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THE LOCAL NASH PROBLEM ON ARC FAMILIES OF SINGULARITIES

by Shihoko ISHII (*)

Abstract. — This paper shows the affirmative answer to the local Nash problem for a toric singularity and analytically pretoric singularity. As a corollary we obtain the affirmative answer to the local Nash problem for a quasi-ordinary singularity.

Résumé. — Cet article présente la réponse positive au problème du Nash local pour une singularité torique ainsi que pour une singularité analytiquement prétorique. Il en résulte comme corollaire une réponse affirmative au problème du Nash local pour une singularité quasi ordinaire.

1. Introduction

The Nash problem was posed by John F. Nash in his preprint in 1968, which was later published as [12] in 1995. The problem in his paper reads in two ways:

(1) the bijectivity of the map from the set of the families of arcs passing through "the singular locus" to the set of the essential divisors over "the singular locus"
(2) the bijectivity of the map from the set of the families of arcs passing through "a singular point" to the essential divisors over "the singular point"

For convenience sake, we call the former the Nash problem and the latter the local Nash problem. For a variety with an isolated singularity, the two problems coincide.

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In case of a 2-dimensional normal (therefore isolated) singularity, the problem is studied in [9], [15], [17]. The Nash problem for general dimension is studied in [7], [8]. (More detailed information about the known facts will be given in 2.5 in the second section.) In this paper we study the local Nash problem. We show the affirmative answer to the local Nash problem for every point of a toric variety and also for an analytically pretoric singularity. As a corollary we obtain the affirmative answer to the local Nash problem for a quasi-ordinary singularity.

This paper is organized as follows: In the second section, we introduce the Nash map, the local Nash map, the Nash problem and the local Nash problem. In the third section, we show the affirmative answer to the local Nash problem for a toric variety. In the fourth section we show the affirmative answer to the local Nash problem for an analytically pretoric singularity. As a corollary we obtain the affirmative answer for a quasi-ordinary singularity.

In this paper we work on schemes over an algebraically closed field k of arbitrary characteristic. All k-schemes are assumed to be pure dimensional excellent schemes over k. All reduced k-schemes are moreover assumed to have open dense regular locus. By a regular k-scheme we mean a k-scheme with every local ring regular.

2. The Nash problem and the local Nash problem

Definition 2.1. — *Let X be a scheme over k and $K \supset k$ a field extension. A morphism $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ is called an* arc *of X. We denote the closed point of $\operatorname{Spec} K[[t]]$ by 0 and the generic point by η.*

For a k-scheme X, the arc space X_∞ is characterized by the following property ([18]):

Proposition 2.2. — *Let X be a k-scheme. Then*

$$\operatorname{Hom}_k(Y, X_\infty) \simeq \operatorname{Hom}_k(Y \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]], X)$$

for an arbitrary k-scheme Y, where $Y \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ means the formal completion of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ along the subscheme $Y \times_{\operatorname{Spec} k} \{0\}$.

2.1. By thinking of the case $Y = \operatorname{Spec} K$ for an extension field K of k, we see that K-valued points of X_∞ correspond to arcs $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ bijectively. Based on this, we denote the K-valued point corresponding to an arc $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ by the same symbol α. The canonical projection $X_\infty \longrightarrow X$, $\alpha \mapsto \alpha(0)$ is denoted by π_X.

A morphism $\varphi : Y \longrightarrow X$ of varieties induces a canonical morphism $\varphi_\infty : Y_\infty \longrightarrow X_\infty$, $\alpha \mapsto \varphi \circ \alpha$.

Definition 2.3. — *Let X be a reduced k-scheme and* $\operatorname{Sing} X$ *the singular locus of X, i.e., the set of the points whose local rings are not regular. Recall that we assume that all reduced k-schemes are pure dimensional excellent schemes and have the open dense regular locus. An irreducible component C of $\pi_X^{-1}(\operatorname{Sing} X)$ is called a* Nash component *of X if C is not contained in $(\operatorname{Sing} X)_\infty$. (In [8] a Nash component is called a "good component".) Let x be a (not necessarily closed) point of X. An irreducible component C of $\overline{\pi_X^{-1}(x)}$ is called a* local Nash component *of (X, x) if C is not contained in $(\operatorname{Sing} X)_\infty$.*

Here, we note that every irreducible component of $\pi_X^{-1}(\operatorname{Sing} X)$ is a Nash component if k is of characteristic zero ([8, Lemma 2.12]).

2.2. Assume that $\varphi : Y \longrightarrow X$ is a proper morphism which is an isomorphism away from $\operatorname{Sing} X$. Let α be the generic point of a Nash component or of a local Nash component. Then $\alpha(\eta)$ is outside of $\operatorname{Sing} X$, therefore it is lifted to Y by the isomorphism φ. Then, by the valuative criterion of properness α can be uniquely lifted to an arc of Y. This property is essential for our arguments in this paper.

Lemma 2.4. — *Let X be an integral k-scheme and x an analytically irreducible point of X, i.e., $\widehat{\mathcal{O}}_{X,x}$ is an integral domain. Let $\widehat{X}$ be* $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$. *Then, the canonical morphism $\iota_\infty : \widehat{X}_\infty \longrightarrow X_\infty$ induces an isomorphism $\pi_{\widehat{X}}^{-1}(x) \simeq \pi_X^{-1}(x)$, where the closed point of $\widehat{X}$ is also denoted by x.*

Proof. — First, note that the canonical morphism $\iota : \widehat{X} \longrightarrow X$ gives the morphism $\iota_\infty : \widehat{X}_\infty \longrightarrow X_\infty$ whose restriction gives $\iota_\infty : \pi_{\widehat{X}}^{-1}(x) \longrightarrow \pi_X^{-1}(x)$. We may assume that $X = \operatorname{Spec} A$ for a k-algebra A. Let $\pi_X^{-1}(x) = \operatorname{Spec} R$. By Proposition 2.2, the inclusion $\pi_X^{-1}(x) \subset X_\infty$ induces a homomorphism $\mathcal{O}_{X,x} \longrightarrow R[[t]]$ which sends the maximal ideal of $\mathcal{O}_{X,x}$ to the ideal (t). Then, we get the homomorphism of projective limits

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,x} & \longrightarrow & R[[t]] \\ \| & & \| \\ \varprojlim \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^m & & \varprojlim R[[t]]/(t)^m. \end{array}$$

Again by Proposition 2.2, this homomorphism gives a morphism $\pi_X^{-1}(x) \longrightarrow \widehat{X}_\infty$ whose image is in $\pi_{\widehat{X}}^{-1}(x)$. This is the inverse morphism of $\iota_\infty : \pi_{\widehat{X}}^{-1}(x) \longrightarrow \pi_X^{-1}(x)$. □

Lemma 2.5. — *Let X be a regular k-scheme and E an irreducible regular closed subset of X. Then $\pi_X^{-1}(E)$ is an irreducible closed subset of X_∞.*

Proof. — We may assume that $X = \operatorname{Spec} A$ for an integral domain A. As $\mathcal{O}_{X,p}$ is a regular local ring for every $p \in X$, we have $\widehat{\mathcal{O}}_{X,p} = k(p)[[x_1, \dots, x_n]]$ for some indeterminates $x_1, \dots, x_n$, where $k(p)$ is the residue field of $\mathcal{O}_{X,p}$. If we put $\widehat{X} = \operatorname{Spec} \widehat{\mathcal{O}}_{X,p}$, this shows that $\pi_X^{-1}(p) = \pi_{\widehat{X}}^{-1}(p)$ is irreducible for every $p \in X$. Therefore, it is sufficient to prove that $\pi_X^{-1}(p) \subset \overline{\pi_X^{-1}(q)}$ for $p, q \in X$ with $p \in \overline{\{q\}}$ and $\overline{\{q\}}$ regular. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the prime ideals in $\mathcal{O}_{X,p}$ corresponding to p and q, respectively. Then, we may assume that $\mathfrak{p} = (x_1, .., x_r, x_{r+1}, .., x_n)$ and $\mathfrak{q} = (x_1, .., x_r)$. Let α be the generic point of $\pi_X^{-1}(p)$. Then α induces a local homomorphism $\mathcal{O}_{X,p} \longrightarrow K[[t]]$ and this can be extended to a local homomorphism

$$\alpha^* : \widehat{\mathcal{O}}_{X,p} = k(p)[[x_1, .., x_r, x_{r+1}, .., x_n]] \longrightarrow K[[t]].$$

Define

$$\Lambda^* : k(p)[[x_1, .., x_r, x_{r+1}, .., x_n]] \longrightarrow K[[\lambda_{r+1}, \dots, \lambda_n, t]]$$

by

$$\Lambda^*(x_i) = \alpha^*(x_i) \text{ for } i = 1, .., r \text{ and}$$
$$\Lambda^*(x_i) = \lambda_i + \alpha^*(x_i) \text{ for } i = r+1, .., n.$$

Here $\lambda_{r+1}, \dots, \lambda_n$ are indeterminates. The restriction of this map onto A gives a family of arcs $\Lambda : \operatorname{Spec} K[[\lambda_{r+1}, \dots, \lambda_n]] \longrightarrow X_\infty$. Let $0'$ and η' be the closed point and the generic point of $\operatorname{Spec} K[[\lambda_{r+1}, \dots, \lambda_n]]$, respectively. Denote the quotient field of $K[[\lambda_{r+1}, \dots, \lambda_n]]$ by $K((\lambda_{r+1}, \dots, \lambda_n))$. Then $\Lambda(0') = \alpha$ and $\beta := \Lambda(\eta') : \operatorname{Spec} K((\lambda_{r+1}, \dots, \lambda_n))[[t]] \longrightarrow X$ is an arc in $\pi_X^{-1}(q)$, since $\beta^{*-1}((t)) = \mathfrak{q} \cap A$. This yields that $\alpha \in \overline{\{\beta\}} \subset \overline{\pi_X^{-1}(q)}$. □

We note that if X is a non-singular variety, $\pi_X^{-1}(E)$ is always irreducible for an irreducible subset E.

Definition 2.6. — *A birational morphism $\varphi : Y \longrightarrow X$ of reduced k-schemes is a morphism which gives a bijection between the sets of the irreducible components of Y and X and the restriction of φ on each irreducible component is birational.*

Let X be a reduced k-scheme, $\psi : X_1 \longrightarrow X$ a proper birational morphism from a normal k-scheme X_1 and $E \subset X_1$ an irreducible exceptional divisor of ψ. Let $\varphi : X_2 \longrightarrow X$ be another proper birational morphism from a normal k-scheme X_2. The birational map $\varphi^{-1} \circ \psi : X_1 \dashrightarrow X_2$ is *defined on a (nonempty) open subset* E^0 *of* E. *The closure of* $(\varphi^{-1} \circ \psi)(E^0)$ *is called the* center *of* E *on* X_2.

We say that E *appears in* φ *(or in* X_2*), if the center of* E *on* X_2 *is also a divisor. In this case the birational map* $\varphi^{-1} \circ \psi : X_1 \dashrightarrow X_2$ *is a local isomorphism at the generic point of* E *and we denote the birational transform of* E *on* X_2 *again by* E. *For our purposes* $E \subset X_1$ *is identified with* $E \subset X_2$. *Such an equivalence class is called an* exceptional divisor over X.

An exceptional divisor E *over* X *is called an* exceptional divisor over (X, x) *for a point* $x \in X$ *if the center of* E *on* X *is* $\overline{\{x\}}$.

Definition 2.7. — *Let* X *be a reduced* k*-scheme. In this paper, by a* resolution *of the singularities of* X *we mean a proper birational morphism* $\varphi : Y \longrightarrow X$ *with a regular* k*-scheme* Y *such that the restriction* $Y \setminus \varphi^{-1}(\operatorname{Sing} X) \longrightarrow X \setminus \operatorname{Sing} X$ *is an isomorphism.*

The existence of a resolution for a reduced k*-scheme* X *is a difficult problem. For a variety over a field of characteristic zero the existence of a resolution was proved by Hironaka* [5]. *But for a general reduced* k*-scheme it is still an open problem. From now on, we always assume the existence of a resolution.*

Definition 2.8. — *An exceptional divisor* E *over a reduced* k*-scheme* X *is called an* essential divisor *over* X *if for every resolution* $\varphi : Y \longrightarrow X$ *the center of* E *on* Y *is an irreducible component of* $\varphi^{-1}(\operatorname{Sing} X)$. *The center of an essential divisor over* X *on a resolution* Y *is called an* essential component on Y.

For a point $x \in X$ *an exceptional divisor* E *over* (X, x) *is called an* essential divisor over (X, x) *if for every resolution* $\varphi : Y \longrightarrow X$ *the center of* E *on* Y *is an irreducible component of* $\overline{\varphi^{-1}(x)}$. *The center of an essential divisor over* (X, x) *on a resolution* Y *is called an* essential component over (X, x) *on* Y.

Remark 2.9. — Take an integral scheme X and a point $x \in X$. There are canonical bijections:

$$\{\text{essential divisors over } X\} \simeq \{\text{essential components on a resolution } Y\},$$

$$\{\text{essential divisors over } (X, x)\} \simeq \{\text{essential components over } (X, x) \text{ on a resolution } Y\}.$$

Indeed, for an essential divisor E, let $\Phi(E)$ be the center of E on Y. Then we have a map Φ from the set of essential divisors to the set of the essential components. Conversely, for an essential component C on Y, take the blow-up $\tilde{Y} \longrightarrow Y$ with the center C and let E be the unique exceptional divisor which is mapped onto C. Then E is an essential divisor whose center on Y is C.

2.3. The Nash problem. Let $\varphi : Y \longrightarrow X$ be a resolution of the singularities of a reduced k-scheme X such that $\varphi^{-1}(\operatorname{Sing} X)$ is a union of non-singular divisors. Let $\varphi^{-1}(\operatorname{Sing} X) = \bigcup_j E_j$ be the decomposition into irreducible components. Let $\{C_i\}$ be the Nash components of X. Then the morphism $\varphi_\infty : \bigcup_j \pi_Y^{-1}(E_j) \longrightarrow \bigcup_i C_i$ is dominant and bijective outside $(\operatorname{Sing} X)_\infty$ by 2.2. As $\pi_Y^{-1}(E_j)$'s are irreducible by Lemma 2.5, for each C_i there is unique E_{j_i} such that $\pi_Y^{-1}(E_{j_i})$ is dominant over C_i. In [12] Nash proved that this E_{j_i} is an essential divisor over X (for the proof see also [8, Theorem 2.15]). This map

$$\mathcal{N} : \{ \text{ Nash components } \} \longrightarrow \{ \text{ essential divisors over } X\},\ C_i \mapsto E_{j_i}$$

is called the Nash map. Obviously this map is injective and the Nash problem asks if this map is bijective.

2.4. The local Nash problem. Let $\varphi : Y \longrightarrow X$ be a resolution of the singularities of a reduced k-scheme such that $\overline{\varphi^{-1}(x)}$ is a union of non-singular divisors. Let $\overline{\varphi^{-1}(x)} = \cup_j E_j$ be the decomposition into irreducible components. Let $\{C_i\}$ be the local Nash components of (X, x). Then the morphism $\varphi_\infty : \bigcup_j \pi_Y^{-1}(E_j) \longrightarrow \bigcup_i C_i$ is dominant and injective outside $(\operatorname{Sing} X)_\infty$ by 2.2. As $\pi_Y^{-1}(E_j)$'s are irreducible, for each C_i there is a unique E_{j_i} such that $\pi_Y^{-1}(E_{j_i})$ is dominant to C_i. By the following lemma, this E_{j_i} is an essential divisor over (X, x). This map

$$\begin{aligned} \ell\mathcal{N} : \{ &\text{ local Nash components of } (X,x)\} \\ &\longrightarrow \{ \text{ essential divisors over } (X,x)\},\ C_i \mapsto E_{j_i} \end{aligned}$$

is called the local Nash map. Obviously this map is injective and the local Nash problem asks if this map is bijective.

If $x \in X$ is a unique singularity on X, then the Nash problem for X is the same as the local Nash problem for (X, x).

Lemma 2.10. — *Under the notation above, E_{j_i} is an essential divisor over (X, x).*

Proof. — Let $\psi : Y' \longrightarrow X$ be any resolution. Let E'_{j_i} be the center of E_{j_i} on Y'. Then, $C_i = \overline{\psi_\infty \pi_{Y'}^{-1}(E'_{j_i})}$. Let E' be an irreducible component of $\overline{\psi^{-1}(x)}$ containing E'_{j_i}. Then

$$C_i = \overline{\psi_\infty \pi_{Y'}^{-1}(E'_{j_i})} \subset \overline{\psi_\infty \pi_{Y'}^{-1}(E')},$$

where the last term is in $\overline{\pi_X^{-1}(x)}$. As C_i is an irreducible component of $\overline{\pi_X^{-1}(x)}$, the above inclusion is the equality. By the bijectivity of ψ_∞ outside $(\operatorname{Sing} X)_\infty$ the generic points α and α' of $\psi_\infty \pi_{Y'}^{-1}(E'_{j_i})$ and $\psi_\infty \pi_{Y'}^{-1}(E')$, respectively, must coincide, which yields that the generic points of E'_{j_i} and E' coincide, because $E'_{j_i} = \overline{\{\alpha(0)\}}$ and $E' = \overline{\{\alpha'(0)\}}$. □

2.5. Known facts on the Nash problem. An essential divisor, which is a slightly different notion from ours, is studied by Catherine Bouvier and Gérard Gonzalez-Sprinberg in [2]. The idea of the proof of a theorem in this paper is very useful for our discussion. The Nash problem is affirmatively answered for A_n-singularities by John F. Nash [12], for a minimal singularity on a surface by Ana Reguera [17] and for a sandwiched surface singularity By Monique Lejeune-Jalabert and Ana Reguera [9], [16]. Recently the author was announced that the affirmative answer is proved for a D_n-singularity on a surface by Camille Plenat. Camille Plenat and Popescu-Pampu [15] proved the affirmative answer to certain non-rational singularities with combinatorial conditions. The Nash problem is affirmatively answered also for a toric variety of arbitrary dimension in [8]. But affirmative answer does not hold for a general singularity. The same paper [8] gives a counter example of dimension 4, therefore we have counter examples for dimension higher than 4 by making the product with a non-singular variety. For dimension 2 and 3 the problem is still open. These are all for a normal variety. We should note that, this problem for a non-normal variety is not automatically reduced to the case of the normalized variety. In spite of that, for a non-normal toric variety the Nash problem is affirmatively proved in [7]. A non-normal toric variety has much stronger properties than just the fact that its normalization is a toric variety.

As a normal surface singularity is isolated, all results on the Nash problem for a normal surface singularity are the results on the local Nash problem. The counter example to the Nash problem given in [8] is an isolated singularity, therefore it is also a counter example to the local Nash problem. Hence, the next step to study is to know in which category the local Nash problem (or the Nash problem) is affirmative.

Now we close this section with the following basic lemma, which implies that a Nash component and a local Nash component are "fat" in terms of [7].

Lemma 2.11. — *Let C be a Nash component of an integral k-scheme X or a local Nash component of (X,x) for a point x of an integral k-scheme X. Let $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ be the generic point of C. Then, $\alpha(\eta)$ is* *the generic point of X, which is equivalent to that the corresponding ring homomorphism $\alpha^* : \Gamma(U, \mathcal{O}_X) \longrightarrow K[[t]]$ is injective, where U is an affine open neighborhood of $\alpha(0)$.*

Proof. — We prove the statement for a local Nash component. The other case is essentially the same. Let C be a local Nash component of (X, x) and E an essential divisor over (X, x) corresponding to C. Let $\varphi : Y \longrightarrow X$ be a resolution of the singularities of X, on which the divisor E appears. Then $\overline{\varphi_\infty(\pi_Y^{-1}(E))} = C$. As $\pi_Y^{-1}(E)$ is an irreducible cylinder on a regular scheme Y, it is not contained in the arc space of any proper closed subscheme of Y. Therefore, the generic point β of $\pi_Y^{-1}(E)$ sends the generic point of $\operatorname{Spec} K[[t]]$ to the generic point of Y. Hence, the generic point $\varphi_\infty\beta$ of C also sends the generic point of $\operatorname{Spec} K[[t]]$ to the generic point of X. □

3. The local Nash problem for a toric variety

In this section we prove the local Nash problem for a toric variety. First we remark some basic notion of the arc space of a toric variety. Here, we use the notation and terminologies of [3]. Let M be the free abelian group $\mathbb{Z}^n$ $(n \geqslant 1)$ and N its dual $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively. The canonical pairing $\langle\ ,\ \rangle : N \times M \longrightarrow \mathbb{Z}$ extends to $\langle\ ,\ \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \longrightarrow \mathbb{R}$. The group ring $\mathbb{C}[M]$ is generated by monomials $x^{\mathbf{m}}$ $(\mathbf{m} \in M)$ over $\mathbb{C}$. A cone in N is a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$.

3.1. Let X be an affine toric variety defined by a cone σ in N. In [6], for $v \in \sigma \cap N$ we define

$$T_\infty^X(v) = \{\alpha \in X_\infty \mid \alpha(\eta) \in T,\ \operatorname{ord}_t \alpha^*(x^u) = \langle v, u \rangle \text{ for } u \in M\},$$

where T denotes the open orbit and also the torus acting on X. The set $T_\infty^X(v)$ is an irreducible locally closed subset of X_∞ which is not contained in $(\operatorname{Sing} X)_\infty$([6]).

Let τ be the face of σ such that $v \in \tau^o$, where τ^o means the relative interior of τ. Then for every $\alpha \in T_\infty^X(v)$, we have that $\alpha(0) \in orb(\tau)$ ([8, Proposition 3.9]).

In $\sigma \cap N$ we define an order $\leqslant_\sigma$ as follows:

$$v \leqslant_\sigma v' \Leftrightarrow v' - v \in \sigma.$$

Then the following is obtained in [6].

PROPOSITION 3.1 ([6]). — *Let X be an affine toric variety defined by a cone σ in N. For $v, v' \in \sigma \cap N$, the relation $v \leqslant_\sigma v'$ holds if and only if $\overline{T^X_\infty(v)} \supset T^X_\infty(v')$.*

THEOREM 3.2. — *Let X be an affine toric variety and x a point of X. Then the local Nash map:*

$$\ell\mathcal{N} : \{ \text{ local Nash components of } (X, x)\} \longrightarrow \{ \text{ essential divisors over } (X, x)\}$$

is bijective.

Proof. — Let σ be the cone defining X. We divide the proof into two cases.

Case 1: The closure $\overline{\{x\}}$ is an invariant set.

In this case, $\overline{\{x\}}$ is $orb(\tau)$ for a face τ of σ in a neighborhood of x. In this neighborhood, $X = X' \times T'$ for an affine toric variety X' and a torus T'. Then, $\overline{\{x\}} \simeq \{x'\} \times T'$, where the point x' is the closed orbit of X'. Therefore, a local Nash component of (X, x) is of the following type:

$$\text{(a local Nash component of } (X', x')) \times T'_\infty.$$

This shows that the number of the local Nash components of (X, x) is that of (X', x'). On the other hand, the product of T' and a resolution of X' is a resolution of X in the neighborhood of x. Therefore, an essential divisor over (X, x) is of type: the product of T' and an essential divisor over (X', x'). This implies the number of the essential divisors over (X, x) is less than or equal to that over (X', x'). Hence we can reduce the problem to the case that x is the closed orbit.

Let x be the closed orbit $orb(\sigma)$ in X. We claim that

$$\pi_X^{-1}(x) = \bigcup_{v \in \sigma^o \cap N} \overline{T^X_\infty(v)}.$$

For every $\alpha \in T^X_\infty(v)$ with $v \in \sigma^o \cap N$, it follows $\alpha(0) \in orb(\sigma) = \{x\}$ as we remark in 3.1. This implies that $\alpha \in \pi_X^{-1}(x)$. For the opposite inclusion, it is sufficient to prove that the generic point α of an irreducible component C of $\pi_X^{-1}(x)$ is contained in $T^X_\infty(v)$ for some $v \in \sigma^o \cap N$. Let $\varphi : Y \longrightarrow X$ be an equivariant resolution. Then the induced map $\varphi_\infty : Y_\infty \longrightarrow X_\infty$ is surjective ([6, Proposition 3.2]). Therefore, there exists a lifting $\tilde{\alpha} \in Y_\infty$ of α. Let $\tilde{\alpha}(0) \in orb(\tau)$ for some cone τ in the fan of Y. Let E be the closure of $orb(\tau)$ in Y. As Y is non-singular, $\pi_Y^{-1}(E)$ is irreducible. Let β be the generic point of $\pi_Y^{-1}(E)$. Since the generic point of $\pi_Y^{-1}(E)$ is fat (see 2.11), $\beta(\eta)$ is the generic point of Y therefore it is in T. The inclusion

$\alpha = \varphi_\infty(\tilde{\alpha}) \in \varphi_\infty(\pi_Y^{-1}(E))$ yields the inclusion

$$C \subset \overline{\varphi_\infty(\pi_Y^{-1}(E))}$$

and this inclusion is an equality, because both are irreducible closed subsets of $\pi_X^{-1}(x)$ and C is an irreducible component of $\pi_X^{-1}(x)$. Hence, $\alpha = \varphi_\infty(\beta)$ and therefore $\alpha(\eta) \in T$. By this we have a ring homomorphism

$$\alpha^* : \mathbb{C}[\sigma^\vee \cap M] \longrightarrow K[[t]]$$

which is extended to

$$\alpha^* : \mathbb{C}[M] \longrightarrow K((t)).$$

Defining $v : M \longrightarrow \mathbb{Z}$ by $v(u) = \mathrm{ord}_t\, \alpha^*(x^u)$, we obtain $v \in \sigma \cap N$. By $\alpha(0) = x$, we have $v \in \sigma^o \cap N$. Therefore, we have that $\alpha \in T_\infty^X(v)$.

Now, noting that $T_\infty^X(v)$ is not contained in $(\mathrm{Sing}\, X)_\infty$, we see that every irreducible component of $\pi_X^{-1}(x)$ is a local Nash component of (X, x) and it is a maximal element of $\{\overline{T_\infty^X(v)} \mid v \in \sigma^o \cap N\}$. Then, by Proposition 3.1 the local Nash components of (X, x) are

$$\{\overline{T_\infty^X(v)} \mid v \ \text{ minimal in}\, \sigma^o \cap N\}.$$

On the other hand, an essential divisor over (X, x) is the same as "composantes essentielles" in [1] and the characterization theorem of composante essentielle in [1, §2.3] shows that

$$\{D_v \mid v \ \text{ minimal in}\, \sigma^o \cap N\}$$

is the set of composantes essentielles over (X, x), where D_v is the divisor corresponding to the one-dimensional cone $\mathbb{R}_{\geqslant 0} v$. (This can be proved also in the similar way as the proof of [8, Lemma 3.15].) This shows the local Nash map is bijective.

Here, we should note that the proof in [1, §2.3] shows that the essential divisors over (X, x) in the category of all resolutions coincides with that in the category of all equivariant resolutions.

Case 2: The closure $\overline{\{x\}}$ is not an invariant set.

To prove this case, we need the following lemma

Lemma 3.3. — *Let $\varphi : Y \longrightarrow X$ be an equivariant resolution of a toric variety. Let* $\mathrm{orb}(\tau)$ *be an orbit in X and $Z \subset Z'$ irreducible invariant closed subsets of* $\varphi^{-1}(\mathrm{orb}(\tau))$ *. If $Z \neq Z'$ then $Z \cap \varphi^{-1}(\Sigma) \neq Z' \cap \varphi^{-1}(\Sigma)$ for a subset $\Sigma \subset$* $\mathrm{orb}(\tau)$.

Proof. — As Z and Z' are invariant closed subsets of $\varphi^{-1}(\mathrm{orb}(\tau))$, there are lower dimensional toric varieties $Z_0 \subset Z_0'$ such that $Z \simeq Z_0 \times \mathrm{orb}(\tau)$, $Z' \simeq Z_0' \times \mathrm{orb}(\tau)$ and the restrictions of the morphism φ on Z, Z' are the projections to the second factors. Then $Z \cap \varphi^{-1}(\Sigma) \simeq Z_0 \times \Sigma$ and $Z' \cap \varphi^{-1}(\Sigma) \simeq Z'_0 \times \Sigma$. Hence, $Z \neq Z'$ implies $Z_0 \neq Z'_0$ and therefore $Z \cap \varphi^{-1}(\Sigma) \neq Z' \cap \varphi^{-1}(\Sigma)$. □

Now we start the proof for Case 2. Take the face $\tau < \sigma$ such that $x \in \mathrm{orb}(\tau)$. Let $\Sigma = \mathrm{orb}(\tau) \cap \overline{\{x\}}$ and let x_τ be the generic point of $\overline{\mathrm{orb}(\tau)}$. Then, we can prove that

$$\#\{\text{essential divisors over } (X,x)\} \leqslant \#\{\text{essential divisors over } (X,x_\tau)\}.$$

In order to prove this, it is sufficient to prove that for a fixed equivariant resolution $\varphi : Y \longrightarrow X$,

$$\#\{\text{essential components over } (X,x) \text{ on } Y\} \leqslant \#\{\text{essential components over } (X,x_\tau) \text{ on } Y\}.$$

Let $\varphi^{-1}(\mathrm{orb}(\tau)) = \bigcup_{i=1}^r V_i$ be the irreducible decomposition. Let $\Sigma_i = V_i \cap \varphi^{-1}(\Sigma)$. Then, $\varphi^{-1}(\Sigma) = \bigcup_{i=1}^r \Sigma_i$ is the irreducible decomposition. An essential component over (X,x_τ) on Y is one of $\overline{V_i}$, and an essential component over (X,x) on Y is one of $\overline{\Sigma_i}$. By taking a suitable φ we may assume that $\overline{V_i}$'s are divisors.

It is sufficient to prove that if $\overline{V_i}$ is not an essential component over (X,x_τ) on Y, then $\overline{\Sigma_i}$ is not an essential component over (X,x) on Y. If $\overline{V_i}$ is not an essential component over (X,x_τ) on Y, there is an equivariant resolution $\psi : Y' \longrightarrow X$ such that the center $\overline{V'_i}$ of V_i is strictly contained in an invariant irreducible component $\overline{V'}$ of $\overline{\psi^{-1}(x_\tau)}$. Let $V'_i = \overline{V'_i} \cap \psi^{-1}(\mathrm{orb}(\tau))$ and $V' = \overline{V'} \cap \psi^{-1}(\mathrm{orb}(\tau))$. Then, by Lemma 3.3, the strict inclusion $V'_i \subset V'$ yields the strict inclusion

$$V'_i \cap \psi^{-1}(\Sigma) \subset V' \cap \psi^{-1}(\Sigma). \tag{1}$$

Let $g : \tilde{Y} \longrightarrow Y$ be an equivariant morphism such that $\varphi \circ g$ is a resolution of the singularities of X and there is a morphism $h : \tilde{Y} \longrightarrow Y'$. As g is equivariant and the minimal invariant closed subset containing $\overline{\Sigma_i}$ is $\overline{V_i}$, there is a unique irreducible component $\widetilde{\Sigma}_i$ of $g^{-1}(\overline{\Sigma_i})$ mapped onto $\overline{\Sigma_i}$. Here, we note that $\widetilde{\Sigma}_i \subset \overline{V_i}$, where we use the same notation for the divisors $\overline{V_i} \subset Y$ and its proper transform on $\tilde{Y}$. Let D be an exceptional divisor over (X,x) whose center on Y is $\overline{\Sigma_i}$. Then the center of D on $\tilde{Y}$ is $\widetilde{\Sigma}_i$ and therefore the center of D in Y' is $h(\widetilde{\Sigma}_i)$ which is in $\overline{V'_i \cap \psi^{-1}(\Sigma)}$. By the strict inclusion (1), $h(\widetilde{\Sigma}_i)$ is contained in another component $\overline{V' \cap \psi^{-1}(\Sigma)}$. Therefore, $\overline{\Sigma_i}$ is not an essential component over (X,x) on Y.

Next, we claim that

$$\#\{\text{Nash components of } (X,x)\} = \#\{\text{Nash components of } (X,x_\tau)\}.$$

This is proved as follows: At a neighborhood of x, $X \simeq X' \times T'$ and $\mathrm{orb}(\tau) = \{0\} \times T'$, where T' is a torus of lower dimension, X' is a suitable toric variety with the closed point orbit 0. We can write $\Sigma = \{0\} \times \Sigma'$, where $\Sigma' \subset T'$ is an irreducible closed subset. Then, $\pi_X^{-1}(\mathrm{orb}(\tau)) = \pi_{X'}^{-1}(0) \times (T')_\infty$ and $\pi_X^{-1}(\Sigma) = \pi_{X'}^{-1}(0) \times \pi_{T'}^{-1}(\Sigma')$. Therefore, the numbers of irreducible components of both subsets are the same as the number of irreducible components of $\pi_{X'}^{-1}(0)$.

Now, using the affirmative answer to the local Nash problem for (X, x_τ) and the injectivity of the local Nash map, we obtain the bijectivity of the local Nash map for (X, x). □

4. The local Nash problem for an analytically pretoric singularity

In [7], a pretoric variety is defined and affirmative answer to the Nash problem for a pretoric variety is proved. In this section we introduce an analytically pretoric singularity and give an affirmative answer to the local Nash problem for this singularity. A good example of a pretoric variety is a non-normal toric variety, while a good example of an analytically pretoric singularity is an analytically irreducible quasi-ordinary singularity.

Definition 4.1. — *Let $\mathcal{O}$ be an integral domain which is the completion of a local ring essentially of finite type over k. Let $X = \mathrm{Spec}\,\mathcal{O}$. The closed point of X is denoted by x. A singularity (X, x) is called an analytically pretoric singularity if the following is satisfied: Let $N = \mathbb{Z}^n$ and M the dual of N. There exist an n-dimensional cone σ in N and a sublattice $M' \subset M$ of finite index. There is a sequence of injective local homomorphisms*

$$k[[\sigma^\vee \cap M']] \xrightarrow{\rho^*} \mathcal{O} \xrightarrow{\nu^*} k[[\sigma^\vee \cap M]],$$

such that

(1) $\nu^* \circ \rho^* : k[[\sigma^\vee \cap M']] \longrightarrow k[[\sigma^\vee \cap M]]$ *is the canonical injection,*
(2) $k[[\sigma^\vee \cap M]]$ *is the integral closure of $\mathcal{O}$ in its quotient field, and*
(3) *Let $\nu : \mathrm{Spec}\,k[[\sigma^\vee \cap M]] \longrightarrow \mathrm{Spec}\,\mathcal{O}$ be the morphism corresponding to ν^*. The restriction of ν onto* $\mathrm{Spec}\,k[[\sigma^\vee \cap M]][M]$ *is an isomorphism onto its image.*

Example 4.2. — One important example of analytically pretoric singularity is an analytically irreducible quasi-ordinary singularity. A quasi-ordinary singularity is first introduced by J. Lipman [10], [11] and studied by Y-N. Gau [4], K. Oh [13] and P. D. González Pérez [14] and others.

We call a singularity (X,x) a quasi-ordinary singularity if it is a hypersurface singularity in $(\mathbb{C}^{n+1},0)$ and there is a finite covering $\rho:(X,x)\longrightarrow(\mathbb{C}^n,0)$ whose discriminant locus is contained in a germ wise in a normal crossing divisor on $\mathbb{C}^n$. P. D. González Pérez [14] proved that if (X,x) is an analytically irreducible quasi-ordinary singularity, then it satisfies the conditions of our analytically pretoric singularity.

4.1. Let (X,x) be an analytically pretoric singularity. Under the notation in Definition 4.1, we denote $\operatorname{Spec} k[\sigma^\vee\cap M]$ and $\operatorname{Spec} k[[\sigma^\vee\cap M]]$ by W and $\widehat{W}$, respectively. We denote $\operatorname{Spec} k[\sigma^\vee\cap M']$ and $\operatorname{Spec} k[[\sigma^\vee\cap M']]$ by Z and $\widehat{Z}$, respectively. By the definition of analytically pretoric singularity, we obtain the following diagram:

$$\widehat{W}\xrightarrow{\ \nu\ }X\xrightarrow{\ \rho\ }\widehat{Z}.$$

We also obtain that $\rho\circ\nu$ induces an equivariant morphism $W\longrightarrow Z$ of toric varieties. Let $w\in W$ and $z\in Z$ be the closed points orbits. We denote the closed point of $\widehat{W}$ and $\widehat{Z}$ by the same symbols w and z. Then they correspond to the point $x\in X$ by the morphism ν and ρ.

As $T^W_\infty(v)\subset\pi_W^{-1}(w)$ for a point $v\in\sigma^o\cap N$, we have $T^W_\infty(v)\subset\pi^{-1}_{\widehat{W}}(w)\subset\widehat{W}$ by Lemma 2.4. In the same way, for $v\in\sigma^o\cap N'$, where N' is the dual of M', we obtain that $T^Z_\infty(v)\subset\pi^{-1}_{\widehat{Z}}(z)\subset\widehat{Z}$.

Definition 4.3. — *For $v\in\sigma^o\cap N$, define the subset $T^X_\infty(v)$ by the image $\nu_\infty(T^W_\infty(v))$.*

Lemma 4.4. — *Let (X,x) be an analytically pretoric singularity. Under the notation in 4.1, we obtain the following*

(i) *The restriction $T^W_\infty(v)\longrightarrow T^X_\infty(v)$ of ν_∞ is bijective for every $v\in\sigma^o\cap N$.*

(ii) *The restriction $T^W_\infty(v)\longrightarrow T^Z_\infty(v)$ of $(\rho\circ\nu)_\infty$ is surjective for every $v\in\sigma^o\cap N$.*

Proof. — The surjectivity of (i) follows from the definition of $T^X_\infty(v)$. The injectivity follows from the valuative criterion of properness, as $\nu:\widehat{W}\longrightarrow X$ is proper and the image of η by every arc in $T^X_\infty(v)$ is in the open set on which ν is isomorphic (see (3) in Definition 4.1). As $W\longrightarrow Z$ is the equivariant quotient morphism of toric varieties by a finite group N'/N, (ii) follows from [7, Lemma 5.6, (ii)]. □

Lemma 4.5. — *For two points $v,v'\in\sigma^o\cap N$, the following are equivalent*

(i) $v\leqslant_\sigma v'$,

(ii) $\overline{T^X_\infty(v)} \supset T^X_\infty(v')$.

Proof. — If $v \leqslant_\sigma v'$, then by Proposition 3.1, $\overline{T^W_\infty(v)} \supset T^W_\infty(v')$. Hence, it follows that

$$\overline{\nu_\infty(T^W_\infty(v))} \supset \nu_\infty(T^W_\infty(v')),$$

which implies (ii).

Conversely, if $\overline{T^X_\infty(v)} \supset T^X_\infty(v')$, then $\overline{\rho_\infty(T^X_\infty(v))} \supset \rho_\infty(T^X_\infty(v'))$ which is the inclusion

$$\overline{T^Z_\infty(v)} \supset T^Z_\infty(v').$$

Again by Proposition 3.1, it follows $v \leqslant_\sigma v'$. □

Lemma 4.6. — *Let v be a minimal element in $\sigma^o \cap N$ with respect to the order $\leqslant_\sigma$. Then $\overline{T^X_\infty(v)}$ is a local Nash component of (X, x).*

Proof. — As $T^X_\infty(v)$ is irreducible, we can take a Nash component C of (X, x) containing $T^X_\infty(v)$. Let α be the generic point of C, then the image $\alpha(\eta)$ of the generic point η of $\operatorname{Spec} K[[t]]$ is the generic point of $\widehat{X}$ by Lemma 2.11. Then α can be uniquely lifted to an arc $\tilde{\alpha} : \operatorname{Spec} K[[t]] \longrightarrow \widehat{W}$ by the valuative criterion of properness. As $\tilde{\alpha}(\eta)$ is the generic point of $\widehat{W}$, $\tilde{\alpha}(\eta)$ is mapped to the generic point of W. Then, there exists $v' \in \sigma^o \cap N$ such that $\tilde{\alpha} \in T^W_\infty(v')$. Since $\overline{\nu_\infty T^W_\infty(v')} \supset \overline{\{\nu_\infty(\tilde{\alpha})\}} = \overline{\{\alpha\}} \supset T^X_\infty(v)$, We obtain

$$\overline{T^X_\infty(v')} \supset T^X_\infty(v).$$

By Lemma 4.5 and the minimality of v, it follows that $v = v'$ and $C = \overline{T^X_\infty(v)}$. □

Lemma 4.7. — *Let $\nu : \widehat{W} \longrightarrow X$ be the normalization of a reduced k-scheme X and for a singular closed point $x \in X$, $\nu^{-1}(x)$ be one closed point w. Then, an essential divisors over (X, x) is an essential divisors over $(\widehat{W}, w)$.*

Proof. — Let E be an essential divisor over (X, x). Let $\psi : \widetilde{W} \longrightarrow \widehat{W}$ be a resolution of the singularities of $\widehat{W}$. Then the composite $\varphi = \nu \circ \psi : \widetilde{W} \longrightarrow X$ is a resolution of the singularities of X and the center of E on $\widetilde{W}$ is an irreducible component of $\varphi^{-1}(x) = \psi^{-1}(w)$. □

Lemma 4.8. — *Let $w \in W$ be a closed point of a variety and let $\widehat{W} = \operatorname{Spec} \widehat{\mathcal{O}}_{W,w}$. Denote the closed point of $\widehat{W}$ again by w. Then an essential divisor over $(\widehat{W}, w)$ is an essential divisor over (W, w).*

Proof. — Let E be an essential divisor over $(\widehat{W}, w)$. Then E is regarded as an exceptional divisor over (W, w). Indeed, for a resolution $\varphi : Y \longrightarrow W$ such that $\varphi^{-1}(w)$ is a divisor, the base change $\varphi' : Y \times_W \widehat{W} \longrightarrow \widehat{W}$ is a resolution of the singularities of $\widehat{W}$ with ${\varphi'}^{-1}(w) = \varphi^{-1}(w)$. As E appears in ${\varphi'}^{-1}(w)$ as a component, we can identify E with the corresponding exceptional divisor over (W, w). Let $\psi : Y' \longrightarrow W$ be any resolution of the singularities of W and $\psi' : Y' \times_W \widehat{W} \longrightarrow \widehat{W}$ the induced resolution of the singularities of $\widehat{W}$ which is the base change. Now, as E is an essential divisor over $(\widehat{W}, w)$, the center of E on $Y' \times_W \widehat{W}$ is an irreducible component of ${\psi'}^{-1}(w) = \psi^{-1}(w)$. □

THEOREM 4.9. — *Let (X, x) be an analytically pretoric singularity. Then the local Nash map :*

$$\ell\mathcal{N} : \{\textit{local Nash components of } (X,x)\} \longrightarrow \{\textit{essential divisors over } (X,x)\}$$

is bijective.

Proof. — Consider the following diagram:

$$\left\{\begin{array}{c}\text{minimal elements}\\ \text{in}\\ v \in \sigma^o \cap N\end{array}\right\} \xrightarrow{\Phi_1} \left\{\begin{array}{c}\text{local Nash}\\ \text{components}\\ \text{of } (X,x)\end{array}\right\} \xrightarrow{\ell\mathcal{N}} \left\{\begin{array}{c}\text{essential}\\ \text{divisors}\\ \text{over } (X,x)\end{array}\right\}$$

$$\xrightarrow{\Phi_2} \left\{\begin{array}{c}\text{essential}\\ \text{divisors}\\ \text{over } (\widehat{W},w)\end{array}\right\} \xrightarrow{\Phi_3} \left\{\begin{array}{c}\text{essential}\\ \text{divisors}\\ \text{over } (W,w)\end{array}\right\} \xrightarrow{\Phi_4} \left\{\begin{array}{c}\text{minimal elements}\\ \text{in}\\ v \in \sigma^o \cap N\end{array}\right\}.$$

The map Φ_1 is defined by $v \mapsto \overline{T^X_\infty(v)}$ and it is injective by Lemma 4.6. The local Nash map $\ell\mathcal{N}$ is injective as noted in 2.4. The canonical map Φ_2 is injective by Lemma 4.7. The canonical map Φ_3 is injective by Lemma 4.8. The map Φ_4 sends D_v to v and it is bijective by Bouvier's characterization of "composante essentielle"([1]), where D_v is the invariant divisor $\overline{\mathrm{orb}(\mathbb{R}_{\geqslant 0} v)}$. Hence the composite of all maps is an injective map from a finite set to itself and therefore all maps are bijective. □

For the final result, we need the following lemma.

LEMMA 4.10. — *Let $\mathcal{O}$ be the completion of a local ring essentially of finite type over k by the maximal ideal. Let $X = \operatorname{Spec} \mathcal{O}$. The closed point of X is denoted by x. Assume that X is reduced and $X = \bigcup_{i=1}^r X_i$ is the decomposition into irreducible components. If the local Nash map is bijective for (X_i, x) $(i = 1, ..., r)$, then the local Nash map is bijective for (X, x).*

Proof. — Note that $\pi_X^{-1}(x) = \bigcup_{i=1}^r \pi_{X_i}^{-1}(x)$. First we claim that

$$\{\text{local Nash components of } (X,x)\} = \bigsqcup_{i=1}^{r} \{\text{local Nash components of } (X_i,x)\}.$$

Let C be a local Nash component of (X,x) and α the generic point of C. As $\alpha(\eta) \in X \setminus \operatorname{Sing} X \subset \bigsqcup_{i=1}^{r} \left(X_i \setminus \bigcup_{j \neq i} X_j \right)$, there is unique i such that $\alpha(\eta) \in X_i$. Then $\alpha \in \pi_{X_i}^{-1}(x)$, therefore $C \subset \pi_{X_i}^{-1}(x)$ and C is a local Nash component of (X_i, x).

Conversely let C be a local Nash component of (X_i, x) and α the generic point of C. Then $\alpha(\eta)$ is the generic point of X_i by Lemma 2.11. Hence, $\alpha(\eta) \notin \operatorname{Sing} X$. Let C' be a local Nash component of (X,x) containing C. Then, by the preceding discussion there is unique j such that C' is a local Nash component of (X_j, x). As C' contains an arc α satisfying that $\alpha(\eta)$ is the generic point of X_i, it turns out that $j = i$. Then $C = C'$ and C is a local Nash component of (X,x).

Next we claim that

$$\{\text{essential divisors over } (X,x)\} \subset \bigsqcup_{i=1}^{r} \{\text{essential divisors over } (X_i,x)\}.$$

Let E be an essential divisor over (X,x), then E is an exceptional divisor over (X_i, x) for some i. Let $\varphi_i : Y_i \longrightarrow X_i$ be a resolution of the singularities of X_i. Take a resolution $\varphi_j : Y_j \longrightarrow X_j$ for each $j \neq i$. Then the composite φ

$$Y := \bigsqcup_{j=1}^{r} Y_j \xrightarrow{\bigsqcup \varphi_j} \bigsqcup_{j=1}^{r} X_j \longrightarrow X$$

is a resolution of the singularities of X. As E is an essential divisor over (X,x), the center of E on Y is an irreducible component of $\varphi^{-1}(x) = \bigsqcup \varphi_i^{-1}(x)$, therefore an irreducible component of $\varphi_i^{-1}(x)$.

Now we obtain the diagram

$$\bigsqcup_{i=1}^{i} \left\{ \begin{array}{c} \text{local Nash} \\ \text{components} \\ \text{of } (X_i,x) \end{array} \right\} = \left\{ \begin{array}{c} \text{local Nash} \\ \text{components} \\ \text{of } (X,x) \end{array} \right\} \xrightarrow{\ell\mathcal{N}} \left\{ \begin{array}{c} \text{essential} \\ \text{divisors} \\ \text{over } (X,x) \end{array} \right\}$$

$$\subset \bigsqcup_{i=1}^{i} \left\{ \begin{array}{c} \text{essential} \\ \text{divisors} \\ \text{over } (X_i,x) \end{array} \right\}.$$

Since the local Nash components of (X_i, x) correspond bijectively to the essential components over (X_i, x) for each i, the composite of all injections of above diagram is bijective. Therefore all maps are bijective. □

Now we obtain the following final statement.

Corollary 4.11. — *Let (X,x) be a quasi-ordinary singularity. Then the local Nash map for (X,x) is bijective.*

Proof. — A quasi-ordinary singularity (X, x) is decomposed into analytically irreducible quasi-ordinary singularities (X_i, x) $(i = 1, .., r)$. As each (X_i, x) is an analytically pretoric singularity, the statement follows from Theorem 4.9 and Lemma 4.10. Professor Patrick Popescu-Pampu kindly informed the author that quasi-ordinary singularity is equivalent to analytically pretoric singularity by virtue of his paper "On the Analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity" Duke Math. J. 124(1) (2004), 67-103. □

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