{e(t_\ell ax)-1}{e(ax)-1}\big(e(-(\ell-1)qx)-1+1\big).$$

Hence we may write

$$G(x) = \frac{e(qx)-1}{e(ax)-1}\sum_{j=1}^{\ell-1} e(-jqx)\left(e(t_j ax)-1\right) \\ + \frac{e(t_\ell ax)-1}{e(ax)-1} e(-(\ell-1)qx) =: G_1 + G_2.$$

Let $S := \{qn+\alpha(r) : r = 0,1,\dots,t_\ell - 1 \text{ and } n = 0,1,\dots,mq-1\}$. Also let

$$f(x) := \sum_{j\in S} e(jx) = \sum_{n=0}^{mQ-1} e(nqx)G(x).$$

This will be the desired concentrated sum of exponentials. Thus for each $p>1$, we must estimate the ratio

$$r := \left(\int_{J\cap E} |f(x)|^p\,dx \Big/ \int_0^1 |f(x)|^p\,dx\right)^{1/p}. \tag{2.14}$$

Sixth Step: General case: lower estimation of the numerator.

We now estimate the numerator of the ratio (2.14).

As in the fourth step, we may assume without loss of generality that J is centered at $\frac{k}{q}$ and has length $\frac{2}{mq}$, where $m=q$. If $x\in J$, write $x =: \frac{k}{q}+y$, so that $|y| \leqslant \frac{1}{mq}$. Then

$$G(x) = \sum_{r=0}^{t_\ell -1} e(\frac{r}{q})e(\alpha(r)y),$$

so that

$$G(x) = \sum_{r=0}^{t_\ell-1} e(\frac{r}{q}) + \sum_{r=0}^{t_\ell-1} e(\frac{r}{q})O(\frac{1}{m}) = \sum_{r=0}^{t_\ell-1} e(\frac{r}{q}) + O(\frac{t_\ell}{m}) \\ = \frac{e(t_\ell/q)-1}{e(1/q)-1} + O(\frac{\omega q}{m}).$$

Thus on J we have

$$|G(x)| = \left|\frac{\sin\pi\,(t_\ell/q)}{\sin\pi\,(1/q)}\right| + O(\frac{\omega q}{m}) = \left|\frac{\sin\pi\,(\omega + (\lceil\omega q\rceil - \omega q)\,/q)}{\sin\pi\,(1/q)}\right| + O(\frac{\omega q}{m}) \\ = \frac{\sin\pi\omega}{\pi}q + O(1)$$

since $m \geqslant q$. Also $\sum_{j=0}^{mQ-1} e(qjx) = D_{mQ}(qx)$, so applying Minkowski's inequality to the numerator in the ratio (2.14), we have

$$\begin{aligned}\left(\int_J |f(x)|^p\,dx\right)^{1/p} &= \left(\int_J \left|\frac{\sin\pi\omega}{\pi}q + O(1)\right|^p |D_{mQ}(qx)|^p\,dx\right)^{1/p}\\ &\geqslant \frac{\sin\pi\omega}{\pi}q\left(\int_J |D_{mQ}(qx)|^p\,dx\right)^{1/p} - O\left(\int_J |D_{mQ}(qx)|^p\,dx\right)^{1/p}.\end{aligned}$$

The same reasoning that led to equation (2.6) above now produces the following estimate for the numerator of (2.14):

$$\left(\int_J |f(x)|^p\,dx\right)^{1/p} \geqslant \frac{\sin\pi\omega}{\pi}q^{1-1/p}\Delta - O(q^{-1/p}\Delta),$$

where $\Delta = \left(\int_0^1 |D_{mQ}(u)|^p\,du\right)^{1/p}$. Taking (2.11) into account brings us to

$$\left(\int_{J\cap E} |f|^p\,dx\right)^{1/p} \geqslant (1-\epsilon)\,\frac{\sin\pi\omega}{\pi}q^{1-1/p}\delta_p^{1/p}\,(mQ)^{1-1/p}\,. \tag{2.15}$$

Seventh Step. General case: upper estimation of the denominator.

We now estimate the denominator of the ratio (2.14). To do this we will need the following lemmas.

Lemma 2.2. — *Let $p > 1$, $0 < \theta < 1$, and*

$$K_{\theta,N} = \left\{(x,y) \in \mathbb{Z}^2 : x + \theta^{-1}y \leqslant N, x \geqslant 0, y \geqslant 0\right\}.$$

Then for arbitrary $N \geqslant 4$,

$$\int_0^1\int_0^1 \left|\sum_{(m,n)\in K_{\theta,N}} e(mx+ny)\right|^p dxdy \leqslant C_pN^{2p-2}$$

uniformly with respect to θ and N.

See [3] for the proof of this. That proof extends (and is much indebted to) work of Yudin and Yudin [15] for $p = 1$ to higher values of p. (*Cf.* [13].)

Lemma 2.3. — *Let $p > 1$, $0 < \theta < 1$, and*

$$L_{\theta,N} = \left\{(x,y) \in \mathbb{Z}^2 : x + \theta^{-1}y \leqslant N, x > 0, y \geqslant 0\right\}.$$

Then for arbitrary $N \geqslant 4$,

$$\int_0^1\int_0^1 \left|\sum_{(m,n)\in L_{\theta,N}} e\,(mx+ny)\right|^p dxdy \leqslant C_pN^{2p-2}$$

uniformly with respect to θ and N.

Proof. — This is immediate from the last lemma because $K_{\theta,N}$ is the disjoint union of $L_{\theta,N}$ and $\{(0,n) : n = 0, 1, \ldots, \lfloor \theta N \rfloor\}$; and by Lemma 2.1,

$$\int_0^1 \int_0^1 \left| \sum_{n=0}^{\lfloor \theta N \rfloor} e\left(0x + ny\right) \right|^p dxdy = \int_0^1 \left| D_{\lfloor \theta N \rfloor}\left(y\right) \right|^p dy \ll N^{p-1} < N^{2p-2}.$$

□

To study the denominator of the ratio (2.14), because of Minkowski's inequality

$$\|f\|_p \leqslant \left\| \frac{\sin \pi mqQx}{\sin \pi qx} G_1 \right\|_p + \left\| \frac{\sin \pi mqQx}{\sin \pi qx} G_2 \right\|_p =: A + B, \tag{2.16}$$

it is enough to estimate these last two terms separately. For the first of these we have

$$\begin{aligned}
A^p &= \int_0^1 \left| \frac{\sin \pi mqQx}{\sin \pi qx} \right|^p \left| \frac{\sin \pi qx}{\sin \pi ax} \right|^p \left| \sum_{n=1}^{\ell-1} e(-nqx)\left(e(t_n ax) - 1\right) \right|^p dx \\
&= \int_0^1 |\sin \pi mqQx|^p \left| \sum_{n=1}^{\ell-1} e(-nqx) \frac{e(t_n ax) - 1}{\sin \pi ax} \right|^p dx \\
&= \int_0^1 |\sin \pi mqQx|^p \left| \sum_{n=1}^{\ell-1} e(-nqx) D_{t_n}\left(ax\right) \right|^p dx \\
&\leqslant \int_0^1 \left| \sum_{n=1}^{\ell-1} e(-nqx) D_{t_n}\left(ax\right) \right|^p dx \\
&= \sum_{j=0}^{q-1} \int_{j/q}^{(j+1)/q} \left| \sum_{n=1}^{\ell-1} e(-nqx) D_{t_n}\left(ax\right) \right|^p dx \\
&= \sum_{j=0}^{q-1} \int_0^{1/q} \left| \sum_{n=1}^{\ell-1} e(-nx - ajn) D_{t_n}\left(ax + aj/q\right) \right|^p dx \\
&= \sum_{j=0}^{q-1} \int_0^{1/q} \left| \sum_{n=1}^{\ell-1} e(-nx) D_{t_n}\left(ax + aj/q\right) \right|^p dx \\
&= \sum_{j=0}^{q-1} \int_0^{1/q} \left| \sum_{n=1}^{\ell-1} e(-nx) D_{t_n}\left(ax + j/q\right) \right|^p dx \\
&= \frac{1}{q} \sum_{j=0}^{q-1} \int_0^1 \left| \sum_{n=1}^{\ell-1} e(-ny/q) D_{t_n}\left(ay/q + j/q\right) \right|^p dy
\end{aligned}$$

where the last five steps are justified by the identity

$$[0,1] = \cup_{j=0}^{q-1} \left[j/q, (j+1)/q\right],$$

the substitutions $x \to x + j/q$, the periodicity of $e(x)$ being 1, the fact that $t \to at$ is a one to one correspondence on $\mathbb{Z}_q$, and the substitution $x = y/q$. Summarizing,

$$A^p \leqslant \frac{1}{q} \sum_{j=0}^{q-1} \int_0^1 \left|P_y\left(ay/q + j/q\right)\right|^p dy$$

where $P_y(x) = \sum_{n=1}^{\ell-1} e(-ny) D_{t_n}(x)$. But a theorem of Marcinkiewicz and Zygmund [16] asserts

$$\frac{1}{q} \sum_{j=0}^{q-1} \left|P_y\left(ay/q + j/q\right)\right|^p \ll \int_0^1 \left|P_y(x)\right|^p dx.$$

We may now apply Lemma 2.3 since the index set of $P_y(x)$ is $L_{\theta,N}$ with $\theta = a/q$ and $N = (\ell - 1)\, q/a$ to get

$$A^p \ll \int_0^1 \int_0^1 \left|P_y(x)\right|^p dx dy \ll q^{2p-2},$$

$$A < Cq^{2-2/p}. \tag{2.17}$$

Passing to the other term, we have

$$\begin{aligned} B^p &= \int_0^1 \left|\frac{\sin \pi m q Q x}{\sin \pi q x}\right|^p \left|\frac{\sin \pi t_\ell a x}{\sin \pi a x}\right|^p \left|e(-(\ell-1)qx)\right|^p dx \\ &= \int_0^1 \left|D_{mQ}(qx)\right|^p \left|D_{\lceil \omega q \rceil}(ax)\right|^p dx. \end{aligned}$$

This is estimated in a very similar way to the way D was estimated above, but with a couple of twists.

$$B^p = \sum_{j=0}^{q-1} \int_{\frac{j}{q}}^{\frac{j+1}{q}} \left|D_{mQ}(qx)\right|^p \left|D_{\lceil \omega q \rceil}(ax)\right|^p dx.$$

Let $x = y + \frac{j}{q}, dy = dx$, to get

$$B^p = \sum_{j=0}^{q-1} \int_0^{\frac{1}{q}} \left|D_{mQ}(qy + j)\right|^p \left|D_{\lceil \omega q \rceil}\left(ay + \frac{ja}{q}\right)\right|^p dy.$$

Since $|D_{mQ}(qy + j)| = |D_{mQ}(qy)|$ for any integer j,

$$B^p = \sum_{j=0}^{q-1} \int_0^{\frac{1}{q}} \left|D_{mQ}(qy)\right|^p \left|D_{\lceil \omega q \rceil}\left(ay + \frac{ja}{q}\right)\right|^p dy.$$

Let $t = qy$ to get

$$B^p = \frac{1}{q}\sum_{j=0}^{q-1}\int_0^1 |D_{mQ}(t)|^p \left|D_{\lceil\omega q\rceil}\left(\frac{at}{q}+\frac{ja}{q}\right)\right|^p dt.$$

Interchange sum and integral:

$$B^p = \int_0^1 |D_{mQ}(t)|^p \frac{1}{q}\sum_{j=0}^{q-1}\left|D_{\lceil\omega q\rceil}\left(\frac{at}{q}+\frac{ja}{q}\right)\right|^p dt.$$

Now as j varies between 0 and $q-1$, so does ja, modulo q. Thus the inner sum of this term may be written as

$$\sum_{j=0}^{q-1}\left|D_{\lceil\omega q\rceil}\left(\frac{at}{q}+\frac{j}{q}\right)\right|^p.$$

The supremum of this over all $t \in [0,1]$ is the same as

$$\sup_{x\in[0,\frac{1}{q})}\sum_{j=0}^{q-1}\left|D_{\lceil\omega q\rceil}\left(x+\frac{j}{q}\right)\right|^p,$$

so recalling that $\Delta^p = \int_0^1 |D_{mQ}(t)|^p\,dt$, we have

$$B^p \leqslant \Delta^p \frac{1}{q}\sup_{x\in[0,\frac{1}{q})}\sum_{j=0}^{q-1}\left|D_{\lceil\omega q\rceil}\left(x+\frac{j}{q}\right)\right|^p.$$

Since $\left|D_{\lceil\omega q\rceil}\right|$ is even and has period 1,

$$B^p \leqslant 2\Delta^p \frac{1}{q}\sup_{x\in[0,\frac{1}{q})}\sum_{j=0}^{\frac{q-1}{2}}\left|D_{\lceil\omega q\rceil}\left(x+\frac{j}{q}\right)\right|^p.$$

Break the sum into two pieces using the standard estimates $\left|D_{\lceil\omega q\rceil}(x)\right| \leqslant \lceil\omega q\rceil$ when $j \leqslant \left\lfloor\frac{1}{\omega}\right\rfloor$ and,

$$\sup_{x\in[0,\frac{1}{q})}\left|D_{\lceil\omega q\rceil}(x+\frac{j}{q})\right| \leqslant 1/\sin\left(\frac{\pi j}{q}\right) \leqslant \frac{q}{2j}$$

when $j \in [\lfloor 1/\omega \rfloor, (q-1)/2]$. We have

$$\begin{aligned}
B^p &\leqslant 2\Delta^p \frac{1}{q} \left\{ \sum_{j=0}^{\lfloor \frac{1}{\omega} \rfloor} \lceil \omega q \rceil^p + \sum_{j=\lfloor \frac{1}{\omega} \rfloor + 1}^{\frac{q-1}{2}} \left(\frac{q}{2j}\right)^p \right\} \\
&\leqslant 2\Delta^p \frac{1}{q} \left\{ (\lfloor 1/\omega \rfloor + 1) \lceil \omega q \rceil^p + \left(\frac{q}{2}\right)^p \int_{\lfloor 1/\omega \rfloor}^{\infty} \frac{dx}{x^p} \right\} \\
&= 2\Delta^p q^{p-1} \omega^p \left\{ (\lfloor 1/\omega \rfloor + 1) (\lceil \omega q \rceil / \omega q)^p + \frac{\lfloor 1/\omega \rfloor}{p-1} \left(\frac{1}{2}\right)^p \left(\frac{1/\omega}{\lfloor 1/\omega \rfloor}\right)^p \right\}.
\end{aligned}$$

Clearly $\lceil \omega q \rceil / (\omega q) \leqslant 2$, whence

$$B^p \leqslant 2^{p+1} \Delta^p q^{p-1} \omega^p \left(\lfloor 1/\omega \rfloor + 1 + \frac{\lfloor 1/\omega \rfloor}{p-1} \rho \right), \tag{2.18}$$

where

$$\rho = \left(\frac{1}{4} \frac{1/\omega}{\lfloor 1/\omega \rfloor} \right)^p .$$

Eighth Step. General case: estimation of the ratio.

Combine the last inequality with inequalities (2.15)–(2.18) to obtain

$$\begin{aligned}
r &\geqslant \frac{(1-\epsilon) \dfrac{\sin \pi\omega}{\pi} q^{1-1/p} \delta_p^{1/p} (mQ)^{1-1/p}}{C q^{2-2/p} + \Delta q^{1-1/p} \omega 2^{1+1/p} \left(\lfloor 1/\omega \rfloor + 1 + \dfrac{\lfloor 1/\omega \rfloor}{p-1} \rho \right)^{1/p}} \\
&= \frac{(1-\epsilon) \dfrac{\sin \pi\omega}{\pi}}{\frac{C}{\delta_p^{1/p}} \left(\frac{q}{mQ}\right)^{1-1/p} + \omega 2^{1+1/p} \left(\lfloor 1/\omega \rfloor + 1 + \dfrac{\lfloor 1/\omega \rfloor}{p-1} \rho \right)^{1/p} + o(1)} \\
&\geqslant \frac{(1-\epsilon) \dfrac{\sin \pi\omega}{\pi}}{\epsilon + \omega 2^{1+1/p} \left(\lfloor 1/\omega \rfloor + 1 + \frac{\lfloor 1/\omega \rfloor}{p-1} \rho \right)^{1/p} + o(1)},
\end{aligned}$$

where we have used $\Delta \simeq \delta_p^{1/p} (mQ)^{1-1/p}$. Here finally is the choice of Q: since $q/m \leqslant 1$, $Q = Q(\epsilon, p)$ is chosen to make the first term of the denominator less than ϵ. Now since ϵ was arbitrary, we can take it to be zero and then we can take q as large as we need to get rid of the $o(1)$ term. In other words,we have

$$C_p \geqslant \frac{\sin \pi\omega/(\pi\omega)}{2^{1+1/p}\left(\lfloor 1/\omega\rfloor + 1 + \frac{\lfloor 1/\omega\rfloor}{p-1}\left(\frac{1}{4}\frac{1/\omega}{\lfloor 1/\omega\rfloor}\right)^p\right)^{1/p}}. \tag{2.19}$$

Since the numbers ω in the right hand side of inequality (2.19) are easily seen to be everywhere dense in $[0,1/2]$, this ends the proof of Part I of Theorem 1.1.

A little more was proved than what was stated in terms of the constant c_p, in fact:

Remark 2.4. — It is possible to get numerical estimates for particular values of p by picking a value of ω that maximizes the right hand side of this inequality. For example, if $p = 2$, then setting $\omega = .34$ produces $c_2 = .13$, which compares reasonably well with the known fact that $C_2 = .48\ldots$. Since $\omega \in (0,.5)$, $1/\omega \in (2,\infty)$ so that $\rho \leqslant (3/8)^p < 1$. (In the statement of the theorem we have very slightly degraded estimate (2.19) by substituting $(3/4)^p$ for ρ.) Hence as $p \to \infty$, the denominator of the right hand side tends to $1/2$ and therefore the right side becomes

$$\frac{1}{2}\frac{\sin \pi\omega}{\pi\omega}$$

which may be made as close to 1 as you like by picking ω small enough. In other words, as p tends to ∞, c_p tends to $1/2$. Also our estimate, if sharp, would show that the constant is $O(p-1)$ as $p \searrow 1$, which would be consistent with our conjecture that concentration fails for L^1.

3. Proof of Theorem 1.1; Part II: $C_p \geqslant c_p^*$ (for all $p \geqslant 2$)

First pick $\omega = \omega(p)$ so that

$$\frac{\sin^p \pi\omega}{\omega^{p-1}} = \sup_{t\in(0,1/2]} \frac{\sin^p \pi t}{t^{p-1}}.$$

Let ξ be an irrational point of density of E and fix $\epsilon > 0$. Pick $\delta > 0$ so small that $J := [\xi-\delta, \xi+\delta] (\bmod 1)$ satisfies

$$|J\backslash E| < \epsilon |J|. \tag{3.1}$$

Set $S_\theta := \{n \text{ integer: } 1 \leqslant n \leqslant N, \|n\xi - \theta\| \leqslant \frac{\omega}{2}\}$ and $f_\theta := \sum_{n\in S_\theta} e(nx)$. Equation (7) on page 901 of [4] asserts that

$$\int_0^1 |f_\theta(x)|^2 d\theta \geqslant \left(\frac{\sin \pi\omega}{\pi}\right)^2 |D_N(x-\xi)|^2. \tag{3.2}$$

Combining this with Hölder's inequality,

$$\left(\int_0^1 |f_\theta(x)|^p d\theta\right)^{1/p} \left(\int_0^1 1^{p'} d\theta\right)^{1/p'} \geqslant \left(\int_0^1 |f_\theta(x)|^2 d\theta\right)^{1/2},$$

we get

$$\int_0^1 |f_\theta(x)|^p d\theta \geqslant \left(\int_0^1 |f_\theta(x)|^2 d\theta\right)^{p/2} \geqslant \left(\frac{\sin \pi\omega}{\pi}\right)^p |D_N(x-\xi)|^p.$$

Now integrate this in x over J to get

$$\begin{aligned}\int_0^1 \int_J |f_\theta(x)|^p dx\, d\theta &\geqslant \left(\frac{\sin \pi\omega}{\pi}\right)^p \int_{\xi-\delta}^{\xi+\delta} |D_N(x-\xi)|^p dx \\ &= \left(\frac{\sin \pi\omega}{\pi}\right)^p \int_{-\delta}^{\delta} |D_N(u)|^p du.\end{aligned} \tag{3.3}$$

Recall from the lemma of Section 2 that

$$\int_0^1 |D_N(u)|^p du = \ell_p N^{p-1} + R_p(N),$$

where $\ell_p = (2/\pi)\int_0^\infty |\sin x/x|^p\, dx$ and

$$R_p(N) = \begin{cases} O_p\left(N^{p-3}\right) & \text{if } \ p>3 \\ O(\log N) & \text{if } \ p=3 \\ O_p(1) & \text{if } \ 1<p<3 \end{cases}.$$

We can now make the estimate

$$\begin{aligned}\int_{-\delta}^{\delta} |D_N(u)|^p du &= \int_{-1/2}^{1/2} - 2\int_\delta^{1/2} \\ &= \ell_p N^{p-1} - \frac{2}{\pi^p}\int_\delta^{\pi/2} |\sin N\pi u|^p/u^p du + R_p(N) \\ &\geqslant \ell_p N^{p-1} - \frac{2}{\pi^p}\int_\delta^{\pi/2} 1/u^p du + R_p(N) \\ &= \ell_p N^{p-1} - \frac{2}{(p-1)\pi^p}(1/\delta)^{p-1} + R_p(N).\end{aligned}$$

Putting this into estimate (3.3) yields

$$\begin{aligned}&\int_0^1 \int_J |f_\theta(x)|^p dx d\theta \\ &\qquad \geqslant \left(\frac{\sin \pi\omega}{\pi}\right)^p \left(\ell_p N^{p-1} - \frac{2}{(p-1)\pi^p}(1/\delta)^{p-1} + R_p(N)\right).\end{aligned}$$

Hence there must be at least one θ for which

$$\int_J |f_\theta(x)|^p dx \geqslant \left(\frac{\sin \pi\omega}{\pi}\right)^p \left(\ell_p N^{p-1} - \frac{2}{(p-1)\pi^p}\left(\frac{1}{\delta}\right)^{p-1} + R_p(N)\right). \tag{3.4}$$

Next observe that

$$\operatorname{card} S_\theta = N\omega + \epsilon(N)N, \tag{3.5}$$

where $\epsilon(N) \to 0$ as $N \to \infty$. (To see this one can, for example, trace through the proof of Weyl's theorem given on pages 11–13 of Körner's *Fourier Analysis* [11]. When the interval $[2\pi a, 2\pi b]$ appearing there is translated, the functions f_+ and f_- are also. But translating a function amounts to multiplying its Fourier coefficients by factors of modulus 1, whence it is easy to see that all of the estimates depend only on $b - a$ and not on the value of a.) It follows that

$$\begin{aligned}\int_0^1 |f_\theta(x)|^p dx &= \int_0^1 |f_\theta(x)|^{p-2} |f_\theta(x)|^2 dx \\ &\leqslant (N\omega + \epsilon(N)N)^{p-2} \int_0^1 |f_\theta(x)|^2 dx \\ &= (N\omega + \epsilon(N)N)^{p-2} (N\omega + \epsilon(N)N) \\ &= (N\omega)^{p-1} + \epsilon_1(N)N^{p-1},\end{aligned}$$

where $\epsilon_1(N) \to 0$ as $N \to \infty$. Thus

$$\int_0^1 |f_\theta(x)|^p dx \leqslant (N\omega)^{p-1} + \epsilon_1(N)N^{p-1}. \tag{3.6}$$

It also follows from relations (3.1) and (3.5) that

$$\begin{aligned}&\int_E |f_\theta(x)|^p dx \Big/ \int_0^1 |f_\theta(x)|^p dx \geqslant \int_{J\cap E} |f_\theta(x)|^p dx \Big/ \int_0^1 |f_\theta(x)|^p dx \\ &\quad = \int_J |f_\theta(x)|^p dx \Big/ \int_0^1 |f_\theta(x)|^p dx - \int_{J\setminus E} |f_\theta(x)|^p dx \Big/ \int_0^1 |f_\theta(x)|^p dx.\end{aligned}$$

Denote the last two ratios by I and II respectively. We complete the proof by showing that I is big and that II is small. To estimate I we use relations (3.4) and (3.6).

$$\begin{aligned}I &\geqslant \frac{((\sin \pi\omega)/\pi)^p \left(\ell_p N^{p-1} - 2(p-1)^{-1}\pi^{-p}(1/\delta)^{p-1} + R_p(N)\right)}{(N\omega)^{p-1} + \epsilon_1(N)N^{p-1}} \\ &= \frac{\sin^p \pi\omega}{\pi^p \omega^{p-1}} \left(\ell_p - 2(p-1)^{-1}\pi^{-p}\left(\frac{1}{N\delta}\right)^{p-1}\right) + \epsilon_2(N),\end{aligned}$$

where $\epsilon_2(N) \to 0$ as $N \to \infty$. Since $[0,1] \supset J$, we may use the estimate (3.4) for the denominator of II, obtaining

$$II < \frac{\epsilon|J| \sup_x |f_\theta(x)|^p}{((\sin \pi\omega)/\pi)^p N^{p-1} \left(\ell_p - 2(p-1)^{-1}\pi^{-p} \left(1/(N\delta)\right)^{p-1} + o(1)\right)}$$

so using $|J| = 2\delta$ and the estimate (3.5), we obtain

$$II < \frac{2\epsilon N\delta\omega}{(\sin^p \pi\omega)\, \pi^{-p}\omega^{1-p} \left(\ell_p - 2(p-1)^{-1}\pi^{-p} \left(1/(N\delta)\right)^{p-1}\right)} + \epsilon_3(N),$$

where $\epsilon_3(N) \to 0$ as $N \to \infty$. Combine the estimates for I and II to achieve

$$C_p^* \geqslant \frac{\sin^p \pi\omega}{\pi^p\omega^{p-1}} \left(\ell_p - 2(p-1)^{-1}\pi^{-p} \left(1/(N\delta)\right)^{p-1}\right) \\ - \frac{2\epsilon N\delta\omega}{(\sin^p \pi\omega)\, \pi^{-p}\omega^{1-p} \left(\ell_p - 2(p-1)^{-1}\pi^{-p} \left(1/(N\delta)\right)^{p-1}\right)} - \epsilon_3(N).$$

Given any $\eta > 0$, find M so large that $2(p-1)^{-1}\pi^{-p} \left(1/(N\delta)\right)^{p-1} < \eta$, whenever $N\delta > M$. Then pick $\epsilon > 0$ so small that

$$\frac{2\epsilon(M+1)\omega}{(\sin^p \pi\omega)\, \pi^{-p}\omega^{1-p} (\ell_p - \eta)} < \eta.$$

Next pick $\delta > 0$ so small that estimate (3.1) holds for this ϵ. Finally choose N so that $M < N\delta < M+1$. It then follows from our last estimate for C_p^* that

$$\left(C_p^*\right)^p \geqslant \frac{\sin^p \pi\omega}{\pi^p\omega^{p-1}} (\ell_p - \eta) - \eta - \epsilon_3(N).$$

Since η was arbitrary and since increasing M also shrinks $\epsilon_3(N)$,

$$\left(C_p^*\right)^p \geqslant \frac{\sin^p \pi\omega}{\pi^p\omega^{p-1}} \ell_p = \frac{\sin^p \pi\omega}{\pi^p\omega^{p-1}} \frac{2}{\pi} \int_0^\infty \left|\frac{\sin x}{x}\right|^p dx.$$

Thus Part II of Theorem 1.1 is proved.

4. Does concentration fail when $p = 1$?

CONJECTURE 4.1. — *Concentration fails for L^1. More specifically, there is an absolute constant D such that if $J = [\frac{1}{q} - \frac{1}{mq}, \frac{1}{q} + \frac{1}{mq}]$, where $m > q^2$, then for every idempotent f,*

$$\int_J |f| dx \Big/ \int_0^1 |f| dx \leqslant \frac{D}{\ln q}. \tag{4.1}$$

Define a *special idempotent* to be a idempotent of the form $D_k(lx)D_m(x)$ for positive integers $k, l,$and m. Our main reason for believing this conjecture is that (1) estimates of the type made in Section 2 above show that the conjecture holds for all special idempotents and (2) the special idempotents do provide both the correct asymptotic behavior at $p = \infty$ and also the exact maximizing constant when $p = 2$.

To see that this last point is so, we must sharpen the estimates that we made in Section 2 above. Set $p = 2$ and use the exact calculation

$$\int_0^1 |D_{mq}(qx)D_{\omega q}(x)|^2 dx = m\omega q^2$$

for the denominator in the quantity (2.3), estimate the numerator as was done in Section 2, and then replace d_{mq} by its exact value mq; then our estimate for (2.3) is improved to

$$\frac{mq\left(\left(\sin \pi\omega\right)/\pi\right)^2 q}{mq \cdot \omega q} = \frac{\sin^2 \pi\omega}{\pi \cdot \pi\omega},$$

which shows that, for the appropriate choice of ω, the best possible constant is achievable even if the supremum is taken only over the small subclass of special idempotents.

Conjecture 4.1 is supported even more strongly by evidence that the following conjecture might be true.

Conjecture 4.2. — *Let* $\wp_n := \left\{\sum_{k=0}^{n-1} \epsilon_k e(kx) : \epsilon_k \text{ is } 0 \text{ or } 1\right\}$. *Then there is an absolute constant c such that for every n,*

$$\sup_{f\in\wp_n} \int_{\frac{1}{q}-\frac{1}{mq}}^{\frac{1}{q}+\frac{1}{mq}} |f(x)|dx \Big/ \int_0^1 |f(x)|dx \leqslant c \cdot \sup_{\substack{f \in \wp_n \\ f \text{ special}}} \int_{\frac{1}{q}-\frac{1}{mq}}^{\frac{1}{q}+\frac{1}{mq}} |f(x)|dx \Big/ \int_0^1 |f(x)|dx.$$

We should remark that this conjecture is trivial when $n \leqslant 2$, since the smallest idempotent that is not special is $e(0x)+e(1x)+e(3x)$. It is easy to see that Conjecture 4.2 easily implies Conjecture 4.1. Supporting numerical evidence for Conjecture 4.2 consists primarily of the fact that for those values of q we have looked at, the vast majority of the "best" functions (*i.e.* functions which produced the largest ratio) were special. (If this were always the case, Conjecture 4.2 would hold with $c = 1$, which would be a spectacular result.) However, there are some non-special functions which do beat out the special functions for certain values of n, hence the need for the constant c. For instance, when $q = 6$ the special functions were not always found to be the best. In particular, the function

$$f = e(0x) + e(1x) + e(5x) + e(6x) + e(7x) + e(12x)$$

produces the largest ratio for $n \leqslant 13$, yet this is not a special function. But the best special function for $n \leqslant 13$ is

$$D_2(6x)D_3(x) = e(0x) + e(1x) + e(2x) + e(6x) + e(7x) + e(8x),$$

whose ratio is only .98 of the ratio produced by f. Table 1 lays out the smallest values of c observed for various values of q. It should be noted

q	**c**	**n's studied**	**minimum c at**	**best projection at highest n**
2	1	3,...,14	–	$D_8(2x)$
3	1	3,..., 15	–	$D_6(3x)$
4	1	3,...,16	–	$D_2(x)D_4(4x)$
5	.87	3,...,16	n=15	$D_7(x)(1 - e(2x) + e(4x))$
6	.98	3,...,18	n=13	$D_3(x)D_3(6x)$
7	.87	3,...,16	n=15	$D_2(x)D_3(7x)$
8	.97	3,...,22	n=18	$D_4(x)D_3(8x)$
10	.98	3,...,22	n=22	$D_3(x)(1{+}e(9x){+}e(19x)){+}e(12x)$
12	1	3,...,20	–	$D_6(x)D_2(12x)$
16	1	3,...,20	–	$D_6(x)D_2(15x)$
32	1	3,...,16	–	$D_{15}(x)$

Table 4.1. Some observed upper bounds for c

that some values of q were studied to larger values of n than others. Clearly, the computation time is exponential in n, so going up to say $n = 22$ amounts to computing 2^{22} ratios of integrals.

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