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STABILITY RESULTS FOR HARNACK INEQUALITIES

by Alexander GRIGOR'YAN (*) &
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Contents.

1. Introduction.

The aim of this work is to provide new methods for proving uniform elliptic and parabolic Harnack inequalities on Riemannian manifolds, hence providing new classes of Riemannian manifolds satisfying these inequalities.

A celebrated theorem of Moser [39] says that if $u(x)$ is a non-negative (weak) solution to a uniformly elliptic equation (with measurable coefficients)

$$Lu := \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = 0$$

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in a ball $B(x_0, R)$ in $\mathbb{R}^N$, then u satisfies the Harnack inequality

$$\sup_{B(x_0,\frac{1}{2}R)} u \leq C \inf_{B(x_0,\frac{1}{2}R)} u, \tag{1.1}$$

where $C = C(N, \lambda)$ and λ is the constant of ellipticity of the operator L (that is, all the eigenvalues of the symmetric matrix (a_{ij}) are bounded between λ^{-1} and λ). In [40], Moser improved this result by showing that if a non-negative function $u(t, x)$ solves the parabolic equation $\partial_t u = Lu$ in a cylinder $(0, T) \times B(x_0, R)$ with $T = R^2$, then

$$\sup_{(\frac{1}{4}T,\frac{1}{2}T)\times B(x_0,\frac{1}{2}R)} u \leq C \inf_{(\frac{3}{4}T,T)\times B(x_0,\frac{1}{2}R)} u, \tag{1.2}$$

where C again depends only on N and λ. Obviously, the elliptic Harnack inequality (1.1) follows from the parabolic one (1.2).

Moser's results have proven to be extremely useful for the development of the theory of elliptic and parabolic equations. In particular, Aronson [1] used (1.2) to show that the heat kernel $p^L(t, x, y)$, i.e., the fundamental solution of the parabolic equation $\partial_t u = Lu$, satisfies the inequalities

$$\frac{C_2}{t^{N/2}} \exp\Big(-\frac{|x-y|^2}{C_2 t}\Big) \leq p^L(t, x, y) \leq \frac{C_1}{t^{N/2}} \exp\Big(-\frac{|x-y|^2}{C_1 t}\Big).$$

One of the goals of this paper is to obtain uniform Harnack inequalities beyond the realm of uniform ellipticity. As a simple example that illustrates this point, consider the operator

$$Lu = \frac{1}{a(x)} \operatorname{div}\big(a(x)\nabla u\big) = \frac{1}{a(x)} \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(a(x)\frac{\partial u}{\partial x_i}\Big), \tag{1.3}$$

where a is a smooth positive function in $\mathbb{R}^N$. Deciding whether or not L satisfies the elliptic and/or parabolic Harnack inequalities seems to be a difficult question (if not hopeless in full generality). When a and $1/a$ are bounded, L satisfies the elliptic and parabolic Harnack inequalities (the latter follows from a slight extension of Moser's theorem; see, e.g., [42]). Now, assume instead that there exists a constant C such that for all $R > 0$

$$\sup_{\{R\leq|x|\leq 2R\}} a(x) \leq C \inf_{\{R\leq|x|\leq 2R\}} a(x). \tag{1.4}$$

Then it follows from our Theorem 6.10 that the elliptic Harnack inequality (1.1) for the operator (1.3) is satisfied without further assumptions, whereas Theorem 5.7 shows that the parabolic Harnack inequality (1.2) holds if and only if for all $R > 0$

$$\int_0^R \bar{a}(r) r^{N-1}\,\mathrm{d}r \le C\bar{a}(R)R^N, \tag{1.5}$$

where $\bar{a}(r) := \sup\{a(x) : |x| = r\}$. Moreover, under the conditions (1.4) and (1.5) the heat kernel associated with the operator (1.3) satisfies the estimate

$$\begin{aligned}\frac{C_2}{W(x,y,t)} \exp\Big(-\frac{|x-y|^2}{C_2 t}\Big) \\ \le p^L(t,x,y) \le \frac{C_1}{W(x,y,t)} \exp\Big(-\frac{|x-y|^2}{C_1 t}\Big),\end{aligned} \tag{1.6}$$

where $W(x,y,t) = t^{N/2}\big[\,\bar{a}(|x| + \sqrt{t})\,\bar{a}(|y| + \sqrt{t})\,\big]^{1/2}$ (see Corollary 5.12). For example, if $\bar{a}(r)$ is of order r^α for large r where $\alpha \in \mathbb{R}$, then (1.5) is satisfied if and only if $\alpha > -N$.

This example is a particular case of one of our main results, Theorem 5.7, which treats similar questions on Riemannian manifolds. In the setting of complete Riemannian manifolds, denote by (EHI) and (PHI) the elliptic and parabolic Harnack inequalities analogous respectively to (1.1) and (1.2). In this case, $L = \Delta$ is the Laplace-Beltrami operator and the balls $B(x_0, r)$ are geodesic balls (see Section 2.1). Let us briefly describe two more applications of our techniques, to *model manifolds* and to *manifolds with ends.*

Given a suitable function ψ on $\mathbb{R}_+$, denote by M_ψ a *model manifold*, that is the Riemannian manifold $(\mathbb{R}^N, \mathrm{d}s^2)$ where $\mathrm{d}s^2$ is a complete rotationally invariant Riemannian metric defined in polar coordinates (r,θ) by $\mathrm{d}s^2 = \mathrm{d}r^2 + \psi(r)^2\mathrm{d}\theta^2$ (see Section 3.3 for more details). Assume that ψ satisfies for all $r > 0$

$$\sup_{[r,2r]} \psi \le C \inf_{[r,2r]} \psi. \tag{1.7}$$

Then Proposition 4.10 (respectively, Proposition 6.7) characterizes those ψ for which M_ψ satisfies (PHI) (resp. (EHI)). For example, if $\psi(r) = r^\alpha$ for large r where $\alpha \in \mathbb{R}$, then

$$\text{(EHI)} \iff \alpha \le 1 \quad \text{and} \quad \text{(PHI)} \iff -\frac{1}{N-1} < \alpha \le 1.$$

Hence, for $\alpha \leq -1/(N-1)$, we obtain a collection of Riemannian manifolds for which (EHI) holds but (PHI) does not. Earlier examples satisfying (EHI) but not (PHI) are described in [2], [3] and [12] but our examples are by far the simplest and most explicit.

To put our results on manifolds with ends in perspective, let us recall that Cheng and Yau [9] and Li and Yau [34], respectively, show that complete Riemannian manifolds with non-negative Ricci curvature satisfy (EHI) and (PHI). It is a natural question to ask when (EHI) and/or (PHI) hold assuming that the Ricci curvature is non-negative only outside a compact set. One can also ask the same question under other similar hypotheses, for example under the assumption of asymptotically non-negative sectional curvature (see Section 7.5 for the definition). Any of these hypotheses ensures that the manifold has finitely many ends (see [5], [33], [35]). Harmonic functions on such manifolds have been studied, e.g., by Li and Tam [32], [33] and by Kasue [28], but the literature seems to contain no results on (PHI).

Let M be a complete Riemannian manifold with a finite number of ends $E_1, E_2, \ldots, E_n$ (see Section 7 for definition). Fix a point $o \in M$ and denote by $V(r)$ the volume of the ball $B(o,r)$ and by $V_i(r)$ the volume of $B(o,r) \cap E_i$. The following result absorbs all the techniques developed here and can be considered as a culmination of this work.

Theorem 1.1 (= Corollary 7.14). — *Let M be a complete non-compact Riemannian manifold having either* (a) *asymptotically non-negative sectional curvature or* (b) *non-negative Ricci curvature outside a compact set and finite first Betti number. Then M satisfies* (PHI) *if and only if it satisfies* (EHI). *Moreover*, (PHI) *and* (EHI) *hold if and only if either M has only one end or M has more than one end and the functions V and V_i satisfy for large enough r the conditions $V_i(r) \approx V(r)$ (for all indices i) and*

$$\int_1^r \frac{s\,\mathrm{d}s}{V(s)} \approx \frac{r^2}{V(r)}. \tag{1.8}$$

(Here the relation $f \approx g$ means that the ratio of positive functions f and g is bounded between two positive constants, for a specified range of the arguments.) For example, if $V(r) \approx r^\alpha$, then (1.8) holds if and only if $\alpha < 2$. The conclusion of Theorem 1.1 is new even for manifolds with non-negative sectional curvature outside a compact set, despite the fact that harmonic functions on such manifolds have been intensively studied.

Let us briefly describe the structure of the paper. For the sake of various applications, we work in the more general setup of *weighted manifolds* introduced in Section 2. A weighted manifold is a Riemannian manifold $(M, \boldsymbol{g})$ equipped in addition with a measure μ. Elliptic and parabolic Harnack inequalities are stated for a weighted Laplace operator Δ_μ using the balls defined in terms of the Riemannian distance.

Section 3 develops one of the main technical tools used in this paper – a discretization procedure. Let an open set $U \subset M$ be covered by open sets $A_1, A_2, \ldots, A_n$, and let Γ be a graph that describes in a certain way the combinatorial structure of the covering. Then Theorem 3.7 provides an estimate of the spectral gap of U in terms of the spectral gaps of the A_i's and the discrete spectral gap of the graph Γ.

Section 4 introduces another useful tool – the notions of *remote* and *anchored* balls, which play a crucial role in this paper. A ball $B(x, r)$ is remote with respect to a fixed point $o \in M$ if the distance $d(o, x)$ is much larger than r, and is anchored if $x = o$. We show that a number of interesting properties (used in proofs of Harnack inequalities) hold for all balls whenever they hold for remote and anchored balls.

Section 5 contains our main technical result – Theorem 5.2. Given a point $o \in M$, we consider two geometric hypotheses relative to o, called the *relative connectedness of annuli* (RCA) and the *volume comparison* (VC) (see Definitions 5.1 and 4.3, respectively), which can be effectively verified in many cases of interest. The condition (VC) was introduced by Li and Tam [33] where it also played a significant role. Theorem 5.2 asserts that if M satisfies (VC), (RCA), as well as the parabolic Harnack inequality (PHI) in remote balls, then (PHI) holds in all balls. Note that, in fact, (VC) is a necessary condition for (PHI) whereas (RCA) is close to be necessary.

A major difficulty in the proof of Theorem 5.2 lies in obtaining a spectral gap estimate for anchored balls, assuming that such an estimate holds for all remote balls. This is done by exploiting the discretization technique of Section 3 and condition (RCA).

We use Theorem 5.2 to prove the stability of (PHI) under a change of measure given by $\mathrm{d}\widetilde{\mu} = \sigma^2 \,\mathrm{d}\mu$ where σ is a smooth positive function on M. For example, if $M = \mathbb{R}^N$, μ is the Lebesgue measure and $\sigma^2 = a$, then the operator (1.3) coincides with $\Delta_{\widetilde{\mu}}$. In general, given (RCA) and (PHI) for Δ_μ, we obtain (PHI) for $\Delta_{\widetilde{\mu}}$ provided the function σ^2 satisfies conditions similar to (1.4) and (1.5) (Theorem 5.7).

Section 6 develops results similar to Theorems 5.2, 5.7, but for the elliptic Harnack inequality (EHI). The methods here are elementary and do not require the discretization tools.

Section 7 contains results concerning manifolds with ends. Theorem 7.1 states necessary and sufficient conditions for (PHI) on manifolds with ends assuming that each end satisfies (PHI) and (RCA). The proof is quite involved and uses the results of Section 6. Combining Theorems 7.1 and 5.2 with known results about (VC) and (RCA), we obtain Theorem 1.1.

The dependencies between sections are given on the following diagram (apart from some additional cross-links between subsections):

$$\begin{tikzcd}
2 \arrow[dr] & 3 \arrow[d] & 4 \arrow[dl] \arrow[dr] & \\
& 5 \arrow[dr] & & 6 \arrow[dl] \\
& & 7 &
\end{tikzcd}$$

2. Preliminaries.

2.1. Harnack inequalities on weighted manifolds.

Let $(M, \boldsymbol{g})$ be a smooth Riemannian manifold of dimension N, possibly with boundary δM (we denote the boundary of manifold by δM rather than ∂M since the letter ∂ is reserved to denote a boundary in the topological sense). Given a smooth positive function σ on M, define a measure μ on M by $\mathrm{d}\mu = \sigma^2 \, \mathrm{d}\mu_0$ where μ_0 is the Riemannian measure of the metric $\boldsymbol{g}$. Depending on the context, we will use various notation for weighted manifold, such as $(M, \boldsymbol{g}, \mu)$, (M, μ), or even M.

The *weighted Laplace operator* Δ_μ is a second order differential operator on M given by

$$\Delta_\mu f := \sigma^{-2} \operatorname{div}(\sigma^2 \nabla f) = \operatorname{div}_\mu(\nabla f), \tag{2.1}$$

where ∇ is the Riemannian gradient, div is the Riemannian divergence, and div_μ is the *weighted divergence* defined by $\operatorname{div}_\mu \vec{v} = \sigma^{-2} \operatorname{div}(\sigma^2 \, \vec{v}\,)$. For example, if $\sigma \equiv 1$, then Δ_μ coincides with the Laplace-Beltrami operator $\Delta = \operatorname{div} \circ \nabla$. Note that for all $u, v \in \mathcal{C}_0^2(M)$ we have

$$\int_M v \Delta_\mu u \, \mathrm{d}\mu = -\int_M \boldsymbol{g}(\nabla u, \nabla v) \, \mathrm{d}\mu - \int_{\delta M} \frac{\partial u}{\partial \boldsymbol{n}} v \, \mathrm{d}\mu', \tag{2.2}$$

where $\boldsymbol{n}$ is the inward normal unit vector field on δM, and μ' is the surface area on δM having the density σ^2 with respect to the Riemannian surface area.

The metric $\boldsymbol{g}$ induces a geodesic distance function $d(x,y)$ which turns M into a metric space. We say that a weighted manifold $(M,\boldsymbol{g},\mu)$ is complete if the metric space (M,d) is complete. Let $B(x,r)=\{y\in M : d(x,y)<r\}$ be the open d-ball centered at $x\in M$ and of radius $r>0$. Note that M is complete if and only if all balls are precompact.

We say that a function $u(x)$ defined in a region $\Omega\subset M$ is *harmonic* if $\Delta_\mu u=0$ in Ω. If the boundary δM is non-empty, then we require in addition that u satisfies the Neumann boundary condition on $\delta\Omega := \Omega\cap\delta M$, that is

$$\frac{\partial u}{\partial \boldsymbol{n}}\Big|_{\delta\Omega}=0. \tag{2.3}$$

(Observe that by (2.2) the operator Δ_μ is symmetric with respect to the measure μ for functions satisfying the Neumann boundary condition on δM.) Let $I\subset\mathbb{R}$ be an interval. We say that a function $u(t,x)$ defined in $I\times\Omega$ satisfies the *heat equation* if $\partial_t u=\Delta_\mu u$ in $I\times\Omega$ and u satisfies (2.3) in $I\times\delta\Omega$.

In this context, *Harnack inequalities* relate the properties of solutions of elliptic and parabolic PDE's on M to its metric properties.

Definition 2.1. — Let $\mathcal{F}$ be an arbitrary family of balls in a weighted manifold M. Fix $\eta\in(0,1)$. We say that $\mathcal{F}$ satisfies the *elliptic Harnack inequality* (EHI_η) if there exists a constant C_H such that for any ball $B(x,r)\in\mathcal{F}$, any positive harmonic function in $B(x,r)$ satisfies

$$\sup_{B(x,\eta r)} u\le C_H \inf_{B(x,\eta r)} u. \tag{2.4}$$

We will use the notation (EHI) when the value of the parameter η is unimportant. If $\mathcal{F}$ is the family of all balls in M, then we say that (M,μ) satisfies (EHI_η) (or simply (EHI)).

Clearly, (EHI_η) $\Rightarrow$ ($\mathrm{EHI}_{\eta'}$) when $\eta'<\eta$ for any family $\mathcal{F}$. If we assume that for any $\epsilon\in(0,1)$ there is an integer Q_ϵ such that any ball of radius r can be covered by at most Q_ϵ balls of radius ϵr, then the Harnack inequalities (EHI_η) with different η are all equivalent if $\mathcal{F}$ is the family of all balls in M. This follows from a simple covering argument. This applies in particular when (M,μ) satisfies the doubling volume property (VD) (cf. Definition 2.4 and Example 3.2).

Definition 2.2. — Let $\mathcal{F}$ be an arbitrary family of balls in M. We say that $\mathcal{F}$ satisfies the *parabolic Harnack inequality* (PHI) if there exists a constant C_H such that for any ball $B(x,r) \in \mathcal{F}$, any positive solution u of the heat equation in the cylinder $Q := (0,t) \times B(x,r)$ with $t = r^2$ satisfies

$$\sup_{Q_-} u \le C_H \inf_{Q_+} u, \tag{2.5}$$

where $Q_- = (\frac{1}{4}t, \frac{1}{2}t) \times B(x, \frac{1}{2}r)$ and $Q_+ = (\frac{3}{4}t, t) \times B(x, \frac{1}{2}r)$ (see Figure 1). If $\mathcal{F}$ is the family of all balls in M, we say that (M,μ) satisfies (PHI).

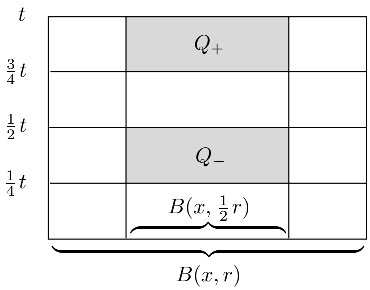

Figure 1. *Cylinder* $Q = (0, t) \times B(x, r)$

Remark 2.3. — One can introduce shrinking parameters (in time and space) in (PHI) in a way that mimics (EHI_η) (see, e.g., [46, Theorem 5.4.3]). It turns out that any two choices of shrinking parameters in (PHI) leads to equivalent inequalities when they are considered for all balls. This follows from Theorem 2.7 below. Clearly, (EHI_η) follows from the proper version of (PHI) involving the same shrinking parameter η in space. Hence (PHI) implies (EHI_η) for any fixed $\eta \in (0, 1)$.

It is well known that (EHI) and (PHI) hold on any complete Riemannian manifold M with non-negative Ricci curvature (see [9] and [34]). In fact, the parabolic Harnack inequality (PHI) can be characterized in terms of the *volume doubling property* and *Poincaré inequality.*

Definition 2.4. — Let $\mathcal{F}$ be a family of d-balls in M. We say that $\mathcal{F}$ satisfies the *volume doubling property* (VD) with constant C_D if, for any ball $B(x,r) \in \mathcal{F}$,

$$\mu\big(B(x,r)\big) \le C_D \mu\big(B(x, \tfrac{1}{2}r)\big). \tag{2.6}$$

If all balls in M satisfy (2.6), then we say that (M,μ) satisfies (VD).

Definition 2.5. — Let $U' \subset U$ be open subsets of M. The *Poincaré constant* $\Lambda(U',U)$ of the couple (U',U) is the smallest positive number such that, for all $f \in \mathcal{C}^1(U)$,

$$\inf_{\xi\in\mathbb{R}} \int_{U'} (f-\xi)^2 \,\mathrm{d}\mu \le \Lambda(U',U) \int_U |\nabla f|^2 \,\mathrm{d}\mu. \tag{2.7}$$

Definition 2.6. — Let $\mathcal{F}$ be a family of d-balls in M. We say that $\mathcal{F}$ satisfies the *Poincaré inequality* (PI) with *parameter* $0<\delta\le 1$ and with constant $C_P>0$ if, for any ball $B(x,r)\in\mathcal{F}$,

$$\Lambda\big(B(x,\delta r),B(x,r)\big) \le C_P\, r^2. \tag{2.8}$$

If all balls in M satisfy (2.8), then we say that (M,μ) satisfies (PI).

For a precompact open set U, the second Neumann eigenvalue (or the *spectral gap*) $\lambda(U)$ is defined by

$$\lambda(U) := \inf\left\{ \frac{\int_U |\nabla f|^2 \,\mathrm{d}\mu}{\int_U |f|^2 \,\mathrm{d}\mu} : f\in\mathcal{C}^1(U)\setminus\{0\},\ \int_U f\,\mathrm{d}\mu = 0 \right\}. \tag{2.9}$$

Comparing with the Definition 2.5, we see that $\Lambda(U,U) = 1/\lambda(U)$. Hence, a family of balls $\mathcal{F}$ satisfies (PI) with parameter $\delta=1$ if and only if, for any ball $B(x,r)\in\mathcal{F}$, $\lambda(B(x,r)) \ge 1/(C_P r^2)$.

It is known (see for example [14] , [6] , [8]) that the heat equation on a weighted manifold M always admits a well defined *heat kernel* $p(t,x,y)$ which, by definition, is the unique minimal positive fundamental solution of the heat equation (if δM is non-empty, then the heat kernel satisfies in addition the Neumann boundary condition on δM).

The notions introduced above are related as follows.

Theorem 2.7. — *For any complete weighted manifold (M,μ) the following properties are equivalent.*

1) *(M,μ) satisfies the parabolic Harnack inequality* (PHI).

2) *(M,μ) satisfies the doubling property* (VD) *and the Poincaré inequality* (PI) *for some/all $\delta\in(0,1]$.*

3) *The heat kernel of (M,μ) satisfies the following two-sided estimate*

$$\frac{c}{V(x,\sqrt{t})}\exp\Big(-C\frac{d^2(x,y)}{t}\Big) \le p(t,x,y) \le \frac{C}{V(x,\sqrt{t})}\exp\Big(-c\frac{d^2(x,y)}{t}\Big) \tag{2.10}$$

for all $x,y\in M$, $t>0$ and some $C,c>0$, where $V(x,r)=\mu(B(x,r))$.

Remark 2.8. — The term $V(x,\sqrt{t}\,)$ in the both sides of (2.10) can be replaced by $(V(x,\sqrt{t}\,)V(y,\sqrt{t}\,))^{1/2}$, using the symmetry of the heat kernel.

The main part of Theorem 2.7 — the implication 2) $\Rightarrow$ 1) — was proved in [17] and [43]. The implication 1) $\Rightarrow$ 2) was proved in [43]. The equivalence 1) $\Leftrightarrow$ 3) goes back to [15] (see also [44] and [46]). The equivalence of the cases $\delta < 1$ and $\delta = 1$ in 2) was proved in [26] (see also [37], [46]).

Theorem 2.7 implies easily that the parabolic Harnack inequality (PHI) is stable under changes of measure $\mathrm{d}\widetilde{\mu} = h^2 \mathrm{d}\mu$ provided h, h^{-1} are positive and bounded, and under changes of metric as long as the new metric $\widetilde{\boldsymbol{g}}$ satisfies $\widetilde{\boldsymbol{g}} \approx \boldsymbol{g}$, that is, $\widetilde{\boldsymbol{g}}$ is quasi-isometric to $\boldsymbol{g}$. In fact, under some weak bounded geometry assumptions, (PHI) is preserved under rough-isometries (see e.g., [11]). Whether or not such a stability holds true for (EHI) is not known yet.

2.2. Technical results concerning volume, Poincaré and Harnack inequalities.

Throughout this section (M,μ) is a weighted manifold. We collect here some useful technical results. The next two lemmas are well known (see, e.g., [17], [46] for proofs). For any set U and $r > 0$ set

$$\mathcal{F}(U,r) = \big\{B(z,s):\ z \in U, s \le r\big\}.$$

If $U = \{x\}$, we write $\mathcal{F}(x,r) = \mathcal{F}(\{x\},r)$.

Lemma 2.9. — *Fix $x \in M$, $r > 0$. Assume that the family of balls $\mathcal{F}(x,r)$ satisfies* (VD) *with constant C_D. Set $\alpha = \log_2 C_D$. Then, for all $0 < s < t \le r$,*

$$\frac{V(x,t)}{V(x,s)} \le C_D\Big(\frac{t}{s}\Big)^{\alpha} \tag{2.11}$$

and, for all $y \in B(x,r)$ and all $0 < s < t$ such that $B(y,t) \subset B(x,r)$,

$$\frac{V(y,t)}{V(x,s)} \le C_D\Big(\frac{t + d(x,y)}{s}\Big)^{\alpha}. \tag{2.12}$$

Lemma 2.10. — *Fix $x \in M$, $r > 0$ and assume that $B = B(x,r) \neq M$. Assume that the family of balls $\mathcal{F}(B,r)$ satisfies* (VD) *with constant C_D.*

Then, for all $0 < s < t \leq r$,

$$\frac{V(x,t)}{V(x,s)} \geq c_1 \Big(\frac{t}{s}\Big)^{\beta} \tag{2.13}$$

where $c_1 = (1 + 1/C_D)^{-1}$ *and* $\beta = \log_3(1 + 1/C_D)$.

Let $U \subset M$ be a precompact open set, and consider the Dirichlet form $D_U^N(u,v)$ obtained as the closure of the form

$$\int_U \boldsymbol{g}(\nabla u, \nabla v)\, \mathrm{d}\mu, \quad u,v \in \mathcal{C}^1(U).$$

Denote by $p_U^N(t,x,y)$, $t > 0$, $x,y \in U$, the heat kernel of the infinitesimal generator of D_U^N. If U has smooth boundary, then the infinitesimal generator of D_U^N can be identified as the weighted Laplacian Δ_μ with the Neumann boundary condition on $\delta U \cup \partial U$. Then p_U^N is the heat kernel of the weighted manifold $\overline{U}$ with the boundary $\delta U \cup \partial U$.

The following lemma is taken from [30] (see also [46]).

Lemma 2.11. — *Let* $U' \subset U \subset M$ *be precompact open sets. Assume that, for some fixed* $t > 0$ *and all* $x,y \in U'$, *we have* $p_U^N(t,x,y) \geq c/\mu(U')$. *Then* $\Lambda(U',U) \leq 2t/c$.

Finally, we will need the following local version of the equivalence 1) $\Leftrightarrow$ 2) in Theorem 2.7.

Theorem 2.12. — *Fix a ball* $B = B(x,r) \subset M$ *and set* $B' = B(x, \frac{1}{2}r)$, $B_c^* = \{z : d(x,z) \leq 2r\}$.

(i) *Assume that* (PHI) *holds for the family* $\{B,B'\}$ *made of the two balls* B,B', *with constant* C_H. *Then the ball* B *satisfies* (VD) *with constant* $C_D = C_H^4$ *and* (PI) *with parameter* $\delta = \frac{1}{2}$ *and constant* $C_P = \frac{5}{2} C_H^5$.

(ii) *Assume that the closed ball* B_c^* *is compact and that the family* $\mathcal{F}(B_c^*,r)$ *satisfies* (VD) *with constant* C_D *and* (PI) *for parameter* $0 < \delta < 1$ *and constant* C_P. *Then* $\mathcal{F}(B,r)$ *satisfies* (PHI) *with constant* C_H *depending only on* C_D, C_P, δ.

Proof. — These results follows from [17], [43], [44] although they are not stated explicitly in this form there. The proof of (ii) is technical and relies on some iteration scheme ([43] uses Moser's well-known scheme whereas [17] uses a different argument, originated from [31]). We only sketch the proof of (i). Consider the function

$$u(t,z) = \begin{cases} \int_{B'} p(t,z,.)\,\mathrm{d}\mu, & \text{if } t>0,\\ 1 & \text{if } t\le 0.\end{cases}$$

One can show that u is a solution of the heat equation in $(-\infty,+\infty)\times B'$. Applying (PHI) in

$$\left(-\tfrac{1}{4}t, \tfrac{3}{4}t\right)\times B' \quad \text{and} \quad \left(\tfrac{1}{2}t, \tfrac{3}{2}t\right)\times B'$$

with $t = (\frac{1}{2}r)^2$ yields

$$1 = u(-0,x) \le C_H^2 \int_{B'} p\left(\tfrac{3}{2}t, x, .\right)\mathrm{d}\mu.$$

Applying (PHI) in $(0,4t)\times B$ to the function $(s,z)\mapsto p(s,x,z)$ gives

$$\int_{B'} p\left(\tfrac{3}{2}t,x,.\right)\mathrm{d}\mu \le C_H\, p(3t,x,x)V\left(x,\tfrac{1}{2}r\right).$$

As the function $s\mapsto p(s,x,x)$ is non-increasing, we obtain

$$p(t,x,x) \ge \frac{C_H^{-3}}{V(x,\frac{1}{2}r)}\cdot \tag{2.14}$$

Now, applying (PHI) in $(0,4t)\times B$ to the heat kernel, we also have

$$V(x,r)\,p(t,x,x) \le C_H \int_{B(x,r)} p(3t,x,.)\,\mathrm{d}\mu \le C_H.$$

This, together with (2.14), gives $V(x,r)\le C_H^4 V(x,\frac{1}{2}r)$ as desired.

The first part of the proof above also applies to the Neumann heat kernel p_B^N and gives

$$p_B^N(t,x,x) \ge \frac{C_H^{-3}}{V(x,\frac{1}{2}r)} = \frac{C_H^{-3}}{\mu(B')}\cdot$$

Applying (PHI) in $(0,4t)\times B$ and $(2t,6t)\times B$, we get for all $y,z\in B'$

$$p_B^N(5t,y,z) \ge \frac{C_H^{-5}}{\mu(B')}\cdot$$

The desired Poincaré inequalities then follow from Lemma 2.11. $\square$

3. Discretization.

3.1. Good coverings.

This section presents a very general discretization scheme. The main result is Theorem 3.7.

DEFINITION 3.1. — Let (M,μ) be a weighted manifold. Let $U,U^{\#}$ be two subsets of M with $U \subset U^{\#}$. Let $\mathcal{A} = \{(A_i,A_i^*,A_i^{\#})\}_{i\in I}$ be a finite family of triplets of sets such that $A_i \subset A_i^* \subset A_i^{\#}$ for any $i \in I$, where I is an index set. We say that $\mathcal{A}$ is a *good covering* of U in $U^{\#}$ with constants Q_1,Q_2 if the following properties are satisfied:

(d0) $U \subset \bigcup_{i\in I} A_i$ and $\bigcup_{i\in I} A_i^{\#} \subset U^{\#}$.

(d1) There exists Q_1 such that, for each $i \in I$

$$\operatorname{card}\big\{j \in I\colon A_j^{\#} \cap A_i^{\#} \neq \emptyset\big\} \leq Q_1. \tag{3.1}$$

(d2) If $d(A_i,A_j) = 0$, then there exists $k = k(i,j) \in I$ such that $A_i \cup A_j \subset A_k^*$.

(d3) There exists Q_2 such that, for all $i,j \in I$, if $d(A_i,A_j) = 0$ and $k = k(i,j)$ as in (d2), then

$$\mu(A_k^*) \leq Q_2 \min\big\{\mu(A_i),\mu(A_j)\big\}. \tag{3.2}$$

Some comments are in order. Condition (d1) implies, in particular, that no more than Q_1 of the sets $A_i^{\#}$ can overlap at a given point. The measure μ plays a role only in (d3). Note that if $d(A_i,A_j) = 0$, then A_i, A_j are contained in $A^*_{k(i,j)}$ and (d3) implies that $\mu(A_i) \approx \mu(A_j) \approx \mu(A^*_{k(i,j)})$.

Example 3.2. — Let U be a precompact set in M. Fix $s > 0$ and set $A_i = B(x_i,s)$ where $\{x_i\}_{i=1}^n$ is a maximal set of points of U at distance at least s from each other (i.e., an s-net in U). Clearly, all sets A_i cover U. Fix constants $b \geq a \geq 3$, set

$$A_i^* = B(x_i, as), \quad A_i^{\#} = B(x_i, bs)$$

and observe that $\mathcal{A} = \{(A_i, A_i^*, A_i^{\#})\}_{i=1}^n$ is a good covering of U in any set $U^{\#}$ containing the set $\{x \in M : d(x,U) < bs\}$ (see Figure 2).

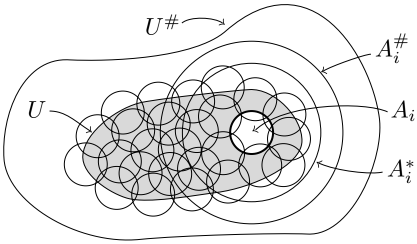

Figure 2. *A good covering by balls*

This is a classical discretization procedure (see e.g., [11], [23], [27]). The constant Q_1 depends on further geometric information, and Q_2 depends of course on the unspecified measure μ. For example, if (M, μ) satisfies (VD), then both Q_1 and Q_2 are bounded in terms of the constant from (VD). Indeed, (3.2) follows immediately from (VD) and Lemma 2.9 since we can set $k(i, j) = i$. To verify (3.1) observe that all balls $A_j^{\#} = B(x_j, bs)$ such that $A_j^{\#} \cap A_i^{\#} \neq \emptyset$ are covered by the ball $B(x_i, 3bs)$. On the other hand, all balls $B(x_j, \frac{1}{2}s)$ are disjoint, and by (VD) and Lemma 2.9, $\mu(B(x_i, 3bs)) \leq CB(x_j, \frac{1}{2}s)$, where C depends only on the doubling constant and b. Therefore, the number of balls $B(x_j, \frac{1}{2}s)$ is at most C, which implies (3.1) with $Q_1 = C$.

Example 3.3. — Fix a point $o \in M$ and let $\{\rho_i\}_{i=1}^{\infty}$ be an increasing sequence of positive numbers. Extend the sequence $\{\rho_i\}$ by setting $\rho_i = 0$ for all $i \leq 0$, and define the sets $A_i, A_i^*, A_i^{\#}$ for all i by

$$A_i = B(o, \rho_i) \setminus B(o, \rho_{i-1}),$$
$$A_i^* = A_i^{\#} = A_{i-1} \cup A_i = B(o, \rho_i) \setminus B(o, \rho_{i-2}).$$

Then $\mathcal{A} = \{(A_i, A_i^*, A_i^{\#})\}_{i=1}^{n}$ is a good covering of $U = B(o, \rho_n)$ in U. Note that $A_i^{\#} \cap A_j^{\#} \neq \emptyset$ implies $|i - j| \leq 1$. Hence we can take $Q_1 = 3$. Clearly, one can take $k(i, i-1) = i$ but the value of Q_2 will depends on the choice of the measure μ.

A modification of this construction gives another good covering with A_i as above whereas

$$A_i^* = A_{i-1} \cup A_i \cup A_{i+1} = B(o, \rho_{i+1}) \setminus B(o, \rho_{i-2}),$$
$$A_i^{\#} = A_{i-1}^* \cup A_i^* \cup A_{i+1}^* = B(o, \rho_{i+2}) \setminus B(o, \rho_{i-3})$$

(see Figure 3). This covering will be used in the proof of the main Theorem 5.2.

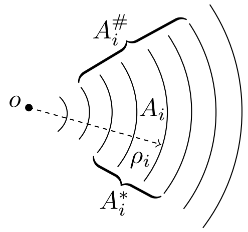

Figure 3. A good covering by annuli

DEFINITION 3.4. — Given a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i\in I}$ of a set U in $U^\#$, define the *continuous Poincaré constant* $\Lambda_c(\mathcal{A})$ of $\mathcal{A}$ by

$$\Lambda_c(\mathcal{A}) = \max\left\{\Lambda(A_i, A_i^*), \Lambda(A_i^*, A_i^\#) : i \in I\right\}. \tag{3.3}$$

In words, $\Lambda_c = \Lambda_c(\mathcal{A})$ is the smallest constant such that, for all $i \in I$, the Poincaré inequalities

$$\int_{A_i} |f - f_{A_i}|^2 \, \mathrm{d}\mu \le \Lambda_c \int_{A_i^*} |\nabla f|^2 \, \mathrm{d}\mu \quad \text{for any } f \in \mathcal{C}^1(A_i^*), \tag{3.4}$$

$$\int_{A_i^*} |f - f_{A_i^*}|^2 \, \mathrm{d}\mu \le \Lambda_c \int_{A_i^\#} |\nabla f|^2 \, \mathrm{d}\mu \quad \text{for any } f \in \mathcal{C}^1(A_i^\#), \tag{3.5}$$

hold true. Here f_A stands for the μ-mean of f over the set A, that is,

$$f_A := \frac{1}{\mu(A)} \int_A f \, \mathrm{d}\mu.$$

When dealing with Poincaré inequalities, it is useful to note that, for any precompact open set A and any $f \in L^1(A)$, we have the identities

$$\begin{aligned} \int_A |f - f_A|^2 \, \mathrm{d}\mu &= \frac{1}{2\mu(A)} \int_A \int_A \left|f(x) - f(y)\right|^2 \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= \inf_{c\in\mathbb{R}} \int_A |f - c|^2 \, \mathrm{d}\mu. \end{aligned} \tag{3.6}$$

In particular, one can replace the left hand sides of (3.4)–(3.5) by the equivalent expressions from (3.6).

Let $(\boldsymbol{V}, \boldsymbol{E})$ be a finite graph, that is, $\boldsymbol{V}$ is a finite set of *vertices*, and $\boldsymbol{E} \subset \boldsymbol{V} \times \boldsymbol{V}$ is a set of *edges*, each edge being a pair of vertices. Let $m(x)$ be a positive function on $\boldsymbol{V}$ considered as a measure on $\boldsymbol{V}$ by setting

$m(U) = \sum_{x\in U} m(x)$ for any set $U \subset \boldsymbol{V}$. Let $m(x,y)$ be a positive function on edges (x,y), which is similarly extended to a measure on $\boldsymbol{E}$ (although we denote the measure on $\boldsymbol{V}$ and on $\boldsymbol{E}$ by the same letter m, a priori these two measures may not be related to each other). It is convenient to extend $m(x,y)$ to a function on all pairs (x,y) by setting $m(x,y) = 0$ whenever $(x,y) \notin \boldsymbol{E}$. We refer to the structure $(\boldsymbol{V}, m)$ as a *weighted graph*, where m here stands for the both measures (note that the function $m(x,y)$ contains information about the edges).

Denote by $\|f\|_p$ the norm of a function f in the space $\ell^p(\boldsymbol{V}, m)$, that is

$$\|f\|_p = \Big(\sum_{x\in\boldsymbol{V}} |f(x)|^p m(x)\Big)^{1/p}.$$

Define an energy form in $\ell^2(\boldsymbol{V}, m)$ by

$$\mathcal{E}(f,f) := \frac{1}{2}\sum_{x,y\in\boldsymbol{V}} \big|f(x) - f(y)\big|^2 m(x,y). \tag{3.7}$$

Analogously to (2.9), the *spectral gap* $\lambda\Big(\boldsymbol{V}, m)$ is defined by

$$\lambda(\boldsymbol{V}, m) = \inf\left\{\frac{\mathcal{E}(f,f)}{\|f\|_2^2} : f \in \ell^2(\boldsymbol{V}, m)\setminus\{0\},\ m(f) = 0\right\}, \tag{3.8}$$

where

$$m(f) = \frac{1}{m(\boldsymbol{V})}\sum_{k\in\boldsymbol{V}} f(k)m(k) \tag{3.9}$$

is the mean value of f with respect to the weight m.

With any good covering we associate a weighted graph $(\boldsymbol{V}, m)$ as follows.

Definition 3.5. — Given a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^{\#})\}_{i\in I}$ of a set U in $U^{\#}$, define the associated graph $(\boldsymbol{V}, \boldsymbol{E})$ by setting

$$\boldsymbol{V} = I \quad \text{and} \quad \boldsymbol{E} = \big\{(i,j) \in \boldsymbol{V}\times\boldsymbol{V} : d(A_i, A_j) = 0\big\}.$$

Define measures on V and E (both called m) as follows:

$$m(i) = \mu(A_i) \qquad \text{on } \boldsymbol{V}, \tag{3.10}$$

$$m(i,j) = \max\big\{m(i), m(j)\big\} \quad \text{on } \boldsymbol{E}. \tag{3.11}$$

Definition 3.6. — Given a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i\in I}$ of a set U in $U^\#$, define the *discrete Poincaré constant* $\Lambda_d(\mathcal{A})$ of $\mathcal{A}$ to be the smallest constant such that the following Poincaré inequality holds on the associated graph $(\boldsymbol{V}, m)$:

$$\sum_{i\in \boldsymbol{V}} \big|f(i) - m(f)\big|^2 m(i) \le \Lambda_d(\mathcal{A})\, \mathcal{E}(f,f), \tag{3.12}$$

where $\mathcal{E}(f,f)$ and $m(f)$ are given by (3.7) and (3.9).

Clearly, the discrete Poincaré constant is related to the spectral gap $\lambda(\boldsymbol{V}, m)$ by

$$\Lambda_d(\mathcal{A}) = \frac{1}{\lambda(\boldsymbol{V}, m)}\cdot \tag{3.13}$$

Theorem 3.7. — *Let $\mathcal{A}$ be a good covering of a set U in $U^\#$. Then*

$$\Lambda(U, U^\#) \le Q_1 \Lambda_c (2 + Q_1^2 Q_2 \Lambda_d), \tag{3.14}$$

where Λ_c and Λ_d are respectively the continuous and discrete Poincaré constants of $\mathcal{A}$ (see Definitions 3.4 *and* 3.6), *and Q_1, Q_2 are defined by* (3.1), (3.2).

The importance of this statement is that it allows us to "glue" Poincaré inequalities. Indeed, (3.14) means that the Poincaré constant of the pair $(U, U^\#)$ is estimated in terms of the Poincaré constants of all pairs (A_k, A_k^*) and $(A_k^*, A_k^\#)$ which form a covering of U in $U^\#$, and in terms of the Poincaré constant of the graph $(\boldsymbol{V}, m)$ that describes the combinatorics of the covering.

Proof. — We need to prove that for any $\phi \in \mathcal{C}^1(U^\#)$

$$\int_U |\phi - \phi_U|^2 \,\mathrm{d}\mu \le Q_1 \Lambda_c (2 + Q_1^2 Q_2 \Lambda_d) \int_{U^\#} |\nabla \phi|^2 \,\mathrm{d}\mu. \tag{3.15}$$

Fix a function $\phi \in \mathcal{C}^1(U^\#)$ and set

$$f(\ell) = \frac{1}{\mu(A_\ell)} \int_{A_\ell} \phi \,\mathrm{d}\mu. \tag{3.16}$$

For a constant c to be chosen later, write

$$\begin{aligned} \int_U |\phi - c|^2 \,\mathrm{d}\mu &\le \sum_{\ell\in \boldsymbol{V}} \int_{A_\ell} |\phi - c|^2 \,\mathrm{d}\mu \\ &\le 2 \sum_{\ell\in \boldsymbol{V}} \int_{A_\ell} |\phi - f(\ell)|^2 \,\mathrm{d}\mu + 2 \sum_{\ell\in \boldsymbol{V}} |f(\ell) - c|^2 m(\ell). \end{aligned} \tag{3.17}$$

The first sum can be estimated using the Poincaré inequality on each A_ℓ. Namely, we have by (3.4) and (3.1)

$$(3.18)\qquad \sum_{\ell\in\boldsymbol{V}}\int_{A_\ell}\left|\phi-f(\ell)\right|^2\mathrm{d}\mu\le\sum_{\ell\in\boldsymbol{V}}\Lambda_c\int_{A^*_\ell}|\nabla\phi|^2\,\mathrm{d}\mu\le Q_1\Lambda_c\int_{U^\#}|\nabla\phi|^2\,\mathrm{d}\mu.$$

To estimate the second sum in (3.17), define $c=m(f)$ by (3.9). Using successively the discrete Poincaré inequality (3.12), definitions (3.16), (3.10), (3.11), condition (3.2), the continuous Poincaré inequality (3.5) with the identities (3.6), we obtain

$$\begin{aligned}
(3.19)\qquad & 2\sum_{\ell\in\boldsymbol{V}}\left|f(\ell)-c\right|^2m(\ell)\\
&\le\Lambda_d\sum_{(i,j)\in\boldsymbol{E}}\left|f(i)-f(j)\right|^2m(i,j)\\
&=\Lambda_d\sum_{(i,j)\in\boldsymbol{E}}\left|\frac{1}{\mu(A_i)}\int_{A_i}\phi\,\mathrm{d}\mu-\frac{1}{\mu(A_j)}\int_{A_j}\phi\,\mathrm{d}\mu\right|^2m(i,j)\\
&=\Lambda_d\sum_{(i,j)\in\boldsymbol{E}}\left|\frac{1}{\mu(A_i)\mu(A_j)}\int_{A_i}\int_{A_j}\left[\phi(x)-\phi(y)\right]\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right|^2m(i,j)\\
&\le\Lambda_d\sum_{(i,j)\in\boldsymbol{E}}\frac{m(i,j)}{m(i)m(j)}\int_{A_i}\int_{A_j}\left|\phi(x)-\phi(y)\right|^2\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
(3.20)\qquad &\le Q_2\Lambda_d\sum_{(i,j)\in\boldsymbol{E}}\frac{1}{\mu(A^*_{k(i,j)})}\int_{A^*_{k(i,j)}}\int_{A^*_{k(i,j)}}\left|\phi(x)-\phi(y)\right|^2\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\le Q_2\Lambda_d\sum_{(i,j)\in\boldsymbol{E}}\Lambda_c\int_{A^\#_{k(i,j)}}\left|\nabla\phi\right|^2\mathrm{d}\mu\\
(3.21)\qquad &\le Q_1^2Q_2\Lambda_d\Lambda_c\sum_{k\in\boldsymbol{V}}\int_{A^\#_k}|\nabla\phi|^2\,\mathrm{d}\mu\\
(3.22)\qquad &\le Q_1^3Q_2\Lambda_d\Lambda_c\int_{U^\#}|\nabla\phi|^2\,\mathrm{d}\mu.
\end{aligned}$$

To obtain (3.20) we have used (3.11) and (3.2), which yield

$$\frac{m(i,j)}{m(i)m(j)}=\frac{\max(m(i),m(j))}{m(i)m(j)}=\frac{1}{\min(m(i),m(j))}\le\frac{Q_2}{\mu(A^*_{k(i,j)})}\cdot$$

To obtain (3.21) we have used that (3.1) implies that each $A^\#_k$ intersects at most Q_1 of the sets A_i, which shows that each $A^\#_k$ covers at most Q_1^2

unions $A_i \cup A_j$. In (3.22) we have used another consequence of (3.1) that at most Q_1 of all $A_k^{\#}$ overlap at any given point.

By (3.6), (3.17), (3.18) and (3.22), we obtain

$$\int_U |\phi - \phi_U|^2 \,\mathrm{d}\mu \le (2Q_1\Lambda_c + Q_1^3 Q_2 \Lambda_d \Lambda_c) \int_{U^{\#}} |\nabla \phi|^2 \,\mathrm{d}\mu,$$

which was to be proved. □

Example 3.7. — Let us show that $\Lambda(U, U^{\#}) < \infty$ whenever U is a non-empty connected precompact open set and $U^{\#}$ is an open set containing the closure $\overline{U}$. Indeed, let $\mathcal{A} = \{(A_i, A_i^*, A_i^{\#})\}_{i=1}^n$ be the following good covering of U (cf. Example 3.2): $A_i = B(x_i, s)$ and $\{x_i\}$ is a maximal s-net in U, and $A_i^* = A_i^{\#} = B(x_i, 3s)$. If s is small enough, then each ball A_i^* is nearly a Euclidean ball. Hence, $\Lambda(A_i, A_i^*) < \infty$ and $\Lambda(A_i^*, A_i^{\#}) < \infty$, which by (3.3) implies $\Lambda_c(\mathcal{A}) < \infty$.

The connectedness of U implies that the union $\cup_i A_i$ is connected, whence it follows that the associated graph $(\boldsymbol{V}, \boldsymbol{E})$ is connected and hence $\lambda(\boldsymbol{V}, m) > 0$. Therefore, $\Lambda_d(\mathcal{A}) < \infty$ and by Theorem 3.7 $\Lambda(U, U^{\#}) < \infty$. A modification of this argument shows that $\Lambda(U, U) < \infty$ whenever U is as above and has a smooth boundary.

3.2. The spectral gap of some finite graphs.

In order to apply successfully our discretization scheme, we need to have at our disposal some Poincaré inequalities on finite graphs. There is by now a large literature on this subject, but we will only need very simple examples.

Let $(\boldsymbol{V}, m)$ be a finite weighted graph. Given a subset U of $\boldsymbol{V}$, the boundary of U is the set of all edges that meet both U and its complement. The Cheeger constant $h(\boldsymbol{V}, m)$ is defined by

$$h(\boldsymbol{V}, m) := \inf \left\{ \frac{m(\partial U)}{m(U)} : U \subset \boldsymbol{V},\ 0 < m(U) \le \tfrac{1}{2} m(\boldsymbol{V}) \right\}. \tag{3.23}$$

By a well-known argument (see e.g., [45, Lemma 3.3.3]), the Cheeger constant h can be equivalently defined by

$$h(\boldsymbol{V}, m) = \inf_f \left\{ \frac{\sum_{x,y} |f(x) - f(y)| \cdot m(x,y)}{\inf_\alpha \sum_x |f(x) - \alpha| \cdot m(x)} \right\}.$$

Moreover, the two constants $h(\boldsymbol{V},m)$ and $\lambda(\boldsymbol{V},m)$ (see (3.8)) compare as follows

$$\frac{h^2(\boldsymbol{V},m)}{8m_0} \leq \lambda(\boldsymbol{V},m) \leq h(\boldsymbol{V},m) \tag{3.24}$$

where

$$m_0 := \max_{x\in\boldsymbol{V}}\left\{\frac{1}{m(x)}\sum_{y\in\boldsymbol{V}} m(x,y)\right\}.$$

The lower bound is a generalization of *the Cheeger inequality* (see e.g., [45, Lemma 3.3.7], [10]).

The notion of a weighted graph contains finite reversible Markov chains as the particular case where the measure $m(x)$ on vertices be related to the measure $m(x,y)$ on edges by $m(x)=\sum_{y\in\boldsymbol{V}} m(x,y)$. However, in our examples, weighted graphs will *not* be associated with reversible Markov chains. Typically we will first be given the measure $m(x)$ on vertices and then define the measure $m(x,y)$ on edges by

$$m(x,y)=\max\{m(x),m(y)\} \tag{3.25}$$

as in (3.11). The following simple examples will be used in this paper.

Example 3.9. — Let $m:\mathbb{N}\to(0,+\infty)$ be a function such that there exist two constants $c_0,\eta_0>0$ for which

$$\begin{cases} c_0\, m(k)\leq m(k+1)\leq c_0^{-1}m(k), & \text{for any } k,\\ m(\ell)\geq c_0(1+\eta_0)^{\ell-k}m(k), & \text{for all } \ell\geq k,\end{cases} \tag{3.26}$$

(in other words, $m(k)$ is an exponentially growing function). Consider the weighted graph $(\boldsymbol{V},m)$ with the vertex set $\boldsymbol{V}=\{0,1,\dots,n\}$, the edge set

$$\boldsymbol{E}=\big\{(i,j)\in\boldsymbol{V}\times\boldsymbol{V}:|i-j|\leq 1\big\}, \tag{3.27}$$

and define the measure on edges by (3.25).

We claim that there exists a constant $c=c(c_0,\eta_0)>0$ such that

$$\lambda(\boldsymbol{V},m)\geq c$$

(observe that this estimate does not depend on n). Indeed, by (3.24) it suffices to show that there exists a constant $C>0$ independent of n such that

$$m(U)\leq Cm(\partial U) \tag{3.28}$$

for any set $U \subset \boldsymbol{V}$ such that $m(U) \leq \frac{1}{2}m(\boldsymbol{V})$. Now, (3.26) implies for any $\ell \in \mathbb{N}$

$$m\big([0,\ell]\big) = \sum_{i=0}^{\ell} m(i) \leq \frac{1}{c_0}\Big(\sum_{i=0}^{\ell}(1+\eta_0)^{-(\ell-i)}\Big)m(\ell) \leq Cm(\ell) \tag{3.29}$$

where $C := c_0^{-1}\eta_0^{-1}(1+\eta_0)$. Setting

$$\ell := \max\big\{i \in \boldsymbol{V} : (i,i+1) \in \partial U\big\}$$

and noticing that either U or $\boldsymbol{V}\setminus U$ is contained in $[0,\ell]$, we obtain by (3.29)

$$\begin{aligned} m(U) &\leq \min\big(m(U), m(\boldsymbol{V}\setminus U)\big) \leq m\big([0,\ell]\big) \\ &\leq Cm(\ell) \leq Cm(\ell,\ell+1) \leq Cm(\partial U). \end{aligned}$$

Example 3.10. — Set $\boldsymbol{V} = \{0,\cdots,n\}$ and let $m : \boldsymbol{V} \to (0,+\infty)$ be a function such that, for some constant $c_0 > 0$ and for any $k = 0,1,\ldots,n-1$,

$$c_0\, m(k) \leq m(k+1) \leq c_0^{-1} m(k). \tag{3.30}$$

Assume further that m is a function with at most one local maximum, that is there exists $k_0 \in \{0,1,\ldots,n\}$ such that

$$\begin{cases} m(k) \leq m(k+1), & \text{if } k < k_0, \\ m(k) \geq m(k+1), & \text{if } k \geq k_0. \end{cases} \tag{3.31}$$

If $k_0 = 0$, then the function $m(k)$ is decreasing on $\boldsymbol{V}$, if $k = n$, then $m(k)$ is increasing, and if $0 < k_0 < n$, then $m(k)$ is increasing on $[0,k_0]$ and decreasing on $[k_0,n]$.

Consider the weighted graph $(\boldsymbol{V},m)$ with the edge set defined by (3.27) and the measure on edges defined by (3.25). It was proved in [13, Proposition 6.3] that

$$\lambda(\boldsymbol{V},m) \geq \frac{c_0}{2(n+1)^2},$$

which gives a very general Poincaré inequality for unimodal distribution.

Remark 3.11. — Note that instead of (3.31) it suffices to assume that $m \approx \widetilde{m}$ where $\widetilde{m}$ satisfies (3.31).

3.3. Two typical examples.

This section illustrates the results of Sections 3.1 and 3.2 by proving some Poincaré inequalities on two simple classes of weighted manifolds.

Example 3.12. — Consider $\mathbb{R}^N$ equipped with the Euclidean metric and the measure

$$\mathrm{d}\mu_\alpha(x) = \left(1+|x|^2\right)^{\alpha/2}\mathrm{d}x, \tag{3.32}$$

where $\alpha \in \mathbb{R}$. Let us show that the family of all balls centered at the origin o satisfies the Poincaré inequality (PI) with parameter $\delta = 1$ (see Definition 2.6). For $r > 1$, let n be the non-negative integer such that $r \in [2^n, 2^{n+1})$. Set $U = B(o,r)$, $\rho_i = 2^{i-n}r$ for $0 \le i \le n$. Consider the good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i=0}^n$ of U in U defined by

$$\begin{cases} A_0 = A_0^* = A_0^\# = B(o,\rho_0), & \\ A_i = B(o,\rho_i)\setminus B(o,\rho_{i-1}), & 1 \le i \le n, \\ A_i^* = A_i^\# = A_{i-1}\cup A_i, & 1\le i\le n. \end{cases} \tag{3.33}$$

Clearly, all the conditions of Definition 3.1 are satisfied with constants Q_1, Q_2 independent of n. Since in each A_i the function $(1+|x|^2)^{\alpha/2}$ is comparable to a constant, the continuous Poincaré constant $\Lambda_c(\mathcal{A})$ is of the same order as the one for the Lebesgue measure, that is $\Lambda_c(\mathcal{A}) \approx r^2$. The weighted graph $(\boldsymbol{V}, m)$ of the covering $\mathcal{A}$ (see Definition 3.5) has the vertex set $\boldsymbol{V} = \{0,1,\dots,n\}$ and the edge set (3.27). The weight $m(i)$ is defined by (3.10), which implies that

$$m(i) = \mu_\alpha(A_i) \approx 2^{i(\alpha+N)}.$$

Under the assumption $\alpha > -N$, the graph $(\boldsymbol{V}, m)$ satisfies all the conditions of Example 3.9 and hence

$$\Lambda_d(\mathcal{A}) = \frac{1}{\lambda(\boldsymbol{V}, m)} \le C,$$

where the constant C does not depend on n. By Theorem 3.7 we conclude that $\Lambda(U,U) \le Cr^2$, which yields (PI) for the central balls.

Remark 3.13. — In fact, the inequality $\Lambda(U,U) \le Cr^2$ holds for all real α but the proof in the case $\alpha \le -N$ requires a certain refinement of Theorem 3.7 that will be developed elsewhere.

Example 3.14. — Let M_ψ be a model manifold, that is $\mathbb{R}^N$ with a complete Riemannian metric $\mathrm{d}s^2$ given in polar coordinates (r,θ) by $\mathrm{d}s^2 = \mathrm{d}r^2 + \psi(r)^2\,\mathrm{d}\theta^2$. This class of examples is considered in more detail in Section 4.4. Here, we study Poincaré inequalities on central balls in M_ψ, in the special case

$$\psi(r) \approx r^\alpha \quad \text{for all } r > 1.$$

Let us set $\lambda_\psi(r) := \lambda(B(o,r))$ where o is the origin, and show that for any real α and all $r > 1$

$$\lambda_\psi(r) \approx r^{-2\max\{\alpha,1\}}. \tag{3.34}$$

First consider the case $\alpha > -1/(N-1)$. Fix $r > 1$ and let n be an integer such that such that $r \in [2^n, 2^{n+1})$. Setting $U = B(o,r)$, we obtain a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i=0}^n$ of $U = B(o,r)$ in U by using the same notation as in (3.33).

Observe that each A_i, $i > 0$, is essentially a piece of a flat cylinder of height 2^i and radius $2^{\alpha i}$, which implies

$$\Lambda_c(\mathcal{A}) \le C_\alpha \max\{r^2, r^{2\alpha}\}. \tag{3.35}$$

The associated graph $(\boldsymbol{V}, m)$ has the vertex set $\boldsymbol{V} = \{0,1,\dots,n\}$, edge set (3.27), and the weight m

$$m(i) = \mu(A_i) \approx 2^{(1+\alpha(N-1))i}.$$

Using Example 3.9 and the fact that $1+\alpha(N-1) > 0$, one checks that $\Lambda_d(\mathcal{A}) \le C_\alpha$. This, together with (3.35) and Theorem 3.7, yields $\Lambda(U,U) \le C_{N,\alpha}\max\{r^2, r^{2\alpha}\}$, which gives the lower bound in (3.34). Tests functions easily yield a matching upper bound.

Consider now the case $\alpha \le -1/(N-1)$. We will use a different good covering of $B(o,r)$ that in fact works for all $\alpha \le \frac{1}{3}$. Fix a constant c_r and define a sequence $\{\rho_i\}_{i\ge1}$ by

$$\rho_i := c_r \sum_{k=1}^{i} k^{1/2} \approx c_r i^{3/2}.$$

Let n be the integer such that

$$\sum_{k=1}^{n-1} k^{1/2} < r \le \sum_{k=1}^{n} k^{1/2},$$

and choose c_r so that $\rho_n = r$. Obviously $c_r \approx 1$. Define a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^{\#})\}_{i=1}^n$ of $B(o,r)$ in itself by

$$\begin{aligned} A_1 &= A_1^* = A_1^{\#} = B(o, \rho_1), \\ A_i &= B(o, \rho_i) \setminus B(o, \rho_{i-1}), \quad 1 < i \le n, \\ A_i^* &= A_i^{\#} = A_{i-1} \cup A_i, \qquad\quad 1 < i \le n. \end{aligned}$$

Note that for this covering $Q_1 \le 4$, $Q_2 \le 6^N$. The associated graph $(\boldsymbol{V}, m)$ has the vertex set $\boldsymbol{V} = \{1, \dots, n\}$, edge set (3.27), and the weight m

$$m(i) = \mu(A_i) \approx i^{1/2} \times (i^{3\alpha/2})^{N-1} \approx i^{[1+3\alpha(N-1)]/2}.$$

We claim that

$$\Lambda_c(\mathcal{A}) \le Cn \quad \text{and} \quad \Lambda_d(\mathcal{A}) \le Cn^2,$$

for some constant $C = C_{N,\alpha}$. The bound on $\Lambda_c(\mathcal{A})$ comes from the fact that the sets A_i, A_i^* are essentially cylinders of height $i^{1/2}$ and radius $i^{3\alpha/2}$, which implies that, as long as $\alpha \le \frac{1}{3}$,

$$\Lambda(A_i, A_i^*) \approx i \quad \text{and} \quad \Lambda(A_i^*, A_i^{\#}) \approx i.$$

The bound on $\Lambda_d(\mathcal{A})$ comes from Example 3.10 (cf. Remark 3.11). Hence, by Theorem 3.7 we obtain

$$\Lambda(U, U) \le Cn^3 \approx Cr^2,$$

whence the lower bound in (3.34) follows. Again, a matching upper bound is easily obtained by test functions.

4. Remote and anchored balls.

The next definition introduces the notions of remote and anchored balls with respect to a closed set.

DEFINITION 4.1. — Fix a parameter $0 < \varepsilon \le 1$ (the *remote parameter*) and a closed set $\Gamma \subset M$ in a metric space (M, d).

1) We say that a ball $B(x,r)$ is *remote* to Γ if $r \le \frac{1}{2}\varepsilon d(\Gamma, x)$.

2) We say that a ball $B(x,r)$ is *anchored* to Γ if $x \in \Gamma$.

The aim of this section is to draw conclusions from hypotheses concerning remote and anchored balls.

4.1. Poincaré inequalities.

We start with Poincaré inequalities for which the result is straightforward.

Proposition 4.2. — *Let (M,μ) be a weighted manifold and let $\Gamma \subset M$ be a closed set. Assume that the Poincaré inequality* (PI) *holds for all anchored and remote balls, with parameter $0 < \delta_0 \leq 1$ and constant $C_P > 0$; that is, for any anchored or remote ball $B(x,r)$*

$$\Lambda\big(B(x,\delta_0 r),B(x,r)\big) \leq C_P r^2.$$

Then (PI) *holds for all balls with parameter $\delta = \varepsilon\delta_0^2/8$ and constant C_P; that is for any ball $B(x,r)$*

$$\Lambda\big(B(x,\delta r),B(x,r)\big) \leq C_P r^2.$$

Here $0 < \varepsilon \leq 1$ is the remote parameter.

Proof. — By definition of the Poincaré constant, we need to prove that for any ball $B(x,r)$ and all functions $f \in \mathcal{C}^1(B(x,r))$

$$\inf_{\xi\in\mathbb{R}} \int_{B(x,\delta r)} (f-\xi)^2 \,\mathrm{d}\mu \leq C_P r^2 \int_{B(x,r)} |\nabla f|^2 \,\mathrm{d}\mu.$$

Consider first the case when $\rho := d(\Gamma,x) \leq \frac{1}{4}\delta_0 r$. Choose a point $o \in \Gamma$ such that $d(o,x) = \rho$. It follows from the inequalities $\delta < \frac{1}{4}\delta_0 < \frac{1}{2}$ that

$$B(x,\delta r) \subset B\big(o,\tfrac{1}{2}\delta_0 r\big) \quad \text{and} \quad B\big(o,\tfrac{1}{2}r\big) \subset B(x,r)$$

(see Figure 4).

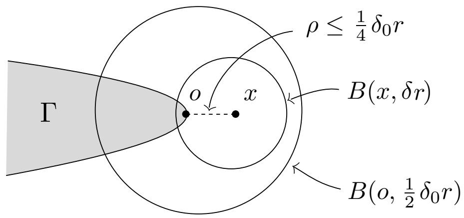

Figure 4. The balls $B(o,\frac{1}{2}\delta_0 r)$ and $B(x,\delta r)$

Thus, applying the Poincaré inequality in the anchored ball $B(o, \frac{1}{2}r)$ we obtain

$$\begin{aligned}\inf_{\xi\in\mathbb{R}}\int_{B(x,\delta r)}(f-\xi)^2\,\mathrm{d}\mu &\le \inf_{\xi\in\mathbb{R}}\int_{B(o,\frac{1}{2}\delta_0 r)}(f-\xi)^2\,\mathrm{d}\mu\\ &\le C_P r^2\int_{B(o,\frac{1}{2}r)}|\nabla f|^2\,\mathrm{d}\mu\\ &\le C_P r^2\int_{B(x,r)}|\nabla f|^2\,\mathrm{d}\mu.\end{aligned}$$

Consider now the case $\rho > \frac{1}{4}\delta_0 r$. Set $s = \frac{1}{8}\varepsilon\delta_0 r$ and notice that $B(x,s)$ is a remote ball (see Figure 5).

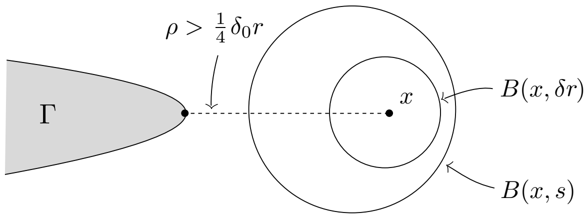

Figure 5. *Remote ball $B(x,s)$ and the ball $B(x,\delta r) = B(x,\delta_0 s)$*

Observing that $\delta r = \frac{1}{8}\varepsilon\delta_0^2 r = \delta_0 s$ and $s < r$, and applying the Poincaré inequality in the remote ball $B(x,s)$, we obtain

$$\begin{aligned}\inf_{\xi\in\mathbb{R}}\int_{B(x,\delta r)}(f-\xi)^2\,\mathrm{d}\mu &= \inf_{\xi\in\mathbb{R}}\int_{B(x,\delta_0 s)}(f-\xi)^2\,\mathrm{d}\mu\\ &\le C_P s^2\int_{B(x,s)}|\nabla f|^2\,\mathrm{d}\mu \le C_P r^2\int_{B(x,r)}|\nabla f|^2\,\mathrm{d}\mu,\end{aligned}$$

which finishes the proof. □

4.2. The volume comparison condition.

The following condition which was introduced in [33] plays an important role in our analysis.

Definition 4.3. — Fix a remote parameter $0 < \varepsilon \le 1$ and a closed set $\Gamma \subset M$ as in Definition 4.1. We say that a weighted manifold (M,μ) satisfies the *volume comparison* condition (VC) with respect to Γ if there exists a positive constant C_V such that, for all $x \in M$ and $o \in \Gamma$ with $d(\Gamma,x) = d(o,x) =: \rho$, we have

$$\mu\big(B(o,\rho)\big) \le C_V \mu\big(B(x,\tfrac{1}{64}\varepsilon\rho)\big), \tag{4.1}$$

Note that the ball $B(o,\rho)$ is anchored to Γ, whereas the ball $B(x,s)$ with $s = \frac{1}{64}\varepsilon\rho$ is remote.

Lemma 4.4. — *If a weighted manifold* (M,μ) *satisfies* (VD), *then it also satisfies* (VC) *with respect to any closed set* $\Gamma \subset M$ *and any remote parameter* $0 < \varepsilon \le 1$. *Conversely, fix a remote parameter* $0 < \varepsilon \le 1$ *and a closed set* $\Gamma \subset M$. *If* (M,μ) *satisfies* (VD) *for anchored and remote balls as well as condition* (VC), *then* (M,μ) *satisfies* (VD) *for all balls.*

Proof. — The first assertion follows immediately from (2.12). Let us prove the second assertion. Given a ball $B(x,r)$ in M, set $\rho = d(\Gamma,x)$ and consider three cases.

Case 1. — If $r \le \frac{1}{2}\varepsilon\rho$, then the ball $B(x,r)$ is remote and hence satisfies (VD) by hypothesis.

Case 2. — If $r \ge 3\rho$, then choose a point $o \in \Gamma$ such that $d(o,x) = \rho$. Using (VD) for anchored balls we obtain

$$\mu\big(B(x,r)\big) \le \mu\big(B(o,\tfrac{4}{3}r)\big) \le C_D^3\mu\big(B(o,\tfrac{1}{6}r)\big) \le C_D^3\mu\big(B(x,\tfrac{1}{2}r)\big), \tag{4.2}$$

that is (VD) for $B(x,r)$.

Case 3. — If $\frac{1}{2}\varepsilon\rho < r < 3\rho$, then (VD) for anchored balls implies

$$\mu\big(B(x,r)\big) \le \mu\big(B(o,4\rho)\big) \le C_D^2\mu\big(B(o,\rho)\big) \tag{4.3}$$

whereas by (VC)

$$\mu\big(B(o,\rho)\big) \le C_V\mu\big(B(x,\tfrac{1}{64}\varepsilon\rho)\big) \le C_V\mu\big(B(x,\tfrac{1}{2}r)\big), \tag{4.4}$$

whence (VD) for $B(x,r)$ follows. □

The following result shows that, given the volumes of anchored and remote balls, one can estimate the volume of any ball.

Corollary 4.5. — *Fix a remote parameter* $0 < \varepsilon \leq 1$ *and a closed set* $\Gamma \subset M$. *If* (M,μ) *satisfies* (VD) *for anchored and remote balls and also* (VC), *then for any non-remote ball* $B(x,r)$ *in* M *we have*

$$\mu\big(B(x,r)\big) \approx \mu\big(B(o,r)\big), \tag{4.5}$$

where o *is a point in* Γ *such that* $d(o,x) = d(\Gamma,x)$.

Proof. — In Case 2 of the previous proof, we have (4.2) which together with (VD) for anchored balls yields $\mu(B(x,r)) \approx \mu(B(o,r))$.

In Case 3 of the previous proof, (4.3) and (4.4) imply $\mu(B(x,r)) \approx \mu(B(o,\rho)) \approx \mu(B(o,r))$. □

Definition 4.6. — Let us say that a point $o \in \Gamma$ of a closed set $\Gamma \subset M$ is *accessible* if for any $r > 0$ there is $x \in M$ such that $d(\Gamma,x) = d(o,x) = r$. We say that Γ is *fully accessible* if Γ is closed and any point $o \in \Gamma$ is accessible.

For example, if (M,d) is complete and non-compact length space, then any set Γ that consists of a single point is fully accessible. Another example of a fully accessible set is a linear submanifold in $\mathbb{R}^N$ of a positive codimension.

Proposition 4.7. — *Fix a remote parameter* $0 < \varepsilon \leq 1$ *and a fully accessible set* $\Gamma \subset M$. *Assume that* (M,μ) *satisfies* (VD) *for remote balls and* (VC). *Then* (M,μ) *satisfies* (VD) *for all balls.*

Remark 4.8. — The difference between this statement and the second part of Lemma 4.4 is that (VD) is no longer assumed for anchored balls but instead we require that Γ is fully accessible. Without the latter condition, the volume doubling may fail for balls deeply inside Γ.

Proof. — By Lemma 4.4, it suffices to prove (VD) for anchored balls, that is for all $r > 0$ and any $o \in \Gamma$

$$\mu\big(B(o,r)\big) \leq C\mu\big(B(o,\tfrac{1}{2}r)\big). \tag{4.6}$$

Let $0 < \varepsilon \leq 1$ be the remote parameter. Observe that for all $0 < s \leq \frac{1}{64}\varepsilon r$ and for any $x \in \Gamma' := \{d(\,.\,,\Gamma) \geq \frac{1}{8}r\}$ the ball $B(x,4s)$ is remote to Γ. Applying hypothesis (VD) for remote balls we obtain, for all s as above and for any two intersecting balls $B(x',s)$, $B(x'',s)$ centered in Γ' ,

$$\mu\big(B(x',s)\big) \leq \mu\big(B(x'',4s)\big) \leq C_D^2\mu\big(B(x'',s)\big). \tag{4.7}$$

Since Γ is fully accessible, for any $o \in \Gamma$ and $r > 0$ there exists a point $x \in M$ such that $d(\Gamma, x) = d(o, x) = r$. By hypothesis (VC), we have

$$\mu\big(B(o,r)\big) \le C_V \mu\big(B(x, \tfrac{1}{64}\varepsilon r)\big).$$

Let γ be a shortest line joining o to x and let y be the point on γ at distance $\frac{1}{4}r$ from o. Since $B(y, \frac{1}{4}r) \subset B(o, \frac{1}{2}r)$, (4.6) will be proved if we show that

$$\mu\big(B(x, \tfrac{1}{64}\varepsilon r)\big) \le C\mu\big(B(y, \tfrac{1}{4}r)\big). \tag{4.8}$$

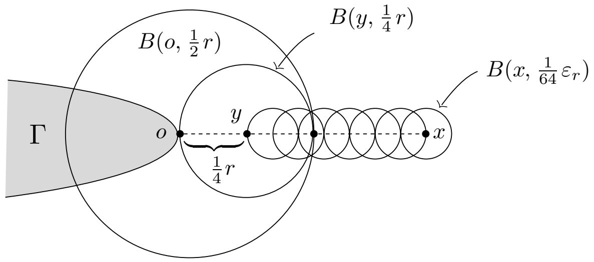

Figure 6. Comparisons of $\mu(B(x, \frac{1}{64}\varepsilon r))$ and $\mu(B(y, \frac{1}{4}r))$

Indeed, covering the segment of γ from y to x by at most $64\varepsilon^{-1}$ remote balls of radius $s = \frac{1}{64}\varepsilon r$ each (see Figure 6) and applying (4.7) for any pair of consecutive balls (note that their centers are in Γ') we obtain (4.8). $\square$

4.3. Radial power weights on $\mathbb{R}^N$.

This section illustrates the use of remote and anchored balls for a simple class of examples. Consider the weighted manifold $(\mathbb{R}^N, \mu_\alpha)$ where $\mathbb{R}^N$ is equipped with its Euclidean metric and the measure μ_α is defined by (3.32), i.e., $\mathrm{d}\mu_\alpha(x) = (1 + |x|^2)^{\alpha/2}\,\mathrm{d}x$ where $\alpha \in \mathbb{R}$. Set $\Gamma = \{o\}$ where o is the origin in $\mathbb{R}^N$ and fix a remote parameter $0 < \varepsilon \le 1$. Observe that $(1 + |x|^2)^{\alpha/2}$ is comparable to a constant on any given remote ball. Hence, for remote balls in $(\mathbb{R}^N, \mu_\alpha)$, the Poincaré inequality (PI) follows from that for the Euclidean metric. For anchored balls, (PI) follows from Example 3.12 provided $\alpha > -N$. Hence, by Proposition 4.2, the space $(\mathbb{R}^N, \mu_\alpha^N)$ satisfies (PI) on all balls, for any $\alpha > -N$ (in fact, as follows from Remark 3.13, $(\mathbb{R}^N, \mu_\alpha^N)$ satisfies (PI) for *all* real α but we will not use this).

Similarly, the family of remote balls in $(\mathbb{R}^N, \mu_\alpha)$ satisfy (VD). Moreover, for any remote ball $B(x,r)$, we have

$$V(x,r) := \mu_\alpha\big(B(x,r)\big) \approx r^N(1+\rho^\alpha),$$

where $\rho := d(\Gamma, x)$. If $B(o,r)$ is an anchored ball, then a simple computation shows that

$$V(o,r) \approx \begin{cases} r^N & \text{if } r \leq 1, \\ r^{N+\alpha} & \text{if } r \geq 1 \text{ and } \alpha > -N, \\ \log(1+r) & \text{if } r \geq 1 \text{ and } \alpha = -N, \\ 1 & \text{if } r \geq 1 \text{ and } \alpha < -N. \end{cases} \tag{4.9}$$

Therefore, condition (VC) holds true if and only if $\alpha > -N$. By Proposition 4.7, $(\mathbb{R}^N, \mu_\alpha)$ satisfies (VD) for all balls if and only if $\alpha > -N$. Thus $(\mathbb{R}^N, \mu_\alpha)$ satisfy (PI) and (VD) if and only if $\alpha > -N$. This and Theorem 2.7 yields the following result.

Proposition 4.9. — *The weighted manifold* $(\mathbb{R}^N, \mu_\alpha)$ *satisfies* (VD), (PI) *and* (PHI) *if and only if* $\alpha > -N$.

Consider the differential operator

$$L_\alpha := \big(1+|x|^2\big)^{-\alpha/2} \sum_{i=1}^N \frac{\partial}{\partial x_i}\Big((1+|x|^2)^{\alpha/2}\frac{\partial}{\partial x_i}\Big) = \Delta + \frac{\alpha x\cdot\nabla}{1+|x|^2},$$

which by (2.1) is the Laplacian of the weighted manifold $(\mathbb{R}^N, \mu_\alpha)$. By Proposition 4.9, (PHI) holds for L_α if and only if $\alpha > -N$. For such α, Corollary 4.5 and (4.9) give that, for any ball $B(x,r)$,

$$\mu_\alpha\big(B(x,r)\big) \approx r^N\big(1+r+|x|\big)^\alpha.$$

By Theorem 2.7, the heat kernel $p_\alpha(t,x,y)$ of L_α satisfies the estimate

$$p_\alpha(t,x,y) \approx \frac{\mathrm{e}^{-c|x-y|^2/t}}{t^{N/2}(1+\sqrt{t}+|x|)^\alpha(1+\sqrt{t}+|y|)^\alpha}, \tag{4.10}$$

where the constant $c > 0$ may take different values in the upper and lower bounds.

Consider now a matrix valued measurable function $x \mapsto (a_{ij}(x))$ defined on $\mathbb{R}^N$. Assume that (a_{ij}) is symmetric and uniformly elliptic, that is, all its eigenvalues are contained in $[\lambda^{-1}, \lambda]$, for some $\lambda \geq 1$. Then, by Proposition 4.9 and the stability results of [SalHar,SalSurv] in the spirit of Theorem 2.7, it follows that the parabolic Harnack inequality (PHI) and the heat kernel bound (4.10) hold true for the operator

$$\left(1+|x|^2\right)^{-\alpha/2}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(\left(1+|x|^2\right)^{\alpha/2}a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

4.4. Model manifolds.

This section develops in detail the case of model manifolds. Here, by a model manifold we mean $\mathbb{R}^N$ equipped with the Riemannian metric given in polar coordinates $(r,\theta)\in(0,+\infty)\times\mathbb{S}^{N-1}$ by

$$\mathrm{d}s^2 = \mathrm{d}r^2+\psi(r)^2\,\mathrm{d}\theta^2, \tag{4.11}$$

where $\mathrm{d}\theta^2$ is the standard metric on $\mathbb{S}^{N-1}$ and ψ is a smooth positive function on $(0,+\infty)$. Clearly, (4.11) defines the metric only away from the origin $o\in\mathbb{R}^N$. The necessary and sufficient conditions under which $\mathrm{d}s^2$ can be smoothly extended to a metric on the entire space $\mathbb{R}^N$ are as follows:

$$\psi(0)=0,\ \ \psi'(0)=1,\quad\text{and}\ \ \psi''(0)=0 \tag{4.12}$$

(see [29], [16, p. 60]). Given a function ψ satisfying (4.12), we denote by M_ψ the model manifold $(\mathbb{R}^N, ds^2)$ and by μ its Riemannian measure, defined by $\mathrm{d}\mu=\psi(r)^{N-1}\,\mathrm{d}r\,\mathrm{d}\theta$. From the point of view of the present paper, the values of ψ for small r are irrelevant, while for large values of r there is no restriction on the function $\psi(r)$ except for being positive and smooth.

For any $r>0$ denote by $K(r)$ the sectional curvature at a point $x=(r,\theta)$ in the direction of any plane in T_xM containing $\partial/\partial r$. It is well known that $K(r)$ satisfies the equation $\psi''+K\psi=0$ (see [16] and references therein), which allows to use model manifolds for curvature comparison techniques (see [16], [8], [7]).

Set $\Gamma=\{o\}$ where $o\in M_\psi$ is the origin, and use this Γ in the definition of anchored and remote balls (see Definition 4.1). Assume that in addition to (4.12) the function ψ satisfies (1.7), that is, for all $r>0$,

$$\sup_{[r,2r]}\psi\le C\inf_{[r,2r]}\psi. \tag{4.13}$$

The volume $V(o,r)$ of an anchored ball $B(o,r)$ is given by

$$V(o,r)=\omega_N\int_0^r\psi(s)^{N-1}\,\mathrm{d}s, \tag{4.14}$$

whereas a remote ball $B(x,r)$ admits the volume estimate

$$V(x,r) \approx \begin{cases} r^N & \text{if } r \leq \psi(\rho), \\ r\psi(\rho)^{N-1} & \text{if } r \geq \psi(\rho), \end{cases}$$

where $\rho = d(o,x)$. Clearly, the family of all remote balls satisfies (VD). On the other hand, it is easy to see that condition (VC) is satisfied if and only if there exists a constant C such that for all $r > 0$

$$\psi(r) \leq Cr \quad \text{and} \quad \int_0^r \psi(s)^{N-1}\,\mathrm{d}s \leq Cr\psi(r)^{N-1}. \tag{4.15}$$

Hence, by Proposition 4.7, (VD) holds for all balls provided (4.13) and (4.15) are satisfied.

Uniformly in x and r, any remote ball $B(x,r)$ is quasi-isometric to a ball in a piece of a flat cylinder. Hence, (PI) holds on remote balls. Let us verify that (4.13) and (4.15) imply the Poincaré inequality

$$\lambda_\psi(r) \geq cr^{-2}, \tag{4.16}$$

for all $r > 1$ and some $c > 0$. Indeed, let n be the integer such that $r \in [2^n, 2^{n+1})$, and define by (3.33) a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i=0}^n$ of $U = B(o,r)$ in U. It follows from (4.13) and the first condition in (4.15) that

$$\Lambda_c(\mathcal{A}) \leq Cr^2$$

(cf. (3.35)). The associated graph $(\boldsymbol{V}, m)$ has vertex set $\boldsymbol{V} = \{0, 1, \ldots, n\}$, edge set (3.27), and weight m given by

$$m(i) = \mu(A_i) \approx 2^i \psi^{N-1}(2^i).$$

By (4.14), (4.13) and (4.15) we obtain $m(i) \approx V(o, 2^i)$. Using the fact that M_ψ satisfies (VD) and Lemma 2.10, we see that $m(i)$ satisfies (3.26), which yields by Example 3.9 $\Lambda_d(\mathcal{A}) \leq C$. Hence, by Theorem 3.7 we have $\Lambda(U,U) \leq Cr^2$, which was to be proved. By Proposition 4.2, it follows that (PI) holds for all balls.

Applying Theorem 2.7 we obtain the following result.

Proposition 4.10. — *Let M_ψ be a model manifold such that ψ satisfies* (4.13) *and* (4.15). *Then M_ψ satisfies* (PI), (VD), *and* (PHI). *Moreover, under the standing assumption* (4.13), *the condition* (4.15) *is necessary and sufficient for* (PHI) *to hold on M_ψ. In particular, if $\psi(r) \approx r^\alpha$ for $r > 1$, then* (PHI) *holds if and only if $-1/(N-1) < \alpha \leq 1$.*

The necessity of (4.15) follows from the fact noted above that (4.15) is necessary for (VC) and hence for (VD) and (PHI).

Remark 4.11. — Given a compact Riemannian manifold Θ of dimension $N-1$ without boundary and a smooth positive function ψ on $[0,+\infty)$ we denote by M_ψ^Θ the manifold $[0,+\infty)\times\Theta$ equipped with the Riemannian metric (4.11), where $\mathrm{d}\theta^2$ is now the Riemannian metric on Θ. Note that M_ψ^Θ is a manifold with boundary. All the results stated in this paper for manifolds M_ψ hold true also for manifolds M_ψ^Θ.

5. Stability results for the parabolic Harnack inequality.

5.1. Parabolic Harnack inequality: from remote balls to all balls.

The examples described above indicate the advantage of performing a discretization based on concentric annuli. For that we need Poincaré inequalities on annuli, which obviously requires some connectivity of the annuli. Consider the following condition.

Definition 5.1. — Fix a constant $C_A>1$ and a point $o\in M$. We say that a metric space (M,d) has *relatively connected annuli* with respect to o, or satisfies condition (RCA), if for any $r\geq C_A^2$ and all $x,y\in M$ such that $d(o,x)=d(o,y)=r$, there exists a continuous path $\gamma\colon[0,1]\to M$ with $\gamma(0)=x$, $\gamma(1)=y$ whose image is contained in $B(o,C_Ar)\setminus B(o,C_A^{-1}r)$.

With this definition, we can state one of the main results of this paper.

Theorem 5.2. — *Let* (M,μ) *be a complete non-compact weighted manifold satisfying* (RCA) *with respect to a point* $o\in M$. *Assume that* (M,μ) *satisfies* (VD) *and* (PI) *for remote balls with respect to* $\Gamma=\{o\}$. *Then* (M,μ) *satisfies* (VD) *and* (PI) *for all balls if and only if it satisfies* (VC).

Remark 5.3. — By Lemma 4.4, (VC) is necessary for (VD). Condition (RCA) is very close to be necessary for (PI) and (VD). Indeed, [24] shows that (RCA) follows from (PI) provided

$$V(x,r)\approx r^Q \quad \text{for some } Q\geq 2 \text{ and for all } x\in M, r>0. \tag{5.1}$$

Condition (5.1), called Q-Ahlfors regularity, is clearly stronger than (VD).

It is known that a connected sum of two Euclidean spaces of dimension $N\geq 2$ satisfies neither (PHI) nor (PI). However, such a connected sum obviously satisfies (VD) and (VC), as well as (PI) and (PHI) on remote balls, whereas (RCA) fails for this manifold. Thus, condition (RCA) cannot be dropped from the hypotheses of Theorem 5.2.

The following result is an immediate consequence of Theorems 5.2 and 2.12.

Corollary 5.4. — *Let* (M,μ) *be a complete non-compact weighted manifold, and let* $o \in M$ *and* $\Gamma = \{o\}$. *Assume that* (M,μ) *satisfies* (PHI) *for remote balls, as well as* (VC) *and* (RCA). *Then* (M,μ) *satisfies* (PHI) *for all balls.*

Remark 5.5. — Assume that (M,μ) is a weighted manifold satisfying (PHI) and (RCA). Let (M_1,μ_1) be another weighted manifold such that the exteriors of some compacts in M and M_1 are isometric as weighted manifolds. Then Corollary 5.4 implies that (M_1,μ_1) also satisfies (PHI). In fact, this result holds true without assuming that M satisfies (RCA). To see this, one should use Theorem 2.7 and the rough isometry techniques developed in [27] , [11].

To prove Theorem 5.2, we have to show that if (M,μ) satisfies the conditions (RCA), (VC), as well as (VD) and (PI) for remote balls then it satisfies (VD) and (PI) for all balls. Under these hypotheses, (VD) for all balls holds by Proposition 4.7. Hence, we are left to prove (PI) for all balls. Let us precede the proof by the following lemma. Recall that following constants are involved in the above hypotheses: the remote parameter ε, C_A from (RCA), C_V from (VC), C_D from (VD) for remote balls, C_P and parameter δ from (PI) for remote balls.

Lemma 5.6. — *Assume that* (M,μ) *satisfies the conditions* (RCA), (VC), *as well as* (VD) *and* (PI) *for remote balls. Let* κ *be any number* $\geq C_A^3$ *and set*

$$U = B(o,\kappa r) \setminus B(o,r) \quad \text{and} \quad W = B(o,\kappa^2 r) \setminus B(o,\kappa^{-1} r).$$

Then $\Lambda(U,W) \leq Cr^2$, *for any* $r \geq C_A^2$, *where* C *depends on the constants from the hypotheses, but does not depend on* r.

Proof. — For any set U^* such that $U \subset U^* \subset W$, we have $\Lambda(U,W) \leq \Lambda(U^*,W)$. Therefore, it suffices to find a set U^* such that $U \subset U^* \subset W$ and

$$\Lambda(U^*,W) \leq Cr^2.$$

By (RCA), for any two points on $\partial B(o,r)$ there is a curve γ connecting these points in $B(o,C_A r) \setminus B(o,C_A^{-1}r)$ (see Figure 7). Fix $s>0$ to be specified below (see (5.2)) and set U^* to be the union of U and the s-neighborhoods of all such curves.

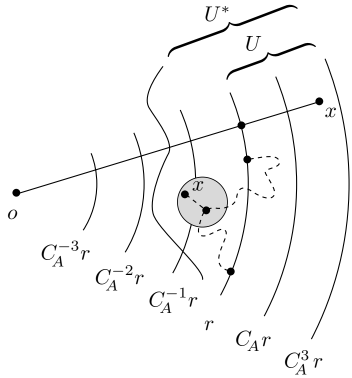

Figure 7. Any point $x \in U^*$ can be connected to a point on $\partial B(o,r)$ by a curve in U^* (the shaded ball has radius s)

This construction ensures that U^* is a connected set (note that the condition (RCA) is used only here). Indeed, let us show that any two points in U^* can be connected by a curve in U^*. Any point $x \in U$ can be connected to a point on $\partial B(o,r)$ by a curve in U just by connecting x to o by a shortest line. Any point $x \in U^* \setminus U$ can be connected to a point on $\partial B(o,r)$ by a curve in U^* because by construction x is a s-neighborhood of a curve connecting in U^* two points on $\partial B(o,r)$ (see Figure 7). Finally, any two points on $\partial B(o,r)$ are connected by a curve in U^* by the definition of U^*.

Define s by

$$s = \frac{1}{2}\varepsilon\delta^3 C_A^{-3} r, \tag{5.2}$$

where $0<\varepsilon\le 1$ is the remote parameter, and $0<\delta\le 1$ is the parameter from the Poincaré inequality for remote balls. Without loss of generality, we can assume that C_A is large enough, for example $C_A>2$. In particular, (5.2) implies $s<(C_A^{-1}-C_A^{-2})r$ whence it follows that

$$U^* \subset B(o,\kappa r) \setminus B(o, C_A^{-2} r) \subset W.$$

Let $\{x_i\}_{i\in I}$ be a maximal set of points of U^* at distance at least s from each other. Consider a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^\#)\}_{i\in I}$ of U^* in W defined by

$$A_i = B(x_i,s), \quad A_i^* = B(x_i,\delta^{-1}s), \quad A_i^\# = B(x_i,\delta^{-2}s).$$

The balls A_i cover U^* by the maximality of $\{x_i\}$, and the fact that $A_i^{\#} \subset W$ follows from (5.2).

By Proposition 4.7, (M,μ) satisfies (VD). Therefore, the constants Q_1 and Q_2 of the covering $\mathcal{A}$ are bounded in terms of the doubling constant (cf. Example 3.2). By the choice of s, all balls $A_i^*, A_i^{\#}$ are remote with respect to $\Gamma = \{o\}$. Hence, by (PI) for remote balls, we have the following estimate of the continuous Poincaré constant of $\mathcal{A}$:

$$\Lambda_c(\mathcal{A}) \le Cs^2 \le Cr^2.$$

To estimate the discrete Poincaré constant $\Lambda_d(\mathcal{A})$, let us first show that the number $n := \operatorname{card} I$ of the points x_i is uniformly bounded above by a constant independent of r. Indeed, all balls $B(x_i, \frac{1}{2}s)$ are disjoint and are contained in $B(o, \kappa r + s)$. Therefore, by (VD), Lemma 2.9, and (5.2),

$$n \le \frac{V(o, \kappa r + s)}{\min_i V(x_i, \frac{1}{2}s)} \le C,$$

where C is independent of r.

The fact that U^* is connected implies that the union $\bigcup_i A_i$ is connected and hence the associated graph $(\boldsymbol{V}, \boldsymbol{E})$ is connected (see Definition 3.5). Alongside the weight $m(i) = \mu(A_i)$ consider the flat weight $m_0(i) \equiv 1$. The spectral gap $\lambda(\boldsymbol{V}, m_0)$ is positive by the connectedness. Since the number of all weighted connected graphs having at most C vertices and equipped with a flat weight is finite, there is a universal lower bound $c > 0$ for the spectral gap of any such graph. Consequently, we have $\lambda(\boldsymbol{V}, m_0) \ge c$. As follows from (VD) and Lemma 2.9, $\mu(A_i) \approx \mu(A_j)$ for all $i, j \in \boldsymbol{V}$, which implies that $\lambda(\boldsymbol{V}, m) \approx \lambda(\boldsymbol{V}, m_0)$. Therefore, we obtain a positive lower bound for $\lambda(\boldsymbol{V}, m)$, which is independent of r, whence it follows that $\Lambda_d(\mathcal{A}) \le C$. By Theorem 3.7, $\Lambda(U^*, W) \le Cr^2$, which was to be proved. □

Proof of Theorem 5.2. — We are left to prove (PI) for anchored balls since then, by Proposition 4.2, (PI) will be true for all balls. Set $\kappa := C_A^3$ and consider the following sets

(5.3) $$A_1 = B(o,\kappa) \quad \text{and} \quad A_i = B(o,\kappa^i) \setminus B(o,\kappa^{i-1}) \text{ for } i > 1,$$

(5.4) $$A_i^* = A_{i-1} \cup A_i \cup A_{i+1},$$

(5.5) $$A_i^{\#} = A_{i-1}^* \cup A_i^* \cup A_{i+1}^*,$$

where we assume $A_i = \emptyset$ for $i \le 0$ (cf. Example 3.3). Given a ball $B(o,r)$ with $r \ge \kappa$ (smaller r can be treated by a compactness argument — cf. Example 3.8) choose an integer n so that $\kappa^{n-1} < r \le \kappa^n$ and consider a good covering $\mathcal{A} = \{(A_i, A_i^*, A_i^{\#})\}_{i=1}^n$ of $B(o,r)$ in $B(o,\kappa^2 r)$: The associated graph $(\boldsymbol{V}, m)$ has the vertex set $\boldsymbol{V} = \{1, 2, \ldots, n\}$, the edge set (3.27), and the weight

$$m(i) = \mu(A_i) \approx V(o, \kappa^i).$$

We claim that m satisfies the conditions (3.26). Indeed, the first condition in (3.26), that is $m(i) \approx m(i+1)$ follows from (VD). By Lemma 2.10, (VD) implies that there exists $c, \beta > 0$ such that $m(\ell) \geq c\kappa^{\beta(\ell-k)} m(k)$, for all non-negative integers $\ell \geq k$, which is exactly the second condition in (3.26). Hence, by Example 3.9, the discrete Poincaré constant $\Lambda_d(\mathcal{A})$ is bounded by a constant C independently of r. By Lemma 3.9, we have $\Lambda(A_i, A_i^*) \leq Cr^2$ and $\Lambda(A_i^*, A_i^{\#}) \leq Cr^2$. By Definition 3.4, we have $\Lambda_c(\mathcal{A}) \leq Cr^2$, and by Theorem 3.7 we obtain

$$\Lambda\big(B(o,r), B(o,\kappa^2 r)\big) \leq Cr^2,$$

which was to be proved. □

5.2. Changes of measure.

Recall that by Theorem 2.7 the parabolic Harnack inequality (PHI) is stable under a change of the measure $\mathrm{d}\mu \mapsto h^2 \mathrm{d}\mu$ provided h, h^{-1} are bounded functions. The next result uses Theorem 5.2 to show that certain changes of weight where h or h^{-1} are unbounded also preserve the validity of (PHI) provided the annuli connectedness condition (RCA) holds. This result is useful in several applications (see, e.g., [21], [22]).

THEOREM 5.7. — *Let (M,μ) be a complete noncompact weighted manifold satisfying* (PHI) *and* (RCA) *with respect to a point $o \in M$. Let h be a positive smooth function on M such that*

(h1) *for all positive integers i*

$$h_i := \sup_{B(o,2^i)\setminus B(o,2^{i-1})} h \leq C \inf_{B(o,2^i)\setminus B(o,2^{i-1})} h. \tag{5.6}$$

Consider the measure $\widetilde{\mu}$ on M defined by $\mathrm{d}\widetilde{\mu} = h^2\,\mathrm{d}\mu$. Then the weighted manifold $(M,\widetilde{\mu})$ satisfies (PHI) *if and only if the following condition holds true*:

(h2) *the numerical sequence $H_i = h_i^2 \mu(B(o,2^i))$ satisfies the inequality $\sum_{i=1}^k H_i \leq CH_k$, for all $k = 1,2,\ldots$.*

Remark 5.8. — If the sequence $\{h_i\}$ is increasing, then the condition (h2) is satisfied automatically. Indeed, using (VD) for (M,μ) and its consequence (2.13) (see Lemma 2.10) and setting $V(r) := \mu(B(o,r))$ we obtain

$$\sum_{i=1}^{k} H_i = \sum_{i=0}^{k} h_i^2 V(2^i) \leq h_k^2 V(2^k) \sum_{i=1}^{k} \frac{V(2^i)}{V(2^k)} \leq CH_k \sum_{i=1}^{k} 2^{(i-k)\beta} \leq CH_k.$$

However, the function $h(x) = V(d(o,x))^{-1/2}$ gives an example for which (h1) is satisfies but not (h2). Indeed, (h1) holds by (VD) but (h2) fails because $H_i \approx 1$.

Remark 5.9. — It is easy to see that the condition (h2) can be equivalently stated as

$$\int_0^r \bar{h}^2(s)\,\mathrm{d}V(s) \leq C\bar{h}^2(r)V(r), \tag{5.7}$$

for all $r \geq 1$, where $\bar{h}(s) := \sup_{\partial B(o,s)} h$.

Remark 5.10. — Consider the weighted manifold (M,μ) where M is the two-sided flat cylinder $\mathbb{R} \times \mathbb{S}$ with coordinates $x = (s,\theta)$ and $\mathrm{d}\mu(x) = \mathrm{d}s\,\mathrm{d}\theta$. Let $h(x) = (1+|s|^2)^{\alpha/4}$. A simple test function argument shows that $(M, h^2\mathrm{d}\mu)$ does not satisfy (PI) if $\alpha \geq 2$. This example shows that the hypothesis (RCA) is essential for the validity of Theorem 5.7.

Proof of Theorem 5.7. — Let us first prove that (h1)–(h2) imply (PHI) for $(M,\widetilde{\mu})$. Since balls are the same on (M,μ) and $(M,\widetilde{\mu})$, the manifold $(M,\widetilde{\mu})$ is complete, non-compact, and satisfies (RCA).

By Theorem 2.7, (M,μ) satisfies (VD) and (PI). It follows from (h1) that $(M,\widetilde{\mu})$ satisfies (VD) and (PI) for remote balls (with respect to the set $\Gamma = \{o\}$ and with remote parameter $\varepsilon = 1$).

Let us show that $(M,\widetilde{\mu})$ satisfies (VC). Indeed, fix a point $x \in M$ such that $\rho := d(o,x) > 1$ (for smaller ρ use a compactness argument) and set $s = \frac{1}{64}\rho$. Let k be a positive integer such that $2^{k-1} \leq \rho < 2^k$. Using (h1), (h2), and (VC) for μ, we obtain

$$\begin{aligned}
\widetilde{\mu}\big(B(o,\rho)\big) &\leq \widetilde{\mu}\big(B(o,1)\big) + \sum_{i=1}^{k} \widetilde{\mu}\big(B(o,2^i) \setminus B(o,2^{i-1})\big) \\
&\leq \mu\big(B(o,1)\big) \sup_{B(o,1)} h^2 + \sum_{i=1}^{k} \mu\big(B(o,2^i)\big) \sup_{B(o,2^i)\setminus B(o,2^{i-1})} h^2 \\
&\leq C \sum_{i=1}^{k} H_i \leq CH_k = C\mu\big(B(o,2^k)\big)h_k^2 \leq C\widetilde{\mu}\big(B(x,s)\big),
\end{aligned}$$

whence (VC) for $(M,\widetilde{\mu})$ follows. By Theorem 5.2, $(M,\widetilde{\mu})$ satisfies (VD) and (PI) for all balls, which implies (PHI) by Theorem 2.7.

Now we show that (h1) and (PHI) for $(M,\widetilde{\mu})$ imply (h2). Indeed, by Theorem 2.7 and Lemma 4.4, $(M,\widetilde{\mu})$ satisfies (VC). Therefore, for any $x \in M$ we have

$$\widetilde{\mu}\big(B(o,\rho)\big) \le C\widetilde{\mu}\big(B(x,s)\big),$$

where $s = \frac{1}{64}\rho$ and $\rho = d(o,x)$. Assuming that $\rho > 1$, choose an integer k such that $2^k \le \rho < 2^{k+1}$. A computation similar to the one above gives

$$\widetilde{\mu}\big(B(o,\rho)\big) \ge c\sum_{i=0}^{k} H_i,$$

whereas $\widetilde{\mu}(B(x,s)) \le CH_k$. Combining these inequalities, we obtain (h2). $\square$

When applying Theorem 5.7 the following elementary lemma comes handy. We omit the proof, which is straightforward and is contained in the above computation.

Lemma 5.11. — *Assume that the hypotheses of Theorem* 5.7 *are satisfied including* (h2). *Then we have, for all $x \in M$ and $r > 0$,*

$$\widetilde{\mu}\big(B(x,r)\big) \approx \mu\big(B(x,r)\big)\bar{h}(\rho + r)^2,$$

where $\rho := d(o,x)$ and $\bar{h}(s) := \sup_{\partial B(o,s)} h$.

Combining Theorem 5.7, Lemma 5.11, and Theorem 2.7, we obtain the following statement.

Corollary 5.12. — *Assume that the hypotheses of Theorem* 5.7 *are satisfied including* (h2). *Then the heat kernel $\widetilde{p}$ of the weighted manifold $(M,\widetilde{\mu})$ satisfies the estimate*

$$\widetilde{p}(t,x,y) \approx \frac{\exp\big(-cd^2(x,y)/t\big)}{(V(x,\sqrt{t}\,)\,V(y,\sqrt{t}\,))^{1/2}\bar{h}(d(o,x)+\sqrt{t}\,)\,\bar{h}(d(o,y)+\sqrt{t}\,)},$$

where $V(x,r) = \mu(B(x,r))$.

6. Elliptic Harnack inequality.

The purpose of the next two subsections is to obtain the elliptic Harnack inequality (EHI) (see Definition 2.1) for *all* balls assuming only that it holds for *some* specific families of balls.

6.1. Elliptic Harnack inequality: from remote balls to all balls.

Recall that the notion of remote and anchored balls has been introduced in Definition 4.1.

LEMMA 6.1. — *Let (M,μ) be a weighted manifold and let $\Gamma \subset M$ be a closed set. Assume that* (EHI_η) *holds for all anchored and remote balls. Then* $(\mathrm{EHI}_{\eta'})$ *holds for all balls with $\eta' = \frac{1}{8}\varepsilon\eta^2$, where $0 < \varepsilon \leq 1$ is the remote parameter.*

Proof. — Let $B(x,r)$ be a ball that is neither anchored nor remote. Then by definition $r > \frac{1}{2}\varepsilon\rho > 0$ where $\rho = d(\Gamma, x)$. Consider first the case when

$$r \geq 2(1 + \eta^{-1})\rho. \tag{6.1}$$

Let $o \in \Gamma$ be a point such that $d(o,x) = \rho$. Then $B(x, \eta' r) \subset B(o, \eta' r + \rho)$ but $B(x,r) \supset B(o, r-\rho)$. Using $\eta' \leq \frac{1}{2}\eta$ and (6.1), it is easy to check that $\eta' r + \rho \leq \eta\big(r - \rho\big)$. Hence, applying (EHI_η) in the anchored ball $B(o, r-\rho)$ we obtain $(\mathrm{EHI}_{\eta'})$ in $B(x,r)$.

Assume now that

$$\frac{1}{2}\varepsilon\rho < r < 2(1 + \eta^{-1})\rho. \tag{6.2}$$

It follows from $\eta' = \frac{1}{8}\varepsilon\eta^2$ and (6.2) that $\eta \cdot \frac{1}{2}\varepsilon\rho > \eta' r$. Hence, applying (EHI_η) in the remote ball $B(x, \frac{1}{2}\varepsilon\rho)$ we obtain $(\mathrm{EHI}_{\eta'})$ in $B(x,r)$. □

Before we proceed further, let us introduce one more notion.

DEFINITION 6.2. — We say that a metric space (M,d) satisfies the *annuli covering condition* (AC_η) with respect to a point $o \in M$ and with parameter $0 < \eta < 1$ if there exist constants $C_A > 1$ and Q such that for any $r \geq C_A^2$ the annulus $\{x : r/C_A \leq d(o,x) \leq C_A r\}$ can be covered by at most Q balls $B(x_i, s_i)$ such that all balls $B(x_i, s_i/\eta)$ are remote to $\Gamma = \{o\}$ (with the remote parameter 1).

When we write (AC) we mean (AC_η) for some η. In what follows we assume that (M, μ) is a complete weighted manifold.

LEMMA 6.3. — *Fix a point* $o \in M$, *set* $\Gamma = \{o\}$, *and assume that* (EHI$_\eta$) *holds for remote balls. Assume further that* (M,μ) *satisfies conditions* (AC_η) *and* (RCA) *with respect to* o *and with constant* C_A. *Then for any* $r > 0$ *and for any non-negative harmonic function* u *in* $B(o,3C_A r) \setminus B(o,r/(3C_A))$,

$$\sup_{\partial B(o,r)} u \le C \inf_{\partial B(o,r)} u. \tag{6.3}$$

Consequently, (EHI) *holds for all balls.*

Proof. — Let us first explain how to obtain (EHI) for all balls assuming that the *annulus Harnack inequality* (6.3) is already proved. By hypothesis, (EHI) is known for remote balls. Hence, by Lemma 6.1, it suffices to prove (EHI) for anchored balls. Let u be a non-negative harmonic function in an anchored ball $B(o, 3C_A r)$. Applying the maximum principle and (6.3), we obtain

$$\sup_{B(o,r)} u = \sup_{\partial B(o,r)} u \le C \inf_{\partial B(o,r)} u = C \inf_{B(o,r)} u,$$

that is (EHI) for anchored balls.

Let us now prove (6.3). For a bounded range of r (6.3) holds just by a compactness argument. Assuming in the sequel that r is large enough, let x (resp. y) be a point on $\partial B(o,r)$ where u attains its maximum (resp. minimum) on $\partial B(o,r)$. By (RCA) there is a continuous path γ connecting x to y in $B(o, C_A r) \setminus B(o, r/C_A)$. By (AC_η), the curve γ can be covered by at most Q balls $B(x_i, s_i)$ such that all balls $B(x_i, s_i/\eta)$ are remote.

Set $r_i = d(o, x_i)$. As $B(x_i, s_i)$ intersects γ and hence the annulus $B(o, C_A r) \setminus B(o, r/C_A)$ we have

$$r_i - s_i \le C_A r \quad \text{and} \quad r_i + s_i \ge C_A^{-1} r.$$

As $B(x_i, s_i/\eta)$ is a remote ball, we have $s_i/\eta \le \frac{1}{2} r_i$. It follows that

$$\frac{2}{3} C_A^{-1} r \le r_i \le 2 C_A r,$$

whence we obtain

$$r_i - s_i/\eta \ge \frac{1}{3} C_A^{-1} r \quad \text{and} \quad r_i + s_i/\eta \le 3 C_A r.$$

In particular, each ball $B(x_i, s_i/\eta)$ is contained in $B(o, 3C_A r) \setminus B(o, r/(3C_A))$. Considering a chain of balls $B(x_i, s_i)$ connecting x to y and applying (EHI$_\eta$) in all balls $B(x_i, s_i/\eta)$ we obtain

$$u(x) \le C_H^Q u(y),$$

whence (6.3) follows. □

Lemma 6.4. — *Fix a point* $o \in M$ *and assume that* (M,μ) *satisfies* (VC) *with respect to* $\Gamma = \{o\}$ *and* (VD) *for anchored balls. Then the condition* (AC_η) *is satisfied with* $\eta = \frac{1}{4}\varepsilon$ *where* ε *is the remote parameter from the condition* (VC).

Proof. — Set $C_A = 2$ and consider an annulus

$$A := \left\{x : \frac{1}{2}r \le d(o,x) \le 2r\right\}.$$

Similarly to Example 3.2, consider a maximal set $\{x_i\}$ of points $x_i \in A$ at distance at least $s := \frac{1}{16}\varepsilon r$ each from other. Clearly, all balls $B(x_i, s)$ cover A.

Set $r_i = d(x_i, o)$. By construction, we have $\frac{1}{2}r_i \le r \le 2r_i$. Therefore, $s/\eta = \frac{1}{4}r \le \frac{1}{2}r_i$, whence we see that the ball $B(x_i, s/\eta)$ is remote, with remote parameter 1. Since $\frac{1}{2}s \ge \frac{\varepsilon}{64}r_i$, we obtain by (VC)

$$\mu\left(B(x_i, \tfrac{1}{2}s)\right) \ge C_V^{-1}\mu\left(B(o, r_i)\right) \ge C_V^{-1}\mu\left(B(o, \tfrac{1}{2}r)\right).$$

On the other hand, all balls $B(x_i, \frac{1}{2}s)$ are disjoint, whence by (VD) for anchored balls

$$\sum_i \mu\left(B(x_i, \tfrac{1}{2}s)\right) \le \mu\left(B(o, 2r + \tfrac{1}{2}s)\right) \le C_V^3\mu\left((B(o, \tfrac{1}{2}r)\right).$$

Therefore, the number of the points x_i is bounded by C_V^4, which finishes the proof. □

Remark 6.5. — Combining Lemma 4.4 with Lemma 6.4 we conclude that (VD) for all balls implies (AC).

Proposition 6.6. — *Fix a point* $o \in M$ *and set* $\Gamma = \{o\}$. *Assume that a complete weighted manifold* (M,μ) *satisfies* (VD) *and* (RCA), *as well as* (EHI) *for remote balls. Then* (M,μ) *satisfies the annulus Harnack inequality* (6.3) *as well as* (EHI) *for all balls.*

Proof. — By Lemma 4.4, (VD) implies (VC) with any remote parameter $0 < \varepsilon \le 1$. By Lemma 6.4, we obtain (AC_η) with parameter $\eta = \frac{1}{4}\varepsilon$. Adjusting ε so that (EHI_η) holds for remote balls with the same η, we finish the proof by Lemma 6.3. □

Proposition 6.6 can be considered as an elliptic analogue of Corollary 5.4 (or Theorem 5.2). Indeed, the former says that

$$(VD) + (\mathrm{RCA}) + \mathrm{remote}(\mathrm{EHI}) \implies (\mathrm{EHI}),$$

whereas the latter says that

$$(\mathrm{VC}) + (\mathrm{RCA}) + \mathrm{remote}(\mathrm{PHI}) \implies (\mathrm{PHI}).$$

However, the difference is that while (VC) is a necessary condition for (PHI), the condition (VD) is not necessary for (EHI).

6.2. Elliptic Harnack inequality for model manifolds.

Consider a model manifold M_ψ where ψ satisfies (1.7), that is

$$\sup_{[r,2r]} \psi \le C \inf_{[r,2r]} \psi, \tag{6.4}$$

for all $r > 0$. This easily implies that (PHI) and (EHI) hold on remote balls with respect to the pole o. Obviously, (RCA) is also satisfied in this case. The annuli covering condition (AC) (see Definition 6.2) is equivalent to

$$\psi(r) \le Cr, \tag{6.5}$$

for all $r > 0$. Thus, by Lemma 6.3, any model manifold satisfying (6.4) and (6.5) satisfies (EHI). In fact, (6.5) is a necessary and sufficient condition for (EHI) as will be proved in the following statement.

Proposition 6.7. — *Let M_ψ be a model manifold with ψ satisfying* (6.4). *Then* (EHI) *holds on M_ψ if and only if ψ satisfies* (6.5). *In particular, if $\psi(r) \approx r^\alpha$ for $r > 1$, then* (EHI) *holds if and only if $\alpha \le 1$.*

Proof. — We only need to prove (EHI) implies (6.5) for all $r > 1$ (for $r \le 1$, (6.5) follows from (4.12)). Fix $r \ge 1$ and consider the Green function $G_{2r}(x, y)$ with the Dirichlet boundary condition in the ball $B(o, 2r)$, defined by

$$G_{2r}(x,y) = \int_0^\infty p_{2r}(t,x,y)\,\mathrm{d}t, \tag{6.6}$$

where $p_{2r}(t, x, y)$ is the heat kernel in $B(o, 2r)$ with the Dirichlet boundary condition. It is not hard to check that

$$G_{2r}(o,y) = \frac{1}{\omega_N} \int_{d(o,y)}^{2r} \frac{1}{\psi(s)^{N-1}}\,\mathrm{d}s$$

(see for example [19, Section 4.2]). In particular, if $d(o, y) = r$, then

$$G_{2r}(o, y) \approx \frac{r}{\psi(r)^{N-1}}. \tag{6.7}$$

Fix an angular component θ and set $y_0 = (r, \theta)$ and $x_0 = (\eta r, \theta)$ where $0 < \eta < 1$ is the parameter from (EHI). By (PHI) on remote balls (which follows from (6.4)) and a standard chaining argument we have

$$p_{2r}(t, x_0, y_0) \geq c p_{2r}\big(\tfrac{1}{2}t, y_0, y_0\big), \quad \text{for all } t \approx r^2. \tag{6.8}$$

By the proof of (2.14), we also have

$$p_{2r}\big(\tfrac{1}{2}t, y_0, y_0\big) \geq \frac{c}{V(y_0, \sqrt{t}\,)}, \quad \text{for all } t \leq \big(\tfrac{1}{16}r\big)^2. \tag{6.9}$$

Combining (6.8), (6.9), and (6.6) we obtain

$$G_{2r}(x_0, y_0) \geq \int_{(r/32)^2}^{(r/16)^2} p_{2r}(t, x_0, y_0)\, \mathrm{d}t \geq c \frac{r^2}{V(y_0, r/16)}. \tag{6.10}$$

Let us show that

$$V\big(y_0, \tfrac{1}{16}r\big) \leq C r^N. \tag{6.11}$$

If $\psi(r) > Cr$ (for a large enough constant C), then the ball $B(y_0, \frac{1}{16}r)$ is quasi-isometric to a Euclidean ball of the same radius, whence (6.11) follows. If $\psi(r) \leq Cr$, then

$$\begin{aligned} V\big(y_0, \tfrac{1}{16}r\big) \leq V(o, 2r) - V\big(o, \tfrac{1}{2}r\big) &= \omega_N \int_{r/2}^{2r} \psi^{N-1}(\xi)\, \mathrm{d}\xi \\ &\leq C\psi(r)^{N-1} r \leq C r^N. \end{aligned}$$

Hence, (6.10) and (6.11) imply

$$G_{2r}(x_0, y_0) \geq \frac{c}{r^{N-2}}. \tag{6.12}$$

Now, $x \mapsto G_{2r}(x, y_0)$ is a non-negative harmonic function in $B(o, r)$. Applying (EHI) to compare its values at o and x_0 we obtain from (6.7) and (6.12)

$$\frac{r}{\psi(r)^{N-1}} \geq \frac{c}{r^{N-2}},$$

whence (6.5) follows. □

Remark 6.8. — Assume that $\psi(r) = r^\alpha$ for $r > 1$ where $\alpha \leq 1$. By (6.4) (PI) holds for all remote balls. By Example 3.14 (PI) holds for all anchored balls. Hence M_ψ satisfies (PI), by Proposition 4.2. Proposition 4.10 shows that M_ψ satisfies (PHI) if and only if $-1/(N-1) < \alpha \leq 1$. Hence, for $\alpha \leq -1/(N-1)$, M_ψ satisfies (EHI) and (PI) but not (PHI). Examples of manifolds satisfying (EHI) but not (PHI) are described in [2], [3], [12]. These examples are much less explicit than M_ψ. To our knowledge, none of the previously known examples satisfies (PI) nor is simply connected. When $N = 2$, the Gauss curvature of M_ψ at $x = (r,\theta)$ equals $-\psi''/\psi\,(r) = cr^{-2}$ which tends to 0 as $r \to \infty$, again unlike any previously known example.

Remark 6.9. — Let us say that a weighted manifold $(M, \boldsymbol{g}, \mu)$ satisfies the *weak Liouville* (resp. *strong Liouville*) *property* if any bounded (resp. positive) harmonic function on M is constant. Liouville properties on model manifolds have been studied by many authors, e.g., [36], [38], [41]. They proved that for M_ψ the weak and strong Liouville properties are equivalent and are satisfied if and only if

$$\int^{+\infty} \Big(\frac{1}{\psi(r)^{N-1}} \int_1^r \psi(s)^{N-3}\,\mathrm{d}s \Big)\,\mathrm{d}r = +\infty. \tag{6.13}$$

If $N = 2$, then (6.13) coincides with the condition

$$\int^{+\infty} \frac{\mathrm{d}s}{\psi(s)} = +\infty, \tag{6.14}$$

which is equivalent to the *parabolicity* of M_ψ (a manifold is called parabolic if any positive *superharmonic* function is constant). From Proposition 6.7 and (6.13), we obtain a class of examples of manifolds having the strong Liouville property but for which (EHI) fails. For instance, in dimension 2, it suffices to take $\psi(r) = r \log r$ (for large r) so that (6.14) holds, whereas (6.5) fails.

6.3. Changes of measure.

THEOREM 6.10. — *Let* (M,μ) *be a complete weighted manifold. Fix* $o \in M$ *and assume that* M *satisfies conditions* (RCA) *and* (AC_η) *(see Definitions 5.1 and 6.2). Let* h *be a positive smooth function on* M *such that one of the following two conditions holds*:

(i) *either* h *satisfies the hypothesis* (h1) *of Theorem* 5.7 *and* (M,μ) *satisfies* (PHI) *for remote balls*;

(ii) *or h is harmonic outside a compact set and* (M,μ) *satisfies* (EHI_η) *for remote balls.*

Define a measure $\widetilde{\mu}$ on M by $\mathrm{d}\widetilde{\mu} = h^2 \mathrm{d}\mu$. *Then* $(M,\widetilde{\mu})$ *satisfies* (EHI).

Remark 6.11. — In the case (i), if (PHI) holds on (M,μ) for all balls, then the hypothesis (AC) can be dropped as it follows from (VD) and hence from (PHI) (see Lemma 6.4 and Theorem 2.7).

Remark 6.12. — The example of Remark 5.10 can be used to show that condition (RCA) is essential for the validity of Theorem 6.10.

Proof. — (i) By Theorem 2.12, (M,μ) satisfies (PI) and (VD) for remote balls. By (h1), the function h varies on any remote ball at most by a constant factor. Therefore, $(M,\widetilde{\mu})$ also satisfies (PI) and (VD) for remote balls. Again by Theorem 2.12, $(M,\widetilde{\mu})$ satisfies (PHI) for remote balls. The latter implies (EHI_η) for remote balls with any parameter $0<\eta<1$ (see Remark 2.3). In particular, take η the same as in (AC_η). Then $(M,\widetilde{\mu})$ satisfies (EHI) for all balls by Lemma 6.3.

(ii) By Lemma 6.3, it suffices to prove (EHI_η) for remote balls in $(M,\widetilde{\mu})$. Let $B(x,r)$ be a remote ball and $\Delta_{\tilde{\mu}} u = 0$, $u \geq 0$ in $B(x,r)$. The Laplace operator $\Delta_{\tilde{\mu}}$ of $(M,\widetilde{\mu})$ has the form

$$\Delta_{\tilde{\mu}} u = h^{-2} \operatorname{div}_\mu (h^2 \nabla u) = \Delta_\mu u + 2h^{-1} \boldsymbol{g}(\nabla h, \nabla u).$$

In particular, in the domain D of harmonicity of h we have $\Delta_\mu h = 0$ whence

$$\Delta_{\tilde{\mu}} u = \Delta_\mu u + 2h^{-1} \boldsymbol{g}(\nabla h, \nabla u) + \Delta_\mu h = h^{-1} \Delta_\mu (hu).$$

Hence, if the ball $B(x,r)$ is contained in D, then we have $\Delta_\mu(hu) = 0$ in $B(x,r)$. By (EHI_η) for remote balls in (M,μ), we obtain

$$\sup_{B(x,\eta r)} (uh) \leq C_H \inf_{B(x,\eta r)} (uh) \quad \text{and} \quad \sup_{B(x,\eta r)} h \leq C_H \inf_{B(x,\eta r)} h,$$

whence $\sup_{B(x,\eta r)} u \leq C_H^2 \inf_{B(x,\eta r)} u$. If $B(x,r)$ is not contained in D, then it intersects a compact set $M \setminus D$ and hence has a bounded radius. In this case, an elliptic Harnack inequality in $B(x,r)$ follows by a compactness argument. □

Corollary 6.13. — *For any complete weighted manifold* (M,μ) *satisfying* (PHI) *and* (RCA), *there exists a smooth positive function h on M such that the manifold* $(M,\widetilde{\mu})$ *with measure* $\mathrm{d}\widetilde{\mu} = h^2 \mathrm{d}\mu$ *satisfies* (EHI) *but not* (PHI).

Proof. — Indeed, take h as in Remark 5.8. Then h satisfies the condition (h1) from Theorem 5.7 but not (h2). By Theorem 5.7, $(M, \widetilde{\mu})$ does not satisfy (PHI), but $(M, \widetilde{\mu})$ satisfies (EHI) by Theorem 6.10 (i) (see also Remark 6.11). □

Example 6.14. — Consider the weighted manifold $(\mathbb{R}^N, \mu_\alpha)$, $N \geq 2$, where $\mathbb{R}^N$ is equipped with the Euclidean metric and measure μ_α is given by $\mathrm{d}\mu_\alpha = (1+|x|^2)^{\alpha/2}\mathrm{d}x$. By Proposition 4.9 or Theorem 5.7, this manifold satisfies (PHI) if and only if $\alpha > -N$. Note that the condition (h2) breaks down exactly for $\alpha \leq -N$. The function $h(x) = (1+|x|^2)^{\alpha/4}$ satisfies (h1) for any real α. Therefore, by Theorem 6.10 (i), the manifold $(\mathbb{R}^N, \mu_\alpha)$ satisfies (EHI) for all real α.

7. Harnack inequalities on manifolds with ends.

7.1. Connected sums of manifolds.

Let $\{M_i\}_{i=1}^n$ be a finite family of non-compact Riemannian manifolds. We say that a Riemannian manifold M is a *connected sum* of the M_i's and write

$$M = M_1 \# M_2 \# \cdots \# M_n \tag{7.1}$$

if, for some compact $K \subset M$ (called *the central part* of M), the exterior $M \setminus K$ is a disjoint union of open sets $E_1, E_2, \ldots, E_n$, such that each E_i is isometric to $M_i \setminus K_i$, for some compact $K_i \subset M_i$ (in fact, we will always identify E_i and $M_i \setminus K_i$). If (M, μ) and (M_i, μ_i) are weighted manifolds, then the isometry is understood in the sense of weighted manifolds, that is it maps measure μ to μ_i. Of course, forming connected sums is not a uniquely defined operation.

Let M be a non-compact manifold and $K \subset M$ be a compact set with smooth boundary such that $M \setminus K$ is a disjoint union of a finite number of connected open sets $E_1, E_2, \ldots, E_n$ which are not precompact (if M has boundary δM, then assume in addition that $\partial K \cap \delta M = \emptyset$). We say that the E_i's are the *ends* of M with respect to K. Consider the closure $\overline{E}_i$ as a manifold with boundary. Then by definition of a connected sum we have $M = \overline{E}_1 \# \overline{E}_2 \# \cdots \# \overline{E}_n$.

Let a weighted manifold (M, μ) be a connected sum of weighted manifolds (M_i, μ_i). Fix points $o \in M$, $o_i \in M_i$ and set

$$V(r) = \mu\big(B(o,r)\big) \quad \text{and} \quad V_i(r) = \mu_i\big(B_{M_i}(o_i, r)\big). \tag{7.2}$$

Consider the following two conditions:

(v1) For all $i = 1, \dots, n$ and all r large enough, $V_i(r) \approx V(r)$.

(v2) For all r large enough,

$$\int_1^r \frac{s\,\mathrm{d}s}{V(s)} \le C \frac{r^2}{V(r)}. \tag{7.3}$$

Condition (v1) means that any two ends have comparable volume growth. Note that, for any bounded range of r, we have always $V_i(r) \approx V(r) \approx r^N$ where $N = \dim M$. Hence, (v1) is equivalent to the fact that $V_i(r) \approx V(r)$ for *all* $r > 0$.

Note that, for $r > 2$, one always has

$$\int_1^r \frac{s\,\mathrm{d}s}{V(s)} \ge \int_{r/2}^r \frac{s\,\mathrm{d}s}{V(s)} \ge \frac{r^2}{4V(r)},$$

which implies that (v2) is equivalent to

$$\int_1^r \frac{s\,\mathrm{d}s}{V(s)} \approx \frac{r^2}{V(r)}. \tag{7.4}$$

Condition (v2) significantly restricts the growth rate of the function V. In particular, it implies

$$\int_1^\infty \frac{s\,\mathrm{d}s}{V(s)} = \infty \tag{7.5}$$

(which is easy to see from (7.4)) and, hence, $V(r) = o(r^2)$ as $r \to \infty$. For example, if $V(r) \approx r^\alpha$ for large r, then (v2) holds if and only if $\alpha < 2$.

THEOREM 7.1. — *Let a weighted manifold (M, μ) be a connected sum of weighted manifolds (M_i, μ_i), $i = 1, \dots, n$, where each M_i is complete, non-compact, and satisfies* (PHI) *and* (RCA) *with respect to a point $o_i \in M_i$.*

(i) *If $n = 1$, then M satisfies* (PHI).

(ii) *Assume that $n \ge 2$ and that* (v1) *and* (v2) *are satisfied. Then M satisfies* (PHI).

(iii) *Assume that $n \ge 2$ and that M satisfies* (EHI). *Then* (v1) *and* (v2) *hold true.*

In particular, if $n \ge 2$, then (PHI) $\Leftrightarrow$ (EHI) $\Leftrightarrow$ (v1) + (v2).

Example 7.2. — Let $n \geq 2$ and let all manifolds $M_1, \ldots, M_n$ be isometric to a model manifold M_ψ of dimension N where $\psi(r) \approx r^\alpha$ for large r. We claim that if $|\alpha| < 1/(N-1)$, then $M = M_1 \# \cdots \# M_n$ satisfies (PHI). Indeed, we have $V_i(r) \approx r^{\alpha(N-1)+1}$ so that (v2) follows from $\alpha < 1/(N-1)$. By Proposition 4.10, each M_i satisfies (PHI) because $-1/(N-1) < \alpha \leq 1$. By Theorem 7.1 (*ii*), we conclude that M satisfies (PHI).

Part (i) of Theorem 7.1 is a consequence of Corollary 5.4 which was already mentioned in Remark 5.5. The proofs of parts (ii) and (iii) are rather involved and are given in Sections 7.3 and 7.4, using an auxiliary material from Section 7.2.

7.2. Flux and capacity.

Fix a couple (D, U) where $U \subset M$ is an open set and $D \subset U$ is a precompact set. For any harmonic function v in $U \setminus \overline{D}$ define the notion of the *flux* of v with respect to the couple (D, U) by

$$\underset{(D,U)}{\operatorname{flux}}\, v := \left| \int_{\partial\Omega} \frac{\partial v}{\partial \boldsymbol{n}} \, \mathrm{d}\mu' \right|,$$

where Ω is any precompact open set with smooth boundary such that $D \Subset \Omega \Subset U$, and $\boldsymbol{n}$ is the inward unit normal vector field on $\partial\Omega$. By the harmonicity of v, the flux does not depend on the choice of Ω. Clearly, we have the following identities

$$\underset{(D,U)}{\operatorname{flux}}\, v = \underset{(D,U)}{\operatorname{flux}}\, (-v) = \underset{(D,U)}{\operatorname{flux}}\, (v + \text{const}\,) = \underset{(D',U')}{\operatorname{flux}}\, v$$

provided $D' \supset D$ and $U' \subset U$.

If in addition U is precompact and v satisfies the boundary conditions $v_{|\partial U} = 0$ and $v_{|\partial D} = 1$, then

$$\underset{(D,U)}{\operatorname{flux}}\, v = \operatorname{cap}(D, U) := \inf_{\substack{\varphi \in \mathcal{C}_0^1(U) \\ \varphi_{|D} \equiv 1}} \int_U |\nabla \varphi|^2 \, \mathrm{d}\mu. \tag{7.6}$$

The following general estimate of capacity was proved in [47]: for all $x \in M$ and $0 < r < \rho$

$$\operatorname{cap}\big(B(x,r), B(x,\rho)\big)^{-1} \geq \frac{1}{2} \int_r^\rho \frac{s \, \mathrm{d}s}{V(x,s)} \cdot \tag{7.7}$$

Assuming in addition that the manifold satisfies (PHI) and $r \le \frac{1}{2}\rho$, one has also an opposite inequality

$$\operatorname{cap}\big(B(x,r),B(x,\rho)\big)^{-1} \le C\int_r^\rho \frac{s\,\mathrm{d}s}{V(x,s)} \tag{7.8}$$

(see [21, Lemma 4.3]). Hence, under the above assumptions we have

$$\operatorname{cap}\big(B(x,r),B(x,\rho)\big)^{-1} \approx \int_r^\rho \frac{s\,\mathrm{d}s}{V(x,s)}\cdot \tag{7.9}$$

Lemma 7.3. — *Let a complete non-compact weighted manifold* (M,μ) *satisfy* (PHI) *and* (RCA) *with respect to a point* o. *Fix positive numbers* r_0 *and* ρ *such that* $\rho > 4r_0$. *Set* $U = B(o,\rho)$, *let* D *be a precompact subset in* $B(o,r_0)$, *and let* v *be a positive harmonic function in* $U \setminus \overline{D}$ *such that*

$$v_{|\partial U} = 0 \quad \text{and} \quad v_{|\partial D} = \text{const.}$$

Then, for any $r \in (2r_0, \frac{1}{2}\rho)$ *and for all* $x \in \partial B(o,r)$, *we have*

$$v(x) \approx \big(\operatorname*{flux}_{(D,U)} v\big)\int_r^\rho \frac{s\,\mathrm{d}s}{V(o,s)}\cdot \tag{7.10}$$

Proof. — Without loss of generality, we can assume $v=1$ on ∂D and extend v to D by setting $v=1$. Set

$$a := \inf_{\partial B(o,r)} v = \inf_{B(o,r)} v \quad \text{and} \quad b := \sup_{\partial B(o,r)} v = \sup_{B(o,\rho)\setminus B(o,r)} v \tag{7.11}$$

and note that by the strong maximum principle $0 < a \le b < 1$. Similarly to the proof of Lemma 6.3, the hypotheses (RCA) and (PHI) imply $b \le Ca$ where the constant C does not depend on r.

For any $0 < t < 1$, consider the set $U_t = \{x \in B(o,\rho) : v(x) > t\}$ (see Figure 8) and observe that by (7.6)

$$\operatorname{cap}(U_t,U) = \operatorname*{flux}_{(U_t,U)} \frac{v}{t} = \frac{1}{t}\operatorname*{flux}_{(D,U)} v.$$

It is clear from (7.11) that $U_b \subset B(o,r) \subset U_a$, whence by the monotonicity of capacity we obtain

$$\frac{\operatorname{flux} v}{b} \le \operatorname{cap}(B(o,r),U) \le \frac{\operatorname{flux} v}{a}$$

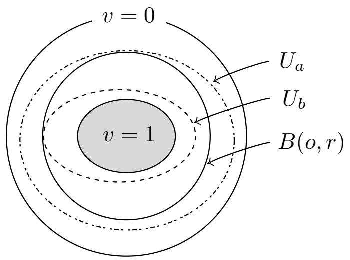

Figure 8. *Sets U_a and U_b*

(here for simplicity we suppress the subscript (D, U) in flux$v$). Since for any $x \in \partial B(o, r)$ we have $v(x) \approx a \approx b$, we obtain

$$v(x) \approx \frac{\operatorname{flux} v}{\operatorname{cap}(B(o,r), B(o,\rho))}. \tag{7.12}$$

Finally, applying (7.9) we obtain (7.10). □

Lemma 7.4. — *Let U be a precompact open subset of M with smooth boundary, and D be a precompact set such that $\overline{D} \subset U$. Let w be a harmonic function in $U \setminus \overline{D}$ such that*

$$w_{|\partial U} = 0 \quad \text{and} \quad w_{|\partial D} = 1, \tag{7.13}$$

and v be a harmonic function in $U \setminus \overline{D}$ such that

$$v_{|\partial U} = 0 \quad \text{and} \quad a \leq v_{|\partial D} \leq b,$$

for positive constants a,b. Then

$$a \operatorname*{flux}_{(D,U)} w \leq \operatorname*{flux}_{(D,U)} v \leq b \operatorname*{flux}_{(D,U)} w. \tag{7.14}$$

Proof. — By the comparison principle, we have $v/b \leq w \leq v/a$. Since all the functions $v/b, w, v/a$ vanish on ∂U, we obtain

$$0 \leq \frac{\partial(v/b)}{\partial \boldsymbol{n}}\Big|_{\partial U} \leq \frac{\partial w}{\partial \boldsymbol{n}}\Big|_{\partial U} \leq \frac{\partial(v/a)}{\partial \boldsymbol{n}}\Big|_{\partial U},$$

where $\boldsymbol{n}$ is the inward normal vector field on ∂U. Clearly, this implies (7.14). □

Lemma 7.5. — *Under the conditions of Lemma 7.3, let us drop the hypothesis that* $v_{|\partial D} = \mathrm{const}$. *Then we still have, for any* $r \in (2r_0, \frac{1}{2}\rho)$ *and for all* $x \in \partial B(o,r)$,

$$C^{-1}\frac{a}{b}\Big(\operatorname*{flux}_{(D,U)} v\Big)\int_r^\rho \frac{s\,\mathrm{d}s}{V(o,s)} \le v(x) \le C\frac{b}{a}\Big(\operatorname*{flux}_{(D,U)} v\Big)\int_r^\rho \frac{s\,\mathrm{d}s}{V(o,s)}, \tag{7.15}$$

where $a = \inf_{\partial D} v$ *and* $b = \sup_{\partial D} v$.

Proof. — Let w be the harmonic function in $U \setminus \overline{D}$ satisfying (7.13). Then by the comparison principle and by Lemma 7.3, we have

$$v(x) \le bw(x) \approx b\big(\operatorname*{flux}_{(D,U)} w\big)\int_r^\rho \frac{s\,\mathrm{d}s}{V(o,s)}\cdot$$

By Lemma 7.4 flux$v \ge a$flux$w$, whence the upper bound in (7.15) follows. The lower bound is proved similarly. □

7.3. Proof of Theorem 7.1: (v1)–(v2) imply (PHI).

In this section $M = M_1\# \cdots \# M_n$ is as in Theorem 7.1, and $n \ge 2$. Assuming that conditions (v1) and (v2) are satisfied, we will prove the parabolic Harnack inequality (PHI) on M.

By Theorem 2.7, each M_i satisfies (VD) and (PI). Fix a point o in the central part of M. Clearly, condition (VD) on M_i and (v1) imply that the function $V(r) := \mu(B(o,r))$ satisfies the doubling property:

$$V(2r) \le V(r), \quad \text{for all } r > 0. \tag{7.16}$$

Set $\Gamma = \{o\}$ and observe that condition (VD) on each M_i implies (VD) for remote balls on M (with the remote parameter $\varepsilon = 1$). In addition, (v1) implies that M satisfies (VC). Hence, by Proposition 4.7, (VD) holds for all balls in M.

For any precompact open set Ω on a weighted manifold $(\mathcal{M},\mu)$ (where $\mathcal{M}$ will be either M or M_i), set

$$\lambda^{(D)}(\Omega) := \inf\left\{\frac{\int|\nabla f|^2\,\mathrm{d}\mu}{\int f^2\,\mathrm{d}\mu} : f \in \mathcal{C}^1(\Omega)\setminus\{0\}\right\},$$

that is, $\lambda^{(D)}(\Omega)$ is the bottom eigenvalue for the Dirichlet problem in Ω. By [17, Theorem 5.1], if $\mathcal{M}$ is complete and satisfies (PHI) then, for any ball B in $\mathcal{M}$ of radius r and for any open set $\Omega \subset B$, the following inequality holds

$$\lambda^{(D)}(\Omega) \ge \frac{c}{r^2}\Big(\frac{\mu(B)}{\mu(\Omega)}\Big)^\alpha, \tag{7.17}$$

with some constant $\alpha, c > 0$ depending on C_H. We refer to (7.17) as the *Faber-Krahn inequality.* Since all manifolds M_i satisfy (PHI), we conclude that all manifolds M_i satisfy the Faber-Krahn inequality.

By a result of [20], if the Faber-Krahn inequality holds on each M_i, then, for any ball $B = B(x,r)$ in M and for any open set $\Omega \subset B$,

$$\lambda^{(D)}(\Omega) \geq \frac{\widetilde{c}}{r^2}\Big(\frac{\widetilde{\mu}(B)}{\mu(\Omega)}\Big)^{\alpha},$$

where $\widetilde{c} > 0$ and

$$\widetilde{\mu}(B) := \begin{cases} \mu(B) & \text{if } B \subset E_i \text{ for some } i \ , \\ \min_i V_i(r) & \text{otherwise.} \end{cases}$$

The hypothesis $V_i(r) \approx V(r)$ allows to replace here $\min_i V_i(r)$ by $V(r)$. Furthermore, if the ball $B(x,r)$ is not contained in any E_i, then it intersects the central part K of M, which implies by (VD) and Lemma 2.9 that $\mu(B(x,r)) \approx \mu(B(o,r)) \approx V(r)$. Therefore, we conclude that $\widetilde{\mu}(B) \approx \mu(B)$ for all balls $B \subset M$, which means that the Faber-Krahn inequality holds also on the manifold M.

By [18, Proposition 5.2], the Faber-Krahn inequality on M implies the following heat kernel estimate:

$$p(t,x,y) \leq \frac{\exp(-cd^2(x,y)/t)}{(V(x,\sqrt{t}\,), V(y,\sqrt{t}\,))^{1/2}}. \tag{7.18}$$

By [25, Theorems 1.2 and 2.6], the upper bound (7.18) together with the elliptic Harnack inequality (EHI) implies (PHI). Hence, we are left to prove that (EHI) holds on (M,μ), which will be done in the rest of proof.

Obviously, M satisfies (EHI) for remote balls because (EHI) holds for each M_i. By Lemma 6.1 it suffices to prove (EHI) for anchored balls in M. Let u be a non-negative harmonic function in an anchored ball $B(o,\rho)$ of a large enough radius ρ. Since each M_i satisfies (EHI), (VD), and (RCA), we have the annulus Harnack inequality (6.3) on each M_i (see Section 6.1 and Proposition 6.6). This implies, for some (large enough) constants C_H and C_A,

$$m_i := \sup_{\partial B(o,r)\cap E_i} u \leq C_H \inf_{\partial B(o,r)\cap E_i} u, \tag{7.19}$$

where $r = \rho/C_A$ (strictly speaking, (6.3) holds on each M_i with respect to the distance d_i on M_i whereas in (7.19) we use the distance d on M;

however, this can be easily handled by increasing the constants C_H and C_A because $d_i \approx d$). By the maximum principle,

$$\sup_{B(o,r)} u = \sup_{\partial B(o,r)} u = \max_i m_i \quad \text{and} \quad \inf_{B(o,r)} u = \inf_{\partial B(o,r)} u \geq C_H^{-1} \min_i m_i.$$

Hence, (EHI) for the ball $B(o,\rho)$ amounts to $\max_i m_i \leq C \min_i m_i$, which will follow if we prove that

$$m_i \approx u(o) \quad \text{for all } i = 1, 2, \dots, n. \tag{7.20}$$

Proof of the lower bound $m_i \geq cu(o)$. — Let r_0 be a constant much larger than the diameter of the central part of M. Assuming that $r \gg r_0$ and $\rho = C_A r$, consider the following subsets of M_i:

$$D_i := (B(o,r_0) \cap E_i) \cup K_i \quad \text{and} \quad U_i = (B(o,\rho) \cap E_i) \cup K_i,$$

where $K_i = M_i \setminus E_i$. In other words, U_i and D_i are open sets in M_i such that their intersections with E_i coincide with the intersections of the balls $B(o,\rho)$ and $B(o,r_0)$ with E_i, respectively. Clearly, we have

$$\partial D_i = \partial B(o,r_0) \cap E_i \quad \text{and} \quad \partial U_i = \partial B(o,\rho) \cap E_i.$$

Let v be the harmonic function in $U_i \setminus \overline{D}_i$ such that

$$v_{|\partial D_i} = 1 \quad \text{and} \quad v_{|\partial U_i} = 0$$

(see Figure 9).

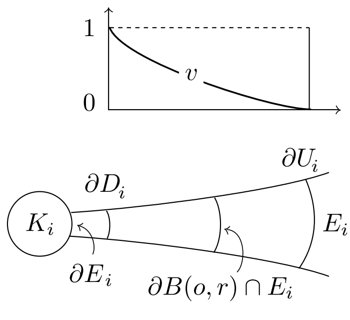

Figure 9. Manifold M_i and function v in $U_i \setminus \overline{D}_i$

By a local Harnack inequality we have $u_{|\partial D_i} \geq cu(o)$. Since also $u_{|\partial U_i} \geq 0$, the comparison principle implies

$$u(x) \geq cu(o)v(x) \quad \text{for any } x \in U_i \setminus \overline{D}_i.$$

Hence, it suffices to show that

$$\inf_{\partial B(o,r)\cap E_i} v \geq c. \tag{7.21}$$

By Lemma 7.3 we obtain, for any $x \in \partial B(o,r) \cap E_i$,

$$v(x) \approx \big(\operatorname*{flux}_{(D_i,U_i)} v\big) \int_r^\rho \frac{s\,\mathrm{d}s}{V_i(s)} \approx \big(\operatorname*{flux}_{(D_i,U_i)} v\big) \frac{\rho^2}{V(\rho)}, \tag{7.22}$$

where we have used $r \approx \rho \approx \rho - r$. Strictly speaking, when applying Lemma 7.3, we have to use $r_i := d_i(x, o_i)$ instead of $r = d(x, o)$; however, since $r_i \approx r$, this only changes the constant multiples in the estimates.

On the other hand, using (7.9), (7.4), and (v1), we obtain

$$\operatorname*{flux}_{(D_i,U_i)} v = \operatorname{cap}_{M_i}(D_i, U_i) \approx \Big(\int_{r_0}^\rho \frac{s\,\mathrm{d}s}{V_i(s)}\Big)^{-1} \approx \frac{V(\rho)}{\rho^2}.$$

Combining with (7.22) we obtain (7.21).

Proof of the upper bound $m_i \leq Cu(o)$. — In this part it will be convenient to redefine m_i as follows

$$m_i := \inf_{\partial B(o,r)\cap E_i} u$$

(by (7.19) this can reduce m_i only by a constant factor). For any $i = 1, 2, \ldots, n$, set

$$D_i = \big(B(o,r_0) \cap E_i\big) \cup K_i \quad \text{and} \quad U_i = (B(o,r) \cap E_i) \cup K_i\,.$$

Clearly, $\partial U_i = \partial B(o,r) \cap E_i$. Fix an index i, and let v be the harmonic function in $B(o,r)$ such that

$$v_{|\partial U_j} = 0 \quad \text{for all } j \neq i, \quad \text{and} \quad v_{|\partial U_i} = 1$$

(see Figure 10). By the comparison principle, we have

$$u(x) \geq m_i v(x) \quad \text{for all } x \in B(o,r).$$

In particular, the required inequality $m_i \leq Cu(o)$ will follow if we prove that $v(o) \geq c$; by a local Harnack inequality, the latter is equivalent to

$$\epsilon := \sup_{B(o,r_0)} v \geq c,$$

where r_0 is as above. Hence, we are left to prove that ε is bounded from below by a positive constant.

Let w be the harmonic function in $B(o,r) \setminus \overline{B(o,r_0)}$ such that

$$w_{|\partial B(o,r)} = 0 \quad \text{and} \quad w_{|\partial B(o,r_0)} = 1.$$

(see Figure 10).

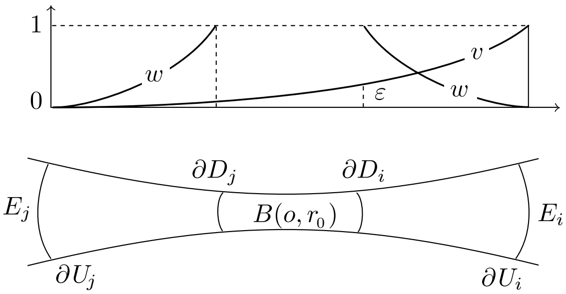

Figure 10. *Functions v and w*

By (7.9), (7.4) and (v1) we have, for any $\ell = 1, 2, \ldots, n$

$$\operatorname*{flux}_{(D_\ell,U_\ell)} w = \operatorname{cap}_{M_\ell}(D_\ell, U_\ell) \approx \left(\int_{r_0}^{r} \frac{s\,\mathrm{d}s}{V_\ell(s)} \right)^{-1} \approx \frac{V(r)}{r^2}\cdot \tag{7.23}$$

By Lemma 7.4,

$$\operatorname*{flux}_{(D_i,U_i)} v = \operatorname*{flux}_{(D_i,U_i)} (1-v) \geq (1-\epsilon) \operatorname*{flux}_{(D_i,U_i)} w \approx (1-\epsilon)\frac{V(r)}{r^2}\cdot \tag{7.24}$$

For any $j \neq i$ we have, again by Lemma 7.4,

$$\operatorname*{flux}_{(D_j,U_j)} v \leq \epsilon \operatorname*{flux}_{(D_j,U_j)} w \approx \epsilon \frac{V(r)}{r^2}\cdot \tag{7.25}$$

Let Ω be any precompact open set with smooth boundary such that $B(o,r_0) \Subset \Omega \Subset B(o,\rho)$, and $\boldsymbol{n}$ is the inward unit normal vector field on $\partial\Omega$. Observe that, by the harmonicity of v,

$$\sum_{j=1}^{n} \int_{\partial\Omega \cap E_j} \frac{\partial v}{\partial \boldsymbol{n}}\, \mathrm{d}\mu' = \int_{\partial\Omega} \frac{\partial v}{\partial \boldsymbol{n}}\, \mathrm{d}\mu' = 0.$$

Since

$$\operatorname*{flux}_{(D_j,U_j)} v = \Big| \int_{\partial\Omega \cap E_i} \frac{\partial v}{\partial \boldsymbol{n}}\, \mathrm{d}\mu' \Big|,$$

this implies

$$\operatorname*{flux}_{(D_i,U_i)} v \leq \sum_{j \neq i} \operatorname*{flux}_{(D_j,U_j)} v. \tag{7.26}$$

Combining (7.24), (7.25), and (7.26), we conclude that ϵ is bounded from below by a positive constant, which was to be proved.

7.4. Proof of Theorem 7.1: (EHI) implies (v1)–(v2).

In this section $M = M_1 \# \ldots \# M_n$ is again as in Theorem 7.1 and $n \geq 2$. We assume that M satisfies (EHI) and we will prove that (v1) and (v2) hold true on M. Let us first observe that each manifold M_i, $i = 1, 2, \ldots, n$, must be *parabolic*, that is, any positive superharmonic function on M_i is constant. Indeed, if there is a non-parabolic end then by [48, Corollary 3.3] there exists a non-constant positive harmonic function on M, which contradicts (EHI).

Using the parabolicity of M_i and (PHI) on M_i we conclude by [21, Lemma 4.5] that there exists a positive harmonic function h_i on E_i, which vanishes on ∂E_i and such that

$$\operatorname*{flux}_{(K_i, M_i)} h_i = \int_{\partial E_i} \frac{\partial h_i}{\partial \boldsymbol{n}} \, d\mu' = 1,$$

where $K_i = M_i \setminus E_i$ and $\boldsymbol{n}$ is the inward unit normal vector field on ∂E_i. Moreover, for this function one has the estimate

$$h_i(x) \approx \int_1^{d(o_i, x)} \frac{s \, \mathrm{d}s}{V_i(s)},$$

provided $d(o_i, x) \gg 1$ (this estimate can also be deduced from Lemma 7.5). Using the notation

$$I_i(r) := \int_1^r \frac{s \, \mathrm{d}s}{V_i(s)},$$

we can write that, for large enough r,

$$h_i(x) \approx I_i(r) \quad \text{for all } x \in \partial B(o, r) \cap E_i,$$

where o is a fixed point in the central part of M (indeed, we have $d(o_i, x) \approx d(o, x) = r$ for large r). Note that $I_i(r) \to \infty$ as $r \to \infty$ because the parabolicity of M_i is equivalent to

$$\int^{\infty} \frac{s \, \mathrm{d}s}{V_i(s)} = \infty$$

(see [19, Theorem 11.1]).

Let us show that, for all $i, j = 1, 2, \ldots, n$ and for large enough r,

$$I_i(r) \approx I_j(r). \tag{7.27}$$

By [48], given two distinct ends E_i, E_j, there exists a harmonic function h_{ij} on M such that h_{ij} is bounded on each E_k, $k \neq i, j$, the function $h_{ij} - h_i$ is bounded on E_i, and the function $h_{ij} + h_j$ is bounded on E_j. Consequently, we obtain, for large enough r,

$$C^{-1} I_i(r) \leq h_{ij}(x) \leq C I_i(r), \qquad \text{if } x \in \partial B(o,r) \cap E_i, \tag{7.28}$$

$$C^{-1} I_j(r) \leq -h_{ij}(x) \leq C I_j(r), \quad \text{if } x \in \partial B(o,r) \cap E_j. \tag{7.29}$$

Set

$$a_{ij}(r) := -\inf_{B(o,r)} h_{ij} = \sup_{B(o,r)} (-h_{ij})$$

and observe that, for all large enough r,

$$-C \leq a_{ij}(r) \leq C I_j(r). \tag{7.30}$$

Indeed, the lower bound in (7.30) follows from

$$\inf_{B(o,r)} h_{ij} \leq \inf_{B(o,r)\cap E_i} h_{ij} \leq \inf_{B(o,r)\cap E_i} (h_i + C) \leq C,$$

and the upper bound in (7.30) follows from (7.29) and from the observation that h_{ij} is bounded from below on any end E_k with $k \neq j$.

Applying (EHI_η) in $B(o,r)$ to the non-negative harmonic function $x \mapsto h_{ij}(x) + a_{ij}(r)$, we obtain

$$\sup_{\partial B(o,\eta r)\cap E_i} h_{ij} + a_{ij}(r) \leq C\Big(\inf_{\partial B(o,\eta r)\cap E_j} h_{ij} + a_{ij}(r)\Big),$$

which together with (7.28), (7.29), and (7.30) yields

$$I_i(\eta r) \leq C I_j(r).$$

Observe that $I_i(\eta r) \approx I_j(r)$ since by Theorem 2.7 each end M_i satisfies (VD). Therefore, we obtain $I_i(r) \leq C I_j(r)$, whence (7.27) follows.

Next we claim that for each i,

$$I_i(r) \approx \frac{r^2}{V_i(r)}, \tag{7.31}$$

which together with (7.27) will clearly imply (v1) and (v2). To prove (7.31), consider a harmonic function u in $B(o,r)$ such that

$$u_{|\partial B(o,r)\setminus E_i} \equiv \text{const} \quad \text{and} \quad u_{|\partial B(o,r)\cap E_i} = 0,$$

where const is chosen so that $.\text{flux}._{(K_i,M_i)} u = 1$. By (EHI) in $B(o,r)$, we have

$$\sup_{B(o,\eta r)} u \leq C \inf_{B(o,\eta r)} u.$$

Let r_0 be a constant much larger than the diameter of the central part of M. Assuming that $r \gg r_0$ we have by Lemma 7.5

$$\sup_{B(o,\eta r)} u \geq \sup_{\partial B(o,r_0)\cap E_i} u \approx \int_{r_0}^{r} \frac{s\,\mathrm{d}s}{V_i(s)} \approx I_i(r),$$

$$\inf_{B(o,\eta r)} u \leq \inf_{\partial B(o,\eta r)\cap E_i} u \approx \int_{\eta r}^{r} \frac{s\,\mathrm{d}s}{V_i(s)} \approx \frac{r^2}{V_i(r)}.$$

Combining together the above three lines, we obtain $I_i(r) \leq Cr^2/V_i(r)$. The opposite inequality follows just from the monotonicity of $V_i(r)$. This finishes the proof of Theorem 7.1.

7.5. Examples involving curvature conditions.

In the context of complete Riemannian manifolds without boundary, after the seminal work of Yau [49], Cheng and Yau [9] proved that any manifold with non-negative Ricci curvature satisfies the elliptic Harnack inequality (EHI). Later, Li and Yau [34] obtained the parabolic version (PHI) for the same class of manifolds. It is natural to try and study manifolds satisfying slightly less stringent curvature conditions. In this spirit, we will consider here the following two classes of Riemannian manifolds.

(a) M has *asymptotically non-negative sectional curvature*, that is there exists a point $o \in M$ and a continuous decreasing function $k:(0,+\infty) \to (0,+\infty)$ satisfying the condition

$$\int^{+\infty} sk(s)\,\mathrm{d}s < \infty$$

and such that the sectional curvature $\mathrm{Sect}(x)$ of M at any point $x \in M$ satisfies $\mathrm{Sect}(x) \geq -k(d(o,x))$.

(b) M has non-negative Ricci curvature outside a compact set and finite first Betti number.

For instance, if $\mathrm{Sect}(x) \geq -C\,d(o,x)^{-\alpha}$ for some $\alpha > 2$, then (a) is satisfied. On the other hand, it is easy to show that (a) implies $\mathrm{Sect}(x) \geq -C\,d(o,x)^{-2}$.

It is clear that the classes (a) and (b) contain manifolds that do not satisfy (PHI), e.g., a connected sum of two n-dimensional Euclidean spaces, $n \geq 2$. Harmonic functions on manifolds of classes (a) and (b) were studied by Li and Tam [32], [33] and by Kasue [28] . In particular, Li and Tam [33] emphasized the role of condition (VC) for the class (b). Corollary 7.14 below characterizes those manifolds in classes (a) and (b) that satisfy (PHI).

Let us recall the following fact which follows from the Gromov-Bishop volume comparison theorem [4] and the gradient estimate of Li and Yau [34].

Proposition 7.6. — *Let M be a complete Riemannian manifold without boundary and o be a fixed point in M. Assume that the Ricci curvature* $\operatorname{Ric}(x)$ *of M satisfies for any $x \in M$ the lower bound*

$$\operatorname{Ric}(x) \geq -C\, d(o,x)^{-2}. \tag{7.32}$$

Then, for any remote parameter $0 < \varepsilon \leq 1$, the family of all remote balls satisfies (VD), (PI), *and* (PHI).

For example, this statement applies to manifolds of the class (a).

In order to use Theorem 5.2 and Corollary 5.4, we need to investigate properties (VC) and (RCA).

Definition 7.7. — We say that a complete Riemannian manifold M has n ends if for sufficiently large compact sets $K \subset M$, the difference $M \setminus K$ has exactly n unbounded components.

Given a Riemannian manifold M with n ends, let us fix a large enough compact set K with smooth boundary such that $M \setminus K$ has n unbounded components $E_1, E_2, \ldots, E_n$. Then we can represent M as a connected sum $M = M_1 \# M_2 \# \cdots \# M_n$ where each M_i is a complete Riemannian manifold such that E_i is isometric to the exterior of a compact in M_i. For example, one can take M_i to be the closure of E_i in M so that M has the boundary $\delta M_i = \partial E_i$. By slightly abusing terminology, we will refer to M_i's as the ends of M.

The following proposition contains already known results (see [28], [33], [5], [35]).

Proposition 7.8. — *Any manifold of class* (a) *or* (b) *has finitely many ends, and each end satisfies* (VC) *and* (RCA).

Example 7.9. — Let M be a complete manifold with a *pole* $o \in M$; that is, the exponential map at o is a diffeomorphism. Clearly, all geodesic spheres around o are connected, which implies (RCA). If M satisfies (7.32), then, by Proposition 7.6, M satisfies (PHI) for remote balls. Hence, by Theorem 5.2, (PHI) on M is equivalent to (VC). If M satisfies the stronger hypothesis (a) instead of (7.32), then, by Proposition 7.8, M satisfies (VC) because M has a single end. Therefore, (PHI) holds on any manifold with a pole satisfying (a).

Observe that by [16, Theorem C], a complete manifold with a pole, having asymptotically non-negative sectional curvature *and* non-positive sectional curvature, is quasi-isometric to $\mathbb{R}^n$. Hence, in this case (PHI) follows also from Moser's theorem.

Another result in this direction is as follows.

Proposition 7.10. — *Let M be a complete Riemannian manifold without boundary having non-negative Ricci curvature outside a compact set. Then M has finitely many ends, say, $M_1, \ldots, M_n$. Furthermore, if an end M_i satisfies* (RCA), *then it also satisfies* (VC).

Proof. — The manifold M has finitely many ends by [5], [35], [33]. By [35], there exist $o \in M$ and a constant Q such that for any $r > 0$ the set $S_r = \{x \in M_i : d(o,x) = r\}$ can be covered by at most Q balls of radius $\frac{1}{8}r$ centered on the set S_r. By [33, Proposition 5.1], there exists $x \in S_r$ such that $V(o,r) \leq CV(x, \frac{1}{5}r)$ with C independent of r. Since M_i has non-negative Ricci curvature outside a compact set and satisfies (RCA), any two balls of radius $\frac{1}{8}r$ centered on S_r have comparable volume, whence condition (VC) follows. □

Remark 7.11. — It is not yet known if in general any end M_i must satisfy (RCA).

Combining with Theorem 5.2, Corollary 5.4, and Proposition 7.6, we obtain the following results.

Corollary 7.12. — *If M is a manifold of class* (a) *or* (b), *then each end of M satisfies* (VD), (PI) *and* (PHI). *In particular, if M has only one end, then M satisfies* (VD), (PI), *and* (PHI).

Corollary 7.13. — *Under the hypotheses of Proposition* 7.10, *if an end M_i satisfies* (RCA), *then M_i satisfies* (VD), (PI), *and* (PHI). *In* *particular, if M has non-negative Ricci curvature outside a compact set and satisfies* (RCA), *then M satisfies* (VD), (PI), *and* (PHI).

We close with a remarkable application of Theorem 7.1, which characterizes manifolds satisfying (PHI) in the classes (a) and (b). This result is new even for manifolds with non-negative sectional curvature outside a compact set.

Corollary 7.14 (= Theorem 1.1). — *Let M be a manifold of class* (a) *or* (b). *Then the following condition are equivalent*:

- *M satisfies* (PHI).
- *M satisfies* (EHI).
- *Either M has only one end or it has more than one end and satisfies the conditions* (v1) *and* (v2) (*see Section 7.1 for the definitions*).

Proof. — If M has one end, then M satisfies (PHI) (and (EHI)) by Corollary 7.12. If M has at least two ends, then the claim follows from Corollary 7.12 and Theorem 7.1. □

For comparison, let us state a similar result for the case when all ends are model manifolds.

Corollary 7.15. — *Assume that M is a complete Riemannian manifold* (*without boundary*) *with sectional curvature satisfying* $\mathrm{Sect}(x) \geq -C\, d(x,o)^{-2}$, *for some $o \in M$. Assume that M has finitely many ends $M_1, \dots, M_n$, $n \geq 2$, and that each end M_i is isometric to a model manifold M_{ψ_i}. Then M satisfies* (PHI) *if and only if for all large enough r and all indices i,j*

- (ψ1) $\quad \psi_i(r) \approx \psi_j(r)$,
- (ψ2) $\quad \displaystyle\int_1^r \frac{\mathrm{d}s}{\psi_i^{N-1}(s)} \leq \frac{Cr}{\psi_i^{N-1}(r)}$,
- (ψ3) $\quad \displaystyle\int_0^r \psi_i^{N-1}(s)\,\mathrm{d}s \leq Cr\psi_i^{N-1}(r)$.

Proof. — Each end M_i obviously satisfies (RCA). By Proposition 7.6, remote balls in M_i (and M) satisfy (PHI). By Theorem 5.2, M_i satisfies (PHI) if and only if M_i satisfies (VC). Given that each M_i satisfies (PHI) and (RCA), by Theorem 7.1 M satisfies (PHI) if and only if M satisfies (v1) and (v2). Hence, M satisfies (PHI) if and only if (v1), (v2), and (VC) are satisfied. Let us verify that

$$(v1) + (v2) + (\mathrm{VC}) \iff (\psi 1) + (\psi 2) + (\psi 3).$$

Indeed, the assumption $\mathrm{Sect}(x) \geq -C\, d(x,o)^{-2}$ implies that

$$\psi_i''(r) \leq C r^{-2} \psi_i(r). \tag{7.33}$$

It is an elementary exercise to show that (7.33) and the positivity of ψ_i imply $|\psi_i'(r)| \leq C r^{-1} \psi_i(r)$, whence

$$\psi_i(r) \approx \psi_i(s) \quad \text{provided } r \approx s. \tag{7.34}$$

Obviously, under (7.34) we have (VC) $\Leftrightarrow$ (ψ3). From (ψ3) and (7.35) we obtain

$$V_i(r) := \omega_N \int_0^r \psi_i(s)^{N-1}\, \mathrm{d}s \approx r \psi_i^{N-1}(r), \tag{7.35}$$

and under (7.35) we clearly have (v1) $\Leftrightarrow$ (ψ1) and (v2) $\Leftrightarrow$ (ψ_2). □

It is easy to see that (ψ2) and (ψ3) do not imply each other. Indeed, if $\psi_i(r) = r^\alpha$ for large r, then (ψ2) is equivalent to $\alpha < 1/(N-1)$ whereas (ψ3) is equivalent to $\alpha > -1/(N-1)$ (cf. Example 7.2). Note that the additional condition (ψ3) appears because the curvature assumption in Corollary 7.15 is (slightly) weaker than condition (a).

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