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Tome 56, nº 3 (2006), p. 561-597.

<http://aif.cedram.org/item?id=AIF_2006__56_3_561_0>

Ann. Inst. Fourier, Grenoble
56, 3 (2006) 561-597

COHOMOLOGY OF DRINFELD SYMMETRIC SPACES AND HARMONIC COCHAINS

by Yacine AÏT AMRANE

ABSTRACT. — Let K be a non-archimedean local field. This article gives an explicit isomorphism between the dual of the special representation of $GL_{n+1}(K)$ and the space of harmonic cochains defined on the Bruhat-Tits building of $GL_{n+1}(K)$, in the sense of E. de Shalit [11]. We deduce, applying the results of a paper of P. Schneider and U. Stuhler [9], that there exists a $GL_{n+1}(K)$-equivariant isomorphism between the cohomology group of the Drinfeld symmetric space and the space of harmonic cochains.

RÉSUMÉ. — Soit K un corps local non-archimédien. Ce papier donne un isomorphisme explicite entre le dual de la représentation spéciale de $GL_{n+1}(K)$ et l'espace des cocycles harmoniques définis sur l'immeuble de Bruhat-Tits de $GL_{n+1}(K)$, au sens de E. de Shalit [11]. Nous déduisons, en appliquant les résultats d'un papier de P. Schneider et U. Stuhler [9], qu'il existe un isomorphisme $GL_{n+1}(K)$-équivariant entre le groupe de cohomologie de l'espace symétrique de Drinfeld et l'espace des cocycles harmoniques.

Introduction

Let K be a non-archimedean local field, i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let n be a fixed natural number. Let $\widetilde{G}$ denote $GL_{n+1}(K)$, let $\widetilde{P}$ be its upper triangular Borel subgroup, and let $\overline{S}$ denote the set of fundamental reflexions s_i, $1 \leqslant i \leqslant n$, in the linear Weyl group $\overline{W}$ of $\widetilde{G}$. Let $\Delta = \{1, \dots, n\}$. For each subset I of Δ, let $\widetilde{P}_I$ be the parabolic subgroup of $\widetilde{G}$ generated by $\widetilde{P}$ and the reflexions s_i, $i \in I$.

Let M be a commutative ring on which $\widetilde{G}$ acts trivially. For any $I \subseteq \Delta$, we denote by $C^\infty(\widetilde{G}/\widetilde{P}_I, M)$ the space of locally constant functions on $\widetilde{G}/\widetilde{P}_I$ with values in M. The action of $\widetilde{G}$ on $C^\infty(\widetilde{G}/\widetilde{P}_I, M)$ comes from

Keywords: Drinfeld symmetric spaces, cohomology, Bruhat-Tits buildings, harmonic cochains, special representations.
Math. classification: 22E50, 20E42.

left translations on $\widetilde{G}/\widetilde{P}_I$. For any integer k, $0 \leqslant k \leqslant n$, if J_k denotes the subset $\{1, \dots, n-k\}$ of Δ, the k-special representation of $\widetilde{G}$ is defined to be the $M[\widetilde{G}]$-module:

$$\mathrm{Sp}^k(M) = \frac{C^\infty(\widetilde{G}/\widetilde{P}_{J_k}, M)}{\sum_{j=n-k+1}^{n} C^\infty(\widetilde{G}/\widetilde{P}_{J_k \cup \{j\}}, M)}.$$

In case $k = n$, we get the ordinary Steinberg representation $\mathrm{Sp}^n(M) = \mathrm{St}^n(M)$.

The n-dimensional Drinfeld symmetric space over K is the complement $\Omega^{(n+1)}$ in $\mathbb{P}^n$ of the union of all the K-rational hyperplanes. The group $\widetilde{G}$ acts on $\Omega^{(n+1)}$.

The symmetric space $\Omega^{(n+1)}$ has been introduced by Drinfeld, [6], who showed that it is endowed with a structure of a rigid analytic variety. In the one dimensional case ($n = 1$) when K is of positive characteristic $p > 0$, Drinfeld computed the first étale cohomology group of $\Omega^{(2)}$ and proved that there are $\widetilde{G}$-isomorphisms:

$$(0.1) \qquad H^1_{\mathrm{et}}(\Omega^{(2)} \otimes_K \mathbf{C}, L) \cong \mathrm{Hom}(\mathrm{St}^1(\mathbb{Z}), L) \cong \mathfrak{H}\mathrm{arm}^{1,1}(\mathbb{Z}, L)$$

where $\mathbf{C}$ is the completion of an algebraic closure of K, L a finite abelian group whose order is prime to p, and $\mathfrak{H}\mathrm{arm}^{1,1}(\mathbb{Z}, L)$ is the space of L-valued harmonic cochains defined on the oriented (or pointed) edges of the Bruhat-Tits tree. ([6], see also [8]).

In their paper [9], P. Schneider and U. Stuhler generalized the first isomorphism in (0.1) to the case of any characteristic of the base field and to any dimension. Indeed, they studied the cohomology groups of $\Omega^{(n+1)}$ for any cohomology theory satisfying certain natural axioms. They proved the existence of a canonical $\widetilde{G}$-equivariant isomorphism, *cf.* [9, §4, Cor.17]):

$$(0.2) \qquad SS: \ H^\bullet(\Omega^{(n+1)}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Sp}^\bullet(\mathbb{Z}),\, L)$$

where $\mathcal{F}$ is a complex of sheaves on the category of smooth separated rigid analytic varieties over K equipped with a suitable Grothendieck topology, and L is the cohomology of the point $H^0(\mathrm{Spec}(K), \mathcal{F})$.

If K is of characteristic zero, the isomorphism of Schneider and Stuhler above, applied to rigid De-Rham cohomology, gives a $\widetilde{G}$-isomorphism

$$(0.3) \qquad SS_{dR}: \ H^\bullet_{dR}(\Omega^{(n+1)}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Sp}^\bullet(\mathbb{Z}),\, K).$$

Let M be a commutative ring as above. Let L be an M-module on which $\widetilde{G}$ acts linearly. For each k, $0 \leqslant k \leqslant n$, denote by $\mathfrak{H}\mathrm{arm}^k(M, L)$ the space of L-valued harmonic cochains defined over the free M-module generated by the pointed k-cells of the Bruhat-Tits building associated to $\widetilde{G}$, see def. 2.1. In zero characteristic, E. de Shalit, who introduced in [11] the notion of harmonic cochains we use here, proved that there is a $\widetilde{G}$-equivariant isomorphism:

$$dS : H^{\bullet}_{dR}(\Omega^{(n+1)}) \xrightarrow{\simeq} \mathfrak{H}\mathrm{arm}^{\bullet}(\mathbb{Z}, K). \tag{0.4}$$

This isomorphism, together with the isomorphism (0.3), gives a $\widetilde{G}$-equivariant isomorphism in characteristic zero:

$$SS_{dR} \circ dS^{-1} : \mathfrak{H}\mathrm{arm}^{\bullet}(\mathbb{Z}, K) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Sp}^{\bullet}(\mathbb{Z}), K). \tag{0.5}$$

In this paper, we shall construct explicitly, for K of arbitrary characteristic, the isomorphism (0.5) above between the harmonic cochain spaces and the K-dual spaces of the special representations.

The main result in this paper is the following theorem which generalizes also the isomorphism of Drinfeld mentioned above, to arbitrary n:

Theorem 0.1. — *3.3 Let K be a non-archimedean local field of arbitrary characteristic. Let M and L be as above. Then, for each k, $0 \leqslant k \leqslant n$, there is an explicit $\widetilde{G}$-equivariant isomorphism:*

$$\mathfrak{H}\mathrm{arm}^{k}(M, L) \cong \mathrm{Hom}_{M}(\mathrm{Sp}^{k}(M), L).$$

As a corollary, together with the isomorphism (0.2) above we obtain the following:

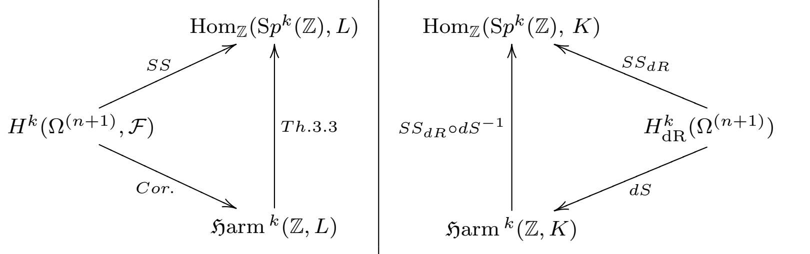
Corollary 0.2. — *Let K be a non-archimedean local field of arbitrary characteristic. Let $\mathcal{F}$ and L be as in the situation of the isomorphism (0.2), and as in [9]. For any k, $0 \leqslant k \leqslant n$, we have the following $\widetilde{G}$-equivariant isomorphism:*

$$H^{k}(\Omega^{(n+1)}, \mathcal{F}) \cong \mathfrak{H}\mathrm{arm}^{k}(\mathbb{Z}, L).$$

In particular, in the case of étale cohomology, this isomorphism allows us to express the étale cohomology groups of $\Omega^{(n+1)}$ in terms of harmonic cochains which are of combinatorial nature.

Let us summarize the $\widetilde{G}$-isomorphisms we have seen so far by the following commutative diagrams:

K of any characteristic | $\mathrm{char}(K) = 0$



where $L = H^0(\mathrm{Spec}(K), \mathcal{F})$ | we have $K = H^0_{\mathrm{dR}}(\mathrm{Spec}(K))$.

The results of this paper were announced without proofs in [2]. The reader will find more detailed proofs in [1].

Here is the plan of this article. We use the notations introduced above.

In the first section we give some preliminaries about the Bruhat-Tits building associated to $\widetilde{G}$. For each $I \subseteq \Delta$, let B_I be the standard parahoric subgroup of $\widetilde{G}$ generated by the upper Iwahori subgroup of $\widetilde{G}$ and the fundamental reflexions s_i, $i \in I$. By the Bruhat decomposition, there is a correspondence between the double classes in the affine Weyl group W of $\widetilde{G}$ and the Bruhat cells in $\widetilde{G}$. By the Iwasawa decomposition and with techniques inspired from Bourbaki [4], we prove that there is a canonical one-to-one correspondence between the double classes in the linear Weyl group $\overline{W}$ and the Iwasawa cells in $\widetilde{G}$. By these correspondences and by using decompositions in the Weyl group $\overline{W}$ into double classes modulo special subgroups, we deduce decompositions of certain subsets of $\widetilde{G}$ into a disjoint union of Bruhat and Iwasawa cells respectively.

In the second section, we recall the definition of harmonic cochains given by E. de Shalit. We also recall the relationship, given by P. Schneider and U. Stuhler, in [9], between the special representations and the parahoric subgroups. Next, we define, for each $I \subseteq \Delta$, a subset C_I of $\widetilde{G}$ which is a product of standard parahoric subgroups. Finally, by using the Iwasawa decomposition, we prove that the characteristic functions of the open compact subsets $C_I \widetilde{P}_{J_k}/\widetilde{P}_{J_k}$ of $\widetilde{G}/\widetilde{P}_{J_k}$, $I \subseteq \Delta$, viewed in $\mathrm{S}p^k(M)$, have properties that are close to those of harmonic cochains.

In section 3, we prove the main theorem which gives an explicit isomorphism between duals of the special representations and harmonic cochain spaces. In this isomorphism, the characteristic functions of the subsets

$C_I\widetilde{P}_{J_k}/\widetilde{P}_{J_k}$ correspond to the standard cells σ_I pointed at the fundamental vertex, stabilized by the standard parahoric subgroups B_I (under the action of $\widetilde{G}$ on its Bruhat-Tits building).

1. Bruhat-Tits building and decompositions in $\widetilde{G}$

1.1. Bruhat-Tits building

>From now on, K will be a non-archimedean local field, O its valuation ring, π a uniformizing parameter and κ the residue field of K. We denote by $\widetilde{G}$ the K-valued points of the connected reductive linear algebraic group GL_{n+1}.

For general properties of buildings, see [5] and [7]. An introduction to the Bruhat-Tits building of $\widetilde{G}$ with pointed cells is given in [11].

The Bruhat-Tits building (pointed cells). Let V be the standard vector space K^{n+1}. A lattice in V is a free O-submodule Λ of V of rank $n+1$. The Bruhat-Tits building of $\widetilde{G}$ may be described as a simplicial complex $\mathfrak{I}$ whose vertices are the dilation classes of lattices. More precisely, two lattices Λ and Λ' are in the same class if $\Lambda' = \lambda\Lambda$ for some $\lambda \in K^*$. The class of Λ is a vertex v and is denoted $v = [\Lambda]$. For k, $0 \leqslant k \leqslant n$, a k-cell σ in $\mathfrak{I}$ is a set of $k+1$ vertices $\{[\Lambda_0], [\Lambda_1], \ldots, [\Lambda_k]\}$ such that:

$$\cdots \supsetneq \Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_k \supsetneq \pi\Lambda_0 \supsetneq \cdots \tag{1.1}$$

Notice that there is an obvious cyclic ordering (mod. $(k+1)$) on the vertices of σ.

A pointed k-cell of $\mathfrak{I}$ is a pair (σ, v) consisting of a k-cell σ together with a distinguished vertex v of σ. Notice, therefore, that in the case of a pointed cell (σ, v) there is a precise ordering on the vertices. If $v = [\Lambda_0]$ we write:

$$(\sigma, v) = (\Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_k \supsetneq \pi\Lambda_0). \tag{1.2}$$

For each k, $0 \leqslant k \leqslant n$, let $\widehat{\mathfrak{I}}^k$ be the set of pointed k-cells of $\mathfrak{I}$.

The action of $\widetilde{G}$. For a fixed basis of the vector space V, the action of $\widetilde{G}$ on V is given by the matrix product ug^{-1} where $u \in V$ is considered as a line matrix with respect to the basis of V. This action induces an action of $\widetilde{G}$ on the vertex set of the building $\mathfrak{I}$ by $g.v = [\Lambda g^{-1}]$. Thus, $\widetilde{G}$ acts on the cells by acting on their vertices.

The type of a pointed cell. (*cf.* [11, § 1.1].) Let $\sigma = (\Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_k \supsetneq \pi\Lambda_0) \in \widehat{\mathfrak{I}}^k$ be a pointed k-cell. The type of σ is defined as follows:

$$t(\sigma) = (d_1, \ldots, d_{k+1})$$

where $d_i = \dim_\kappa \Lambda_{i-1}/\Lambda_i$ for each $i = 1, \ldots, k+1$ (here, we suppose $\Lambda_{k+1} = \pi\Lambda_0$). The type of a pointed k-cell is preserved by the action of $\widetilde{G}$. Indeed, the action of $\widetilde{G}$ preserves the dimension of the κ-vector spaces Λ_{i-1}/Λ_i.

The standard cells. Let $\{u_1, \ldots, u_{n+1}\}$ be the standard basis of $V = K^{n+1}$. Consider, for each $i = 0 \ldots n$, the vertex $v_i^o = [\overset{o}{\Lambda}_i]$ represented by the lattice:

$$\overset{o}{\Lambda}_i = \pi O u_1 \oplus \cdots \oplus \pi O u_i \oplus O u_{i+1} \oplus \cdots \oplus O u_{n+1}.$$

Since the $\overset{o}{\Lambda}_i$, $0 \leqslant i \leqslant n$, satisfy (1.1), we have an n-cell $\sigma_\emptyset = \{v_0^o, v_1^o, \ldots, v_n^o\}$ called the fundamental chamber of $\mathfrak{I}$.

Now, once and for all, fix $\Delta = \{1, \ldots, n\}$. For each $I \subseteq \Delta$ such that $\Delta - I = \{i_1 < \cdots < i_k\}$, we have a k-cell

$$\sigma_I = \{v_0^o, v_{i_1}^o, \ldots, v_{i_k}^o\}. \tag{1.3}$$

The σ_I, $I \subseteq \Delta$, are called the standard cells of the Bruhat-Tits building $\mathfrak{I}$. These cells are the faces of the fundamental chamber $\sigma_\emptyset$ having v_0^o as vertex, called the fundamental vertex of $\mathfrak{I}$.

We denote by $\widetilde{T}$ the standard maximal torus of $\widetilde{G}$ of diagonal matrices and by $\widetilde{N}$ its normalizer in $\widetilde{G}$. Since the Weyl group $\overline{W} = \widetilde{N}/\widetilde{T}$ of $\widetilde{G}$ with respect to $\widetilde{T}$ is isomorphic to the permutation group $\mathcal{S}_{n+1}$, $\overline{W}$ is generated by the set $\overline{S} = \{s_i,\, i \in \Delta\}$ of the reflexions s_i which correspond to the transpositions $(i, i+1) \in \mathcal{S}_{n+1}$. We have the following lemma:

Lemma 1.1. — *Let y_i, $0 \leqslant i \leqslant n$, be the diagonal matrix*

$$y_i = \operatorname{diag}(\overbrace{1, \ldots, 1}^{\text{i times}}, \pi, \ldots, \pi)$$

and let $w_i = (s_i s_{i+1} \cdots s_n)(s_{i-1} s_i \cdots s_{n-1}) \cdots (s_1 s_2 \cdots\ s_{n-i+1}) \in \overline{W}$. We have:

$$(\sigma_\emptyset, v_i^o) = y_i w_i (\sigma_\emptyset, v_0^o).$$

If $(\sigma, v_{i_j}^o) = (v_{i_j}^o, \ldots, v_{i_k}^o, v_{i_0}^o, v_{i_1}^o, \ldots, v_{i_{j-1}}^o)$ is a face of the pointed chamber $(\sigma_\emptyset, v_{i_j}^o)$, where $0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n$ and $0 \leqslant j \leqslant k$, then

$$(\sigma, v_{i_j}^o) = y_{i_j} w_{i_j} (\sigma_{\widehat{I}_{i_j}}, v_0^o)$$

where $\Delta - \widehat{I}_{i_j} = \{i_{j+1} - i_j < \cdots < i_k - i_j < n + 1 + i_0 - i_j < \cdots < n + 1 + i_{j-1} - i_j\}$.

Proof. — The vertices of the fundamental chamber are $v_l^o = [\overset{o}{\Lambda}_l]$. We can easily check that the representants $\overset{o}{\Lambda}_l$ of these vertices satisfy:

$$\overset{o}{\Lambda}_l y_i w_i = \begin{cases} \overset{o}{\Lambda}_{n+1+l-i} & \text{if} \quad 0 \leqslant l \leqslant i-1 \\ \overset{o}{\Lambda}_{l-i}\pi & \text{if} \quad i \leqslant l \leqslant n. \end{cases}$$

Therefore, by taking into account the way in which $\widetilde{G}$ acts on the vertices of $\mathfrak{I}$, it follows that:

$$w_i^{-1}y_i^{-1}v_l^o = \begin{cases} v_{n+1+l-i}^o & \text{if} \quad 0 \leqslant l \leqslant i-1 \\ v_{l-i}^o & \text{if} \quad i \leqslant l \leqslant n, \end{cases}$$

hence $w_i^{-1}y_i^{-1}(\sigma_\emptyset, v_i^o) = (\sigma_\emptyset, v_0^o)$ and, if $(\sigma, v_{i_j}^o)$ and $\widehat{I}_{i_j}$ are as in the lemma, we have $w_{i_j}^{-1}y_{i_j}^{-1}(\sigma, v_{i_j}^o) = (\sigma_{\widehat{I}_{i_j}}, v_0^o)$. □

Since the action of $\widetilde{G}$ is transitive on the chambers of $\mathfrak{I}$, the lemma above shows that $\widetilde{G}$ acts transitively on the pointed k-cells of a given type. Furthermore, if we denote by t_I the type of the pointed standard k-cell (σ_I, v_0^o) and by $\widehat{\mathfrak{I}}^{k,t_I}$ the set of all pointed k-cells of type t_I, we have $\widehat{\mathfrak{I}}^k = \coprod_{I\subseteq\Delta} \widehat{\mathfrak{I}}^{k,t_I}$, where the disjoint union is taken over the subsets $I \subseteq \Delta$ such that $\Delta - I$ is of cardinal k. Notice, therefore, that for k fixed, there are exactly $\binom{n}{k}$ types of pointed k-cells.

Remark 1.2. — For each $I \subseteq \Delta$, let B_I be the pointwise stabilizer in $\widetilde{G}$ of the standard cell σ_I, or equivalentely the stabilizer of the pointed standard cell (σ_I, v_0^o). The first assertion of the lemma 1.1 shows that, for every i, $0 \leqslant i \leqslant n$, we have

$$y_i w_i B = B y_i w_i, \tag{1.4}$$

where $B = B_\emptyset$.

1.2. Bruhat and Iwasawa decomposition in $\widetilde{G}$

1.2.1. The Bruhat decomposition

The parabolic subgroups of $\widetilde{G}$. Let $\widetilde{P}$ be the upper triangular Borel subgroup of $\widetilde{G}$. A parabolic subgroup of $\widetilde{G}$ is a closed subgroup which contains a Borel subgroup. The subgroups which contain $\widetilde{P}$ are said to be special; these subgroups are completely determined by the subsets I of Δ. Indeed, if for each $I \subseteq \Delta$, we let W_I be the subgroup of $\overline{W}$ generated by the s_i, $i \in I$, it has been shown that the subset

$$\widetilde{P}_I = \widetilde{P}W_I\widetilde{P} \qquad (:= \widetilde{P}\widetilde{N}_I\widetilde{P} \quad \text{where} \quad \widetilde{N}_I \subseteq \widetilde{N} \ \text{ is such that } \ \widetilde{N}_I/\widetilde{T} = W_I)$$

is a subgroup of $\widetilde{G}$ containing $\widetilde{P}$, and that every subgroup of $\widetilde{G}$ containing $\widetilde{P}$ is a certain $\widetilde{P}_I$ for $I \subseteq \Delta$. Note that $\widetilde{P} = \widetilde{P}_\emptyset$.

The parahoric subgroups of $\widetilde{G}$. For each $I \subseteq \Delta$, we denote by $\overset{\circ}{B}_I$ the open compact subgroup of $\widetilde{G}(O)$ which is the inverse image of the standard parabolic subgroup $\widetilde{P}_I(\kappa)$ of $\widetilde{G}(\kappa)$ by the map "reduction mod. π": $\widetilde{G}(O) \to \widetilde{G}(\kappa)$. The parahoric subgroups of $\widetilde{G}$ are the conjugates in $\widetilde{G}$ of the $\overset{\circ}{B}_I$, $I \subseteq \Delta$. Note that we have $B_I = \overset{\circ}{B}_I K^*$.

The Bruhat decomposition. Let us recall, for each $I \subseteq \Delta$, the following Bruhat decomposition (*cf.* [5, ch. V], [4] or [7]):

(1.5)

$$B_I = BW_IB = \coprod_{w \in W_I} BwB \qquad \text{resp.} \qquad \widetilde{P}_I = \widetilde{P}W_I\widetilde{P} = \coprod_{w \in W_I} \widetilde{P}w\widetilde{P}.$$

As a consequence of the Bruhat decomposition, we obtain the following proposition:

Proposition 1.3. — *Let $I_1, I_2 \subseteq \Delta$. The map which to $W_{I_1}wW_{I_2}$ associates $B_{I_1}wB_{I_2}$ for $w \in \overline{W}$ is a one-to-one correspondence:*

$$W_{I_1}\backslash\overline{W}/W_{I_2} \xrightarrow{\sim} B_{I_1}\backslash\widetilde{G}(O)K^*/B_{I_2}.$$

Proof. — *cf.* [4, ch. IV, § 2.5, rem. 2]. □

1.2.2. The Iwasawa decomposition

In the following, we shall use the same techniques as in [4, ch. IV, § 2,2] and use the generalized Iwasawa decomposition (see for example [7, th. 17.6]):

$$\widetilde{G} = \coprod_{w \in \overline{W}} Bw\widetilde{P} \tag{1.6}$$

to prove theorem 1.7 below, which gives an analogous result to the proposition 1.3.

Lemma 1.4. — *Let $w \in \overline{W}$ and $j \in \Delta$. We have the following inclusions:*

1. $w\widetilde{P}s_j \subseteq Bw\widetilde{P} \cup Bws_j\widetilde{P}$
2. $s_jBw \subseteq Bw\widetilde{P} \cup Bs_jw\widetilde{P}$.

Proof. — Indeed, by putting $B' = w^{-1}Bw$ in the first inclusion (resp. $\widetilde{P}' = w\widetilde{P}w^{-1}$ in the second inclusion) we have to show:

$$\widetilde{P}s_j \subseteq B'\widetilde{P} \cup B's_j\widetilde{P} \qquad (\text{resp. } s_jB \subseteq B\widetilde{P}' \cup Bs_j\widetilde{P}').$$

The canonical basis of K^{n+1} being $\{u_1, \dots, u_{n+1}\}$, let $\widetilde{G}_j$ be the subgroup of $\widetilde{G}$ consisting of the elements which fix the u_i for $i \neq j, j+1$ and which fix the plane spanned by u_j and u_{j+1}. Put $\widetilde{G}_j(O) = \widetilde{G}_j \cap \widetilde{G}(O)$. So (*cf.* [4, ch.IV, §2.2]), we have $\widetilde{G}_j(k)\widetilde{P}(k) = \widetilde{P}(k)\widetilde{G}_j(k)$ for any base field k, hence, for $k = K$ (resp. $k = \kappa$) we get $\widetilde{G}_j\widetilde{P} = \widetilde{P}\widetilde{G}_j$ (resp. $\widetilde{G}_j(O)B = B\widetilde{G}_j(O)$, by lifting the equality $\widetilde{G}_j(\kappa)\widetilde{P}(\kappa)=\widetilde{P}(\kappa)\widetilde{G}_j(\kappa)$ to $\widetilde{G}(O)$ and multiplying then by K^*). Therefore, it's enough to prove:

$$\widetilde{G}_j \subseteq (B'\cap \widetilde{G}_j)(\widetilde{P}\cap \widetilde{G}_j)\cup (B'\cap \widetilde{G}_j)s_j(\widetilde{P}\cap \widetilde{G}_j)$$
$$(\text{resp.}\quad \widetilde{G}_j \subseteq (B\cap \widetilde{G}_j)(\widetilde{P}'\cap \widetilde{G}_j)\cup (B\cap \widetilde{G}_j)s_j(\widetilde{P}'\cap \widetilde{G}_j)\,).$$

By identifying $\widetilde{G}_j$ with GL_2, the proof may be completed as in [loc. cit.], except that we use the Iwasawa decomposition instead of the Bruhat decomposition. □

Corollary 1.5. — *Let $u_1,\dots,u_d\in\overline{S}$ and $w\in\overline{W}$. We have:*

1. $w\widetilde{P}u_1\dots u_d \subseteq \bigcup_{(l_1,\dots,l_p)} Bwu_{l_1}\dots u_{l_p}\widetilde{P}$
2. $u_1\dots u_d Bw \subseteq \bigcup_{(l_1,\dots,l_p)} Bu_{l_1}\dots u_{l_p}w\widetilde{P}$

where $(l_1,\dots,l_p)$ runs through the increasing sequenses (including the empty sequence) in $[\![1,d]\!]$.

Proof. — Induct on d and use the lemma 1.4 above (see also [4, ch. IV, § 2, lem. 1]). □

Corollary 1.6. — *Let $I_1,I_2\subseteq\Delta$. For each $w\in\overline{W}$, we have $B_{I_1}w\widetilde{P}_{I_2}=BW_{I_1}wW_{I_2}\widetilde{P}$.*

Proof. — Let I_1,I_2 and w be as above. Let $w'=u'_1\cdots u'_{d_1}\in W_{I_1}$ and $w''=u''_1\cdots u''_{d_2}\in W_{I_2}$. We have:

$$Bw'B.Bw\widetilde{P}.\widetilde{P}w''\widetilde{P}=Bu'_1\cdots u'_{d_1}Bw\widetilde{P}u''_1\cdots u''_{d_2}\widetilde{P},$$

therefore, the corollary 1.5 gives $Bw'B.Bw\widetilde{P}.\widetilde{P}w''\widetilde{P}\subseteq BW_{I_1}wW_{I_2}\widetilde{P}$, and if we take the union as w' and w'' run through W_{I_1} and W_{I_2} respectively, one obtains:

$$B_{I_1}w\widetilde{P}_{I_2}\subseteq BW_{I_1}wW_{I_2}\widetilde{P}.$$

The other inclusion is obvious. □

Theorem 1.7. — *Let $I_1,I_2\subseteq\Delta$. The map $\overline{W}\to B_{I_1}\backslash\widetilde{G}/\widetilde{P}_{I_2}$ which to w associates $B_{I_1}w\widetilde{P}_{I_2}$, induces a one-to-one map:*

$$W_{I_1}\backslash\overline{W}/W_{I_2}\xrightarrow{\sim}B_{I_1}\backslash\widetilde{G}/\widetilde{P}_{I_2}.$$

Proof. — The generalized Iwasawa decomposition (1.6) shows that the map $w\mapsto Bw\widetilde{P}$ is bijective from $\overline{W}$ on the set $B\backslash\widetilde{G}/\widetilde{P}$, so, by the corollary 1.6, the surjective map $\overline{W}\xrightarrow{\sim}B\backslash\widetilde{G}/\widetilde{P}\twoheadrightarrow B_{I_1}\backslash\widetilde{G}/\widetilde{P}_{I_2}$ induces the following surjective map:

$$W_{I_1}\backslash\overline{W}/W_{I_2}\longrightarrow B_{I_1}\backslash\widetilde{G}/\widetilde{P}_{I_2}.$$

In order to prove that this map is injective, it is enough to prove the following property:

for any $w, w'\in\overline{W}$, $B_{I_1}w\widetilde{P}_{I_2}=B_{I_1}w'\widetilde{P}_{I_2}$, if and only if, $W_{I_1}wW_{I_2}=W_{I_1}w'W_{I_2}$.

Indeed, suppose $B_{I_1}w\widetilde{P}_{I_2}\cap B_{I_1}w'\widetilde{P}_{I_2}\neq\emptyset$, so there exist $b\in \overset{\circ}{B}_{I_1}$ and $p\in\widetilde{P}_{I_2}$ with $bwp=w'$. This implies $p=w^{-1}b^{-1}w'\in\widetilde{G}(O)\cap\widetilde{P}_{I_2}\subseteq B_{I_2}$ and hence $B_{I_1}wB_{I_2}\cap B_{I_1}w'B_{I_2}\neq\emptyset$ which, by Proposition 1.3, gives $W_{I_1}wW_{I_2}=W_{I_1}w'W_{I_2}$. □

Remark 1.8. — Let $I_1, I_2\subseteq\Delta$. Recall (see [4, ch. IV, § 2.5, prop. 2]), that for Bruhat cells, we have a similar formula to the formula in the corollary 1.6, that is:

$$B_{I_1}wB_{I_2}=BW_{I_1}wW_{I_2}B. \tag{1.7}$$

Now, let $I_1, I_2\subseteq\Delta$ such that for each $i\in I_1$, $j\in I_2$, we have $|i-j|\geqslant 2$ (which gives $s_is_j=s_js_i$). Then, since every element in W_{I_1} commutes with every element in W_{I_2}, we get:

$$W_{I_1\cup I_2}=W_{I_1}.W_{I_2}=W_{I_2}.W_{I_1}.$$

The equality (1.7), for $w=1$, gives then:

$$B_{I_1\cup I_2}=B_{I_1}.B_{I_2}=B_{I_2}.B_{I_1}. \tag{1.8}$$

Notice also that, for each $I\subseteq\Delta$, one gets (see also [9, lem. 14 (ii), § 4]):

$$B_I\widetilde{P}_I=B\widetilde{P}_I=B_I\widetilde{P}. \tag{1.9}$$

1.2.3. Decomposition in the Weyl group $\overline{W}$

For each $r, r'\in\Delta=\{1,\dots,n\}$ such that $r\leqslant r'+1$, we set $w_r^{r'}=s_rs_{r+1}\dots s_{r'}$, ($w_{r'+1}^{r'}=1$). Using the Coxeter relations in the Weyl group $\overline{W}$:

$$\begin{cases} s_l^2=1 & \text{for} \quad l=1,\dots,n\\ s_{l_1}s_{l_2}=s_{l_2}s_{l_1} & \text{for} \quad 1\leqslant l_1<l_2-1\leqslant n-1\\ s_ls_{l+1}s_l=s_{l+1}s_ls_{l+1} & \text{for} \quad l=1,\dots,n-1, \end{cases} \tag{1.10}$$

it is easy to show that we have:

$$s_lw_r^{r'}=w_r^{r'}s_{l-1} \quad \text{for every } r, r' \text{ and } l \text{ such that} \quad r+1\leqslant l\leqslant r'\leqslant n. \tag{1.11}$$

In all what follows, for any integers a and b such that $a\leqslant b+1$, we denote by $[\![a,b]\!]$ the set of all integers j such that $a\leqslant j\leqslant b$. It is the empty set in the case $a=b+1$.

Proposition 1.9. — *Let k be an integer such that $1 \leqslant k \leqslant n$. Let $w \in \overline{W}$. Then:*

(1) *(1.] For each integer j such that $n-k+2 \leqslant j \leqslant n$, we have the decompositions:*

$$BwB_{J_k\cup\{j\}} = BwB_{J_k} \amalg Bws_jB_{J_k} \text{ and } Bw\widetilde{P}_{J_k\cup\{j\}} = Bw\widetilde{P}_{J_k} \amalg Bws_j\widetilde{P}_{J_k}.$$

2. *For $j = n-k+1$, we have the following decompositions:*

$$BwB_{J_k\cup\{n-k+1\}} = \coprod_{r\in[\![1,n-k+2]\!]} Bww_r^{n-k+1}B_{J_k}$$

and

$$Bw\widetilde{P}_{J_k\cup\{n-k+1\}} = \coprod_{r\in[\![1,n-k+2]\!]} Bww_r^{n-k+1}\widetilde{P}_{J_k}.$$

Proof. — We use proposition 1.7 and theorem 1.3 to conclude respectively the first and the second decomposition in 1., and also in 2., from the decomposition into left cosets in $\overline{W}$.

1. It is easy to check that the sets W_{J_k} and $s_jW_{J_k}$ are the only different left cosets in $W_{J_k\cup\{j\}}$ modulo W_{J_k} since we have $n-k+2 \leqslant j \leqslant n$.

2. To prove that the only left cosets in $W_{J_k\cup\{n-k+1\}}$ modulo W_{J_k} are the $w_r^{n-k+1}W_{J_k}$, with $1 \leqslant r \leqslant n-k+2$, it suffices to prove more generally that for any a_1 such that $1 \leqslant a_1 \leqslant n-k+2$ we have the property:

$\mathbf{A_1}$: For each $w \in W_{[\![a_1,n-k+1]\!]}$, there is an $r \in [\![a_1, n-k+2]\!]$ so that

$$wW_{J_k} = w_r^{n-k+1}W_{J_k}.$$

We do this by induction on the length $l(w)$ of w in $W_{[\![a_1,n-k+1]\!]}$, where the length is defined with respect to the generators $\overline{S}$. If $l(w) = 0$ then $w = 1$, so the equality in $\mathbf{A_1}$ holds trivially with $r = n-k+2$. Now let $w \in W_{[\![a_1,n-k+1]\!]}$ be such that $l(w) = d+1$. Therefore, if $\iota \in [\![a_1, n-k+1]\!]$ is such that $l(s_\iota w) = d$, by induction, there is an $r \in [\![a_1, n-k+2]\!]$ so that: $wW_{J_k} = s_\iota s_\iota wW_{J_k} = s_\iota w_r^{n-k+1}W_{J_k}$. To complete the proof of $\mathbf{A_1}$, we study several cases depending on ι and r:

- $\iota \leqslant r-2$: the elements s_ι and w_r^{n-k+1} commute and $s_\iota \in W_{J_k}$. Therefore,

$$s_\iota w_r^{n-k+1}W_{J_k} = w_r^{n-k+1}s_\iota W_{J_k} = w_r^{n-k+1}W_{J_k}.$$

- $\iota = r-1$ (resp. $\iota = r$): we have:

$$s_\iota w_r^{n-k+1}W_{J_k} = w_{r-1}^{n-k+1}W_{J_k} \quad (\text{resp.} \quad s_\iota w_r^{n-k+1}W_{J_k} = w_{r+1}^{n-k+1}W_{J_k}).$$

- $r+1 \leqslant \iota \leqslant n-k+1$: by (1.11) we have $s_\iota w_r^{n-k+1} = w_r^{n-k+1} s_{\iota-1}$, and since $r \leqslant \iota - 1 \leqslant n-k$ then $s_{\iota-1} \in W_{J_k}$. Therefore,
$$s_\iota w_r^{n-k+1} W_{J_k} = w_r^{n-k+1} s_{\iota-1} W_{J_k} = w_r^{n-k+1} W_{J_k}.$$

To prove that the left cosets $w_r^{n-k+1} W_{J_k}$, $1 \leqslant r \leqslant n-k+2$, are different will be done more generally in the proof of the next proposition. □

Proposition 1.10. — *Let k, $1 \leqslant k \leqslant n$. Let $a_1, \ldots, a_k$ be such that $1 \leqslant a_1 \leqslant \cdots \leqslant a_k$. Assume, furthermore, that $a_\iota \leqslant n-k+\iota+1$ for any $\iota = 1, \ldots, k$. We have the decompositions:*
$$B_{[\![a_k,n]\!]} \cdots B_{[\![a_1,n-k+1]\!]} B_{J_k} = \coprod_{(r_1,\ldots,r_k)} B w_{r_k}^n \cdots w_{r_1}^{n-k+1} B_{J_k}$$
and
$$B_{[\![a_k,n]\!]} \cdots B_{[\![a_1,n-k+1]\!]} \widetilde{P}_{J_k} = \coprod_{(r_1,\ldots,r_k)} B w_{r_k}^n \cdots w_{r_1}^{n-k+1} \widetilde{P}_{J_k}$$
where $(r_1, \ldots, r_k)$ runs through the set $\prod_{\iota=1}^k [\![a_\iota, n-k+\iota+1]\!]$.

Proof. — As above, by Proposition 1.7 and Theorem 1.3, it's enough to prove that one has the following decomposition:
$$W_{[\![a_k,n]\!]} \cdots W_{[\![a_1,n-k+1]\!]} W_{J_k} = \coprod_{(r_1,\ldots,r_k)} w_{r_k}^n \cdots w_{r_1}^{n-k+1} W_{J_k} \tag{1.12}$$
with $(r_1, \ldots, r_k)$ running through the set $\prod_{\iota=1}^k [\![a_\iota, n-k+\iota+1]\!]$.

To prove the equality, it suffices to prove by induction on m that, for $m = 1, \ldots, k$, the following holds:

$\mathbf{A_m}$: for each $w \in W_{[\![a_m,n-k+m]\!]}$
and each $(r_1, \ldots, r_{m-1}) \in \prod_{\iota=1}^{m-1} [\![a_\iota, n-k+\iota+1]\!]$,

there is an $(r'_1, \ldots, r'_m) \in \prod_{\iota=1}^{m} [\![a_\iota, n-k+\iota+1]\!]$ so that we have an equality:
$$w w_{r_{m-1}}^{n-k+m-1} \cdots w_{r_1}^{n-k+1} W_{J_k} = w_{r'_m}^{n-k+m} \cdots w_{r'_1}^{n-k+1} W_{J_k}.$$
The proof of $\mathbf{A_1}$ is above. Assume that $\mathbf{A_m}$ holds for $m \leqslant k-1$ and let us show $\mathbf{A_{m+1}}$. We have to prove that for any $w \in W_{[\![a_{m+1},n-k+m+1]\!]}$ and any $(r_1, \ldots, r_m) \in \prod_{\iota=1}^m [\![a_\iota, n-k+\iota+1]\!]$, there exists $(r'_1, \ldots, r'_{m+1}) \in \prod_{\iota=1}^{m+1} [\![a_\iota, n-k+\iota+1]\!]$ so that:
$$w w_{r_m}^{n-k+m} \cdots w_{r_1}^{n-k+1} W_{J_k} = w_{r'_{m+1}}^{n-k+m+1} \cdots w_{r'_1}^{n-k+1} W_{J_k}. \tag{1.13}$$
We prove (1.13) by induction on the length $l(w)$ of $w \in W_{[\![a_{m+1},n-k+m+1]\!]}$. If $l(w) = 0$ then $w = 1$ and (1.13) holds trivially. Assume that (1.13) is

true when $l(w) = d$. Let $w \in W_{[\![a_{m+1}, n-k+m+1]\!]}$ be such that $l(w) = d+1$. Therefore, if $j \in [\![a_{m+1}, n-k+m+1]\!]$ is such that $l(s_j w) = d$, there exists $(r'_1, \ldots, r'_{m+1})$ in $\prod_{\iota=1}^{m+1} [\![a_\iota, n-k+\iota+1]\!]$, by the induction hypothesis, so that:

$$s_j w w_{r_m}^{n-k+m} \cdots w_{r_1}^{n-k+1} W_{J_k} = w_{r'_{m+1}}^{n-k+m+1} \cdots w_{r'_1}^{n-k+1} W_{J_k}$$

and hence

$$ww_{r_m}^{n-k+m} \cdots w_{r_1}^{n-k+1} W_{J_k} = s_j w_{r'_{m+1}}^{n-k+m+1} \cdots w_{r'_1}^{n-k+1} W_{J_k}. \tag{1.14}$$

There are several cases depending on j and r'_{m+1}:

- $a_{m+1} \leqslant j \leqslant r'_{m+1} - 2 \leqslant n-k+m$: we have $s_j w_{r'_{m+1}}^{n-k+m+1} = w_{r'_{m+1}}^{n-k+m+1} s_j$. Since $s_j w_{r'_m}^{n-k+m} \in W_{[\![a_m, n-k+m]\!]}$, by induction $(\mathbf{A_m})$, there exists $(r''_1, \ldots, r''_m)$ in $\prod_{\iota=1}^{m} [\![a_\iota, n-k+\iota+1]\!]$ such that:

$$s_j w_{r'_m}^{n-k+m} \cdots w_{r'_1}^{n-k+1} W_{J_k} = w_{r''_m}^{n-k+m} \cdots w_{r''_1}^{n-k+1} W_{J_k}.$$

- $j = r'_{m+1} - 1$: we have $s_j w_{r'_{m+1}}^{n-k+m+1} = w_{r'_{m+1}-1}^{n-k+m+1}$.
- $j = r'_{m+1} \leqslant n-k+m+1$: we have $s_j w_{r'_{m+1}}^{n-k+m+1} = w_{r'_{m+1}+1}^{n-k+m+1}$.
- $a_{m+1} + 1 \leqslant r'_{m+1} + 1 \leqslant j \leqslant n-k+m+1$: we have $s_j w_{r'_{m+1}}^{n-k+m+1} = w_{r'_{m+1}}^{n-k+m+1} s_{j-1}$. Since $a_m \leqslant a_{m+1} \leqslant j-1 \leqslant n-k+m$, we have $s_{j-1} w_{r'_m}^{n-k+m} \in W_{[\![a_m, n-k+m]\!]}$ and in the same way as in the first case, one gets:

$$s_{j-1} w_{r'_m}^{n-k+m} \cdots w_{r'_1}^{n-k+1} W_{J_k} = w_{r''_m}^{n-k+m} \cdots w_{r''_1}^{n-k+1} W_{J_k}.$$

Thus, together with (1.14), there exists $(r''_1, \ldots, r''_{m+1})$ in $\prod_{\iota=1}^{m+1} [\![a_\iota, n-k+\iota+1]\!]$ such that:

$$ww_{r_m}^{n-k+m} \cdots w_{r_1}^{n-k+1} W_{J_k} = w_{r''_{m+1}}^{n-k+m+1} w_{r''_m}^{n-k+m} \cdots w_{r''_1}^{n-k+1} W_{J_k}.$$

This completes the proof of (1.13) and also the proof of $\mathbf{A_m}$, $1 \leqslant m \leqslant k$.

Let us prove now that the union in (1.12) is a disjoint union. Deny and assume that there are two different elements $(r_1, \ldots, r_k)$ and $(r'_1, \ldots, r'_k)$ in $\prod_{\iota=1}^{k} [\![a_\iota, n-k+\iota+1]\!]$ such that:

$$w_{r_k}^{n} \cdots w_{r_1}^{n-k+1} W_{J_k} = w_{r'_k}^{n} \cdots w_{r'_1}^{n-k+1} W_{J_k}.$$

Put $j_0 = \max\{j,\ 1 \leqslant j \leqslant k \mid r_j \neq r'_j\}$, without loss of generality we can even assume $r_{j_0} > r'_{j_0}$. Therefore, since $w_{r_{j_0-1}}^{n-k+j_0-1} \cdots w_{r_1}^{n-k+1} \in$

$W_{[\![1,n-k+j_0-1]\!]}$, by multiplying the formula above by $(w^n_{r_k}\cdots w^{n-k+j_0}_{r_{j_0}})^{-1}$ on the left and by $W_{[\![1,n-k+j_0-1]\!]}$ on the right, we get:

$$W_{[\![1,n-k+j_0-1]\!]} = (w^{n-k+j_0}_{r_{j_0}})^{-1} w^{n-k+j_0}_{r'_{j_0}} W_{[\![1,n-k+j_0-1]\!]},$$

hence $(w^{n-k+j_0}_{r_{j_0}})^{-1} w^{n-k+j_0}_{r'_{j_0}} \in W_{[\![1,n-k+j_0-1]\!]}$. As we assumed $r_{j_0} > r'_{j_0}$, it follows by (1.11):

$$(w^{n-k+j_0}_{r_{j_0}})^{-1} w^{n-k+j_0}_{r'_{j_0}} = w^{n-k+j_0-1}_{r'_{j_0}} s_{n-k+j_0} (w^{n-k+j_0-1}_{r_{j_0}-1})^{-1}.$$

As $w^{n-k+j_0-1}_{r'_{j_0}}, (w^{n-k+j_0-1}_{r_{j_0}-1})^{-1} \in W_{[\![1,n-k+j_0-1]\!]}$, this implies that s_{n-k+j_0} lies in $W_{[\![1,n-k+j_0-1]\!]}$, a contradiction. □

2. Harmonic cochains and special representations

Through all this section, we fix a commutative ring M and an M-module L. Assume that $\widetilde{G}$ acts trivially on M and that L is endowed with an M-linear $\widetilde{G}$-action.

2.1. Harmonic cochains

This paragraph concerns some technical lemmas which will be useful to prove the main theorem below (Theorem 3.3). Recall that $\widehat{\mathfrak{I}}^k$ denotes the set of pointed k-cells of the Bruhat-Tits building, see §1.1. From now on, we sometimes denote by σ a pointed cell (σ, v) when it is clear that it is pointed and which vertex is distinguished.

Let us recall the definition of harmonic cochains given by E. de Shalit ([11, def. 3.1]).

Definition 2.1. — *Let k be an integer such that $0 \leqslant k \leqslant n$. A k-harmonic cochain with values in the M-module L is a homomorphism $\mathfrak{h} \in \mathrm{Hom}_M(M[\widehat{\mathfrak{I}}^k], L)$ which satisfies the following conditions:*

(HC1) *If $\sigma = (v_0, v_1, \ldots, v_k) \in \widehat{\mathfrak{I}}^k$ is a k-pointed cell and if $\sigma' = (v_1, \ldots, v_k, v_0)$ is the same cell but pointed at v_1, see §1.1, then*

$$\mathfrak{h}(\sigma) = (-1)^k \mathfrak{h}(\sigma').$$

(HC2) *Fix a pointed $(k-1)$-cell $\eta \in \widehat{\mathfrak{I}}^{k-1}$, fix a type t of pointed k-cells, and consider the set $\mathcal{B}(\eta, t) = \{\sigma \in \widehat{\mathfrak{I}}^k;\ \eta < \sigma \text{ and } t(\sigma) = t\}$. Then*

$$\sum_{\sigma \in \mathcal{B}(\eta,t)} \mathfrak{h}(\sigma) = 0.$$

(HC3) *Let* $k \geqslant 1$. *Fix* $\sigma = (\Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_k \supsetneq \pi\Lambda_0) \in \widehat{\mathfrak{I}}^k$ *and fix an index* $0 \leqslant j \leqslant k$. *Let* $\mathcal{C}(\sigma, j)$ *be the collection of all* $\sigma' = (\Lambda'_0 \supsetneq \Lambda'_1 \supsetneq \cdots \supsetneq \Lambda'_k \supsetneq \pi\Lambda'_0) \in \widehat{\mathfrak{I}}^k$ *for which* $\Lambda'_i = \Lambda_i$ *if* $i \neq j$, $\Lambda_j \supseteq \Lambda'_j \supsetneq \Lambda_{j+1}$ *and* $\dim_\kappa \Lambda'_j/\Lambda_{j+1} = 1$. *Then*

$$\mathfrak{h}(\sigma) = \sum_{\sigma' \in \mathcal{C}(\sigma,j)} \mathfrak{h}(\sigma').$$

(HC4) *Let* $\sigma = (v_0, v_1, \ldots, v_{k+1}) \in \widehat{\mathfrak{I}}^{k+1}$. *Let* $\sigma_j = (v_0, \ldots, \hat{v}_j, \ldots, v_{k+1}) \in \widehat{\mathfrak{I}}^k$. *Then*

$$\sum_{j=0}^{k} (-1)^j \mathfrak{h}(\sigma_j) = 0.$$

For any k, $0 \leqslant k \leqslant n$, we denote by $\mathfrak{Harm}^k(M, L)$ the space of k-harmonic cochains with values in the M-module L. In case $k = 0$, the condition **(HC4)** shows that

$$\mathfrak{Harm}^0(M, L) \cong L. \tag{2.1}$$

The action of $\widetilde{G}$. The action of $\widetilde{G}$ on $\mathfrak{Harm}^k(M, L)$ is induced from its natural action on $\mathrm{Hom}_M(M[\widehat{\mathfrak{I}}^k], L)$, namely $(g.\mathfrak{h})(\sigma) = g.\mathfrak{h}(g^{-1}\sigma)$ for any $\mathfrak{h} \in \mathfrak{Harm}^k(M, L)$, any $g \in \widetilde{G}$, and any $\sigma \in \widehat{\mathfrak{I}}^k$.

To shorten notation, for $I \subseteq \Delta$, $r' \in \Delta$ and $r \in I \cup \{r'\}$, set $I_r^{r'} = (I \cup \{r'\}) - \{r\}$.

Lemma 2.2. — *Let* $I \subseteq \Delta$ *such that* $\Delta - I = \{i_1 < \cdots < i_k\}$ *and let* j *such that* $1 \leqslant j \leqslant k$. *Let* $\mathfrak{h} \in \mathrm{Hom}_M(M[\widehat{\mathfrak{I}}^k], L)$ *satisfy the condition* **(HC3)**. *Then:*

1. *If* l *is such that* $j + 1 \leqslant l \leqslant k$, *then for each* $\sigma \in \mathfrak{C}(\sigma_{I \cup \{i_j\}}, l - 1)$ *there is* $g_l^\sigma \in \widetilde{G}$, *so that we have the following:*

$$\sum_{\sigma \in \mathcal{B}(\sigma_{I \cup \{i_j\}}, t_I)} \mathfrak{h}(\sigma) = \sum_{\sigma \in \mathcal{C}(\sigma_{I \cup \{i_j\}}, l-1)} \left(\sum_{\sigma' \in \mathcal{B}(\sigma_{I^{i_l}_{i_{l+1}-1} \cup \{i_j\}}, t_{I^{i_l}_{i_{l+1}-1}})} \mathfrak{h}(g_l^\sigma . \sigma') \right).$$

2. *For* $j = l$, *there is an integer* m *such that*

$$\sum_{\sigma \in \mathcal{B}(\sigma_{I \cup \{i_j\}}, t_I)} \mathfrak{h}(\sigma) = m \left(\sum_{\sigma' \in \mathcal{B}(\sigma_{t_{I^{i_j}_{i_{j+1}-1}} \cup \{i_{j+1}-1\}}, t_{I^{i_j}_{i_{j+1}-1}})} \mathfrak{h}(\sigma') \right).$$

Proof. — First, let l be such that $j \leqslant l \leqslant k$. Since $\mathfrak{h}$ satisfies the condition **(HC3)**, it follows

$$\sum_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \mathfrak{h}(\sigma) = \sum_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \sum_{\sigma'\in\mathcal{C}(\sigma,l)} \mathfrak{h}(\sigma'). \tag{2.2}$$

Now, let us prove the two assertions of the corollary:

1. Assume that $j+1 \leqslant l \leqslant k$. It is not difficult to show that we have:

$$\coprod_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \mathcal{C}(\sigma,l) = \coprod_{\sigma\in\mathcal{C}(\sigma_{I\cup\{i_j\}},l-1)} \mathcal{B}(\sigma, t_{I^{i_l}_{i_{l+1}-1}}). \tag{2.3}$$

Combining this with (2.2) we get

$$\sum_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \mathfrak{h}(\sigma) = \sum_{\sigma\in\mathcal{C}(\sigma_{I\cup\{i_j\}},l-1)} \sum_{\sigma'\in\mathcal{B}(\sigma,t_{I^{i_l}_{i_{l+1}-1}})} \mathfrak{h}(\sigma'). \tag{2.4}$$

The action of $\widetilde{G}$ being transitive on the set of pointed cells of a given type, for each $\sigma\in\mathcal{C}(\sigma_{I\cup\{i_j\}}, l-1)$ there is $g_l^\sigma\in\widetilde{G}$ so that $\sigma = g_l^\sigma(\sigma_{I^{i_l}_{i_{l+1}-1}\cup\{i_j\}}, v_0^o)$, which implies $\mathcal{B}(\sigma, t_{I^{i_l}_{i_{l+1}-1}}) = g_l^\sigma.\mathcal{B}(\sigma_{I^{i_l}_{i_{l+1}-1}\cup\{i_j\}}, t_{I^{i_l}_{i_{l+1}-1}})$. Consequently (2.4) can be written as follows

$$\sum_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \mathfrak{h}(\sigma) = \sum_{\sigma\in\mathcal{C}(\sigma_{I\cup\{i_j\}},l-1)} \left(\sum_{\sigma'\in\mathcal{B}(\sigma_{I^{i_l}_{i_{l+1}-1}\cup\{i_j\}},t_{I^{i_l}_{i_{l+1}-1}})} \mathfrak{h}(g_l^\sigma.\sigma') \right).$$

2. Assume that $l=j$. This assertion being trivial if $i_{j+1} = i_j + 1$ we can suppose $i_{j+1} - i_j \geqslant 2$. It is not difficult to show that we have the following equality by using the definition of the different sets involved:

$$\bigcup_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \mathcal{C}(\sigma,j) = \mathcal{B}(\sigma_{I\cup\{i_j\}}, t_{I^{i_j}_{i_{j+1}-1}}). \tag{2.5}$$

Note that for each $\sigma' \in \mathcal{B}(\sigma_{I\cup\{i_j\}}, t_{I^{i_j}_{i_{j+1}-1}})$, there is exactly m distinct cells $\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}}, t_I)$ so that $\sigma'\in\mathcal{C}(\sigma,j)$. In total, with (2.2), we get

$$\sum_{\sigma\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_I)} \mathfrak{h}(\sigma) = m \left(\sum_{\sigma'\in\mathcal{B}(\sigma_{I\cup\{i_j\}},t_{I^{i_j}_{i_{j+1}-1}})} \mathfrak{h}(\sigma') \right).$$

Now, to complete the proof notice that we have the obvious equality $I\cup\{i_j\} = I^{i_j}_{i_{j+1}-1}\cup\{i_{j+1}-1\}$. □

Lemma 2.3. — *Let $I \subseteq \Delta$ with $\Delta - I = \{i_1 < \cdots < i_k\}$ and let j be an integer such that $1 \leqslant j \leqslant k$. Then*

1. $\mathcal{B}(\sigma_{I\cup\{i_j\}}, t_I) = B_{I\cup\{i_j\}}.(\sigma_I, v_0^o)$.
2. $\mathcal{C}(\sigma_I, j) = B_I.(\sigma_{I^{i_j}_{i_{j+1}-1}}, v_0^o)$.

Proof. — In both equalities, we prove that the left hand side set is contained in the right hand side set. The inverse inclusions are obvious.

1. Let $\sigma \in \mathcal{B}(\sigma_{I\cup\{i_j\}}, t_I)$. Since $t(\sigma) = t_I$ and since the action of $\widetilde{G}$ is transitive on the pointed cells of a given type, there exists $b \in \widetilde{G}$ such that $\sigma = b.(\sigma_I, v_0^o)$. Therefore, $(\sigma_{I\cup\{i_j\}}, v_0^o)$ and $b(\sigma_{I\cup\{i_j\}}, v_0^o)$ are pointed faces of the same cell σ. Being also of the same type, we necessarily have $(\sigma_{I\cup\{i_j\}}, v_0^o) = b.(\sigma_{I\cup\{i_j\}}, v_0^o)$. Hence $b \in B_{I\cup\{i_j\}}$.

2. Let $\sigma \in \mathcal{C}(\sigma_I, j)$, then, $\sigma = (v_0^o,\ldots, v_{i_{j-1}}^o, v_{i_{j+1}-1}, v_{i_{j+1}}^o,\ldots, v_{i_k}^o)$ is the pointed cell obtained from (σ_I, v_0^o) by replacing the vertex $v_{i_j}^o = [\Lambda_{i_j}^0]$ by another vertex $v_{i_{j+1}-1} = [\Lambda_{i_{j+1}-1}]$ with $\Lambda_{i_{j+1}-1} \subsetneq \Lambda_{i_j}^0$ and $\dim_\kappa(\Lambda_{i_{j+1}-1}/\Lambda_{i_{j+1}}^0)$ $= 1$. We have:

$$t(\sigma) = t_{I^{i_j}_{i_{j+1}-1}}.$$

On the other hand, the pointed $(k+1)$-cell

$$\sigma' = (v_0^o, \ldots, v_{i_{j-1}}^o, v_{i_j}^o, v_{i_{j+1}-1}, v_{i_{j+1}}^o, \ldots, v_{i_k}^o)$$

lies in $\mathcal{B}(\sigma_I, \mathrm{t}_{I\setminus\{i_{j+1}-1\}})$, thus, by (1) above, there exists $b \in B_I$ such that $\sigma' = b.(\sigma_{I\setminus\{i_{j+1}-1\}}, v_0^o)$. Acting b^{-1} on the following obvious relation:

$$(\sigma_{I\cup\{i_j\}}, v_0^o) < \sigma < \sigma' = b.(\sigma_{I\setminus\{i_{j+1}-1\}}, v_0^o)$$

we obtain

$$(\sigma_{I\cup\{i_j\}}, v_0^o) < b^{-1}\sigma < (\sigma_{I\setminus\{i_{j+1}-1\}}, v_0^o).$$

Since $t(b^{-1}\sigma) = t(\sigma) = t_{I^{i_j}_{i_{j+1}-1}}$, this clearly forces $b^{-1}\sigma = (\sigma_{I^{i_j}_{i_{j+1}-1}}, v_0^o)$. □

2.2. Special representations

Let X be a locally compact space. We denote by $C^\infty(X, M)$ (resp. $C_c^\infty(X, M)$) the set of locally constant functions on X with values in M (resp. those which, moreover, are compactly supported). Notice that if X is compact then we have:

$$C_c^\infty(X, M) = C^\infty(X, M).$$

The sets $C^\infty(X,M)$ and $C_c^\infty(X,M)$ are naturally endowed with M-module structures. Recall, *cf.* [3, lemma 4], that if X is locally compact, metrizable and totally discontinuous space then:

$$C_c^\infty(X,M) = C_c^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} M. \tag{2.6}$$

The group $\widetilde{G}$ is a locally compact topological group with topological structure induced from the topology of the non-archimedean field K. We know, that for any $I \subseteq \Delta$, the homogeneous space $\widetilde{G}/\widetilde{P}_I$ is compact with respect to the quotient topology.

The action of $\widetilde{G}$. For any $I \subseteq \Delta$, the action of $\widetilde{G}$ on $C^\infty(\widetilde{G}/\widetilde{P}_I, M)$ and $C_c^\infty(\widetilde{G}/B_I, M)$ is induced by its action by left translations on respectively $\widetilde{G}/\widetilde{P}_I$ and $\widetilde{G}/B_I$.

Let $I \subseteq \Delta$. For any subset H of $\widetilde{G}$, we denote by $\chi_{H\widetilde{P}_I} \in C^\infty(\widetilde{G}/\widetilde{P}_I, M)$ (resp. $\chi_{HB_I} \in C_c^\infty(\widetilde{G}/B_I, M)$) the characteristic function of $H\widetilde{P}_I/\widetilde{P}_I$ (resp. HB_I/B_I).

Proposition 2.4. — *(P. Schneider and U. Stuhler) The $M[\widetilde{G}]$-module $C^\infty(\widetilde{G}/\widetilde{P}_I, M)$ is generated by the characteristic function $\chi_{B_I\widetilde{P}_I}$.*

Proof. — See [9, §4, prop. 8' and cor. 9'] and use (2.6) above. □

Remark 2.5. — For any $I_1 \subseteq I_2 \subseteq \Delta$, we have natural commutative diagrams of $M[\widetilde{G}]$-monomorphisms

$$\begin{array}{ccc} & C^\infty(\widetilde{G}/\widetilde{P}, M) & \\ \nearrow & & \nwarrow \\ C^\infty(\widetilde{G}/\widetilde{P}_{I_2}, M) & \rightarrow & C^\infty(\widetilde{G}/\widetilde{P}_{I_1}, M) \end{array} \quad \text{and} \quad \begin{array}{ccc} & C_c^\infty(\widetilde{G}/B, M) & \\ \nearrow & & \nwarrow \\ C_c^\infty(\widetilde{G}/B_{I_2}, M) & \rightarrow & C_c^\infty(\widetilde{G}/B_{I_1}, M). \end{array}$$

Definition 2.6. — *Let k be an integer with $0 \leqslant k \leqslant n$ and let J_k be the subset $[\![1, n-k]\!]$ of Δ. A k-special representation of $\widetilde{G}$ is the $M[\widetilde{G}]$-module:*

$$\mathrm{Sp}^k(M) = \frac{C^\infty(\widetilde{G}/\widetilde{P}_{J_k}, M)}{\sum_{j=n-k+1}^n C^\infty(\widetilde{G}/\widetilde{P}_{J_k\cup\{j\}}, M)}.$$

In case $k = n$, this is the ordinary Steinberg representation. Notice also that, in case $k = 0$, this is the trivial representation:

$$\mathrm{Sp}^0(M) \cong M. \tag{2.7}$$

Relation to the parahoric groups. In order to interpret the special representation $\mathrm{Sp}^k(M)$ in terms of parahoric subgroups, we recall, following [9, §4], that we have a surjective map:

$$H: \quad C_c^\infty(\widetilde{G}/B, M) \longrightarrow C^\infty(\widetilde{G}/\widetilde{P}, M)$$

defined by $H(\varphi) = \sum_{g \in \widetilde{G}/B} \varphi(g) g.\chi_{B\widetilde{P}}$. Recall also that this map induces, for any $I \subseteq \Delta$, a surjective map:

$$H_I : \ C_c^{\infty}(\widetilde{G}/B_I, M) \ \longrightarrow \ C^{\infty}(\widetilde{G}/\widetilde{P}_I, M)$$

whose kernel is the $M[\widetilde{G}]$-submodule of $C_c^{\infty}(\widetilde{G}/B_I, M)$ generated by the functions $\chi_{By_iB_I} - \chi_{B_I}$, $0 \leqslant i \leqslant n$, and where y_i is the diagonal matrix introduced in lemma 1.1. This leads to the following proposition:

Proposition 2.7. — *For any k, $0 \leqslant k \leqslant n$, we have a canonical isomorphism of $M[\widetilde{G}]$-modules*

$$H_{J_k} : \frac{C_c^{\infty}(\widetilde{G}/B_{J_k}, M)}{\mathfrak{R}_{J_k}} \cong \mathrm{Sp}^k(M),$$

where $\mathfrak{R}_{J_k}$ is the $M[\widetilde{G}]$-submodule of $C_c^{\infty}(\widetilde{G}/B_{J_k}, M)$ generated by the functions $\chi_{B_{J_k}s_jB_{J_k}} + \chi_{B_{J_k}}$, $n-k+1 \leqslant j \leqslant n$, and the functions $\chi_{By_iB_{J_k}} - \chi_{B_{J_k}}$, $0 \leqslant i \leqslant n$.

2.3. Harmonic cochains and special representations

2.3.1. Definition of new sets C_I of $\widetilde{G}$.

For each $I \subseteq \Delta$, for each $r'_1, \ldots, r'_m \in \Delta$ and each $r_1, \ldots, r_m \in I \cup \{r'_1, \ldots, r'_m\}$, we set:

$$I_{r_1,\ldots,r_m}^{r'_1,\ldots,r'_m} = (I \cup \{r'_1, \ldots, r'_m\}) - \{r_1, \ldots, r_m\}.$$

Let us fix an integer k such that $1 \leqslant k \leqslant n$ and denote again J_k the subset $[\![1, n-k]\!]$ of Δ.

Let $I \subseteq \Delta$ be such that $\Delta - I = \{i_1 < \cdots < i_k\}$. For every $m = 1, \ldots, k$, we necessarily have $i_m \leqslant n-k+m$, therefore the integers $i_1, \ldots, i_m$ lie in the subset $J_k \cup \{n-k+1, \ldots, n-k+m\} = [\![1, n-k+m]\!]$ of Δ. Hence, for any $m = 1, \ldots, k$, if we put $i_0 = 0$, one can see easily that we have:

$$(2.8) \quad {J_k}_{i_1, \ldots, i_m}^{n-k+1, \ldots, n-k+m} = \left(\coprod_{\iota=1}^{m} [\![i_{\iota-1}+1, i_\iota - 1]\!] \right) \amalg [\![i_m + 1, n-k+m]\!].$$

Moreover, if we put $m = k$ in this formula, we can see that ${J_k}_{i_1, \ldots, i_k}^{n-k+1, \ldots, n} = I$.

Now, if $I \subseteq \Delta$ is such that $\Delta - I = \{i_1 < \cdots < i_k\}$, we write:

(2.9)

$$\overset{\circ}{C}_I = \overset{\circ}{B}_{{J_k}_{i_1, \ldots, i_k}^{n-k+1, \ldots, n}} \cdots \overset{\circ}{B}_{{J_k}_{i_1}^{n-k+1}} \overset{\circ}{B}_{J_k} \text{ and } C_I = B_{{J_k}_{i_1, \ldots, i_k}^{n-k+1, \ldots, n}} \cdots B_{{J_k}_{i_1}^{n-k+1}} B_{J_k}.$$

The set $\overset{\circ}{C_I}$ is compact open in $\widetilde{G}$ and we clearly have $C_I = \overset{\circ}{C_I} K^*$, see §1.2.1. Hence, the set $C_I \widetilde{P}_{J_k}/\widetilde{P}_{J_k} = \overset{\circ}{C_I} \widetilde{P}_{J_k}/\widetilde{P}_{J_k}$ is compact open in the homogeneous space $\widetilde{G}/\widetilde{P}_{J_k}$.

Theorem 2.8. — *If for each $I \subseteq \Delta$ such that $\Delta - I = \{i_1 < \cdots < i_k\}$, we define $\mathfrak{C}_I = \prod_{\iota=1}^{k} [\![i_\iota + 1, n-k+\iota+1]\!]$, then we have the following decompositions:*

$$\chi_{C_I} = \sum_{\underline{r}\in\mathfrak{C}_I} \chi_{Bw^n_{r_k}\cdots w^{n-k+1}_{r_1} B_{J_k}} \quad \text{and} \quad \chi_{C_I \widetilde{P}_{J_k}} = \sum_{\underline{r}\in\mathfrak{C}_I} \chi_{Bw^n_{r_k}\cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}},$$

where $\underline{r}$ denote the k-tuple $(r_1, \ldots, r_k)$.

Proof. — For any $m = 1, \ldots, k$, the expression (2.8) above shows that $J_k{}^{n-k+1,\ldots,n-k+m}_{\ i_1\ \ ,\ldots,\ \ i_m}$ decomposes as a union of intervals which satisfy pairwise the hypothesis of the assertion (1.8) given in Remark 1.8. Thus, an easy induction on m by using assertion (1.8) proves that we have:

$$C_I = B_{[\![i_k+1,n]\!]} \cdots B_{[\![i_1+1,n-k+1]\!]} B_{J_k}. \tag{2.10}$$

Next, since we have $B_{J_k}\widetilde{P}_{J_k} = B\widetilde{P}_{J_k}$ which is given by the assertion (1.9) of the same remark, we conclude from (2.10) that we also have:

$$C_I \widetilde{P}_{J_k} = B_{[\![i_k+1,n]\!]} \cdots B_{[\![i_1+1,n-k+1]\!]} \widetilde{P}_{J_k}. \tag{2.11}$$

Finally, by Proposition 1.10, we deduce from (2.10) and (2.11) the following respective decompositions

$$C_I = \coprod_{\underline{r}\in\mathfrak{C}_I} Bw^n_{r_k}\cdots w^{n-k+1}_{r_1} B_{J_k} \quad \text{and} \quad C_I \widetilde{P}_{J_k} = \coprod_{\underline{r}\in\mathfrak{C}_I} Bw^n_{r_k}\cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k},$$

and the theorem follows. □

For convenience, we will call $\mathfrak{C}_I$ the index set associated to the decomposition of C_I.

2.3.2. About vanishing in $\mathrm{S}p^k(M)$.

The propositions below give a method which allows us to know whether certain elements vanish in $\mathrm{S}p^k(M)$.

Proposition 2.9. — *Let $w \in \overline{W}$ and $w' \in W_{[\![n-k+2,n]\!]}$ (In case $k = n$, consider $w' \in \overline{W}$). We have:*

$$\chi_{BwB_{J_k}} - (-1)^{l(w')} \chi_{Bww'B_{J_k}} \in \sum_{j=n-k+1}^{n} C_c^\infty(\widetilde{G}/B_{J_k\cup\{j\}}, M)$$

and

$$\chi_{Bw\widetilde{P}_{J_k}} - (-1)^{l(w')}\chi_{Bww'\widetilde{P}_{J_k}} \in \sum_{j=n-k+1}^{n} C^\infty(\widetilde{G}/\widetilde{P}_{J_k\cup\{j\}}, M).$$

Proof. — We prove the second assertion. Let $u_1, \dots, u_d \in \overline{S}$ such that $w' = u_1 \cdots u_d$ is a reduced expression ($d = l(w')$). We have:

$$\chi_{Bw\widetilde{P}_{J_k}} - (-1)^d \chi_{Bww'\widetilde{P}_{J_k}} = \sum_{r=1}^{d}(-1)^{r-1}(\chi_{Bwu_1\cdots u_{r-1}\widetilde{P}_{J_k}} + \chi_{Bwu_1\cdots u_r\widetilde{P}_{J_k}}).$$

The expression $w' = u_1 \cdots u_d$ being reduced and since $w' \in W_{[\![n-k+2,n]\!]}$, we deduce that for each $r = 1, \dots, d$, there is an integer j such that $n-k+2 \leqslant j \leqslant n$ (in case $k = n$, j is such that $1 \leqslant j \leqslant n$) and $u_r = s_j$. Thus, by Proposition 1.9, we have:

$$\chi_{Bwu_1\cdots u_{r-1}\widetilde{P}_{J_k}} + \chi_{Bwu_1\cdots u_r\widetilde{P}_{J_k}} = \chi_{Bwu_1\cdots u_{r-1}\widetilde{P}_{J_k\cup\{j\}}} \in C^\infty(\widetilde{G}/\widetilde{P}_{J_k\cup\{j\}}, M).$$

The proof of the first assertion is similar. □

PROPOSITION 2.10. — *Let $w, w' \in \overline{W}$ and let a, b be two integers such that $1 \leqslant a \leqslant b \leqslant n$. Suppose furthermore that we have $s_b w' = w' s_{b'}$, where b' is an integer such that $n-k+2 \leqslant b' \leqslant n$. Then*

$$\sum_{(r_1,r_2)} \chi_{Bww^b_{r_2}w^{b-1}_{r_1}w'B_{J_k}} = \sum_{l=0}^{b-a}\sum_{r=a}^{b-l} \chi_{Bww^b_r w^{b-1}_{b-l}w'B_{J_k\cup\{b'\}}} \in C_c^\infty(\widetilde{G}/B_{J_k\cup\{b'\}}, M),$$

and also

$$\sum_{(r_1,r_2)} \chi_{Bww^b_{r_2}w^{b-1}_{r_1}w'\widetilde{P}_{J_k}} = \sum_{l=0}^{b-a}\sum_{r=a}^{b-l} \chi_{Bww^b_r w^{b-1}_{b-l}w'\widetilde{P}_{J_k\cup\{b'\}}} \in C^\infty(\widetilde{G}/\widetilde{P}_{J_k\cup\{b'\}}, M)$$

where the pair (r_1, r_2) runs through the set $[\![a,b]\!] \times [\![a,b+1]\!]$.

Proof. — First, if we put $b = a + m$, by induction on $m \geqslant 0$ we prove easily that the set $[\![a,b]\!] \times [\![a,b+1]\!]$ decomposes into a disjoint union as follows:

$$[\![a,b]\!] \times [\![a,b+1]\!] = \coprod_{l=0}^{b-a} \left([\![a,b-l]\!] \times \{b-l+1\} \amalg \{b-l\} \times [\![a,b-l]\!]\right).$$

Now, let us prove the second formula of the proposition. Since (r_1, r_2) runs through $[\![a,b]\!] \times [\![a,b+1]\!]$, from the decomposition above we get:

$$\sum_{(r_1,r_2)} \chi_{Bww^b_{r_2} w^{b-1}_{r_1} w' \widetilde{P}_{J_k}} = \sum_{l=0}^{b-a} \Big(\sum_{r_1=a}^{b-l} \chi_{Bww^b_{b-l+1} w^{b-1}_{r_1} w' \widetilde{P}_{J_k}} + \sum_{r_2=a}^{b-l} \chi_{Bww^b_{r_2} w^{b-1}_{b-l} w' \widetilde{P}_{J_k}} \Big).$$

On the other hand, by using the formula (1.11) together with the hypothesis of the proposition, we obtain:

$$w^b_{b-l+1} w^{b-1}_{r_1} w' = w^b_{b-l+1} w^b_{r_1} s_b w' = w^b_{r_1} w^{b-1}_{b-l} s_b w' = w^b_{r_1} w^{b-1}_{b-l} w' s_{b'}.$$

By replacing in the first sum the right hand side of the equality above, we obtain:

$$\sum_{(r_1,r_2)} \chi_{Bww^b_{r_2} w^{b-1}_{r_1} w' \widetilde{P}_{J_k}} = \sum_{l=0}^{b-a} \sum_{r=a}^{b-l} (\chi_{Bww^b_r w^{b-1}_{b-l} w' s_{b'} \widetilde{P}_{J_k}} + \chi_{Bww^b_r w^{b-1}_{b-l} w' \widetilde{P}_{J_k}}).$$

Finally, since $n-k+2 \leqslant b' \leqslant n$, by Proposition 1.9, we have

$$\chi_{Bww^b_r w^{b-1}_{b-l} w' s_{b'} \widetilde{P}_{J_k}} + \chi_{Bww^b_r w^{b-1}_{b-l} w' \widetilde{P}_{J_k}} = \chi_{Bww^b_r w^{b-1}_{b-l} w' \widetilde{P}_{J_k \cup \{b'\}}}.$$

The proof of the first formula is similar. □

2.3.3. Harmonicity in $\mathrm{S}\mathbf{p}^{\mathbf{k}}(\mathbf{M})$

The following proposition and its corollary below show that the characteristic functions χ_{C_I} and $\chi_{C_I \widetilde{P}_{J_k}}$ have properties that are somehow similar to those of harmonic cochains. First, we will need the following technical lemma:

Lemma 2.11. — *Let i be an integer such that $0 \leqslant i \leqslant n$. Let $r_1, \cdots, r_k$ be integers such that for each $j = 1, \cdots, k$, we have $1 \leqslant r_j \leqslant n-k+j+1-i$. Then*

$$w_i w^n_{r_k} \cdots w^{n-k+1}_{r_1} = w^n_{r_k+i} \cdots w^{n-k+1}_{r_1+i} w^{n-k}_i \cdots w^{n-i+1-k}_1, \tag{2.12}$$

where $w_i = w^n_i w^{n-1}_{i-1} \cdots w^{n-i+1}_1$ (cf. lemma 1.1).

Proof. — Let us first prove that, if a, a', b, b' are integers satisfying $1 \leqslant a \leqslant a' \leqslant b \leqslant b' \leqslant n$, we have the equality:

$$w^b_a w^{b'}_{a'} = w^{b'}_{a'+1} w^{b-1}_a. \tag{2.13}$$

Indeed, we can write $w_a^b = w_a^{a'} w_{a'+1}^b$. On the other hand, by using the formula (1.11), we get $w_{a'+1}^b w_{a'}^{b'} = w_{a'}^{b'} w_{a'}^{b-1}$. Thus, the left hand side of the equality (2.13) can be written as follows:

$$w_a^b w_{a'}^{b'} = w_a^{a'} w_{a'}^{b'} w_{a'}^{b-1}. \tag{2.14}$$

Since we have $w_a^{a'} w_{a'}^{b'} = w_a^{a'-1} w_{a'+1}^{b'} = w_{a'+1}^{b'} w_a^{a'-1}$ and $w_a^{a'-1} w_{a'}^{b-1} = w_a^{b-1}$, substituting these into (2.14) we get (2.13).

Now, in order to establish the formula (2.12) in the lemma, apply the identity (2.13) with $w_a^b = w_1^{n-i+1}$ and $w_{a'}^{b'} = w_{r_j}^{n-k+j}$ for each $j = k, \dots, 1$, and in that order. Next, proceed in the same way with $w_a^b = w_\iota^{n-i+\iota}$ for $\iota = 2, \dots, i$ respectively. □

Proposition 2.12. — *Let $I \subseteq \Delta$ with $\Delta - I = \{i_1 < \cdots < i_k\}$. Let i_{k+1} be such that $i_k \lneq i_{k+1} \leqslant n+1$ and let $\widehat{I}_1 \subseteq \Delta$ be such that*

$$\Delta - \widehat{I}_1 = \{i_2 - i_1 < \cdots < i_k - i_1 < i_{k+1} - i_1\}.$$

In $\mathrm{Sp}^k(M)$, we have the equalities (the first equality is seen in $\mathrm{Sp}^k(M)$ through the isomorphism H_{J_k} given by Proposition 2.7):

1. $$\chi_{C_I} = \sum_{\underline{r} \in \mathfrak{C}_I^0} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} B_{J_k}} + \sum_{t=1}^{k-1} (-1)^{k-t-1} \sum_{\underline{r} \in \mathfrak{C}_I^{t,k-t-1}} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} B_{J_k}} + \sum_{\underline{r} \in \mathfrak{C}_I^k} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} B_{J_k}}.$$
2. $$\chi_{C_I \widetilde{P}_{J_k}} = \sum_{\underline{r} \in \mathfrak{C}_I^0} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} \widetilde{P}_{J_k}} + \sum_{t=1}^{k-1} (-1)^{k-t-1} \sum_{\underline{r} \in \mathfrak{C}_I^{t,k-t-1}} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} \widetilde{P}_{J_k}} + \sum_{\underline{r} \in \mathfrak{C}_I^k} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} \widetilde{P}_{J_k}}.$$
3. $$y_{i_1} w_{i_1} \chi_{C_{\widehat{I}_1} \widetilde{P}_{J_k}} = (-1)^k \sum_{\underline{r} \in \mathfrak{C}_I^0} \chi_{B w_{r_k}^n \cdots w_{r_1}^{n-k+1} \widetilde{P}_{J_k}}$$

where we have set $\mathfrak{C}_I^0 = \prod_{\iota=1}^{k} [\![i_\iota + 1, i_{\iota+1}]\!]$, $\mathfrak{C}_I^k = \left(\prod_{\iota=1}^{k-1} [\![i_\iota + 1, n-k+\iota+1]\!] \right) \times [\![i_{k+1} + 1, n+1]\!]$, *and for each t such that $1 \leqslant t \leqslant k-1$ we have set*

$$\mathfrak{C}_I^{t,k-t-1} = \left(\prod_{\iota=1}^{t-1} [\![i_\iota + 1, n-k+\iota+1]\!] \right) \times \left(\prod_{\iota=t}^{k-1} [\![i_{\iota+1} + 1, n-k+\iota+1]\!] \right) \times [\![i_k + 1, i_{k+1}]\!].$$

Proof. — The proofs of (1) and (2) are similar, we will prove (1) and (3).
1. In fact, it's enough to prove that the equality holds in $C_c^\infty(\widetilde{G}/B_{J_k}, M)$ modulo the $M[\widetilde{G}]$-submodule $\sum_{j=n-k+1}^{n} C_c^\infty(\widetilde{G}/B_{J_k\cup\{j\}}, M)$. Recall that for any $I \subseteq \Delta$, to the subset C_I of $\widetilde{G}$ we have associated a set $\mathfrak{C}_I$ so that we have the following decomposition (*cf.* Theorem 2.8):

$$\chi_{C_I} = \sum_{\underline{r}\in\mathfrak{C}_I} \chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}}.$$

On the other hand, if $\mathfrak{C}_I^0$ and $\mathfrak{C}_I^k$ are as in the statement of this proposition and if for each $t = 1, \ldots, k-1$, we put

$$\mathfrak{C}_I^t = \left(\prod_{\iota=1}^{t-1}[\![i_\iota+1, n-k+\iota+1]\!]\right) \times [\![i_{t+1}+1, n-k+t+1]\!] \times \left(\prod_{\iota=t+1}^{k}[\![i_\iota+1, i_{\iota+1}]\!]\right),$$

then it is not difficult to show that $\mathfrak{C}_I$ is a disjoint union of the $\mathfrak{C}_I^t$, $0 \leqslant t \leqslant k$. Hence:

$$\chi_{C_I} = \sum_{t=0}^{k}\sum_{\underline{r}\in\mathfrak{C}_I^t} \chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}}. \tag{2.15}$$

Now, let t be such that $1 \leqslant t \leqslant k-1$. For each $t' = 0, \ldots, k-t-2$, put:

$$\mathfrak{C}_I^{t,t'} = \left(\prod_{\iota=1}^{t-1}[\![i_\iota+1, n-k+\iota+1]\!]\right) \times \left(\prod_{\iota=t}^{t+t'}[\![i_{\iota+1}+1, n-k+\iota+1]\!]\right) \times [\![i_{t+t'+1}+1, n-k+t+t'+2]\!] \times \left(\prod_{\iota=t+t'+2}^{k}[\![i_\iota+1, i_{\iota+1}]\!]\right),$$

and for $t' = n-k-1$, let $\mathfrak{C}_I^{t,n-k-1}$ as in the statement of the proposition. Also, for each $t' = 0, \ldots, k-t-1$, put:

$$\mathfrak{D}_I^{t,t'} = \left(\prod_{\iota=1}^{t-1}[\![i_\iota+1, n-k+\iota+1]\!]\right) \times \left(\prod_{\iota=t}^{t+t'}[\![i_{\iota+1}+1, n-k+\iota+1]\!]\right) \times \left(\prod_{\iota=t+t'+1}^{k}[\![i_\iota+1, i_{\iota+1}]\!]\right),$$

and for $t' = k-t$, let $\mathfrak{D}_I^{t,k-t} = \emptyset$. Notice that the interval which corresponds to $\iota = t+t'+1$ in $\mathfrak{C}_I^{t,t'}$ can be decomposed as follows:

$$\llbracket i_{t+t'+1}+1, n-k+t+t'+2\rrbracket = \llbracket i_{t+t'+1}+1, i_{t+t'+2}\rrbracket \amalg \llbracket i_{t+t'+2}+1, n-k+t+t'+2\rrbracket,$$

therefore, for each t', $0 \leqslant t' \leqslant k-t-1$, we have $\mathfrak{C}_I^{t,t'} = \mathfrak{D}_I^{t,t'} \amalg \mathfrak{D}_I^{t,t'+1}$. Consider the alternating sum over t' as follows:

$$\sum_{t'=0}^{k-t-1}(-1)^{t'}\sum_{\underline{r}\in\mathfrak{C}_I^{t,t'}}\chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}} = \sum_{t'=0}^{k-t-1}(-1)^{t'}\sum_{\underline{r}\in\mathfrak{D}_I^{t,t'}\amalg\mathfrak{D}_I^{t,t'+1}}\chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}}.$$

In the right hand side of this equality, we see that all the sums over the $\mathfrak{D}_I^{t,t'}$, $t' = 1,\ldots,k-t-1$, cancel each other. What remains is the sum over $\mathfrak{D}_I^{t,0} = \mathfrak{C}_I^t$ and the sum over $\mathfrak{D}_I^{t,k-t} = \emptyset$. Therefore we get:

$$\sum_{\underline{r}\in\mathfrak{C}_I^t}\chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}} = \sum_{t'=0}^{k-t-1}(-1)^{t'}\sum_{\underline{r}\in\mathfrak{C}_I^{t,t'}}\chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}}. \tag{2.16}$$

Notice that in $\mathfrak{C}_I^{t,t'}$, for each $t' = 0,\ldots k-t-2$, the two intervals which correspond to the indices $t+t'$ and $t+t'+1$ are of the form of those in Proposition 2.10 with $a = i_{t+t'+1}+1$ and $b = n-k+t+t'+1$, and if we put $w' = w_{r_{t+t'-1}}^{n-k+t+t'-1}\cdots w_{r_1}^{n-k+1}$, it is clear that we have $s_bw' = w's_b$. Therefore, in $\mathrm{Sp}^k(M)$, we have (for each $t = 1,\ldots,k-1$):

$$\sum_{\underline{r}\in\mathfrak{C}_I^t}\chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}} = (-1)^{k-t-1}\sum_{\underline{r}\in\mathfrak{C}_I^{t,k-t-1}}\chi_{Bw_{r_k}^n\cdots w_{r_1}^{n-k+1}B_{J_k}}.$$

Finally, substituting this into (2.15) establishes the formula.

3. Set $\mathfrak{C}_{\widehat{I_1}}^0 = \prod_{\iota=1}^k\llbracket i_\iota - i_1+1, i_{\iota+1}-i_1\rrbracket$, and for each t such that $1 \leqslant t \leqslant k$,

$$\mathfrak{C}_{\widehat{I_1}}^t = \left(\prod_{\iota=1}^{t-1}\llbracket i_{\iota+1}-i_1+1, n-k+\iota+1\rrbracket\right)\times\llbracket i_t-i_1+1, n-k+t+1\rrbracket \times\left(\prod_{\iota=t+1}^{k}\llbracket i_\iota-i_1+1, i_{\iota+1}-i_1\rrbracket\right)$$

and

$$\mathfrak{D}^t_{\widehat{I_1}} = \prod_{\iota=1}^{t-1} [\![i_{\iota+1} - i_1 + 1, n-k+\iota+1]\!] \times \prod_{\iota=t}^{k} [\![i_\iota - i_1 + 1, i_{\iota+1} - i_1]\!] .$$

We proceed as in the proof of the formula (2.16) above. Notice that for each $t = 1, \ldots, k$, we have $\mathfrak{C}^t_{\widehat{I_1}} = \mathfrak{D}^t_{\widehat{I_1}} \amalg \mathfrak{D}^{t+1}_{\widehat{I_1}}$. Therefore by considering the following alternating sum:

$$\sum_{t=1}^{k} (-1)^{k-t} \sum_{\underline{r} \in \mathfrak{C}^t_{\widehat{I_1}}} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}},$$

all the sums over the $\mathfrak{D}^t_{\widehat{I_1}}$ cancel each other, except the sum over $\mathfrak{D}^1_{\widehat{I_1}} = \mathfrak{C}^0_{\widehat{I_1}}$ and the sum over $\mathfrak{D}^{k+1}_{\widehat{I_1}} = \mathfrak{C}_{\widehat{I_1}}$. The alternating sum above gives then:

$$(-1)^{k-1} \sum_{\underline{r} \in \mathfrak{C}^0_{\widehat{I_1}}} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}} + \sum_{\underline{r} \in \mathfrak{C}_{\widehat{I_1}}} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}}.$$

Next, the two expressions above being equal, by taking the sum over $\mathfrak{C}^0_{\widehat{I_1}}$ to the other side of the equality, we get:

$$\sum_{\underline{r} \in \mathfrak{C}_{\widehat{I_1}}} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}} = \sum_{t=0}^{k} (-1)^{k-t} \sum_{\underline{r} \in \mathfrak{C}^t_{\widehat{I_1}}} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}}.$$

By Theorem 2.8, the left hand side of this equality corresponds to the decomposition of the characteristic function $\chi_{C_{\widehat{I_1}} \widetilde{P}_{J_k}}$. Therefore, by acting with the element $y_{i_1} w_{i_1}$, we get:

$$y_{i_1} w_{i_1} \chi_{C_{\widehat{I_1}} \widetilde{P}_{J_k}} = \sum_{t=0}^{k} (-1)^{k-t} \sum_{\underline{r} \in \mathfrak{C}^t_{\widehat{I_1}}} y_{i_1} w_{i_1} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}}. \tag{2.17}$$

In this equality (2.17), for each $t = 1, \ldots, k$, the sum over $\mathfrak{C}^t_{\widehat{I_1}}$ is trivial in $\mathrm{S}p^k(M)$. Indeed, when $t = 1$ this follows from Proposition 1.9.(2) and when t is such that $2 \leqslant t \leqslant k$ this follows from Proposition 2.10. Thus, in $\mathrm{S}p^k(M)$, we have the equality:

$$y_{i_1} w_{i_1} \chi_{C_{\widehat{I_1}} \widetilde{P}_{J_k}} = (-1)^k \sum_{\underline{r} \in \mathfrak{C}^0_{\widehat{I_1}}} y_{i_1} w_{i_1} \chi_{Bw^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}}. \tag{2.18}$$

Finally, by (1.4) and since $\widetilde{T}$ is a normal subgroup of $\widetilde{N}$ (hence, for each $w \in \overline{W}$, there is $y \in \widetilde{T} \subseteq \widetilde{P}_{J_k}$ such that $y_{i_1} w = wy$), we have:

$$y_{i_1} w_{i_1} B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k} = B w_{i_1} w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}.$$

Therefore, by Lemma 2.11, for any $\underline{r} \in \mathfrak{C}^0_{\widehat{I_1}}$ we have:

$$y_{i_1} w_{i_1} B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k} = B w^n_{r_k+i} \cdots w^{n-k+1}_{r_1+i} \widetilde{P}_{J_k}.$$

We conclude:

$$\sum_{\underline{r} \in \mathfrak{C}^0_{\widehat{I_1}}} y_{i_1} w_{i_1} \chi_{B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}} = \sum_{\underline{r} \in \mathfrak{C}^0_I} \chi_{B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}}.$$

This, with (2.18), completes the proof. □

Corollary 2.13. — *Let I, i_{k+1} and $\widehat{I_1}$ as in the proposition above. Assume that $i_{k+1} = n+1$. Then, in $\mathrm{Sp}^k(M)$ we have the identity*

$$\chi_{C_I \widetilde{P}_{J_k}} = (-1)^k y_{i_1} w_{i_1} \chi_{C_{\widehat{I_1}} \widetilde{P}_{J_k}}.$$

Proof. — Under the assumption $i_{k+1} = n+1$, in the proposition 2.12 we have $\mathfrak{C}^k_I = \emptyset$ and for each $t = 1, \ldots, k-1$

$$\mathfrak{C}^{t,k-t-1}_I = \left(\prod_{\iota=1}^{t-1} [\![i_\iota + 1, n-k+\iota+1]\!]\right) \times \left(\prod_{\iota=t}^{k-1} [\![i_{\iota+1} + 1, n-k+\iota+1]\!]\right) \times [\![i_k + 1, n+1]\!],$$

on the other hand, we can write:

$$\sum_{\underline{r} \in \mathfrak{C}^{t,k-t-1}_I} \chi_{B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}} = \sum_{(r_1,\ldots,r_{k-2})} \sum_{(r_{k-1},r_k)} \chi_{B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}}$$

where $(r_1, \ldots, r_{k-2})$ runs through the set $(\prod_{\iota=1}^{t-1} [\![i_\iota + 1, n-k+\iota+1]\!]) \times (\prod_{\iota=t}^{k-2} [\![i_{\iota+1} + 1, n-k+\iota+1]\!])$, and where the pair (r_{k-1}, r_k) runs through the cartesian product of the two last intervals of $\mathfrak{C}^{t,k-t-1}_I$, i.e. $[\![i_k + 1, n]\!] \times [\![i_k + 1, n+1]\!]$. Thus, for each $(r_1, \ldots, r_{k-2})$, by applying Proposition 2.10 to the pair (r_{k-1}, r_k) (it is clear that, if $w' = w^{n-2}_{r_{k-2}} \ldots w^{n-k+1}_{r_1}$, then $s_n w' = w' s_n$), we deduce that

$$\sum_{(r_{k-1},r_k)} \chi_{B w^n_{r_k} \ldots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}} \in C^\infty(\widetilde{G}/\widetilde{P}_{J_k \cup \{n\}}, M),$$

then for any $t = 1, \ldots, k-1$, we have:

$$\sum_{\underline{r} \in \mathfrak{C}^{t,k-t-1}_I} \chi_{B w^n_{r_k} \cdots w^{n-k+1}_{r_1} \widetilde{P}_{J_k}} \in \sum_{j=n-k+1}^{n} C^\infty(\widetilde{G}/\widetilde{P}_{J_k \cup \{j\}}, M).$$

Consequently, from the second formula in Proposition 2.12, it follows that in $\mathrm{Sp}^k(M)$ we have the equality:

$$\chi_{C_I \widetilde{P}_{J_k}} = \sum_{\underline{r}\in\mathfrak{C}_I^0} \chi_{Bw_{r_k}^n \cdots w_{r_1}^{n-k+1}\widetilde{P}_{J_k}}$$

which we have just proved when $k > 1$ and which is obvious for $k = 1$, the sets $\mathfrak{C}_I^{t,k-t-1}$ being empty. By combining this with the third formula in Proposition 2.12, we obtain the equality in the corollary. □

3. Main theorem

The two propositions that follow are in preparation for the proof of Theorem 3.3.

PROPOSITION 3.1. — *Let $1 \leqslant k \leqslant n$. Let $\varphi \in \mathrm{Hom}_M(\mathrm{Sp}^k(M), L)$. The map $\mathfrak{h}_\varphi \in \mathrm{Hom}_M(M[\widehat{\mathfrak{I}}^k], L)$ defined by $\mathfrak{h}_\varphi(g(\sigma_I, v_0^o)) = \varphi(g\chi_{C_I\widetilde{P}_{J_k}})$ for any $g \in \widetilde{G}$ and any $I \subseteq \Delta$ such that $|\Delta - I| = k$, satisfies the harmonicity conditions.*

Proof. — Since $\widetilde{G}$ acts transitively on the pointed cells of a given type, we only need to show that $\mathfrak{h}_\varphi$ satisfies the conditions of harmonicity on the standard pointed cells.
(HC1) Let $I \subseteq \Delta$ be such that $\Delta - I = \{i_1 < \cdots < i_k\}$. Let $\widehat{I_1}$ be such that $\Delta - \widehat{I_1} = \{i_2 - i_1 < \cdots < i_k - i_1 < n + 1 - i_1\}$. By Lemma 1.1 we have $(\sigma_I, v_{i_1}^o) = y_{i_1}w_{i_1}(\sigma_{\widehat{I_1}}, v_0^o)$, and by Corollary 2.13 we have:

$$\varphi(\chi_{C_I\widetilde{P}_{J_k}}) = (-1)^k\varphi(y_{i_1}w_{i_1}\chi_{C_{\widehat{I_1}}\widetilde{P}_{J_k}}).$$

Therefore,

$$\mathfrak{h}_\varphi(\sigma_I, v_0^o) = (-1)^k\mathfrak{h}_\varphi(\sigma_I, v_{i_1}^o).$$

Since we will need **(HC3)** in order to prove **(HC2)**, we will first prove **(HC3)**.
(HC3) First, notice that if $(\sigma, v_0) = (v_0, v_1 \ldots, v_k) \in \widehat{\mathfrak{I}}^k$, since $\mathfrak{h}_\varphi$ satisfies **(HC1)**, we have:

$$\mathfrak{h}_\varphi(\sigma, v_0) = (-1)^{(j+1)k}\mathfrak{h}_\varphi(\sigma, v_{j+1}). \tag{3.1}$$

Notice also that for each integer j, $0 \leqslant j \leqslant k$, we have a bijective correspondence:

$$\mathcal{C}((\sigma, v_0), j) \xrightarrow{\sim} \mathcal{C}((\sigma, v_{j+1}), k), \tag{3.2}$$

which sends the pointed cell (σ', v_0) (or (σ', v_0') in case $j = 0$) to the pointed cell (σ', v_{j+1}). Thus, by (3.1) and (3.2), we have:

$$\sum_{\sigma' \in \mathcal{C}((\sigma, v_0), j)} \mathfrak{h}_\varphi(\sigma') = (-1)^{(j+1)k} \sum_{\sigma' \in \mathcal{C}((\sigma, v_{j+1}), k)} \mathfrak{h}_\varphi(\sigma'). \tag{3.3}$$

Now, because of this formula, we need only to prove that $\mathfrak{h}_\varphi$ satisfies **(HC3)** in case $j = k$. That is, if $I \subseteq \Delta$ is such that $\Delta - I = \{i_1 < \cdots < i_k\}$, we have to prove:

$$\sum_{\sigma \in \mathcal{C}((\sigma_I, v_0^o), k)} \mathfrak{h}_\varphi(\sigma) = \mathfrak{h}_\varphi(\sigma_I, v_0^o).$$

By the second point of Lemma 2.3 and by the definition of $\mathfrak{h}_\varphi$ it suffices to prove that we have:

$$\coprod_{b \in B_I B_{I_n^{i_k}} / B_{I_{i_n}^{i_k}}} bC_{I_n^{i_k}} \widetilde{P}_{J_k} = B_I C_{I_n^{i_k}} \widetilde{P}_{J_k} = C_I \widetilde{P}_{J_k}. \tag{3.4}$$

Both equalities are obvious (for the second equality see the definition of C_I). The union is disjoint. Indeed, take $b \in B_I$ such that $bC_{I_n^{i_k}} \widetilde{P}_{J_k} \cap C_{I_n^{i_k}} \widetilde{P}_{J_k} \neq \emptyset$. By the formula (2.11) in the proof of Theorem 2.8, for each $\iota = 1, \ldots k-1$, there exist $b_\iota, b'_\iota \in B_{[\![i_\iota + 1, n-k+\iota]\!]} \subseteq K^* \widetilde{G}(O)$ and $p \in \widetilde{P}_{J_k}$ such that

$$bb_{k-1} \ldots b_1 = b'_{k-1} \ldots b'_1 p.$$

This implies $p \in K^* \widetilde{G}(O) \cap \widetilde{P}_{J_k} = B_{J_k}$ and hence $b = b'_{k-1} \ldots b'_1 p b_1^{-1} \ldots b_{k-1}^{-1} \in B_{I'}$, where I' is the subset of Δ defined as follows:

$$I' = J_k \cup \bigcup_{\iota=0}^{k-1} [\![i_\iota + 1, n-k+\iota]\!].$$

But, since $n \notin I'$ then $I \cap I' \subseteq I_n^{i_k}$. We therefore have $b \in B_{I_n^{i_k}}$.

(HC2) Let $I \subseteq \Delta$ be such that $\Delta - I = \{i_1 < \cdots < i_k\}$. Since $\mathfrak{h}_\varphi$ satisfies the condition **(HC3)** proved above, we can apply lemma 2.2. By this lemma, together with **(HC1)**, we need only to consider the case $i_k = n$. Therefore we have to prove:

$$\sum_{\sigma \in \mathcal{B}((\sigma_{I \cup \{n\}}, v_0^o), t_I)} \mathfrak{h}_\varphi(\sigma) = 0.$$

By the first point of Lemma 2.3 and by the definition of $\mathfrak{h}_\varphi$ it suffices to show that in $\mathrm{S}p^k(M)$ we have:

$$\sum_{b \in B_{I \cup \{n\}} / B_I} b\chi_{C_I \widetilde{P}_{J_k}} = 0. \tag{3.5}$$

First, notice that we have the following obvious equality. Similar arguments as in the proof of (3.4) show that the union in the right hand side is disjoint:

$$B_{I\cup\{n\}}C_I\widetilde{P}_{J_k} = \coprod_{b\in B_{I\cup\{n\}}/B_I} bC_I\widetilde{P}_{J_k}. \tag{3.6}$$

On the other hand, if we proceed similarly as in the proof of Theorem 2.8 and by using Proposition 1.10, we show that we have the decomposition:

$$B_{I\cup\{n\}}C_I\widetilde{P}_{J_k} = \coprod_{\underline{r}\in\mathfrak{C}_I^n} Bw_{r_k}^n w_{r_{k-1}}^{n-1}\cdots w_{r_1}^{n-k+1}\widetilde{P}_{J_k} \tag{3.7}$$

where $\mathfrak{C}_I^n = \left(\prod_{\iota=1}^{k-1}[\![i_\iota+1, n-k+\iota+1]\!]\right)\times[\![i_{k-1}+1, n+1]\!]$. Therefore, combining (3.6) with (3.7) we get:

$$\sum_{b\in B_{I\cup\{n\}}/B_I}\chi_{bC_I\widetilde{P}_{J_k}} = \sum_{\underline{r}\in\mathfrak{C}_I^n}\chi_{Bw_{r_k}^n w_{r_{k-1}}^{n-1}\cdots w_{r_1}^{n-k+1}\widetilde{P}_{J_k}}.$$

Finally since the pair (r_{k-1}, r_k) runs through the set $[\![i_{k-1}+1, n]\!]\times[\![i_{k-1}+1, n+1]\!]$ it follows from Proposition 2.10 (or from Proposition 1.9 in case $k=1$) that we have:

$$\sum_{b\in B_{I\cup\{n\}}/B_I}\chi_{bC_I\widetilde{P}_{J_k}}\in C^\infty(\widetilde{G}/\widetilde{P}_{J_k\cup\{n\}}, M).$$

This finishes the proof of (3.5).

(HC4) Let $I\subseteq\Delta$ be such that $\Delta - I = \{i_1<\cdots<i_k\}$. Assume $i_k<n$ and let i_{k+1} be such that $i_k<i_{k+1}\leqslant n$. We have to show that:

$$\sum_{j=0}^{k+1}(-1)^j\mathfrak{h}_\varphi(v_0^o, v_{i_1}^o,\ldots,\widehat{v_{i_j}^o},\ldots,v_{i_{k+1}}^o) = 0.$$

Let $\widehat{I}_1\subseteq\Delta$ be such that $\Delta-\widehat{I}_1 = \{i_2-i_1<\cdots<i_{k+1}-i_1\}$, *cf.* Lemma 1.1. By this lemma and by the definition of $\mathfrak{h}_\varphi$, it suffices to prove that in $\mathrm{Sp}^k(M)$ we have:

$$y_{i_1}w_{i_1}\chi_{C_{\widehat{I}_1}\widetilde{P}_{J_k}} + \sum_{j=1}^{k+1}(-1)^j\chi_{C_{I_{i_{k+1}}^{i_j}}\widetilde{P}_{J_k}} = 0. \tag{3.8}$$

Recall that, for each $j = 1, \dots, k+1$, the index set which corresponds to the decomposition of the characteristic function $\chi_{C_{I^{i_j}_{i_{k+1}}}\widetilde{P}_{J_k}}$ is (*cf.* Theorem 2.8):

$$\mathfrak{C}_{I^{i_j}_{i_{k+1}}} = \prod_{\iota=1}^{j-1} [\![i_\iota + 1, n-k+\iota+1]\!] \times \prod_{\iota=j}^{k} [\![i_{\iota+1} + 1, n-k+\iota+1]\!].$$

By combining the identities (2) and (3) in Proposition 2.12, and since we have $\mathfrak{C}_{I^{i_k}_{i_{k+1}}} = \mathfrak{C}^k_I$ and $\mathfrak{C}_{I^{i_{k+1}}_{i_{k+1}}} = \mathfrak{C}_I$, we obtain:

$$y_{i_1} w_{i_1} \chi_{C_{\widehat{I_1}}\widetilde{P}_{J_k}} + (-1)^k \chi_{C_{I^{i_k}_{i_{k+1}}}\widetilde{P}_{J_k}} + (-1)^{k+1} \chi_{C_I \widetilde{P}_{J_k}} = \sum_{j=1}^{k-1} (-1)^j \sum_{\underline{r}\in\mathfrak{C}_I^{j,k-j-1}} \chi_{B w^n_{r_k}\cdots w^{n-k+1}_{r_1}\widetilde{P}_{J_k}}.$$

Substituting this into (3.8) we conclude that:

$$y_{i_1} w_{i_1} \chi_{C_{\widehat{I_1}}\widetilde{P}_{J_k}} + \sum_{j=1}^{k+1} (-1)^j \chi_{C_{I^{i_j}_{i_{k+1}}}\widetilde{P}_{J_k}} = \sum_{j=1}^{k-1} (-1)^j \sum_{\underline{r}\in\mathfrak{D}_{I^{i_j}_{i_{k+1}}}} \chi_{B w^n_{r_k}\cdots w^{n-k+1}_{r_1}\widetilde{P}_{J_k}},$$

where we have set $\mathfrak{D}_{I^{i_j}_{i_{k+1}}} = \mathfrak{C}_{I^{i_j}_{i_{k+1}}} \amalg \mathfrak{C}_I^{j,n-j-1}$. Now, it remains to show that the right hand side in the equality above is trivial in $\mathrm{S}p^k(M)$. It is easy to see that we have:

$$\mathfrak{D}_{I^{i_j}_{i_{k+1}}} = \Big(\prod_{\iota=1}^{j-1} [\![i_\iota+1, n-k+\iota+1]\!] \Big) \times \Big(\prod_{\iota=j}^{k-1} [\![i_{\iota+1}+1, n-k+\iota+1]\!] \Big) \times [\![i_k+1, n+1]\!].$$

Notice that the cartesian product of the two last intervals in $\mathfrak{D}_{I^{i_j}_{i_{k+1}}}$ is $[\![i_k+1, n]\!] \times [\![i_k+1, n+1]\!]$. Thus, we can check easily that we can apply Proposition 2.10 (or Proposition 1.9 in case $k = 1$) to conclude the following:

$$\sum_{\underline{r}\in\mathfrak{D}_{I^{i_j}_{i_{k+1}}}} \chi_{B w^n_{r_k}\cdots w^{n-k+1}_{r_1}\widetilde{P}_{J_k}} \in C^\infty(\widetilde{G}/\widetilde{P}_{J_k\cup\{n\}}, M).$$

This completes the proof of the assertion (3.8). □

The proof of the following proposition has been inspired by case $k = n$ which was treated by P. Schneider and J. Teitelbaum, see [10, page 401, Lemma 10]. The computations are much more complicated for general k, $1 \leqslant k \leqslant n$.

Proposition 3.2. — *Let $\mathfrak{h} \in \mathfrak{H}\mathrm{arm}^k(M,L)$. The map $\psi_{\mathfrak{h}} \in \mathrm{Hom}_M(C_c^\infty(\widetilde{G}/B_{J_k}, M), L)$ defined by $\psi_{\mathfrak{h}}(g\chi_{B_{J_k}}) = \mathfrak{h}(g(\sigma_{J_k}, v_0^o))$, for any $g \in \widetilde{G}$, vanishes on the $M[\widetilde{G}]$-submodule $\mathfrak{R}_{J_k}$ of $C_c^\infty(\widetilde{G}/B_{J_k}, M)$ generated by the functions $\chi_{B_{J_k} s_j B_{J_k}} + \chi_{B_{J_k}}$, $n-k+1 \leqslant j \leqslant n$, and the functions $\chi_{By_iB_{J_k}} - \chi_{B_{J_k}}$, $0 \leqslant i \leqslant n$.*

Proof. — Let j be such that $n-k+1 \leqslant j \leqslant n$. Since $\mathfrak{h}$ is harmonic and then satisfies the condition **(HC2)**, we conclude that:

$$\psi_{\mathfrak{h}}(\chi_{B_{J_k \cup \{j\}}}) = \sum_{b \in B_{J_k \cup \{j\}}/B_{J_k}} \psi_{\mathfrak{h}}(b\chi_{B_{J_k}}) = \sum_{b \in B_{J_k \cup \{j\}}/B_{J_k}} \mathfrak{h}(b(\sigma_{J_k}, v_0^o)) = 0.$$

Therefore $\psi_{\mathfrak{h}}$ is trivial on the $M[\widetilde{G}]$-submodule $\sum_{j=n-k+1}^n C_c^\infty(\widetilde{G}/B_{J_k \cup \{j\}}, M)$ of $C_c^\infty(\widetilde{G}/B_{J_k}, M)$. Let us show that $\psi_{\mathfrak{h}}$ vanishes on the functions $\chi_{By_iB_{J_k}} - \chi_{B_{J_k}}$, $0 \leqslant i \leqslant n$. Since $\mathfrak{h}$ satisfies **(HC3)**, for any $I \subseteq \Delta$ such that $|\Delta - I| = k$, we have:

$$\psi_{\mathfrak{h}}(g.\chi_{C_I}) = \mathfrak{h}(g(\sigma_I, v_0^o)). \tag{3.9}$$

On the other hand, if w_i is as in Lemma 1.1, the assertion (1.4) given in the following Remark 1.2 says that y_iw_i normalizes B, thus:

$$\chi_{By_iB_{J_k}} = \chi_{By_iw_iw_i^{-1}B_{J_k}} = y_iw_i.\chi_{Bw_i^{-1}B_{J_k}}. \tag{3.10}$$

Now, there are two cases depending on i:

• $\underline{0 \leqslant i \leqslant n-k+1}$: observe that w_i^{-1} decomposes in two factors as follows:

$$w_i^{-1} = (w_{n-i+1}^n \cdots w_{n-k-i+2}^{n-k+1}).(w_{n-k-i+1}^{n-k} \cdots w_1^i)$$

and that the second factor $w_{n-k-i+1}^{n-k} \cdots w_1^i$ lies in W_{J_k} and hence in B_{J_k}. Thus, we have:

$$Bw_i^{-1}B_{J_k} = Bw_{n-i+1}^n \cdots w_{n-k-i+2}^{n-k+1}B_{J_k}.$$

So, if for each ι, $1 \leqslant \iota \leqslant k+1$, we consider $i_\iota = n-k-i+\iota$, and if $I \subseteq \Delta$ is such that $\Delta - I = \{i_1 < \cdots < i_k\}$, we clearly have:

$$\chi_{Bw_i^{-1}B_{J_k}} = \sum_{\underline{r} \in \mathfrak{C}_I^0} \chi_{Bw_{r_k}^n \cdots w_{r_1}^{n-k+1}B_{J_k}},$$

with $\mathfrak{C}^0_I = \prod_{\iota=1}^{k} [\![i_\iota+1, i_{\iota+1}]\!]$. Since $\psi_{\mathfrak{h}}$ vanishes on $\sum_{j=n-k+1}^{n} C_c^\infty(\widetilde{G}/B_{J_k\cup\{j\}}, M)$, by the first assertion of Proposition 2.12, we conclude that $\psi_{\mathfrak{h}}$ satisfies:

$$\psi_{\mathfrak{h}}(\chi_{Bw_i^{-1}B_{J_k}}) = \psi_{\mathfrak{h}}\Bigg(\chi_{C_I} - \chi_{C_{I^{i_k}_{i_{k+1}}}} - \sum_{t=1}^{k-1}(-1)^{k-t-1} \sum_{\underline{r}\in\mathfrak{C}_I^{t,k-t-1}} \chi_{Bw^n_{r_k}\cdots w^{n-k+1}_{r_1}B_{J_k}}\Bigg). \tag{3.11}$$

On the other hand, since $\mathfrak{h}$ satisfies **(HC4)**, we have:

$$\sum_{j=0}^{k+1}(-1)^j \mathfrak{h}(v_0^o, v_{i_1}^o, \ldots, \widehat{v_{i_j}^o}, \ldots, v_{i_{k+1}}^o) = 0.$$

Therefore, if $\widehat{I}_1 \subseteq \Delta$ is such that $\Delta - \widehat{I}_1 = \{i_2 - i_1 < \cdots < i_{k+1} - i_1\}$, we have $(v_{i_1}^o, \ldots, v_{i_{k+1}}^o) = y_{i_1} w_{i_1}(\sigma_{\widehat{I}_1}, v_0^o)$ and, by (3.9), the identity above gives:

$$\psi_{\mathfrak{h}}(y_{i_1} w_{i_1} \chi_{C_{\widehat{I}_1}}) + \sum_{t=1}^{k+1}(-1)^t \psi_{\mathfrak{h}}(\chi_{C_{I^{i_t}_{i_{k+1}}}}) = 0. \tag{3.12}$$

Combining (3.11) with (3.12) we conclude that:

$$\psi_{\mathfrak{h}}(\chi_{Bw_i^{-1}B_{J_k}}) = \psi_{\mathfrak{h}}\Bigg((-1)^k y_{i_1} w_{i_1} \chi_{C_{\widehat{I}_1}} + \sum_{t=1}^{k-1}(-1)^{k-t} \sum_{\underline{r}\in\mathfrak{D}_{I^{i_t}_{i_{k+1}}}} \chi_{Bw^n_{r_k}\cdots w^{n-k+1}_{r_1}B_{J_k}}\Bigg),$$

where we have set $\mathfrak{D}_{I^{i_t}_{i_{k+1}}} = \mathfrak{C}_{I^{i_t}_{i_{k+1}}} \amalg \mathfrak{C}_I^{t,k-t-1}$, and where $\mathfrak{C}_{I^{i_t}_{i_{k+1}}}$ is the index set associated to the decomposition of $C_{I^{i_t}_{i_{k+1}}}$ (see Theorem 2.8). By Proposition 2.10, for each $t = 1, \ldots, k-1$, ψ_h vanishes on the sum over $\mathfrak{D}_{I^{i_t}_{i_{k+1}}}$. Therefore:

$$\psi_{\mathfrak{h}}(\chi_{Bw_i^{-1}B_{J_k}}) = (-1)^k \psi_{\mathfrak{h}}(y_{i_1} w_{i_1} \chi_{C_{\widehat{I}_1}})$$

and hence, by (3.10), we have:

$$\psi_{\mathfrak{h}}(\chi_{By_iB_{J_k}}) = (-1)^k \psi_{\mathfrak{h}}(y_i w_i y_{i_1} w_{i_1} \chi_{C_{\widehat{I}_1}}).$$

Recall that we have set $i_1 = n-k-i+1$, therefore $y_i w_i y_{i_1} w_{i_1} C_{\widehat{I}_1} = y_{n-k+1} w_{n-k+1} C_{\widehat{I}_1}$, and hence:

$$\psi_{\mathfrak{h}}(\chi_{By_iB_{J_k}}) = (-1)^k \psi_{\mathfrak{h}}(y_{n-k+1} w_{n-k+1} \chi_{C_{\widehat{I}_1}}). \tag{3.13}$$

Finally, since $\mathfrak{h}$ satisfies **(HC1)**, we have $\mathfrak{h}(\sigma_{J_k}, v^o_{n-k+1}) = (-1)^k \mathfrak{h}(\sigma_{J_k}, v^o_0)$. Thus, by Lemma 1.1 and by (3.9) we get:

$$\psi_{\mathfrak{h}}(y_{n-k+1} w_{n-k+1} \chi_{C_{(\widehat{J_k})_1}}) = (-1)^k \psi_{\mathfrak{h}}(\chi_{B_{J_k}}), \tag{3.14}$$

where $(\widehat{J_k})_1 \subseteq \Delta$ is such that $\Delta - (\widehat{J_k})_1 = \{1 < 2 < \cdots < k\}$. Note that $\widehat{I}_1 = (\widehat{J_k})_1$, therefore by combining (3.13) with (3.14) we get $\psi_{\mathfrak{h}}(\chi_{By_iB_{J_k}}) = \psi_{\mathfrak{h}}(\chi_{B_{J_k}})$.

• $\underline{n - k + 2 \leqslant i \leqslant n}$: simple calculation shows that we have $w_i^{-1} = ww'$ with

$$w = w^{2n-k-i+1}_{n-i+1} \cdots w^{n-k+2}_2 w^{n-k+1}_1 \text{ and } w' = w^n_{2n-k-i+2} \cdots w^{i+1}_{n-k+3} w^i_{n-k+2}.$$

Notice that $w' \in W_{[\![n-k+2,n]\!]}$ and that $l(w) = (n-i+1)(k-1) \pmod 2$. Thus, since ψ_h vanishes on $\sum_{j=n-k+1}^n C_c^\infty(\widetilde{G}/B_{J_k \cup \{j\}}, M)$, by Proposition 2.9 we have:

$$\psi_{\mathfrak{h}}(\chi_{Bw_i^{-1}B_{J_k}}) = (-1)^{(n-i+1)(k-1)} \psi_{\mathfrak{h}}(\chi_{Bw^{2n-k-i+1}_{n-i+1} \cdots w^{n-k+2}_2 w^{n-k+1}_1 B_{J_k}}). \tag{3.15}$$

Next, if we consider $I \subseteq \Delta$ such that $\Delta - I = \{i_1 < \cdots < i_k\}$ with the i_ι, $1 \leqslant \iota \leqslant k$, defined as follows:

$$i_\iota = \begin{cases} \iota & \text{si } 1 \leqslant \iota \leqslant n-i+1, \\ n-k+\iota & \text{si } n-i+2 \leqslant \iota \leqslant n, \end{cases}$$

then, we have the identity:

$$\psi_{\mathfrak{h}}(\chi_{C_I}) = (-1)^{n-i+1} \psi_{\mathfrak{h}}(\chi_{Bw^{2n-k-i+1}_{n-i+1} \cdots w^{n-k+2}_2 w^{n-k+1}_1 B_{J_k}}). \tag{3.16}$$

Indeed, since $\psi_{\mathfrak{h}}$ vanishes on $\sum_{j=n-k+1}^n C_c^\infty(\widetilde{G}/B_{J_k \cup \{j\}}, M)$, by Proposition 1.9 we have:

$$\psi_{\mathfrak{h}}(\chi_{Bw^{2n-k-i+1}_{n-i+1} \cdots w^{n-k+2}_2 w^{n-k+1}_1 B_{J_k}}) = -\psi_{\mathfrak{h}}\left(\sum_{r_1=2}^{n-k+2} \chi_{Bw^{2n-k-i+1}_{n-i+1} \cdots w^{n-k+2}_2 w^{n-k+1}_{r_1} B_{J_k}} \right),$$

and by Proposition 2.10 with (r_1, r_2) running through $[\![2, n-k+2]\!] \times [\![2, n-k+3]\!]$, we conclude that the right hand side in the equality above is equal to:

$$\psi_{\mathfrak{h}}\left(\sum_{r_2=3}^{n-k+3} \sum_{r_1=2}^{n-k+2} \chi_{Bw^{2n-k-i+1}_{n-i+1} \cdots w^{n-k+3}_3 w^{n-k+2}_{r_2} w^{n-k+1}_{r_1} B_{J_k}} \right).$$

Consequently, we have the identity:

$$\psi_{\mathfrak{h}}(\chi_{Bw_{n-i+1}^{2n-k-i+1}\cdots w_2^{n-k+2}w_1^{n-k+1}B_{J_k}}) = (-1)^2\psi_{\mathfrak{h}}\left(\sum_{r_2=3}^{n-k+3}\sum_{r_1=2}^{n-k+2}\chi_{Bw_{n-i+1}^{2n-k-i+1}\cdots w_3^{n-k+3}w_{r_2}^{n-k+2}w_{r_1}^{n-k+1}B_{J_k}}\right).$$

By repeating this process and using Proposition 2.10 successively with the pairs (r_2, r_3), (r_3, r_4), …, (r_{n-i}, r_{n-i+1}) running through the sets $[\![3, n-k+3]\!] \times [\![3, n-k+4]\!]$, $[\![4, n-k+4]\!] \times [\![4, n-k+5]\!]$, …, $[\![n-i+1, 2n-k-i+1]\!] \times [\![n-i+1, 2n-k-i+2]\!]$ respectively, we get:

$$\psi_{\mathfrak{h}}(\chi_{Bw_{n-i+1}^{2n-k-i+1}\cdots w_2^{n-k+2}w_1^{n-k+1}B_{J_k}}) = (-1)^{n-i+1}\psi_{\mathfrak{h}}\left(\sum_{r_{n-i+1}=n-i+2}^{2n-k-i+2}\cdots\sum_{r_2=3}^{n-k+3}\sum_{r_1=2}^{n-k+2}\chi_{Bw_{r_{n-i+1}}^{2n-k-i+1}\cdots w_{r_2}^{n-k+2}w_{r_1}^{n-k+1}B_{J_k}}\right).$$

Notice that the expression in the right hand side above defined by the sums over the r_ι is nothing else than the decomposition of χ_{C_I} given in Theorem 2.8. This proves (3.16).

Finally, combining (3.15) with (3.16) we get

$$\psi_{\mathfrak{h}}(\chi_{Bw_i^{-1}B_{J_k}}) = (-1)^{(n-i+1)k}\psi_{\mathfrak{h}}(\chi_{C_I}),$$

therefore, by using (3.10), we deduce:

$$\psi_{\mathfrak{h}}(\chi_{By_iB_{J_k}}) = (-1)^{(n-i+1)k}\psi_{\mathfrak{h}}(y_iw_i\chi_{C_I}). \tag{3.17}$$

On the other hand, $\mathfrak{h}$ being harmonic, by **(HC1)** we have:

$$\mathfrak{h}(\sigma_{J_k}, v_i^o) = (-1)^{(n-i+1)k}\mathfrak{h}(\sigma_{J_k}, v_0^o). \tag{3.18}$$

>From Lemma 1.1, we have $(\sigma_{J_k}, v_i^o) = y_iw_i(\sigma_{(\widehat{J_k})_i}, v_0^o)$ where $(\widehat{J_k})_i$ is nothing else than I. This, with (3.18), gives:

$$\mathfrak{h}(y_iw_i(\sigma_I, v_0^o)) = (-1)^{(n-i+1)k}\mathfrak{h}(\sigma_{J_k}, v_0^o),$$

and, by (3.9), we deduce:

$$\psi_{\mathfrak{h}}(y_iw_i\chi_{C_I}) = (-1)^{(n-i+1)k}\psi_{\mathfrak{h}}(\chi_{B_{J_k}}). \tag{3.19}$$

Thus, combining (3.17) with (3.19) we get $\psi_{\mathfrak{h}}(\chi_{By_iB_{J_k}}) = \psi_{\mathfrak{h}}(\chi_{B_{J_k}})$. □

THEOREM 3.3. — *For any k, $0 \leqslant k \leqslant n$, there is an $M[\widetilde{G}]$-isomorphism*

$$\mathrm{Hom}_M(\mathrm{Sp}^k(M), L) \cong \mathfrak{Harm}^k(M, L).$$

Proof. — For $k = 0$, by (2.7) and (2.1), both sides of the isomorphism of this theorem are canonically isomorphic to L. Now, let k, $1 \leqslant k \leqslant n$. By Proposition 3.1, the map which to φ associates $\mathfrak{h}_\varphi$ gives a well defined M-homomorphism

$$H^k : \mathrm{Hom}_M(\mathrm{Sp}^k(M),\, L) \to \mathfrak{H}\mathrm{arm}^{\,k}(M, L).$$

This homomorphism is clearly $\widetilde{G}$-equivariant. On the other hand, we have a well defined $M[\widetilde{G}]$-homomorphism:

$$\Phi^k : \quad \mathfrak{H}\mathrm{arm}^{\,k}(M, L) \quad \longrightarrow \quad \mathrm{Hom}_M(\mathrm{Sp}^k(M),\, L)$$

which sends an harmonic cochain $\mathfrak{h}$ to φ_h defined by $\varphi_\mathfrak{h}(g\chi_{B_{J_k}\widetilde{P}_{J_k}}) = \mathfrak{h}(g(\sigma_{J_k}, v_0^o))$ for any $g \in \widetilde{G}$. Indeed, it is easy to check that $\Phi^k = \widetilde{H}_{J_k}^{-1} \circ \Psi^k$, where the $M[\widetilde{G}]$-homomorphism

$$\Psi^k : \quad \mathfrak{H}\mathrm{arm}^{\,k}(M, L) \quad \longrightarrow \quad \mathrm{Hom}_M(C_c^\infty(\widetilde{G}/B_{J_k},\, M)/\mathfrak{R}_{J_k},\, L)$$

which to $\mathfrak{h}$ associates $\psi_\mathfrak{h}$ is given by Proposition 3.2, and $\widetilde{H}_{J_k}$ is the $M[\widetilde{G}]$-isomorphism

$$\widetilde{H}_{J_k} : \quad \mathrm{Hom}_M(\mathrm{Sp}^k(M),\, L) \quad \xrightarrow{\sim} \quad \mathrm{Hom}_M(C_c^\infty(\widetilde{G}/B_{J_k},\, M)/\mathfrak{R}_{J_k},\, L),$$

dual to the isomorphism H_{J_k} given by Proposition 2.7.

Let us prove that Φ^k and H^k are inverse to each other. Let $\mathfrak{h} \in \mathfrak{H}\mathrm{arm}^{\,k}(M, L)$ and let $\mathfrak{h}_{\varphi_\mathfrak{h}}$ be its image by $H^k \circ \Phi^k$. We have $\mathfrak{h}_{\varphi_\mathfrak{h}}(\sigma_{J_k}, v_0^o) = \varphi_\mathfrak{h}(\chi_{B_{J_k}\widetilde{P}_{J_k}}) = \mathfrak{h}(\sigma_{J_k}, v_0^o)$. As we have $\mathfrak{h}_{\varphi_\mathfrak{h}} \in \mathfrak{H}\mathrm{arm}^{\,k}(M, L)$, this proves that $\mathfrak{h} = \mathfrak{h}_{\varphi_\mathfrak{h}}$, by the property **(HC3)**.

On the other hand, if $\varphi \in \mathrm{Hom}_M(\mathrm{S}p^k(M),\, L)$ and if $\varphi_{\mathfrak{h}_\varphi}$ is its image by $\Phi^k \circ H^k$, then we have $\varphi_{\mathfrak{h}_\varphi}(\chi_{B_{J_k}\widetilde{P}_{J_k}}) = \mathfrak{h}_\varphi(\sigma_{J_k}, v_0^o) = \varphi(\chi_{B_{J_k}\widetilde{P}_{J_k}})$. We are done. □

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Manuscrit reçu le 18 mai 2005,
accepté le 23 août 2005.

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