



# ANNALES

DE

# L'INSTITUT FOURIER

Emil HOROZOV

**Calogero-Moser spaces and an adelic  $W$ -algebra**

Tome 55, n° 6 (2005), p. 2069-2090.

[http://aif.cedram.org/item?id=AIF\\_2005\\_\\_55\\_6\\_2069\\_0](http://aif.cedram.org/item?id=AIF_2005__55_6_2069_0)

© Association des Annales de l'institut Fourier, 2005, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

## CALOGERO-MOSER SPACES AND AN ADELIC W-ALGEBRA

by Emil HOROZOV

---

### 0. Introduction.

In this paper we give an alternative description of the phase spaces of Calogero-Moser particle systems. In recent years they turned to be connected to several objects: rational solutions of KP-hierarchy [1, 26]; Wilson’s adelic Grassmannian [38, 39]; isomorphism classes of right ideals of the Weyl algebra [11, 9], *etc.* to mention few of them.

The motion of Calogero-Moser particle [12, 31] systems is governed by the Hamiltonians

$$H_n = \frac{1}{2} \sum_i^n p_i^2 - \sum_{i < j} \frac{1}{(x_i - x_j)^2}.$$

In the original papers on the subject the variables are real and collisions ( $x_i = x_j$ ) are avoided; however here we work with complex variables and do allow collisions, which will be explained below (see [38, 25] for more details).

In [25] Kazhdan, Kostant and Sternberg used Hamiltonian reduction (in the opposite direction) to easily determine explicitly the Calogero-Moser flows. We briefly recall their construction. Consider the sets  $\overline{\mathcal{C}}_n 0, = 1, 2 \dots$  of all pairs of complex matrices  $X, Z$  subject to the condition:

$$[X, Z] + I$$

---

*Keywords:* Fock spaces, bispectral operators, Sato’s theory for KP hierarchy.  
*Math. classification:* 37K30, 37K35.

has rank one. Here  $I$  is the identity matrix. Let  $C_n$  be the quotient space  $C_n = \overline{C}_n/GL(n, \mathbb{C})$ , where the group  $GL(n, \mathbb{C})$  acts by simultaneous conjugation. Denote by  $C'_N$  the subspace of pairs, such that  $X$  is diagonalizable. In that case the pair can be conjugated to a pair of the form

$$X = \text{diag}(x_1, \dots, x_n), \quad Z_{ii} = \alpha_i, \quad Z_{ij}, \quad \text{for } i \neq j.$$

A matrix  $Z$  of that form is called *Calogero-Moser matrix*. The main result of [25] is that the Calogero-Moser flows are quotients of the simple flows  $(X, Z) \rightarrow (X - tZ, Z)$ . The last flows obviously make sense for any pair of matrices. For this reason the Calogero-Moser flows can be continued on the singular locus and the spaces  $C_n$  are completions of  $C'_N$ .

Our main objective in this paper will be to describe the spaces  $C_n$  in terms of representations of a suitable Lie algebra. This will be done by passing through an intermediate object - the so called *bispectral operators*.

Bispectral operators have been introduced by F. A. Grünbaum in his work on medical imaging [18] ( see also [15]). An ordinary differential operator  $L(x, \partial_x)$  is called bispectral if there exists an infinite-dimensional family of eigenfunctions  $\psi(x, z)$ , which are also eigenfunctions of another differential operator  $\Lambda(z, \partial_z)$  in the spectral parameter  $z$ , *i.e.* for which the following identities hold

$$\begin{aligned} L(x, \partial_x)\psi(x, z) &= f(z)\psi(x, z), \\ \Lambda(z, \partial_z)\psi(x, z) &= \theta(x)\psi(x, z), \end{aligned}$$

with some non constant functions  $f(z)$  and  $\theta(x)$ . G. Wilson [38] has classified all bispectral operators of rank one (see the next section for more details). Using slightly different terminology than in [38], they are all operators with rational coefficients that are Darboux transformations of operators with constant coefficients. Sato's theory associates with each operator (or rather with the maximal algebra of operators that commute with it) a plane in Sato's Grassmannian. The set of all planes corresponding to the rank one bispectral algebras of operators has been called by G. Wilson *an adelic Grassmannian* and denoted by  $\text{Gr}^{\text{ad}}$ . Originally G. Wilson has characterized the rank one bispectral algebras  $A$  as those whose spectral curve  $\text{Spec}A$  is rational and its singularities are only cusps. Then in [39] he found an isomorphism between the disjoint union  $\bigcup_{n \geq 0} C_n$  and  $\text{Gr}^{\text{ad}}$ .

In a different development [8] we have characterized those of bispectral algebras whose spectral curve has only one cusp in terms of representations of  $W_{1+\infty}$ -algebra. More precisely we have built certain bosonic highest weight modules of  $W_{1+\infty}$ . Denote the module corresponding to the rank

one case by  $\mathcal{M}_0$ . Then the tau-functions of the bispectral operators (with the above restriction) lie in  $\mathcal{M}_0$  and vice versa - all the tau functions in the module are tau-functions of bispectral operators.

A natural question (see [8]) is if a similar result holds for the entire set of rank one bispectral operators. The present paper gives an affirmative answer to this question. Obviously one has first to point out a suitable generalization of the  $W_{1+\infty}$ -algebra. The most natural candidate does the job - the algebra we look for is a central extension of the algebra of differential operators with rational coefficients. We call this new algebra *an adelic W-algebra*. Then we proceed as in [5, 8]. We construct a bosonic representation  $\mathcal{M}^{\text{ad}}$  which is similar to a highest weight representation. Then our main result is the following

**THEOREM 0.1.** — *If an element  $\tau \in \mathcal{M}^{\text{ad}}$  is a tau-function then the corresponding plane belongs to  $\text{Gr}^{\text{ad}}$ . Conversely, if  $W \in \text{Gr}^{\text{ad}}$  then  $\tau_W \in \mathcal{M}^{\text{ad}}$ .*

Returning to the realization of  $\text{Gr}^{\text{ad}}$  as Calogero-Moser spaces we obtain immediately

**THEOREM 0.2.** — *The points of the Calogero-Moser spaces are in 1:1 correspondence with the tau-functions in  $\mathcal{M}^{\text{ad}}$ .*

For other interpretations see [9, 10].

It would be interesting to find analogs of the present results for other bispectral operators - both continuous and discrete. For example there are particle systems connected to the bispectral operators from [34, 23]. Even more intriguing would be to consider particle systems coming from discrete bispectral operators [19, 20, 21]. In this respect it seems to me that the results of P. Iliev [24] will be very helpful.

Many of the constructions in the present paper are similar to those of [8]. We skip some of the auxiliary results but repeat (with less details) the main steps as there are some differences. The organization of the paper is the following. Section 1. contains preliminaries on Sato's Grassmannian, Darboux transformations, bispectral operators,  $W_{1+\infty}$ -algebra. In Section 2. we introduce the adelic W-algebra together with a bosonic representation  $\mathcal{M}^{\text{ad}}$ . In Section 3. we show that the tau-functions in the module  $\mathcal{M}^{\text{ad}}$  correspond to planes in  $\text{Gr}^{\text{ad}}$ . In Section 4. we give the inverse result.

*Acknowledgments.* — I wish to thank P. van Moerbeke and L. Haine

for a number of remark and questions that helped me improve the paper. This work was partially supported by Grant MM-1403/04 of NFSR of Bulgarian Ministry of Education.

## 1. Preliminaries.

Here we have collected some facts and notation needed throughout the paper. In particular we recall Sato's theory, Darboux transforms and the bispectral problem,  $W_{1+\infty}$ -algebra.

### 1.1. Sato's theory of KP-hierarchy.

In this subsection we recall some facts and notation from Sato's theory of KP-hierarchy [36, 13, 35] needed in the paper. We use the approach of V. Kac and D. Peterson based on infinite wedge products (see e.g. [27]) and the survey paper by P. van Moerbeke [36].

Consider the infinite-dimensional vector space of formal series

$$\mathbb{V} = \left\{ \sum_{k \in \mathbb{Z}} a_k v_k \mid a_k = 0 \text{ for } k \gg 0 \right\}.$$

Sato's Grassmannian  $\text{Gr}$  (more precisely - its big cell) [36, 13] consists of all subspaces ("planes")  $W \subset \mathbb{V}$  which have an admissible basis

$$w_k = v_k + \sum_{i < k} w_{ik} v_i, \quad k = 0, 1, 2, \dots$$

Then define the *fermionic Fock space*  $F^{(0)}$  consisting of formal infinite sums of semi-infinite wedge monomials

$$v_{i_0} \wedge v_{i_1} \wedge \dots$$

such that  $i_0 < i_1 < \dots$  and  $i_k = k$  for  $k \gg 0$ . The wedge monomial

$$\psi_0 = v_0 \wedge v_1 \wedge \dots$$

plays a special role and is called *the vacuum*. The plane that corresponds to it will be denoted by  $W_0$ . There exists a well known linear isomorphism, called a *boson-fermion correspondence*:

$$(1) \quad \sigma: F^{(0)} \rightarrow B,$$

(see [29]), where  $B = \mathbb{C}[[t_1, t_2, \dots]]$  is the bosonic Fock space.

To any plane  $W \in \text{Gr}$  one naturally associates a state  $|W\rangle \in F^{(0)}$  as follows

$$|W\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \dots,$$

where  $w_0, w_1, \dots$ , form an admissible basis. One of the main objects of Sato's theory is the *tau-function* of  $W$  defined as the image of  $|W\rangle$  under the boson-fermion correspondence (1)

$$\tau_W(t) = \sigma(|W\rangle) = \sigma(w_0 \wedge w_{-1} \wedge w_{-2} \wedge \dots).$$

It is a formal power series in the variables  $t_1, t_2, \dots$ , i.e. an element of  $B := \mathbb{C}[[t_1, t_2, \dots]]$ . In particular the tau-function corresponding to the vacuum  $\psi_0$  is  $\tau_0 \equiv 1$ . Using the tau-function one can define the other important function connected to  $W$  - the *Baker* or *wave function*

$$(2) \quad \Psi_W(t, z) = e^{\sum_{k=1}^{\infty} t_k z^k} \frac{\tau_W(t - [z^{-1}])}{\tau_W(t)},$$

where  $[z^{-1}]$  is the vector  $(z^{-1}, z^{-2}/2, \dots)$ . Introducing the vertex operator

$$X(t, z) = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k}\right)$$

the above formula (2) can be written as

$$\Psi_W(t, z) = \frac{X(t, z)\tau(t)}{\tau(t)}.$$

We often use the formal series  $\Psi_W(x, z) = \Psi_W(t, z)|_{t_1=x, t_2=t_3=\dots=0}$ , which we call again wave function. The wave function, corresponding to the vacuum is

$$\Psi_0(x, z) = e^{xz}.$$

The wave function  $\Psi_W(x, z)$  contains the whole information about  $W$  and hence about  $\tau_W$ , as the vectors  $w_{-k} = \partial_x^k \Psi_W(x, z)|_{x=0}$  form an admissible basis of  $W$  (if we take  $v_k = z^k$  as a basis of  $\mathbb{V}$ ).

### 1.2. Darboux transforms and bispectral operators.

We shall recall a version of Darboux transform from [6]

DEFINITION 1.1. — We say that a plane  $W$  (or the corresponding wave function  $\Psi_W(x, z)$ , the tau-function  $\tau_W$ ) is a Darboux transformation of the vacuum (respectively - of the wave function  $\Psi_0(x, z)$ , the tau-function

$\tau_0$ ) iff there exist polynomials  $f(z)$ ,  $g(z)$  and differential operators  $P(x, \partial_x)$ ,  $Q(x, \partial_x)$  such that

$$\begin{aligned}\Psi_W(x, z) &= \frac{1}{g(z)} P(x, \partial_x) \Psi_0(x, z), \\ \Psi_0(x, z) &= \frac{1}{f(z)} Q(x, \partial_x) \Psi_W(x, z).\end{aligned}$$

The Darboux transformation is called *polynomial* iff the operators  $P(x, \partial_x)$  and  $Q(x, \partial_x)$  have rational coefficients.

Obviously

$$Q(x, \partial_x) P(x, \partial_x) \Psi_0 = g(z) f(z) \Psi_0,$$

denoting the polynomial  $g(z)f(z)$  by  $h(z)$  and recalling that  $\Psi_0 = e^{xz}$  we see that

$$Q(x, \partial_x) P(x, \partial_x) = h(\partial_x).$$

On the other hand the wave function  $\Psi_W$  is an eigen-function of the differential operator

$$L(x, \partial_x) = P(x, \partial_x) Q(x, \partial_x).$$

Notice that the operator  $L$  is a traditional Darboux transform of the operator  $h(\partial_x)$ , which justifies the terminology of the definition. We will also say that the operator  $L$  is a *polynomial Darboux transform* of the operator  $\partial_x$ .

We shall need a second definition of the polynomial Darboux transformation. In the above notation let the polynomial  $h(\partial_x)$  factorize as:

$$h(\partial_x) = \prod_{j=1}^m (\partial_x - \lambda_j)^{d_j},$$

where  $\lambda_j$  are the different roots with multiplicities  $d_j$ . Then the kernel of  $h(\partial_x)$  is given by

$$\ker h(\partial_x) = \bigoplus_{j=1}^m W_j,$$

where

$$W_j = \{e^{\lambda_j x}, x e^{\lambda_j x}, \dots, x^{d_j-1} e^{\lambda_j x}\}.$$

DEFINITION 1.2. — *The Darboux transform is polynomial iff the kernel of  $P$  has the form*

$$\ker P = \bigoplus_{j=1}^m K_j,$$

where  $K_j$  is a linear subspace of  $W_j$ .

The equivalence of the two definitions can be found in [7]. Each nonzero element  $f \in K_j$  will be called (after Wilson) *condition supported at  $\lambda_j$* . The Darboux transform will be called *monomial* iff all the conditions are supported at one point. Finally we recall the bispectral involution  $b$ , which in this case maps the operators with polynomial coefficients in the  $x$ -variable into operators with polynomial coefficients in the  $z$ -variable by the formulas

$$b(\partial_x) = z, \quad b(x) = \partial_z,$$

i.e. in this case  $b$  is the formal Fourier transform. It will be used when the differential operators are applied to  $\Psi_0$  as follows:

$$\partial_x \Psi_0 = z \Psi_0, \quad x \Psi_0 = \partial_z \Psi_0$$

We end this subsection with the following important result of G. Wilson [38]:

**THEOREM 1.3.** — *Any polynomial Darboux transform of  $\partial_x$  is a rank one bispectral operator and vice versa.*

This theorem is formulated by G. Wilson in a different terminology. See [22] for an exposition using Darboux transforms.

Following G. Wilson we will call the set of all planes  $W \subset \text{Gr}$  that are polynomial Darboux transforms of  $W_0$  *the adelic Grassmannian* and denote it by  $\text{Gr}^{\text{ad}}$ . In another paper [39] G. Wilson proved that there is a bijection

$$\beta : \bigcup_{n \geq 0} C_n \rightarrow \text{Gr}^{\text{ad}}$$

between the union of all Calogero-Moser spaces and the adelic Grassmannian. Thus it is enough to prove Theorem 0.2.

### 1.3. $W_{1+\infty}$ -algebra.

In this subsection we recall the definition of  $W_{1+\infty}$ , and some of its bosonic representations introduced in [5]. For more details see [28].

The algebra  $w_\infty$  of the additional symmetries of the KP-hierarchy is isomorphic to the Lie algebra of regular polynomial differential operators on the circle



$$w_\infty \equiv \mathcal{D} = \text{span}\{z^\alpha \partial_z^\beta \mid \alpha, \beta \in \mathbb{Z}, \beta \geq 0\}.$$

It was introduced in [16, 33] and was extensively studied by many authors (see, e.g. [3, 14, 17, 32], *etc.* ). Its unique central extension is denoted by  $W_{1+\infty}$ .

Denote by  $c$  the central element of  $W_{1+\infty}$  and by  $W(A)$  the image of  $A \in \mathcal{D}$  under the natural embedding  $\mathcal{D} \hookrightarrow W_{1+\infty}$  (as vector spaces). The algebra  $W_{1+\infty}$  has a basis

$$c, J_k^l = W(-z^{l+k} \partial_z^l), \quad l, k \in \mathbb{Z}, l \geq 0.$$

In [5] we constructed a family of highest weight modules of  $W_{1+\infty}$ . Here we need the most elementary one of them, for which the next theorem is an easy exercise.

**THEOREM 1.4.** — *The function  $\tau_0$  satisfies the constraints*

$$J_k^l \tau_0 = 0, \quad k \geq 0, l \geq 0,$$

$$W(z^{-k} P_k(D_z) D_z^l) \tau_0 = 0, \quad k > 0, l \geq 0,$$

where  $P_k(D_z) = \prod_{j=0}^{k-1} (D_z - j)$ ,  $D_z = z \partial_z$ .

The first constraint means that  $\tau_0$  is a highest weight vector with highest weight  $\lambda(J_0^l) = 0$  of a representation of  $W_{1+\infty}$  in the module

$$\mathcal{M}_0 = \text{span}\left\{ J_{k_1}^{l_1} \cdots J_{k_p}^{l_p} \tau_0 \mid k_1 \leq \cdots \leq k_p < 0 \right\}.$$

One easily checks that the central charge  $c = 1$ . The second constraint yields that the module  $\mathcal{M}_0$  is quasifinite, *i.e.* it is finite-dimensional in each level.

## 2. An adelic W-algebra.

The adelic W-algebra is a Lie algebra that we intend to introduce in analogy with  $W_{1+\infty}$ . Most of the definitions and constructions are similar to those of  $W_{1+\infty}$ . For that reason we list the facts but skip much of the arguments.

Instead of the Lie algebra  $w_\infty$  of regular operators on the circle we start with the Lie algebra  $\mathcal{RD}$  of differential operators with rational coefficients on the complex line. We are going to use the following basis of  $\mathcal{RD}$ :

- 1)  $z^{n+l}\partial_z^l, \quad n \in \mathbb{Z}, \quad l \geq 0;$
- 2)  $(z - a)^{-n+l}\partial_z^l, \quad -n + l < 0, \quad l \geq 0, \quad a \in \mathbb{C} - \{0\}.$

Usually we shall consider the elements from  $\mathcal{RD}$  as differential operators with coefficients that are Laurent series in  $z^{-1}$  by expanding  $(z-a)^{-n+l}$  around infinity. We would like to construct natural representations of  $\mathcal{RD}$ . We shall work with the space  $\mathbb{V}$  where  $v_k = z^k$ . Obviously  $\mathcal{RD}$  acts naturally on  $\mathbb{V}$ . Then we can associate with each operator  $A \in \mathcal{RD}$  an infinite matrix having only finite number of diagonals below the principal one but having eventually infinite number above it. In other words the matrix  $(a_{i,j})$ , associated with  $A$ , has the property that  $a_{i,j} = 0$  for  $i - j \gg 0$ . The Lie algebra of such matrices will be denoted by  $a'_\infty$ . It can be considered as a completion of the algebra  $a_\infty$  of matrices having only finite number of diagonals (see [29]). Now we explain how to construct representations in the fermionic Fock space  $F^{(0)}$ . We recall that in the case considered here  $F^{(0)}$  consists of formal series of semi-infinite wedge monomials:

$$z^{i_0} \wedge z^{i_1} \wedge z^{i_2} \wedge \dots,$$

with  $i_0 < i_1 < \dots$  and  $i_k = k$  for  $k \gg 0$ . We can define the action of  $A \in a'_\infty$  by the standard definition (see [29]). First for matrices with only finite number of entries define

$$\begin{aligned} r(A)(z^{i_0} \wedge z^{i_1} \wedge \dots) &= Az^{i_0} \wedge z^{i_1} \wedge \dots \\ &\quad + z^{i_0} \wedge Az^{i_1} \wedge \dots \\ &\quad \dots \end{aligned}$$

It is easy to check that if  $A \in a'_\infty$  has no entries on the main diagonal  $r(A)$  still makes sense, the image being infinite formal series. For matrices with infinite number of entries on the main diagonal the above definition is no longer meaningful. For that reason we need to modify it as follows. We put

$$\begin{aligned} \hat{r}(E_{i,j}) &= r(E_{i,j}) \quad \text{for } i \neq j \quad \text{or } i = j > 0; \\ \hat{r}(E_{i,i}) &= r(E_{i,i}) - Id \quad \text{for } i \leq 0. \end{aligned}$$

See [29] for more details.

This defines a representation of the central extension  $a'_\infty \oplus \mathbb{C}c$ . The corresponding central extension of the subalgebra  $\mathcal{RD}$  of  $a'_\infty$  will be called *adelic W-algebra*. We will use the notation  $W^{ad}$ . The terminology and the notation are chosen to be similar to those of the adelic Grassmannian  $Gr^{ad}$ . The main result of the present paper naturally connects the two objects.

We shall describe in some more details  $W^{ad}$ . By  $W(A)$  we shall denote the image of the element  $A \in \mathcal{RD}$  under the natural embedding  $\mathcal{RD} \subset W^{ad}$

(as vector spaces). Then for  $a \in \mathbb{C}$ ,  $l \geq 0$ ,  $n \in \mathbb{Z}$  put

$$(3) \quad J_n^l(a) = W(-(z - a)^{n+l} \partial_z^l).$$

For  $a = 0$  we also shall use the notation  $J_n^l = J_n^l(0)$ . When  $a$  is fixed the above operators (3) together with the central charge  $c$  form a copy of  $W_{1+\infty}$ , which we shall denote by  $W_{1+\infty}(a)$ . Recall that  $W_{1+\infty}(a)$  has a grading: the elements  $J_n^l(a)$  have weight  $n$ . The elements with nonnegative grading are common for all  $a$ . In fact the common part is much larger: for all  $n + l \geq 0$  the elements  $J_n^l(a)$  are common. Thus we have the following basis for  $W^{\text{ad}}$ :

- 1)  $J_n^l(0)$ ,  $l \geq 0$   $n + l \geq 0$ ;
- 2)  $J_n^l(a)$ ,  $n + l < 0$ ,  $a \neq 0$ ;
- 3)  $c$ .

In complete analogy to the case of  $W_{1+\infty}$  we can construct a representation of  $W^{\text{ad}}$  in the Fock spaces using the vacuum. We formulate the needed properties in the bosonic picture.

**THEOREM 2.1.** — *The tau-function  $\tau_0$  satisfies the following constraints*

- 1)  $J_n^l \tau_0 = 0$ ,  $l \geq 0$ ;  $n \geq 0$
- 2)  $W((z - a)^{-k} P_k((z - a) \partial_z))((z - a) \partial_z)^l \tau_0 = 0$ ,

where  $P_k(u) = u(u - 1) \cdots (u - k + 1)$ .

We set

$$W_-^{\text{ad}} = \text{span}\{J_n^l, \quad a \in \mathbb{C}, \quad n < 0\}.$$

Then define the  $W_-^{\text{ad}}$ -module  $\mathcal{M}^{\text{ad}}$  by

$$\mathcal{M}^{\text{ad}} = \text{span}\{J_{n_1}^{l_1}(a_1) \cdots J_{n_m}^{l_m}(a_m) \tau_0, \}$$

where  $n_j + l_j < 0$  for  $a_j \neq 0$  and  $n_j < 0$  for  $a_j = 0$ .

**COROLLARY 2.2.** — *The vector space  $\mathcal{M}^{\text{ad}}$  is a space of representation of the Lie algebra  $W^{\text{ad}}$ .*

*Proof.* — We need only to check that  $J_n^l(J_{n_1}^{l_1}(a_1) \cdots J_{n_m}^{l_m}(a_m) \tau_0) \in \mathcal{M}^{\text{ad}}$ . We proceed by induction. Let  $m = 1$ . Then we have

$$J_n^l J_{n_1}^{l_1}(a_1) = J_{n_1}^{l_1}(a_1) J_n^l + \sum b_k J_{n_k}^{l_k}(a_1) + \sum J_{n_s}^{l_s},$$

where the sums are finite. Then this expression when applied to  $\tau_0$  by Corollary 2.2 obviously gives an element of  $\mathcal{M}^{\text{ad}}$ . Proceeding by induction we can push the elements of the type  $J_n^l$  to the right and again use the theorem. □

### 3. Tau-functions in the module $\mathcal{M}^{\text{ad}}$ .

In this section we prove the first part of Theorem 0.1. More precisely we will prove

**THEOREM 3.1.** — *If  $\tau$  is a tau-function in the module  $\mathcal{M}^{\text{ad}}$  then it is polynomial Darboux transform of  $\tau_0$ .*

Before giving the proof of the theorem we shall recall some results from [8] as well as their modifications for the case of the adelic  $W$ -algebra. We start with the commutation of the elements of  $W^{\text{ad}}$  with the vertex operator  $X(t, z)$ . First we consider elements from  $W_{1+\infty}(0)$ . The following result has been proved in [8]

**LEMMA 3.2.** — *Let  $X(t, z)$  be the vertex operator. Then*

$$X(t, z)J_k^l = \left( J_k^l + lJ_k^{l-1} + \delta_{k,0}\delta_{l,0} - z^{k+l}\partial_z^l \right) X(t, z).$$

See [8] for the proof. In order to give the analog of the above relation for  $J_n^l(a)$  when  $a \neq 0$  we shall introduce some notation. Put

$$(4) \quad J_{n,-1}^l(a) = W(-z^{-1}(z-a)^{n+l}\partial_z^l).$$

The analog of Lemma 3.2 is

**LEMMA 3.3.** — *The commutation relation of the vertex operator  $X(t, z)$  with  $J_n^l(a)$  is*

$$(5) \quad X(t, z)J_n^l(a) = \left( J_n^l(a) + lJ_{n+1,-1}^{l-1}(a) - (z-a)^{n+l}\partial_z^l \right) X(t, z).$$

*Proof.* — The proof is a direct consequence of Lemma 3.2. We shall present  $J_n^l(a)$  as infinite series:

$$J_n^l(a) = W(-(z-a)^{n+l}\partial_z^l) = W\left(\sum_{j=0}^{\infty} \alpha_j z^{n+l-j}\partial_z^l\right) = \sum_{j=0}^{\infty} \alpha_j J_{n-j}^l.$$

Now apply Lemma 3.2 to (3.4) and recall that  $n + l < 0$ . We get

$$X(t, z)J_n^l(a) = \sum_{j=0}^{\infty} \alpha_j (J_{n-j}^l + lJ_{n-j}^{l-1} - z^{n-j+l}\partial_z^l)X(t, z).$$

Then using the notation (4) we can rewrite the last formula as (5). □

We also need the following result from [8] which we reformulate for the present situation.

LEMMA 3.4. — *Let  $\tau \in \mathcal{M}^{\text{ad}}$ . Then  $\tau = u\tau_0$  where  $u$  is an element of the universal enveloping algebra of  $W_-^{\text{ad}}$  of the form:*

$$u = \sum b_{k,l,a} J_{-k_1}^{l_1}(a_1) \cdots J_{-k_p}^{l_p}(a_p) J_{-k_{p+1}}^{l_{p+1}} \cdots J_{-k_{p+r}}^{l_{p+r}},$$

where  $l_j < k_j$ .

Finally we need some facts about the so called *adjoint involution* [37]. It is an involution defined on the planes of Sato’s Grassmannian. Introduce after [13] a non-degenerate bilinear form  $B$  on the space  $\mathbb{V}$  of formal Laurent series in  $z^{-1}$  as follows

$$B(f, g) = -\text{Res}_{\infty} f(z)g(-z), \quad f, g \in \mathbb{V}.$$

If  $V$  is a plane in  $\text{Gr}$  then define  $aV$  to be the plane orthogonal to  $V$  with respect to the bilinear form  $B$ , i.e.

$$aV = \{g(z) \in \mathbb{V} \mid B(f, g) = 0 \quad \text{for all } f \in \mathbb{V}\}.$$

Obviously the adjoint involution has meaning on related objects. On tau-functions it acts as

$$a(\tau_V(t_1, t_2, \dots)) = \tau_V(t_1, -t_2, \dots, (-1)^{n-1}t_n \dots).$$

We can continue this action on the bosonic Fock space by linearity. We need to continue it also on elements of  $W^{\text{ad}}$ .

In [8] we proved that for  $U \in W_{1+\infty}$  it is well defined demanding

$$a(U\tau) = a(U)a(\tau), \quad \tau \in B.$$

We will show that the same result holds for  $U \in W^{\text{ad}}$ . The above definitions show that

$$a(J_k^0) = (-1)^{n-1}J_k^0.$$

Instead of working with the generators  $J_k^l$  of  $W_{1+\infty}$  we will use fields (of dimension  $l$ ) or generating functions:

$$J^l(z) = \sum_{k \in \mathbb{Z}} J_k^l z^{-k-l-1}.$$

In order to recall results from [8] we shall introduce other fields. Following [4] put

$$(7) \quad V^l(z) = \binom{2l}{l} \sum_{k=0}^l (-1)^k \binom{2l-k}{l} \partial^k J^{l-k}(z).$$

Then the modes  $V_k^l$  of the fields  $V^l(z)$ :

$$V^l(z) = \sum_{k \in \mathbb{Z}} V_k^l z^{-k-l-1}$$

obviously also form a basis (together with  $c$ ) of  $W_{1+\infty}$ . In [8] we have proved the following:

LEMMA 3.5. — *The adjoint involution  $a$  acts on  $V_k^l$  by the formula*

$$(8) \quad a(V^l(z)) = V^l(-z).$$

Now we want to continue the adjoint involution  $a$  to the adelic  $W$ -algebra. We have to define it only for  $J_n^l(b)$  with  $l+n < 0$  and  $b \neq 0$ . But first we shall compute  $a(J_n^l)$ . We have

LEMMA 3.6. — *The adjoint involution  $a$  acts on  $J_n^l$  by the formula*

$$(9) \quad a(J_n^l) = (-1)^{n+l+1} \left( J_n^l + \sum_{s=1}^{l-1} \beta_{l,s} (n+l+1) \cdots (n+l-s+1) J_n^{l-s} \right)$$

with some constants  $\beta_{l,s}$  that do not depend on  $n$ .

*Proof.* — We will use induction on  $l$ . For  $l = 0$  we have

$$V^0(z) = J^0(z).$$

Hence in this case (9) holds. Suppose we have proved the formula for some  $l$ . From (7) we have

$$(10) \quad V_n^{l+1} = \sum_{s=0}^{l+1} \alpha_{l+1,s} (n+l-s+2) \cdots (n+l+1) J_n^{l+1-s}$$

with some constants  $\alpha_{l+1,s}$  that do not depend on  $n$  and  $\alpha_{l+1,0} \neq 0$ . Also by (8) we have

$$(11) \quad a(V_n^{l+1}) = (-1)^{n+l+2} V_n^{l+1}.$$

Now plug in (11) the expression for  $V_n^{l+1}$  from (10). Then use linearity and the induction hypothesis. This proves (9). □

The above formula (9) allows us to write down an expression for  $a(J_n^l(b))$  and hence to continue the involution on  $W^{\text{ad}}$ . Recall that  $J_n^l(b)$  should be considered as infinite sum and for each summand  $a$  is already defined by (9).

LEMMA 3.7. — *Let  $n + l < 0$  and  $b \in \mathbb{C}$  be arbitrary. Then the adjoint involution is well defined for  $a(J_n^l(b))$ . More precisely we have* (12)

$$a(J_n^l(b)) = (-1)^{k+l-1} \left( J_n^l(-b) + \sum_{s=1}^l \gamma_{l,s}(n+l) \cdots (n+l-s+1) J_n^{l-s}(-b) \right).$$

*Proof.* — We have

$$J_n^l(b) = \sum_{j=0}^{\infty} \mu_j W(-z^{n+l-j} \partial_z^l)$$

where  $\mu_j$  are the coefficients of the expansion around infinity of  $(z - b)^{n+l}$ :

$$(z - b)^{n+l} = \sum_{j=0}^{\infty} \mu_j z^{n+l-j}.$$

Formula (9) gives

$$\begin{aligned} a\left(\sum_{j=0}^{\infty} \mu_j J_{n-j}^l\right) &= \sum_{j=0}^{\infty} \mu_j a(J_{n-j}^l) \\ &= \sum \mu_j (-1)^{n+l-j-1} \left( J_{n-j}^l + \sum_{s=1}^l \beta_{l,s}(n+l) \right. \\ &\quad \left. \cdots (n+l-s+1) J_{n-j}^{l-s} \right). \end{aligned}$$

Let us consider the sum

$$(13) \quad \sum \mu_j (-1)^{n+l-j-1} (n+l) \cdots (n+l+s-1) J_{n-j}^{l-s}.$$

It is equivalent to

$$\sum \mu_j (-1)^{n+l-j-1} (n+l) \cdots (n+l+s-1) W(-z^{n-j+l-s} \partial_z^{l-s}).$$

In order to simplify the notation we drop  $-\partial^{l-s}$  and the sign  $W$ . So we have

$$\begin{aligned} &\sum \mu_j (-1)^{n+l-j-1} (n+l) \cdots (n+l+s-1) z^{n-j+l-s} \\ &= (-1)^{n+l-1+s} \partial_z^s (z+b)^{k+l} \\ &= (-1)^{n+l+s-1} (n+l) \cdots (n+l-s+1) (z+b)^{n+l-s}. \end{aligned}$$

This shows that (13) is equivalent to

$$(-1)^{n+l+s-1} J_n^{l-s}(-b).$$

Thus we obtain (12). □

Now we are ready to give the proof of Theorem 3.1.

*Proof.* — of Theorem 3.1. Let  $\tau_W = u\tau_0$  where  $u$  is an element of the universal enveloping algebra of the form from Lemma 3.4. We express the wave function  $\Psi_W(t, z)$  in terms of  $u$ :

$$(14) \quad \Psi_W(t, z) = \frac{X(t, z)u\tau_0}{u\tau_0} \Big|_{t_1=x, t_2=\dots=0}.$$

Using Lemma 3.2 and Lemma 3.3 we commute  $u$  and  $X(t, z)$  to obtain

$$\Psi_W = \frac{U(t, z)X(t, z)\tau_0}{u\tau_0} \Big|_{t_1=x, t_2=\dots=0},$$

where

$$\begin{aligned} U(t, z) = & \sum b_{k,l,a} \left( J_{-k_1}^{l_1}(a_1) + l_1 J_{-k_1+1,-1}^{l_1-1} - (z - a_1)^{-k_1+l_1} \partial_z^{l_1} \right) \\ & \dots \left( J_{-k_p}^{l_p}(a_p) + l_p J_{-k_p+1,-1}^{l_p-1} - (z - a_p)^{-k_p+l_p} \partial_z^{l_p} \right) \\ & \dots \left( J_{-k_{p+1}}^{l_{p+1}} + l_{p+1} J_{-k_{p+1}}^{l_{p+1}-1} + \delta_{k_{p+1},0} \delta_{l_{p+1},0} - z^{-k_{p+1}+l_{p+1}} \partial_z^{l_{p+1}} \right) \\ & \dots \left( J_{-k_{p+r}}^{l_{p+r}} + l_{p+r} J_{-k_{p+r}}^{l_{p+r}-1} + \delta_{k_{p+r},0} \delta_{l_{p+r},0} - z^{-k_{p+r}+l_{p+r}} \partial_z^{l_{p+r}} \right). \end{aligned}$$

Now we use that

$$J_{-k}^l \Big|_{t_1=x, t_2=t_3=\dots=0} = x^k \delta_{l+1,k} \quad \text{if } l < k.$$

The point is that there are no differentiations but only multiplications by powers of  $x$ . From the above formula we can derive for  $J_{-k}^l(a)$  and for  $J_{-k+1,-1}^l(a)$  the following ones:

$$\begin{aligned} J_{-k}^l(a) \Big|_{t_1=x, t_2=t_3=\dots=0} &= x^k \delta_{l+1,k}, \\ J_{-k+1,-1}^l(a) \Big|_{t_1=x, t_2=t_3=0} &= 0. \end{aligned}$$

Both formulas follow from the expansion of the l.h.s. as infinite series and the fact that  $l < k$ .

In this way we get

$$\begin{aligned} U(t, z) \Big|_{t_1=x, t_2=t_3=\dots=0} &= \sum b_{k,l,a} \left( x^{k_1} \delta_{l_1+1,k_1} - (z - a_1)^{-k_1+l_1} \partial_z^{l_1} \right) \\ & \dots \left( x^{k_{p+r}} (\delta_{l_{p+r}+1,k_{p+r}} + l_{p+r} \delta_{l_{p+r},k_{p+r}}) - z^{-k_{p+r}+l_{p+r}} \partial_z^{l_{p+r}} \right) \\ &= g(z)^{-1} P(x, z, \partial_z), \end{aligned}$$



where  $g(z)$  is a polynomial with roots at  $a_j$  and 0 and  $P$  is a differential operator in  $z$  with polynomial coefficients in  $x$  and  $z$ . In the same way  $u|_{t_1=x, t_2=t_3=\dots=0} = p(x)$  is a polynomial in  $x$ . Thus (14) is equivalent to

$$\Psi_W = \frac{P(x, z, \partial_z)\Psi_0}{g(z)p(x)}$$

or using the bispectral involution we finally obtain:

$$(15) \quad \Psi_W = \frac{P_1(x, \partial_x)\Psi_0}{g(z)},$$

where  $P_1$  is an operator with rational coefficients.

We need also to express  $\Psi_0(x, z)$  in terms of  $\Psi_W(x, z)$ . We shall use the adjoint involution  $a$ . From Lemma 3.7 we know that  $a(\tau_W) = a(u)\tau_0$  is a tau-function in the module  $\mathcal{M}^{\text{ad}}$ . The last formula (15) gives

$$(16) \quad \Psi_{aW}(x, z) = \frac{P_2(x, \partial_x)\Psi_0}{g_2(z)}$$

with some operator  $P_2$  and a polynomial  $g_2(z)$ . We shall use the following simple lemma:

LEMMA 3.8. — *If the wave functions  $\Psi_W$  and  $\Psi_V$  satisfy*

$$\Psi_W(x, z) = \frac{P(x, \partial_x)\Psi_V(x, z)}{g(z)}$$

then

$$\Psi_{aV}(x, z) = \frac{P^*(x, \partial_x)\Psi_{aW}(x, z)}{g(-z)}.$$

where  $P^*$  is the formal adjoint operator of  $P$ .

The lemma and (16) imply

$$\Psi_0(x, z) = \frac{P_2^*(x, \partial_x)\Psi_W}{g_2(z)},$$

which together with (15) gives the proof of the theorem. □

### 4. The planes of the adelic Grassmannian.

In this section we shall prove the inverse of Theorem 3.1.

THEOREM 4.1. — *If a plane  $W \in \text{Gr}^{\text{ad}}$  then the corresponding tau-function  $\tau_W \in \mathcal{M}^{\text{ad}}$ .*

*Proof.* — To fix the notation let the Darboux transform be given by

$$(17) \quad \Psi_W(x, z) = \frac{P(x, \partial_x)\Psi_0(x, z)}{g(z)}$$

$$(18) \quad \Psi_0(x, z) = \frac{Q(x, \partial_x)\Psi_W(x, z)}{f(z)},$$

where  $Q \circ P = h(\partial_x)$  with some polynomial  $h$ . Let  $\lambda_1, \dots, \lambda_m$  be the different points, where the conditions are supported. Then we can suppose that the polynomial  $h$  is:

$$h(z) = \prod_{j=0}^m (z - \lambda_j)^{d_j}.$$

Denote the degree of  $h$  by  $d$ . Let the number of the conditions supported at the point  $\lambda_j$  be  $r_j$ . Then

$$g(z) = \prod_{j=0}^m (z - \lambda_j)^{r_j}.$$

Put also  $\deg g(z) = r = r_1 + \dots + r_m$ . Let  $\{\Phi_i\}_{i=1, \dots, d}$  be the standard basis of  $\ker h(z)$ , i.e.

$$\{\Phi_i\} = \bigcup_{j=1}^m \{e^{\lambda_j x}, \dots, x^{d_j-1} e^{\lambda_j x}\}.$$

Denote by  $f_1, \dots, f_r$  the functions forming the kernel of the operator  $P$ , i.e. defining the Darboux transform (17)–(18). Then

$$f_l(x) = \sum_{i=1}^d a_{l,i} \Phi_i(x), \quad l = 1, \dots, r.$$

Denote by  $A$  the matrix formed by the above coefficients, i.e.

$$A = (a_{l,i}), \quad l = 1, \dots, r, \quad i = 1, \dots, d.$$

For any  $r$ -element subset  $I\{i_1, \dots, i_r\} \subset \{1, \dots, d\}$  denote by  $A^I$  the following minor of  $A$

$$A^I = (a_{l,i_k})|_{l,k=1, \dots, r}.$$

Put  $\Phi = \{\Phi_{i_1, \dots, i_r}\}$  and

$$\Psi_I(x, z) = \frac{Wr(\Phi_I, \Psi_0)}{g(z)Wr(\Phi_I)}.$$

We need the following formula from [6]:

$$\Psi_W(x, z) = \frac{\sum_I \det A^I Wr(\Phi_I)\Psi_I(x, z)}{\sum_I \det^I Wr(\Phi_I)}.$$

Notice that for any  $I$  we have

$$Wr(\Phi_I, \Psi_0) = e^x \sum \lambda_j r_j P_I(x, \partial_x) = e^x \sum \lambda_j r_j \sum_{j=0}^r p_{I,j}(x) \partial_x^j e^{xz},$$

where  $p_{I,j}(x)$  are polynomials. Also we have:

$$Wr(\Phi_I) = e^x \sum \lambda_j r_j q_I(x),$$

where the polynomial  $q_I(x) = p_{I,r}(x)$ . Notice that the exponential factor is the same everywhere. Then we have

$$\Psi_W = \frac{\sum_I \det A^I P_I(x, \partial_x) e^{xz}}{g(z) \sum_I \det^I q_I(x)}.$$

Among the subsets  $I$  there is one that corresponds to the set of following functions from the kernel of  $h(\partial_x)$

$$\begin{aligned} \tilde{f}_1(x) &= e^{\lambda_1 x}, \dots, \tilde{f}_{r_1}(x) = x^{r_1-1} e^{\lambda_1 x} \\ &\vdots \\ \tilde{f}_{r-r_m+1} &= e^{\lambda_m x}, \dots, \tilde{f}_r = x^{r_m-1} e^{\lambda_r x}. \end{aligned}$$

Denote this subset by  $I_0$ . Notice that  $P_{I_0} = \sum_{j=0}^r \beta_j \partial_x^j$ , where  $\beta_j \in \mathbb{C}$ , i.e.  $P_{I_0}$  is an operator with constant coefficients. It is easy to check that  $P_{I_0} \equiv g(\partial_x)$ . Introduce the matrix  $A_0$  as follows. Let  $I_0 = (i_1^0 < i_2^0 < \dots < i_r^0)$ . Then let  $A_0 = (a_{j,i} \delta_{j,i_k^0})$ . Now consider  $A$  as a deformation of  $A_0$ :

$$A(\epsilon) = \epsilon A + (1 - \epsilon) A_0.$$

Obviously  $a_{j,i}(\epsilon) = a_{j,i}$  if  $i \in I_0$  and  $a_{j,i} = \epsilon a_{j,i}$  otherwise.

Let us consider first the case when  $\det A_0^I \neq 0$ . We can assume that  $\det A_0^I = 1$ . Then the wave function  $\Psi_{W(\epsilon)}$  reads:

$$\Psi_{W(\epsilon)} = \frac{(g(\partial_x) + \sum_{I \neq I_0} \det A^I(\epsilon) P_I(x, \partial_x)) \Psi_0}{g(z) (1 + \sum_{I \neq I_0} \det A^I(\epsilon) q_I(x))}.$$

Using the bispectral involution and dividing by  $g(z)$  we can write the above formula as

$$\Psi_{W(\epsilon)} = \frac{\left(1 + \sum \det A^I \frac{\tilde{P}_I(z, \partial_z)}{g(z)}\right) \Psi_0}{1 + \sum \det A^I(\epsilon) q_I(x)}.$$

Notice that

$$\tilde{P}_I(z, \partial_z) = \sum_{j=0}^r z^j p_{I,j}(\partial_z) = \sum \tilde{p}_{I,j}(z) \partial_z^j.$$

Here  $\deg \tilde{p}_{I,j} \leq r$ . We also have

$$\lim_{z \rightarrow \infty} \frac{\tilde{p}_{I,0}(z)}{g(z)} = p_{I,r}(0) = q_I(0).$$

So we can put

$$1 + \sum \det A^I(\epsilon) \frac{\tilde{P}_I(z, \partial_z)}{g(z)} = 1 + \sum \det A^I(\epsilon) q_I(0) + \sum \det A^I(\epsilon) Q_I(z, \partial_z),$$

where  $W(Q_I(z, \partial_z)) \in W_-^{\text{ad}}$ .

Having in mind that  $\det A^I(\epsilon)$  is a polynomial in  $\epsilon$  without a free term, we can write  $\Psi_{W(\epsilon)}$  in the following form:

$$\Psi_{W(\epsilon)} = \frac{(1 + \sum_{j=1}^s \alpha_j \epsilon^j + \sum_{j=1}^s \beta_j \epsilon^j Q_j(z, \partial_z)) \Psi_0(z, x)}{1 + \sum_{j=1}^s \alpha \epsilon^j + \sum_{j=1}^s \gamma_j \epsilon^j \tilde{q}_j(x)}$$

with  $\tilde{q}_j(0) = 0$ . Now expand the above expression around  $\epsilon = 0$  and then use once again the bispectral involution to get rid of  $x$ -dependence. We get

$$\Psi_{W(\epsilon)} = (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z, \partial_z)) \Psi_0.$$

The important fact here is that all the operators  $P_j(z, \partial_z) \in W_-^{\text{ad}}$ . The standard basis of  $W_0$  is given by  $w_k = \partial_x^k \Psi_0 = z^k, k = 0, 1, \dots$ . We need to find expression for the basis of  $W(\epsilon)$ . We have

$$\partial_x^k \Psi_{W(\epsilon)} = (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z, \partial_z)) w_k.$$

Using the boson-fermion correspondence  $\sigma$  we get the tau-function  $\tau_{W(\epsilon)}$ :

$$\begin{aligned} \tau_{W(\epsilon)} &= \sigma \left( (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z, \partial_z)) w_0 \wedge (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z, \partial_z)) w_1 \wedge \dots \right) \\ &= \tau_0 + \epsilon r(P_1) \tau_0 + \epsilon^2 \left( r(P_2) + \frac{1}{2} r(P_1)^2 - \frac{1}{2} r(P_1^2) \right) \tau_0 + \dots \end{aligned}$$

Notice that the coefficients at the powers of  $\epsilon$  are polynomials in  $r(P_k^j)$ , applied to  $\tau_0$ . Hence all of them belong to the  $W^{\text{ad}}$ -module  $\mathcal{M}^{\text{ad}}$ . We shall show that the entire series belong to it. Once again we need a formula from [6] - this time for the tau-function. We have

$$\tau_{W(\epsilon)} = \frac{1 + \sum_{I \neq I_0} \det A^I(\epsilon) \Delta_I \tau_I}{1 + \sum_{I \neq I_0} \det A^I(\epsilon) \Delta_I},$$

where  $\Delta_I = W r(\Phi_I)(0)$ . We shall use the fact that the tau-function is defined up to a multiplicative constant. So multiplying  $\tau_{W(\epsilon)}$  by the

denominator, which is a polynomial in  $\epsilon$  we get a polynomial in  $\epsilon$  (the numerator), which, having a finite number of terms that belong to  $\mathcal{M}^{\text{ad}}$  by the above argument, itself belongs to  $\mathcal{M}^{\text{ad}}$  for all  $\epsilon$ . In particular for  $\epsilon = 1$  we get that  $\tau_W \in \mathcal{M}^{\text{ad}}$ .

Next consider the case when  $\det A_0^I = 0$ . Put  $A(\zeta) = \zeta A + (1 - \zeta)A_0$ . For all but a finite number of values of  $\zeta$  the expression  $\det A(\zeta) \neq 0$ . The argument above shows that for this values of  $\zeta$  the corresponding tau-function  $\tau_{W(\zeta)} \in \mathcal{M}^{\text{ad}}$ . Again consider the numerator of  $\tau_{W(\zeta)}$ . It is a polynomial in  $\zeta$ . Hence  $\tau_{W(\zeta)} \in \mathcal{M}^{\text{ad}}$  for all values of  $\zeta$ .  $\square$

## BIBLIOGRAPHY

- [1] M. ADLER, J. MOSER, On a class of polynomials connected with the Korteweg-de Vries equation, *Commun. Math. Phys.*, 61 (1978), 1–30.
- [2] H. AIRAULT, H.P. MCKEAN, J. MOSER, Rational and elliptic solutions to the Korteweg-de Vries equation and related many-body problem, *Comm. Pure Appl. Math.*, 30 (1977), 95–148.
- [3] M. ADLER, T. SHIOTA, P. VAN MOERBEKE, A Lax representation for the vertex operator and the central extension, *Commun. Math. Phys.*, 171 (1995), 547–588.
- [4] B.N. BAKALOV, L.S. GEORGIEV, I.T. TODOROV, A QFT approach to  $W_{1+\infty}$ , In: *New Trends in Quantum Field Theory, Proc. of the 1995 Razlog (Bulgaria) Workshop*, A. Ganchev et al. eds., Heron Press, Sofia (1996), 147–158, hep-th/9512160.
- [5] B. BAKALOV, E. HOROZOV, M. YAKIMOV, Tau-functions as highest weight vectors for  $W_{1+\infty}$  algebra, *J. Phys. A: Math. Gen.*, 29 (1996), 5565–5573, hep-th/9510211.
- [6] B. BAKALOV, E. HOROZOV, M. YAKIMOV, Bäcklund-Darboux transformations in Sato's Grassmannian, *Serdica Math. J.*, 4 (1996), q-alg/9602010.
- [7] B. BAKALOV, E. HOROZOV, M. YAKIMOV, Bispectral algebras of commuting ordinary differential operators, *Comm. Mat. Phys.* 190 (1997), 331–373, q-alg/9602011.
- [8] B. BAKALOV, E. HOROZOV, M. YAKIMOV, Highest weight modules over  $W_{1+\infty}$ , and the bispectral problem, *Duke Math. J.*, 93 (1998), 41–72.
- [9] YU. BEREST, G. WILSON, Automorphisms and ideals of the Weyl algebra, *Math. Ann.*, 318-1 (2000), 127–147.
- [10] YU. BEREST, G. WILSON, Ideal classes of the Weyl algebra and noncommutative projective geometry, arXiv.math.AG/0104240 (2001).
- [11] R.C. CANNINGS, M.P. HOLLAND, Right ideals in rings of differential operators, *J. Algebra*, 167 (1994), 116–141.
- [12] F. CALOGERO, Solution of the one-dimensional  $n$ -body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.*, 12 (1971), 419–436.
- [13] E. DATE, M. JIMBO, M. KASHIWARA, T. MIWA, Transformation groups for soliton equations, In: *Proc. RIMS Symp., Nonlinear integrable systems - Classical and*

- Quantum theory, (Kyoto 1981), M. Jimbo, T. Miwa eds., Singapore: World Scientific, (1983), 39–111.
- [14] L. DICKEY, Soliton equations and integrable systems, Singapore: World Scientific, 1991.
  - [15] J.J. DUISTERMAAT, F.A. GRÜNBAUM, Differential equations in the spectral parameter, *Commun. Math. Phys.*, 103 (1986), 177–240.
  - [16] B. FUCHSTEINER, Master-symmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Progr. Theor. Phys.*, 70-6 (1983), 1508–1522.
  - [17] P.G. GRINEVICH, A.Yu. ORLOV E.I. SCHULMAN, On the symmetries of the integrable system, In: *Modern development of the Soliton theory*, A. Fokas, V.E. Zakharov eds., 1992.
  - [18] P.A. GRÜNBAUM, The limited angle reconstruction problem in computer tomography, *Proc. Symp. Appl. Math.*, 27, AMS, L. Shepp ed. (1982), 43–61.
  - [19] F.A. GRÜNBAUM, L. HAINE, E. HOROZOV, Some functions that generalize the Krall-Laguerre polynomials, *J. Comp. Appl. Math.*, 106-2 (1999), 271–297.
  - [20] F.A. GRÜNBAUM, M. YAKIMOV, Discrete bispectral Darboux transformations from Jacobi operators, *arXiv.mat.CA/0012191* (2000).
  - [21] L. HAINE, P. ILIEV, Commutative rings of difference operators and an adelic flag manifold, *Int. Math. Res. Notices*, 6 (2000), 281–323.
  - [22] E. HOROZOV, Dual algebras of differential operators, in: *Kowalevski property (Montréal)*, CRM Proc. Lecture Notes, Surveys from Kowalevski Workshop on Mathematical methods of Regular Dynamics, Leeds, April 2000, Amer. Math. Soc., Providence (2002).
  - [23] E. HOROZOV, The Weyl algebra, bispectral operators and dynamics of poles in integrable systems, *Reg. & Chaotic Dynamics*, 7-4 (2002), 399–424.
  - [24] PL. ILIEV, Algèbres commutatives d'opérateurs aux  $q$ -différences et systèmes de Calogero-Moser, *C. R. Sci. Paris, Série I*, 329 (1999), 877–882.
  - [25] D. KAZHDAN, B. KOSTANT, S. STERNBERG, Hamiltonian group actions and dynamical systems of Calogero type, *Comm. Pure Appl. Math.*, 31 (1978), 481–507.
  - [26] I.M. KRICHEVER, On Rational solutions of Kadomtsev-Petviashvili equation and integrable systems of  $N$  particles on the line, *Funct. Anal. Appl.*, 12-1, (1978), 76–78 (Russian), 59–61 (English).
  - [27] V.G. KAC, D.H. PETERSON, Spin and wedge representations of infinite-dimensional Lie algebras and groups, *Proc. Natl. Acad. Sci. USA*, 78 (1981), 3308–3312.
  - [28] V.G. KAC, A. RADUL, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Commun. Math. Phys.*, 157 (1993), 429–457, hep-th/9308153.
  - [29] V.G. KAC, A. RAINA, Bombay lectures on highest weight representations of infinite dimensional Lie algebras, *Adv. Ser. Math. Phys.*, 2, Singapore: World Scientific, 1987.
  - [30] A. KASMAN, Bispectral KP solutions and linearization of Calogero-Moser particle systems, *Commun. Math. Phys.*, 172 (1995), 427–448.
  - [31] J. MOSER, Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. Math.*, 16 (1975), 197–220.

- [32] A.Yu. ORLOV, Vertex operators,  $\bar{\partial}$ -problem, symmetries, variational identities and Hamiltonian formalism for  $2 + 1$  integrable systems, In: Proc. Kiev Intern. Workshop, Plasma theory and non-linear and turbulent processes in Physics, Baryakhtar ed., Singapore: World Scientific, 1988.
- [33] A.Yu. ORLOV, E.I. SCHULMAN, Additional symmetries for integrable and conformal algebra representation, *Lett. Math. Phys.*, 12 (1989), 171–179.
- [34] M. ROTHSTEIN, Explicit formulas for the Airy and Bessel involutions in terms of Calogero-Moser pairs, In: *The Bispectral Problem*, J. Harnad and A. Kasman ed. CRM Proceedings and Lecture Notes, AMS, Providence, 1998.
- [35] M. SATO, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, *RIMS Kokyuroku*, 439 (1981), 30–40.
- [36] G. SEGAL, G. WILSON, Loop Groups and equations of KdV type, *Publ. Math. IHES*, 61 (1985), 5–65.
- [37] P. VAN MOERBEKE, , Integrable foundations of string theory, CIMPA-Summer school at Sophia-Antipolis (1991), in: *Lectures on integrable systems*, O. Babelon et al. eds., Singapore: World Scientific (1994), 163–267.
- [38] G. WILSON, Bispectral commutative ordinary differential operators, *J. Reine Angew. Math.*, 442 (1993), 177–204.
- [39] G. WILSON, Collisions of Calogero-Moser particles and an adelic Grassmannian (with an appendix by I. G. Macdonald), *Invent. Math.*, 133 (1998), 1–41.

Emil HOROZOV,  
Bulgarian Academy of Science  
Institute of Mathematics and Informatics  
Acad. G. Bonchev Str. Block 8  
1113 Sofia (Bulgaria)  
horozov@fmi.uni-sofia.bg  
horozov@math.bas.bg