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Baker domains for Newton's method

BAKER DOMAINS FOR NEWTON'S METHOD

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ABSTRACT. — For an entire function f let $N(z) = z - f(z)/f'(z)$ be the Newton function associated to f. Each zero ξ of f is an attractive fixed point of N and is contained in an invariant component of the Fatou set of the meromorphic function N in which the iterates of N converge to ξ. If f has an asymptotic representation $f(z) \sim \exp(-z^n)$, $n \in \mathbb{N}$, in a sector $|\arg z| < \varepsilon$, then there exists an invariant component of the Fatou set where the iterates of N tend to infinity. Such a component is called an invariant Baker domain.

A question in the opposite direction was asked by A. Douady: if N has an invariant Baker domain, must 0 be an asymptotic value of f? X. Buff and J. Rückert have shown that the answer is positive in many cases.

Using results of Balašov and Hayman, it is shown that the answer is negative in general: there exists an entire function f, of any order between $\frac{1}{2}$ and 1, and without finite asymptotic values, for which the Newton function N has an invariant Baker domain.

RÉSUMÉ. — Pour une fonction entière f soit $N(z) = z - f(z)/f'(z)$ la fonction de Newton associée à f. Chaque zéro ξ de f est un point fixe attractif de N et appartient à une composante invariante de l'ensemble de Fatou de la fonction méromorphe N dans laquelle les itérées de N convergent vers ξ. Si f a une représentation asymptotique $f(z) \sim \exp(-z^n)$, $n \in \mathbb{N}$, dans un secteur $|\arg z| < \varepsilon$, alors il existe une composante invariante de l'ensemble de Fatou de N dans laquelle les itérées de N tendent vers l'infini. Une composante avec cette proprieté est appelée un domaine invariant de Baker.

Une question dans la direction réciproque a été posée par A. Douady : si N a un domaine invariant de Baker, est-ce que 0 doit être une valeur asymptotique de f ? X. Buff et J. Rückert ont démontré que la réponse est affirmative dans beaucoup de cas.

En utilisant des résultats de Balašov et Hayman, on prouve que la réponse est négative en général : il existe une fonction entière f, d'ordre arbitraire entre $\frac{1}{2}$ et 1, et sans valeurs finies asymptotiques, pour laquelle il existe un domaine invariant de Baker de la fonction de Newton N.

1. Introduction and result

Let f be an entire function. Newton's method for finding the zeros of f consists of iterating the function

$$N(z) := z - \frac{f(z)}{f'(z)}.$$

If ξ is a zero of f, then $N(\xi) = \xi$ and $|N'(\xi)| < 1$, so there is an N-invariant domain U containing ξ in which the iterates N^k of N converge to ξ as $k \to \infty$. (Here N-invariance of U means that $N(U) \subset U$.)

There may also be N-invariant domains in which the iterates of N tend to ∞. A simple example is given by $f(z) = P(z) \exp Q(z)$ where P and Q are polynomials, with Q nonconstant. Then N is rational. Moreover, in the terminology of complex dynamics, the point at ∞ is a fixed point of N of multiplier 1, and the iterates of N tend to ∞ in the Leau petals associated to this fixed point.

If f does not have the above form, then N is transcendental; see [2] for an introduction to the iteration theory of transcendental meromorphic functions. A maximal N-invariant domain where the iterates of N tend to ∞ is called an *invariant Baker domain*.

A simple example (cf. [3]) is given by functions f for which $f(z) \sim \exp(-z^n)$ as $z \to \infty$ in some sector $|\arg z| < \varepsilon$. Then

$$N(z) = z + (1/n + o(1))z^{1-n}$$

and this implies that $N^k|_U \to \infty$ as $k \to \infty$ for some N-invariant domain U containing all sufficiently large positive real numbers. Note that $f(x) \to 0$ as $x \to +\infty$, $x \in \mathbb{R}$. Thus 0 is an asymptotic value of f, the positive real axis being an asymptotic path. Figuratively speaking one might say that Newton's method believes that there is a zero of f at $+\infty$, and thus it yields a domain U containing all sufficiently large positive real numbers such that $N^k(z) \to +\infty$ for $z \in U$ as $k \to \infty$.

The question arises whether an entire function f must always have 0 as an asymptotic value if N has an invariant Baker domain. This question was raised by A. Douady and has been brought to our attention by J. Rückert. It has been shown by X. Buff and J. Rückert [4] that the answer to this question is positive in situations much more general than those given above. However, we shall show that this is not always the case.

THEOREM. — *There exists an entire function f without finite asymptotic values such that $N(z) = z - f(z)/f'(z)$ has an invariant Baker domain.*

Moreover, f can be chosen to be of any preassigned order strictly between $\frac{1}{2}$ and 1.

We explain the basic idea of the construction. Using functions of the type introduced by S. K. Balašov [1], in §2 we construct an entire function f of order less than 1 (and in fact of any preassigned order strictly between $\frac{1}{2}$ and 1) which satisfies

$$f(z) \sim \sqrt[q]{z}$$

for some integer q and some branch of the q-th root as $z \to \infty$ in the spiralling region

$$S := \left\{ re^{ic \log r + i\theta} : r > 1, |\theta| < \theta_0 \right\},$$

where $c := \pi / \log(q-1)$ and $0 < \theta_0 < \pi$. Here the relation between c and q is such that S is invariant under $z \mapsto -pz$ where $p := q - 1$. We show in §3 that

$$\frac{f'(z)}{f(z)} \sim \frac{1}{qz}$$

so that

$$N(z) = z - \frac{f(z)}{f'(z)} \sim -pz.$$

In fact, we have an explicit error estimate in this asymptotic equality, which yields that S contains an N-invariant domain in which the iterates of N tend to ∞. Hence N has an invariant Baker domain. Finally we show in §4, using the Denjoy-Carleman-Ahlfors Theorem, that f has no finite asymptotic values.

2. The construction of f

Let (a_k) be a sequence of complex numbers tending to infinity. For $r > 0$ let $n(r)$ be the number of a_k, taking account of repetition, in $|z| \leqslant r$. Let

$$\rho := \limsup_{r \to \infty} \frac{\log n(r)}{\log r}.$$

Equivalently, ρ is the exponent of convergence of the sequence (a_k). It is well known that the canonical product Π whose zeros are the a_k has order ρ; that is,

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, \Pi)}{\log r}$$

where $M(r, \Pi) := \max_{|z|=r} |\Pi(z)|$ is the maximum modulus. There are standard results concerning the asymptotic behavior of Π if all a_k lie on one ray and

$$(2.1) \qquad\qquad n(r) \sim \Delta r^\rho$$

for some $\Delta > 0$ as $r \to \infty$. These results have been extended by Balašov [1] to the case where the a_k lie on a logarithmic spiral, say

$$(2.2) \qquad\qquad a_k \in \left\{ re^{ic \log r} : r \geqslant 1 \right\},$$

where $c > 0$. We quote only a simplified version of Balašov's result [1, Theorem 1], as this suffices for our purposes.

LEMMA 1. — *Let (a_k) be a sequence satisfying (2.1) and (2.2). Suppose that ρ is not an integer. Let Π be the canonical product formed with the a_k. Then*

$$\lim_{r \to \infty} \frac{\log \Pi(re^{ic \log r + i\theta})}{r^\rho} = -\frac{2\pi i \Delta \exp\left(i\rho\theta/(1+ic)\right)}{1 - \exp\left(i2\pi\rho/(1+ic)\right)}$$

for $0 < \theta < 2\pi$ and a suitable branch of the logarithm, the convergence being uniform for $\varepsilon \leqslant \theta \leqslant 2\pi - \varepsilon$ if $\varepsilon > 0$. In particular,

$$
\begin{aligned}
(2.3) \qquad & \lim_{r \to \infty} \frac{\log |\Pi(re^{ic \log r + i\theta})|}{r^\rho} \\
& = -2\pi\Delta \operatorname{Re}\left(\frac{i \exp\left(i\rho\theta/(1+ic)\right)}{1 - \exp\left(i2\pi\rho/(1+ic)\right)} \right) \\
& =: h(\theta).
\end{aligned}
$$

Now let $\tfrac{1}{2} < \rho < 1$ and $\Delta > 0$. Choose $p \in \mathbb{N}$ such that

$$(2.4) \qquad \mu := \frac{\rho}{1+c^2} := \frac{\rho}{1+(\pi/\log p)^2} > \frac{1}{2},$$

thus defining $c := \pi/\log p$. Note that since $\tfrac{1}{2} < \mu < \rho < 1$ we have $c < 1$ and hence $p > \exp(\pi) > 23$. Let (a_k) be a sequence satisfying (2.1) and (2.2) and let Π be the canonical product formed with the a_k so that (2.3) holds.

A series of elementary modifications of Π will produce the function f of our theorem.

A computation shows that

$$
\begin{aligned}
h(0) \;&=\; -2\pi\Delta \operatorname{Re}\left(\frac{i}{1 - \exp\left(i2\pi\rho/(1+ic)\right)} \right) \\
&=\; \frac{2\pi\Delta \exp(2\pi\mu c)}{\left|1 - \exp\left(i2\pi\rho/(1+ic)\right)\right|^2} \sin(2\pi\mu).
\end{aligned}
$$

Since $\frac{1}{2} < \mu < 1$ we thus have $h(0) < 0$. Hence there exists $\theta_0 > 0$ such that $h(\theta) < 0$ for $|\theta| < \theta_0$. For $0 < \varepsilon < \theta_1 < \theta_0$ we thus deduce from (2.3) that there exists $\eta_0 > 0$ such that

$$(2.5) \qquad \log|\Pi(re^{ic\log r+i\theta})| \leqslant -\eta_0 r^\rho \quad \text{for} \;\; \varepsilon \leqslant |\theta| \leqslant \theta_1,$$

provided r is sufficiently large.

We show that an estimate of this type also holds for $|\theta| < \varepsilon$. In order to do so, we use a standard estimate which in slightly different form can be found in [6, p. 548] or [8, p. 117].

LEMMA 2. — *Let $D \subset \mathbb{C}$ be an unbounded domain. For $r > 0$ such that the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$ intersects D, let $r\theta(r)$ be the linear measure of the intersection. Let $\theta^*(r) := \theta(r)$ if $C_r \not\subset D$ and let $\theta^*(r) := \infty$ and thus $1/\theta^*(r) := 0$ if $C_r \subset D$.*

Suppose that $u : \overline{D} \to [-\infty, \infty)$ is continuous in $\overline{D}$ and subharmonic in D. Suppose also that u is bounded above on ∂D, but not bounded above in D. Let $0 < \kappa < 1$ and let $R > 0$ be such that C_R intersects D. Then $B(r, u) := \max_{|z|=r} u(z)$ satisfies

$$\log B(r, u) \geqslant \pi \int_R^{\kappa r} \frac{dt}{t\theta^*(t)} - O(1)$$

as $r \to \infty$.

We may assume that ε in (2.5) is chosen such that $0 < \varepsilon < \pi/2$. We consider the spiralling domain

$$D := \left\{ re^{ic\log r+i\theta} : r > 1, |\theta| < \varepsilon \right\}$$

and the function

$$u(z) := \log|\Pi(z)| + \eta_0|z|^\rho.$$

Then u is continuous in $\overline{D}$, subharmonic in D and bounded above on ∂D. We claim that u is also bounded above in D. Otherwise, on applying Lemma 2 and noting that $\theta^*(r) = 2\varepsilon$ we find that

$$\log B(r, u) \geqslant \pi \int_R^{\kappa r} \frac{dt}{2\varepsilon t} - O(1) = \frac{\pi}{2\varepsilon}\log r - O(1) > \log r$$

and thus

$$\log M(r, \Pi) = B(r, u) - \eta_0 r^\rho > r - \eta_0 r^\rho > \frac{r}{2}$$

for large r. This implies that the order of Π is at least 1, a contradiction.

Thus u must be bounded above in D, and this, together with (2.5), implies that if $0 < \eta_1 < \eta_0$, then

$$(2.6) \qquad \log|\Pi(re^{ic\log r+i\theta})| \leqslant -\eta_1 r^\rho \quad \text{for} \;\; |\theta| \leqslant \theta_1$$

and sufficiently large r.

Let L be the natural parametrization of the logarithmic spiral on which the a_k lie; that is, $L : [1, \infty) \to \mathbb{C}$, $L(t) = te^{ic \log t}$. Then $\Pi(L(t)) \to 0$ as $t \to \infty$ by (2.6). Thus there exists $t_0 > 1$ such that $|\Pi(L(t))| < |\Pi(L(t_0))|$ for $t > t_0$.

The function f of our theorem will now be defined as follows. We put $z_0 := L(t_0)$ and define $g_0(z) := \Pi(z + z_0)$. Next we put $q := p + 1$ and $g_1(z) := g_0(z^q)$, and define $\sigma : [0, \infty) \to \mathbb{C}$ by $\sigma(t) = \sqrt[q]{L(t_0 + t) - z_0}$, for some fixed branch of the root. We then define

$$(2.7) \qquad g_2(z) := \int_0^z g_1(\zeta)^n d\zeta = \int_0^z \Pi(\zeta^q + z_0)^n d\zeta,$$

where $n \in \mathbb{N}$. It will follow easily that

$$a := \int_\sigma g_1(z)^n dz$$

is finite for all $n \in \mathbb{N}$, and using a result of W. K. Hayman [5, Lemma 1] we will see that $a = a(n) \neq 0$ if n is sufficiently large. For such n we then define $g_3(z) := g_2(z)/az$ and note that g_3 is of the form $g_3(z) = g_4(z^q)$ for some entire function g_4. The function claimed in the theorem is

$$f(z) := zg_4(z)^{q-1}.$$

We remark that we introduced z_0 and n only to ensure that $a \neq 0$. In a generic situation we could probably define g_2 directly via (2.7) with $z_0 = 0$ and $n = 1$.

To prove that f has the desired properties, we determine the asymptotic behavior of the g_j and f in spiralling regions similar to D. We first note from (2.6) that if $0 < \eta_2 < \eta_1$ and if $0 < \theta_2 < \theta_1$, then

$$\log |g_0(re^{ic \log r + i\theta})| \leqslant -\eta_2 r^\rho \quad \text{for } |\theta| \leqslant \theta_2$$

and sufficiently large r. This implies that if $|\theta| \leqslant \theta_2/q$ and if r is sufficiently large, then

$$\log |g_1(re^{ic \log r + i\theta})| = \log |g_0(r^q e^{ic \log(r^q) + iq\theta})| \leqslant -\eta_2 r^{q\rho}.$$

With

$$S_1 := \left\{ re^{ic \log r + i\theta} : r > 1, |\theta| < \frac{\theta_2}{q} \right\}$$

we thus find that

$$|g_1(z)| \leqslant \exp\left(-\eta_2 |z|^{q\rho}\right)$$

if $z \in S_1$ is sufficiently large. Moreover, $\sigma(t) \in S_1$ for large t, since for suitably chosen branches of the argument we have

$$
\begin{aligned}
\arg \sigma(t) &= \frac{1}{q} \arg(L(t_0 + t) - z_0) \\
&= \frac{1}{q} \arg L(t_0 + t) + o(1) \\
&= \frac{c}{q} \log |L(t_0 + t)| + o(1) \\
&= c \log |\sigma(t)| + o(1)
\end{aligned}
$$

as $t \to \infty$. From this it is straightforward to deduce that the integral defining a converges for all $n \in \mathbb{N}$. In order to show that $a \neq 0$ for large n, we use the following result of W. K. Hayman [5, Lemma 1].

LEMMA 3. — *Let γ be a Jordan arc in $\mathbb{C}$ which tends to ∞ in both directions and let g be holomorphic in a domain containing γ. Suppose that $\int_\gamma |g(z)||dz| < \infty$ and that $|g(z)| \to 0$ as $z \to \infty$ on γ. Suppose also that $|g(z)| \leqslant M$ for z on γ, with equality for a single point z_1 on γ with $g'(z_1) = 0$. Suppose finally that γ cannot be deformed in a neighborhood of z_1 into a curve on which $|g(z)| < M$. Then*

$$
\int_\gamma g(z)^n dz \neq 0
$$

for all sufficiently large integers n.

We remark that since $g'(z_1) = 0$, the set $\{z \in \mathbb{C} : |g(z)| < M\}$ has at least two components whose boundary contains z_1. The condition that γ cannot be deformed in a neighborhood of z_1 into a curve on which $|g(z)| < M$ means that γ passes from one component of this set into another component at z_1.

We apply Lemma 3 with $g := g_1$, the curve γ parametrized as $\gamma : \mathbb{R} \to \mathbb{C}$,

$$
\gamma(t) := \begin{cases} \sigma(-t) & \text{if } t \leqslant 0, \\ \sigma^*(t) := e^{2\pi i/q}\sigma(t) & \text{if } t > 0, \end{cases}
$$

and $z_1 := \gamma(0) = 0$. Since by the choice of z_0 we have

$$
|g_1(\sigma^*(t))| = |g_1(\sigma(t))| = |\Pi(L(t_0 + t))| < |\Pi(L(t_0))| = |g_1(\sigma(0))|
$$

for $t > 0$, it follows that $|g_1(\gamma(t))| < |g_1(z_1)|$ for $t \neq 0$. Moreover, $g'_1(z_1) = g'_1(0) = 0$, and thus the hypotheses of Lemma 3 are satisfied. Since

$$
\int_\gamma g_1(z)^n dz = -\int_\sigma g_1(z)^n dz + \int_{\sigma^*} g_1(z)^n dz = \left(-1 + e^{2\pi i/q}\right) \int_\sigma g_1(z)^n dz
$$

we conclude from Lemma 3 that $a = \int_\sigma g_1(z)^n dz \neq 0$ for sufficiently large values of n.

Thus $g_2(\sigma(t)) \to a$ as $t \to \infty$. More generally, $g_2(z) \to a$ as $z \to \infty$ in S_1. In fact, if $z \in S_1$ then

$$g_2(z) - a = \int_{\tau_z} g_1(\zeta)^n d\zeta$$

for any path τ_z joining z to ∞ in S_1. For large $z \in S_1$ and a suitable path τ_z we find that

$$|g_2(z) - a| \leqslant \int_{\tau_z} |g_1(\zeta)|^n |d\zeta| \leqslant \int_{\tau_z} \exp\left(-n\eta_2|\zeta|^{q\rho}\right) |d\zeta| \leqslant \exp\left(-\eta_3|z|^{q\rho}\right)$$

for some $\eta_3 > 0$. It follows that if $z \in S_1$ is sufficiently large, then

$$\left| g_3(z) - \frac{1}{z} \right| = \frac{|g_2(z) - a|}{|az|} \leqslant \frac{\exp\left(-\eta_3|z|^{q\rho}\right)}{|az|} \leqslant \exp\left(-\eta_3|z|^{q\rho}\right).$$

Now let

$$S_2 := \left\{ re^{ic \log r + i\theta} : r > 1, |\theta| < \theta_2 \right\}.$$

For $z \in S_2$ we have $\sqrt[q]{z} \in S_1$ for a suitable branch. For large $z \in S_2$ we thus find that

$$\left| g_4(z) - \frac{1}{\sqrt[q]{z}} \right| = \left| g_3(\sqrt[q]{z}) - \frac{1}{\sqrt[q]{z}} \right| \leqslant \exp\left(-\eta_3|z|^\rho\right);$$

i. e. if $z \in S_2$ is sufficiently large, then

(2.8) $$\left| f(z) - \sqrt[q]{z} \right| \leqslant \exp\left(-\eta_4|z|^\rho\right)$$

for some $\eta_4 > 0$ and a suitable branch.

3. Newton's method for f

We choose θ_3 with $0 < \theta_3 < \theta_2$ and define

$$S_3 := \left\{ re^{ic \log r + i\theta} : r > 1, |\theta| < \theta_3 \right\}.$$

Then there exists $\delta > 0$ such that if $z \in S_3$ is sufficiently large, then the closed disk of radius $\delta|z|$ around z is contained in S_2. With $d(z) :=$

$f(z) - \sqrt[q]{z}$ we deduce from (2.8) that if $z \in S_3$ is sufficiently large, then

$$\left| f'(z) - \frac{\sqrt[q]{z}}{qz} \right| = |d'(z)|$$

$$= \frac{1}{2\pi} \left| \int_{|\zeta-z|=\delta|z|} \frac{d(\zeta)}{(\zeta-z)^2} d\zeta \right|$$

$$\leqslant \frac{1}{\delta|z|} \max_{|\zeta-z|=\delta|z|} |d(\zeta)|$$

$$\leqslant \frac{1}{\delta|z|} \exp\left(-\eta_4(1-\delta)^\rho |z|^\rho\right)$$

$$\leqslant \exp\left(-\eta_5|z|^\rho\right)$$

for some $\eta_5 > 0$. Combining this with (2.8) we find that if $z \in S_3$ is sufficiently large, then

$$\left| \frac{f(z)}{f'(z)} - qz \right| \leqslant \exp\left(-\eta_6|z|^\rho\right)$$

where $\eta_6 > 0$. Since $q = p + 1$ we deduce that

$$(3.1) \qquad |N(z) + pz| = \left| z - \frac{f(z)}{f'(z)} + pz \right| = \left| \frac{f(z)}{f'(z)} - qz \right| \leqslant \exp\left(-\eta_6|z|^\rho\right)$$

for large $z \in S_3$. In particular,

$$\big| |N(z)| - p|z| \big| \leqslant \exp\left(-\eta_6|z|^\rho\right)$$

which implies that

$$(3.2) \qquad \big| \log|N(z)| - \log(p|z|) \big| \leqslant \exp\left(-\eta_6|z|^\rho\right)$$

for large $z \in S_3$. Moreover, (3.1) yields

$$(3.3) \qquad |\arg N(z) - \arg(-pz)| \leqslant \exp\left(-\eta_6|z|^\rho\right)$$

for large $z \in S_3$. However, c was chosen such that $c \log p = \pi$, so we deduce from (3.2) and (3.3) that

$$\big| \arg N(z) - c\log|N(z)| \big|$$
$$\leqslant \big| \arg N(z) - \arg(-pz) \big| + \big| \arg(-pz) - c\log(p|z|) \big|$$
$$\quad + \big| c\log(p|z|) - c\log|N(z)| \big|$$
$$\leqslant \big| \arg(-pz) - c\log(p|z|) \big| + (1+c)\exp\left(-\eta_6|z|^\rho\right)$$
$$= \big| \arg z + \pi - c\log p - c\log|z| \big| + (1+c)\exp\left(-\eta_6|z|^\rho\right)$$
$$= \big| \arg z - c\log|z| \big| + (1+c)\exp\left(-\eta_6|z|^\rho\right)$$
$$\leqslant \big| \arg z - c\log|z| \big| + \frac{1}{2|z|}$$

for large $z \in S_3$. Since $p > 23$ we deduce from (3.1) that $|N(z)| > 2|z|$ if $z \in S_3$ and if $|z|$ is sufficiently large, say $|z| > r_0 > 1$. Combining this with the previous estimate, we conclude that if

$$\big| \arg z - c \log |z| \big| < \theta_3 - \frac{1}{|z|},$$

then

$$\big| \arg N(z) - c \log |N(z)| \big| < \theta_3 - \frac{1}{|z|} + \frac{1}{2|z|} = \theta_3 - \frac{1}{2|z|} < \theta_3 - \frac{1}{|N(z)|}$$

if $z \in S_3$ and if $|z|$ is large enough, say $|z| > r_1 > r_0$. This implies that

$$U := \left\{ re^{ic \log r + i\theta} : r > r_1, |\theta| < \theta_3 - \frac{1}{r} \right\}$$

is N-invariant. Since $|N(z)| \geqslant 2|z|$ for $z \in U$, we have $|N^k(z)| \geqslant 2^k|z|$ for $z \in U$ and $k \in \mathbb{N}$. Thus $N^k|_U \to \infty$ as $k \to \infty$. Hence U is contained in an invariant Baker domain of N.

4. Asymptotic values of f

Suppose that f has a finite asymptotic value, say $f(z) \to b \in \mathbb{C}$ as $z \to \infty$ along a curve Γ. The function

$$F(z) := \frac{f(z)^q}{z}$$

is entire since $f(0) = 0$. By (2.8) we have $F(z) \to 1$ as $z \to \infty$ along the logarithmic spiral L while $F(z) \to 0$ as $z \to \infty$ along Γ. Thus F has two finite asymptotic values. By the Denjoy-Carleman-Ahlfors Theorem (see [7, §XI.4.5]), F has order at least 1. On the other hand, F has the same order as f, which has been taken less than 1. This is a contradiction.

Remark. — Our method will produce examples f of any preassigned non-integer order $\rho > 1$, as well as examples with more than one invariant Baker domain. We only sketch the modifications that have to be made.

We again choose ρ and p such that (2.4) is holds. The condition $\mu < 1$ need not be satisfied, and there may be several, say ℓ, intervals where $h(\theta)$ is negative and corresponding spiralling regions $S_1, \ldots, S_\ell$ where $\Pi(z) \to 0$ as $z \to \infty$. It is not difficult to see that ℓ can be any given positive number. For each j, let L_j be a curve starting at 0 which outside the unit circle is a logarithmic spiral in S_j and which inside the unit circle is a straight line from 0 to the corresponding point of the unit circle. Deforming one of the curves L_j if necessary we may assume that there exists $z_0 \in \bigcup_{j=1}^{\ell} L_j$

such that $|\Pi(z_0)| > |\Pi(z)|$ for all $z \in \bigcup_{j=1}^{\ell} L_j$. Defining g_2 by (2.7) for some large n and then f as in §2, we find that $f(z) \sim c_j \sqrt[q]{z}$ for some $c_j \neq 0$ as $z \to \infty$ in S_j. As before, this means that $N(z) \sim -pz$ as $z \to \infty$ in S_j, $j = 1, \ldots, \ell$. We thus obtain an entire function f for which N has ℓ invariant Baker domains. A difference occurs in the proof that f does not have finite asymptotic values. Here we cannot simply appeal to the classical Denjoy-Carleman-Ahlfors Theorem, but instead use that the function f constructed has only ℓ "tracts"; see [6, §8.3].

Balašov's result takes a different form if ρ is an integer, but it seems possible to treat this case along the same lines.

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BIBLIOGRAPHY

[1] S. K. BALAŠOV, "On entire functions of finite order with zeros on curves of regular rotation", *Math. USSR Izvestija* (1973), no. 7, p. 601-627, translation form Izv. Akad. Nauk. SSSR, Ser. Mat. 37 (1973) no 3.

[2] W. BERGWEILER, "Iteration of meromorphic functions", *Bull. Amer. Math. Soc.* (1993), no. 29, p. 151-188.

[3] W. BERGWEILER, F. V. HAESELER, H. KRIETE, H. G. MEIER & N. TERGLANE, "Newton's method for meromorphic functions", in *Complex Analysis and its Applications* (C. C. Yang, G. C. Wen, K. Y. Li & Y. M. Chiang, eds.), Notes Math. Ser., vol. 305, Pitman Res., 1994, p. 147-158.

[4] X. BUFF & J. RÜCKERT, "Virtual immediate basins of Newton maps and asymptotic values", *Int. Math. Res. Not.* (2006), no. 65498, p. 1-18.

[5] W. K. HAYMAN, "On integral functions with distinct asymptotic values", in *Proc. Cambridge Philos. Soc.*, no. 66, 1969, p. 301-315.

[6] ———, *Subharmonic functions*, vol. 2, London Math. Soc. Monographs, no. 20, Academic Press, London, 1989.

[7] R. NEVANLINNA, *Eindeutige analytische Funktionen*, Springer, Göttingen, Heidelberg, 1953.

[8] M. TSUJI, *Potential theory in modern function theory*, New York ed., Maruzen, reprint by Chelsea, 1959.

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