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# AXIOMATIC THEORY OF HARMONIC FUNCTIONS. BALAYAGE

by N. BOBOC, C. CONSTANTINESCU and A. CORNEA

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This paper is devoted to the theory of balayage of non-negative hyperharmonic functions on a locally compact space X on which there is given a sheaf of harmonic functions. The axioms satisfied by this sheaf represents a slightly weakened form of those introduced by H. Bauer [1].

For any non-negative hyperharmonic function $s$ on X and any subset A of X let $R_s^A$ be the greatest lower bound of the set of non-negative hyperharmonic functions on X, which dominate $s$ on A and let $\hat{R}_s^A$ be the function obtained by its lower semi-continuous regularisation. We prove the following relations :

(1) $$R_{s+t}^A = R_s^A + R_t^A;$$

(2) $$R_s^{A\cup B} + R_s^{A\cap B} \leqslant R_s^A + R_s^B;$$

(3) $$A_n \uparrow A,\ s_n \uparrow s \Longrightarrow R_{s_n}^{A_n} \uparrow R_s^A.$$

The same relations hold for $\hat{R}$. We give sufficient conditions for $R_s^A = \hat{R}_s^A$ outside A.

If there exists a large number of potentials on X an dif $\mu$ is a measure for which any finite continuous potential is integrable then there exists for any subset A of X a measure $\mu^A$ such that the relation

(4) $$\int^* s\, d\mu^A = \int^* \hat{R}_s^A\, d\mu$$

holds for any finite continuous potential $s$. We prove that this relation holds also in the following cases:

*a*) X has a countably basis and A, $s$ are arbitrary;

*b*) Brelot's axiom D [3] is fulfilled, $s$ is arbitrary and there exists a series of locally bounded potentials whose sum is positive on A.

*c*) $s$ is arbitrary and A is fine open.

R. M. Hervé [4] has also proved the relations (1) and (4) under supplementary conditions: Brelot's axiom 3 is fulfilled, X has a countable basis and either A is closed or A is open or Brelot's axiom D is fulfilled.

A good many proofs done in this paper were inspired from the classic case or from Brelot's axiomatic theory. The same is true for all concepts used here (e.g. potential, fine topology, quasi-continuity) which coïncide with the usual ones in the classic cases.

In order to facilitate the reading of this paper, we introduced a paragraph of preliminary results. For some of them, however, the proofs are not given here, since they are identical with the classic ones or can be found in the paragraph of preliminaries of [2].

## 1. Preliminaries.

Let X be a locally compact space and $\mathcal{H}$ a sheaf on X of real vector spaces of real continuous functions called *harmonic functions.*

An open relatively compact set U of X is called regular if it has non-empty boundary $\partial U$ and any real continuous function $f$ on $\partial U$ possesses a unique continuous extention to $\overline{U}$, whose restriction $H_f^U$ to U is harmonic, non-negative if $f$ is non-negative. For any regular set U and any $x \in U$ the map $f \to H_f^U(x)$ is a linear non-negative functional on the space of real continuous functions on $\partial U$; we denote by $\omega_x^U$ the measure on U associated with this functional and we call it *harmonic measure.*

A numerical function on an open set U is called *hyper-harmonic* if

*a*) it does not take the value $-\infty$;

*b*) it is lower semi-continuous;

*c*) any point $x \in U$ possesses a neighbourhood $U_s(x) \subset U$ such that for every regular set V, $\overline{V} \subset U_s(x)$, and any $y \in V$

$$s(y) \geqslant \int^* s \, d\omega_y^V.$$

An open set U is called an MP-set if any hyperharmonic function $s$ on U is non-negative if there exists a compact subset $K_s$ of X such that $s$ is non-negative on U-$K_s$ and for any boundary point $x$ of U

$$\liminf_{y \to x} s(y) \geqslant 0.$$

We shall suppose that the sheaf $\mathcal{H}$ satisfies the following axioms:

$H_0$. *For any point $x \in X$ there exists a harmonic function on a neighbourhood of $x$, positive at $x$;*

$H_1$. *The regular sets form a basis of* X;

$H_2$. *The MP-sets form a covering of* X;

$H_3$. *For any open set* U *the least upper bound of any upper directed non-empty set of equally bounded harmonic functions on* U *is harmonic.*

For any regular set V and any bounded (resp. lower bounded) function $f$ on $\partial V$ the function $s$ on V

$$x \to \int^* f \, d\omega_x^V$$

is harmonic (resp. lower semi-continuous and for any regular set W, $\overline{W} \subset V$,

$$s(x) = \int^* s \, d\omega_x^W, \quad x \in W).$$

PROPOSITION 1.1. — *Let* $U_1$, $U_2$ be two *open sets and for any* $\iota \in \{1, 2\}$ *let* $s_\iota$ *be a hyperharmonic function on* $U_\iota$. *If the function s defined on* $U_1 \cup U_2$ *by*

$$s(x) = \inf_{U_\iota \ni x} s_\iota(x).$$

*is lower semi-continuous, then it is hyperharmonic.*

It follows from this proposition that any open subset of an MP-set is also an MP-set. Hence the regular MP-sets form a basis of X and in the point *c*) of the definition of hyperharmonic function one may take, in the role of $U_s(x)$, any MP-set containing $x$, this means independently of $s$.

A numerical function $s$ on an open set U is called *nearly hyperharmonic* if it is locally lower bounded and for any regular MP-set V, $\overline{V} \subset U$, and for any $x \in V$ we have

$$s(x) \geqslant \int^* s \, d\omega_x^V.$$

The greatest lower bound of a locally equally lower bounded set of nearly hyperharmonic functions is also nearly hyperharmonic.

LEMMA 1.1. — *Let $s$ be a nearly hyperharmonic function on* X. *The function $\hat{s}$ equal to*

$$\liminf_{y \to x} s(y)$$

*at any $x \in X$ is hyperharmonic and*

$$\hat{s}(x) = \sup_{V \in \mathfrak{V}_x} \int^* s \, d\omega_x^V = \lim_{V, \mathcal{F}_x} \int^* s \, d\omega_x^V,$$

*where $\mathfrak{V}_x$ is the set of regular* MP-*sets containing $x$ and $\mathcal{F}_x$ is the filter of sections on $\mathfrak{V}_x$, considering $\mathfrak{V}_x$ ordered by the relation $\supset$.*

COROLLARY 1.1. — *If $s_1$, $s_2$ are nearly hyperharmonic functions then $s_1 + s_2$ is also nearly hyperharmonic and*

$$\widehat{s_1 + s_2} = \hat{s}_1 + \hat{s}_2.$$

COROLLARY 1.2. — *If $(s_n)_{n \in N}$ is an increasing sequence of nearly hyperharmonic functions, then $s = \lim\limits_{n \to \infty} s_n$ is also nearly hyperharmonic and*

$$\hat{s} = \lim_{n \to \infty} \hat{s}_n.$$

For any family $\mathcal{S} = (s_\iota)_{\iota \in I}$ of hyperharmonic functions we denote by

$$\vee \mathcal{S} \text{ or } \bigvee_{\iota \in I} s_\iota \quad \left(\text{resp. } \wedge \mathcal{S} \text{ or } \bigwedge_{\iota \in I} s_\iota\right)$$

the least upper bound (resp. the greatest lower bound) of $\mathcal{S}$ in the set of hyperharmonic functions, if it exists.

LEMMA 1.2. — *For any upper directed (resp. locally equally lower bounded) family* $\mathcal{S} = (s_\iota)_{\iota \in I}$ *of hyperharmonic functions* $\bigvee \mathcal{S}$ (resp. $\bigwedge \mathcal{S}$) *exists and*

$$\bigvee \mathcal{S} = \sup_{\iota \in I} s_\iota \quad \Big(\text{resp. } \bigwedge \mathcal{S} = \widehat{\inf_{\iota \in I} s_\iota}\Big).$$

*For any hyperharmonic function* $s$ *we have*

$$s + \bigvee_{\iota \in I} s_\iota = \bigvee_{\iota \in I} (s + s_\iota) \quad \Big(\text{resp. } s + \bigwedge_{\iota \in I} s_\iota = \bigwedge_{\iota \in I} (s + s_\iota)\Big).$$

LEMMA 1.3. — *Let* $x$ *be a point of* X, $(\mathcal{S}_n)_{n \in N}$ *be a sequence of sets of non-negative hyperharmonic functions on* X *and let* $\mathcal{S}$ *be the set of non-negative hyperharmonic functions on* X *which may be written in the form*

$$\sum_{n \in N} s_n, \qquad s_n \in \mathcal{S}_n.$$

*If for any* $n \in N$

$$(\bigwedge \mathcal{S}_n)(x) = 0,$$

*then*

$$(\bigwedge \mathcal{S})(x) = 0.$$

Let us denote

$$A_n = \Big\{ y \in X \,|\, \inf_{s \in \mathcal{S}_n} s(y) = 0 \Big\},$$
$$A = \Big\{ y \in X \,|\, \inf_{s \in \mathcal{S}} s(y) = 0 \Big\}.$$

We have

$$A \supset \bigcap_{n \in N} A_n.$$

Indeed let $y \in \bigcap_{n \in N} A_n$ and $\varepsilon > 0$. There exists for any $n \in N$, an $s_n \in \mathcal{S}_n$ such that

$$\sum_{n \in N} s_n(y) < \varepsilon.$$

Hence $y \in A$.

Let V be a regular neighbourhood of $x$. We have

$$0 \leqslant \int^* (\inf_{s \in \mathcal{S}_n} s)\, d\omega_x^V \leqslant \widehat{\inf_{s \in \mathcal{S}_n} s}\,(x) = (\bigwedge \mathcal{S}_n)(x) = 0.$$

Hence

$$\omega_x^V(X - A_n) = 0,$$
$$\omega_x^V(X - A) \leqslant \omega_x^V\Big(\bigcup_{n \in N} (X - A_n)\Big) = 0,$$
$$\int^* (\inf_{s \in \mathcal{S}} s)\, d\omega_x^V = 0.$$

V being arbitrary, we get

$$(\wedge\mathcal{S})(x) = \widehat{\inf_{s\in\mathcal{S}} s}\,(x) = \sup_{V\in\mathfrak{V}_x} \int^* (\inf_{s\in\mathcal{S}} s)\, d\omega_x^V = 0.$$

Lemma 1.4. — *Let $s_1$, $s_2$ be hyperharmonic functions on* X, $s_1 \geqslant s_2$, *and*

$$s_1(x) + \int^* s_2\, d\omega_x^V \geqslant s_2(x) + \int^* s_1\, d\omega_x^V$$

*for any regular MP-set* V *and any* $x \in V$. *The function $s$ on* X *equal to $s_1 - s_2$ where $s_2$ is finite and equal to $+\infty$ where $s_2$ is infinite, is nearly hyperharmonic and*

$$s_1 = s_2 + \hat{s}.$$

Proposition 1.2. — *Let $(s_\iota)_{\iota\in I}$ be a lower directed family of hyperharmonic functions such that for any regular MP-set* V *and any $y \in V$ we have*

$$s_\iota(y) = \int^* s_\iota\, d\omega_y^V, \quad \iota \in I.$$

*For any point $x \in X$ such that*

$$\inf_{\iota\in I} s_\iota(x) < +\infty$$

*we have*

$$\inf_{\iota\in I} s_\iota(x) = \int^* \inf_{\iota\in I} s_\iota\, d\omega_x^V$$

*for any regular MP-set* V *containing $x$.*

Let us denote

$$s = \inf_{\iota\in I} s_\iota$$

and let V be a regular MP-set containing $x$. Obviously

$$s(x) \geqslant \int^* s\, d\omega_x^V.$$

Hence it is sufficient to prove this proposition only in the case

$$s(x) > -\infty.$$

Let $\iota \in I$ such that

$$s_\iota(x) < +\infty.$$

For any $\varkappa \in I$ such that $s_\varkappa \leqslant s_\iota$ we denote by $t_\varkappa$ the function equal to $s_\iota - s_\chi$ wherever $s_\chi$ is finite and equal to $+\infty$ elsewhere. By the preceding lemma $t_\chi$ is nearly hyperharmonic and

$$s_\iota = s_\chi + \hat{t}_\chi.$$

The family $(\hat{t}_\chi)$ being upper directed its least upper bound $t$ is hyperharmonic and we have

$$s_\iota = s + t.$$

Since $s(x)$ is finite, $t(x)$ is finite. Hence $t$ and $s$ are $\omega_x^{\mathrm{V}}$ integrable and

$$s(x) + t(x) = s_\iota(x) = \int s_\iota \, d\omega_x^{\mathrm{V}} = \int s \, d\omega_x^{\mathrm{V}} + \int t \, d\omega_x^{\mathrm{V}},$$
$$s(x) \leqslant \int s \, d\omega_x^{\mathrm{V}}.$$

## 2. Thin sets and fine topology

We say that a set $\mathrm{A} \subset \mathrm{X}$ is thin at a point $x \in \mathrm{X} - \mathrm{A}$ if either $x \notin \overline{\mathrm{A}}$ or $x \in \overline{\mathrm{A}}$ and there exists a hyperharmonic function $s$ defined on a neighbourhood of $x$ such that

$$s(x) < \liminf_{\mathrm{A} \ni y \to x} s(y).$$

Let U be an open subset of X, $s$ be a hyperharmonic function on U and $\alpha$ be a real number. We denote

$$(\mathrm{U}, s, \alpha) = \{x \in \mathrm{U} \mid s(x) < \alpha\}.$$

The *fine topology* on X is the least fine topology on X for which the sets $(\mathrm{U}, s, \alpha)$ are open. We shall say: fine neighbouhood, fine open set, fine continuous function, etc., instead of neighbourhood, open set, continuous function, etc., with respect to the fine topology.

LEMMA 2.1. (¹) — *Let* A *be a subset of* X *and* $x \in \mathrm{A}$. *A is a fine neighbourhood of $x$ if and only if* X — A *is thin at $x$.*

It is sufficient to prove the lemma for the case $x \in \overline{\mathrm{X} - \mathrm{A}}$. Suppose that X — A is thin at $x$. Then there exists a hyper-

(¹) This lemma shows that the fine topology introduced in this paper coincides in Brelot's axiomatic with the fine topology introduced in [3], p. 139.

harmonic function $s$ defined on a neighbourhood of $x$ and a real number $\alpha$ such that

$$\liminf_{X-A\ni y\to x} s(y) > \alpha > s(x).$$

Let U be a neighbourhood of $x$ such that $s$ is defined on U and

$$s(y) \geqslant \alpha$$

for any $y \in U - A$. Hence

$$x \in (U, s, \alpha) \subset A$$

and A is a fine neighbourhood of $x$.

Suppose now that A is a fine neighbourhood of $x$. Then there exists a finite system $(U_i, s_i, \alpha_i)$, $i = 1, 2, \ldots n$, such that

$$A \supset \bigcap_{i=1}^{n} (U_i, s_i, \alpha_i).$$

Let $s$ be the hyperharmonic function defined on $\bigcap_{i=1}^{n} U_i$,

$$s = \sum_{i=1}^{n} s_i$$

and $\mathfrak{U}$ be an ultrafilter on $X - A$ converging to $x$ such that

$$\lim_{\mathfrak{U}} s = \liminf_{X-A\ni y\to x} s(y).$$

Then there exists an $j$ such that

$$X - (U_j, s_j, \alpha_j) \in \mathfrak{U}.$$

Hence

$$\lim_{\mathfrak{U}} s = \sum_{i=1}^{n} \lim_{\mathfrak{U}} s_i \geqslant \sum_{i=1}^{n} s_i(x) + \alpha_j - s_j(x) > s(x).$$

LEMMA 2.2. — *Let* A *be a fine neighbourhood of* $x$. *There exists a compact set* $K \subset A$ *which is a fine neighbourhood of* $x$.

We may suppose that $x \in \overline{X - A}$. There exists, then, a hyperharmonic function $s$ defined on a neighbourhood of $x$ and a real number $\alpha$ such that

$$\liminf_{X-A\ni y\to x} s(y) > \alpha > s(x).$$

Let K′ be a compact neighbourhood of $x$ such that $s$ is defined on K′ and

$$s > \alpha$$

on K′ — A. The set

$$K = \{y \in K' | s(y) \leqslant \alpha\}$$

fulfils the required conditions.

LEMMA 2.3 ([2]). — *Let $x \in X$, A be a fine neighbourhood of $x$ and $\mathcal{F}_x$ be the filter of sections on the set of all regular sets containing $x$ ordered by the converse inclusion relation. Then*

$$\lim_{V, \mathcal{F}_x} (\omega_x^V)_* (A) = 1,$$

*where $(\omega_x^V)_*$ is the inner measure associated with $\omega_x^V$.*

Let $s$ be a hyperharmonic function defined on a neighbourhood of $x$, $\alpha$, $\beta$ be real numbers such that

$$\alpha < s(x) < \beta < \liminf_{X - A \ni y \to x} s(y),$$

and $u$ be a harmonic function defined on a neighbourhood of $x$ equal to 1 at $x$. There exists a neighbourhood U of $x$ such that

$$s > \alpha u$$

on U and

$$s > \beta u$$

on U — A. We have

$$s(x) - \alpha \geqslant \lim_{V, \mathcal{F}_x} \int (s - \alpha u)\, d\omega_x^V \geqslant \limsup_{V, \mathcal{F}_x} \int_{X-A}^{*} (\beta - \alpha) u \, d\omega_x^V$$
$$= (\beta - \alpha) \limsup_{V, \mathcal{F}_x} \int_{X-A}^{*} u \, d\omega_x^V.$$

$\alpha$ being arbitrary we get

$$\limsup_{V, \mathcal{F}_x} \int_{X-A}^{*} u \, d\omega_x^V = 0.$$

Let $\gamma$ be a real number, $\gamma > 1$. For a sufficiently small U we have

$$\frac{1}{\gamma} < u < \gamma$$

([2]) This lemma was proved in Brelot's axiomatic theory by M. Brelot [3], p. 131 and R. M. Hervé [4], p. 435.

on U. Then

$$\frac{1}{\gamma} - \frac{1}{\gamma}\int_{X-A}^{*} u\, d\omega_x^V \leqslant (\omega_x^V)_* (A) \leqslant \gamma - \gamma \int_{X-A}^{*} u\, d\omega_x^V,$$

for any V, $\overline{V} \subset U$. Hence

$$\frac{1}{\gamma} \leqslant \liminf_{V, \mathcal{F}_x} (\omega_x^V)_*(A) \leqslant \limsup_{V, \mathcal{F}_x} (\omega_x^V)_*(A) \leqslant \gamma.$$

The proof is complete since $\gamma$ is arbitrary.

Lemma 2.4. — *Let $s$ be a nearly hyperharmonic function on X and $x \in X$. Then the fine lower limit of $s$ at $x$ is equal to the lower on X and limit of $s$ at $x$.*

Let $\alpha$ be a real number smaller than the fine lower limit of $s$ at $x$ and A a fine neighbourhood of $x$ such that

$$s > \alpha$$

on A. We have (lemma 1.1)

$$\liminf_{y \to x} s(y) = \lim_{V, \mathcal{F}_x} \int^* s\, d\omega_x^V \geqslant \limsup_{V, \mathcal{F}_x} \int_A^* s\, d\omega_x^V \geqslant \alpha \lim_{V, \mathcal{F}_x} (\omega_x^V)_*(A) = \alpha.$$

Hence the fine lower limit of $s$ at $x$ is not larger than the lower limit of $s$ at $x$. Since the converse inequality is trivial, the assertion is proved.

### 3. Balayage of non-negative hyperharmonic functions

Lef $f$ be a locally lower bounded numerical function on X. We denote by $R_f$ the greatest lower bound of the set of hyperharmonic functions which dominate $f$. $R_f$ is a nearly hyperharmonic function.

Proposition 3.1. — *Let $f$, $g$ be locally lower bounded numerical functions and $(f_\iota)_{\iota \in I}$ be a family of locally lower bounded numerical functions. We have*

*a*) $f \leqslant g \Longrightarrow R_f \leqslant R_g$;

*b*) $R_{f+g} \leqslant R_f + R_g$;

*c*) $f \leqslant g \leqslant R_f \Longrightarrow R_f = R_g$;

*d*) *if for any* $\iota \in I$ $R_{f_\iota}$ *is hyperharmonic then* $R_{\sup\limits_{\iota \in I} f_\iota}$ *is also hyperharmonic and*

$$R_{\sup\limits_{\iota \in I} f_\iota} = \bigvee_{\iota \in I} R_{f_\iota};$$

*e*) *if f is fine lower semi-continuous then* $R_f$ *is hyperharmonic.*

*a*) - *d*) are trivial. *e* follows from lemma 2.4.

PROPOSITION 3.2. — *Let f be a locally lower bounded numerical function,* U *be an open subset of* X *such that* $R_f$ *is locally bounded on* U *and f is harmonic on* U. *Then* $R_f$ *is harmonic on* U.

Let V be a regular MP-set, $\overline{V} \subset U$. If $s$ is a hyperharmonic function on X dominating $f$, then the function on X equal to $s$ on X — V and equal to

$$x \to \int^* s \, d\omega_x^V$$

on V is hyperharmonic and dominates also $f$. We denote by $\mathcal{G}$ the family of hyperharmonic functions on V of the form

$$x \to \int^* s \, d\omega_x^V,$$

where $s$ is a hyperharmonic function on X dominating $f$. Obviously for any $t \in \mathcal{G}$, any regular set W, $\overline{W} \subset V$, and any $x \in W$ we have

$$t(x) = \int^* t \, d\omega_x^V.$$

Since

$$\inf_{t \in \mathcal{G}} t = R_f$$

on V, it follows from proposition 1.2 that $R_f$ is harmonic on V.

*Remark.* — It follows from this proposition and *c*) of the proposition 3.1 that if for a locally lower bounded numerical function $f$ and for a point $x \in X$

$$\limsup_{y \to x} f(y) < \liminf_{y \to x} R_f(y)$$

and $R_f$ is bounded on a neighbourhood of $x$, then $R_f$ is harmonic on a neighbourhood of $x$.

PROPOSITION 3.3. — *Let $f$ be a locally lower bounded numerical function on* X *and $x$ be a point of* X. *If*

$$\limsup_{y \to x} f(y) \leqslant \liminf_{y \to x} R_f(y)$$

*and $R_f$ is bounded on a neighbourhood of $x$ then $R_f$ is upper semi-continuous at $x$.*

Let $\varepsilon$ be a positive number and $u$ be a positive harmonic function on a neighbourhood of $x$ equal to 1 at $x$. Let further V be a regular MP-neighbourhood of $x$ such that $u$ is defined on $\overline{V}$, $R_f$ is bounded on $\overline{V}$ and

$$f \leqslant (\limsup_{y \to x} f(y) + \varepsilon)u, \qquad R_f \geqslant (\liminf_{y \to x} R_f(y) - \varepsilon)u$$

on $\overline{V}$. For any hyperharmonic majorant $s$ of $f$ we denote by $h_s$ the function on V equal to

$$y \to \int^* s \, d\omega_y^V.$$

The function on X equal to $s$ on X — V and equal to $\min(2\varepsilon u + h_s, s)$ on V is a hyperharmonic majorant of $h$. Hence

$$R_f \leqslant 2\varepsilon u + h_s$$

on V. The family $(h_s)_s$ is lower directed and

$$\inf_s h_s \leqslant R_f$$

on V. Since $R_f$ is bounded on V we deduce by proposition 1.2 that the function $\inf_s h_s$ is harmonic. Hence

$$\limsup_{y \to x} R_f(y) \leqslant 2\varepsilon u(x) + \inf_s h_s(x) \leqslant 2\varepsilon + R_f(x).$$

$\varepsilon$ being arbitrary we get

$$\limsup_{y \to x} R_f(y) \leqslant R_f(x).$$

*Remark.* — It follows from propositions 3.1 and 3.3 that if $f$ is a continuous finite function and $R_f$ is locally bounded then $R_f$ is continuous.

Let $s$ be a non-negative hyperharmonic function on X and

A be a subset of X. If $f$ is the function on X equal to $s$ on A and equal to 0 on X — A we denote

$$R_s^A = R_f.$$

*Obviously*

$$\begin{aligned} &R_s^A = s \quad \text{on} \quad A;\\ &A \subset B, \quad s \leqslant t \Longrightarrow R_s^A \leqslant R_t^B;\\ &R_{s+t}^A \leqslant R_s^A + R_t^A;\\ &R_s^{A\cup B} \leqslant R_s^A + R_s^B. \end{aligned}$$

From proposition 3.2 it follows that *if* $R_s^A$ *is locally bounded on an open set* U, $U \cap A = \emptyset$, *then* $R_s^A$ *is harmonic on* U. *If* V *is a regular* MP-*set then*

$$R_s^{X-V} = \hat{R}_s^{X-V}$$

*and*

$$R_s^{X-V}(x) = \int^* s \, d\omega_x^V$$

*for any* $x \in V$.

THEOREM 3.1. — *Let* A *be a fine open subset of* X *and* $s$ *be a non-negative hyperharmonic function on* X. *Then*

*a*) $\hat{R}_s^A = R_s^A$;

*b*) *for any regular* MP-*set* V, $V \cap A = \emptyset$, *we have*

$$R_s^A(x) = \int^* R_s^A \, d\omega_x^V;$$

*c*) *if* $(s_\iota)_{\iota\in I}$ (resp. $(A_\lambda)_{\lambda\in\Lambda}$) *is an upper directed family of non-negative hyperharmonic functions on* X (*resp. fine open subsets of* X) *such that* $s = \bigvee_{\iota\in I} s_\iota$ $\left(\text{resp. } A = \bigcup_{\lambda\in\Lambda} A_\lambda\right)$, *then*

$$\hat{R}_s^A = \bigvee_{\lambda,\iota} \hat{R}_{s_\iota}^{A_\lambda};$$

*d*)
$$\hat{R}_s^A = \bigvee_K \hat{R}_s^K,$$

*where* K *runs through the set of compact subsets of* A.

*a*) follows from proposition 3.1 *e*) since the function on X equal to $s$ on A and equal to 0 on X—A is fine lower semi-continuous. *b*) follows from *a*) and from the fact that the function on X equal to $R_s^A$ on X — V and equal to

$$x \to \int^* R_s^A \, d\omega_x^V$$

on V is a non-negative hyperharmonic and equal to $s$ on A. $c$) follows from proposition 3.1 $d$) since the function on X equal to $s_\iota$ on $A_\lambda$ and equal to 0 on $X - A_\lambda$ is fine lower semi-continuous. $d$) follows from $c$) and lemma 2.2.

LEMMA 3.1. — *Let* A *be a subset of* X *and* $s$ *be a non-negative hyperharmonic function on* X *finite on* A. *Then*

$$R_s^A = \inf_G \hat{R}_s^G,$$

*where* G *runs through the set of fine open sets containing* A.

Let $s'$ be a non-negative hyperharmonic function on X such that

$$s' \geqslant s$$

on A. We denote, for any $\alpha > 1$,

$$G_\alpha = \{x \in X | \alpha s'(x) > s(x)\} \cup \{x \in X | s(x) = 0\}.$$

$G_\alpha$ is a fine neighbourhood of A since $\{x \in X | s(x) = 0\}$ is fine open and

$$\hat{R}_s^{G_\alpha} \leqslant \alpha s'.$$

$\alpha$ being arbitrary we get

$$R_s^A \leqslant \inf_G \hat{R}_s^G \leqslant R_s^A,$$

where G runs through the set of fine open sets containing A.

THEOREM 3.2. — *For any subset* A *of* X *and any two non-negative hyperharmonic functions* $s$, $t$ *on* X *we have*

$$R_{s+t}^A = R_s^A + R_t^A, \qquad \hat{R}_{s+t}^A = \hat{R}_s^A + \hat{R}_t^A.$$

The second equality follows from the first one by corollary 1.1. Since the inequality

$$R_{s+t}^A \leqslant R_s^A + R_t^A$$

is obvious, it is sufficient to prove the converse inequality.

Suppose firstly A fine open. Let K be a compact subset of A and

$$s_K = \hat{R}_s^K.$$

Let $x$ be a boundary point of K, V be a regular MP-neighbourhood of $x$ and $h_V$ the hyperharmonic function on V,

$$y \to \int^* s_K \, d\omega_y^V.$$

We have

$$h_V \leqslant R_{s+t}^A$$

on V. Let $s_V$ be the function on V equal to infinite where $h_V$ is infinite and equal to $R_{s+t}^A - h_V$ elsewhere. By lemma 1.4 $s_V$ is nearly hyperharmonic. Since

$$R_{s+t}^A = s + t \geqslant s_K + t \geqslant h_V + t$$

on $V \cap A$ we have

$$s_V \geqslant t$$

on $V \cap A$. Hence by lemma 2.4

$$\liminf_{V-K \ni y \to x} \hat{s}_V(y) \geqslant \hat{s}_V(x) = \liminf_{V \cap A \ni y \to x} s_V(y) \geqslant \liminf_{V \cap A \ni y \to x} t(y) = t(x).$$

Let $f$ be a non-negative real continuous function on X whose support is contained in the set

$$\{x \in X | s_K(x) > 0\}$$

and $f < s_K$ whenever $f$ is positive. We denote

$$s' = R_f.$$

Obviously

$$R_{s'}^{K_0} \leqslant R_{s+t}^A,$$

where $K_0$ is the fine interior of K. By theorem 3.1 *b*) we have

$$R_{s'}^{K_0}(y) = \int^* R_{s'}^{K_0} \, d\omega_y^W$$

for any regular MP-set W, $W \cap K = \emptyset$. Hence by lemma 1.4 the function $s''$ on $X-K$ equal to infinite wherever $R_{s'}^{K_0}$ is infinite and equal to $R_{s+t}^A - R_{s'}^{K_0}$ elsewhere is nearly hyperharmonic. Let $x$ be a boundary point of K, and V be a regular MP-neighbourhood of $x$ such that

$$h_V \geqslant f$$

on V. The function on X equal to $s_K$ on $X-V$ and equal to $h_V$ on V is a non-negative hyperharmonic function which dominates $f$. Hence

$$h_V \geqslant s' \geqslant R_{s'}^{K_0}$$

on V and therefore

$$s'' \geqslant s_V, \qquad \hat{s}'' \geqslant \hat{s}_V$$

on V — K. We deduce

$$\liminf_{X-K \ni y \to x} \hat{s}''(y) \geqslant t(x).$$

The function $t'$ on X equal to $t$ on K and equal to inf $(t, \hat{s}'')$ on X — K is lower semi-continuous on X and therefore, by proposition 1.1, hyperharmonic. It is obviously non-negative and

$$t' \geqslant t$$

on K. Hence

$$t' \geqslant R_t^K,$$
$$\hat{s}'' \geqslant R_t^K,$$

on X — K. It follows

$$R_{s+t}^A \geqslant R_{s'}^{K_0} + R_t^K \geqslant R_{s'}^{K_0} + R_t^{K_0}$$

on X.

Let $\mathcal{G}$ be the family of the functions $R_f$, where $f$ is a non-negative real continuous function on X, whose support is contained in the set

$$\{x \in X | s_K(x) > 0\}$$

and $f < s_K$ whenever $f$ is positive. We have

$$\bigvee \mathcal{G} = s_K.$$

Since

$$s_K = s$$

on $K_0$ by theorem 3.1 *a*), we get by theorem 3.1 *c*)

$$\bigvee_{s' \in \mathcal{G}} R_{s'}^{K_0} = R_{s_K}^{K_0} = R_s^{K_0}.$$

Hence

$$R_{s+t}^A \geqslant R_s^{K_0} + R_t^{K_0}$$

and, by theorem 3.1 *c*),

$$R_{s+t}^A \geqslant R_s^A + R_t^A.$$

Suppose now A arbitrary and $s + t$ finite on A. Then we have

$$R_{s+t}^A = \inf_G R_{s+t}^G = \inf_G (R_s^G + R_t^G) \geqslant R_s^A + R_t^A,$$

where G runs through the set of fine open sets containing A.

Let us consider now the general case. We denote by B the subset of A where $s + t$ is finite. Let $x$ be a point where $R^A_{s+t}$ is finite and $s'$ be a non-negative hyperharmonic function on X such that

$$s' \geqslant s + t$$

on A and finite at $x$. For any $\varepsilon > 0$ and any non-negative hyperharmonic function $s''$ on X, such that

$$s'' \geqslant s$$

on B, we have

$$s'' + \varepsilon s' \geqslant s$$

on A. Hence

$$s'' + \varepsilon s' \geqslant R^A_s.$$

$\varepsilon$ and $s''$ being arbitrary we get

$$R^B_s(x) \geqslant R^A_s(x).$$

Similarly we get

$$R^B_t(x) \geqslant R^A_t(x).$$

We have, by the above considerations,

$$R^A_{s+t}(x) \geqslant R^B_{s+t}(x) = R^B_s(x) + R^B_t(x) \geqslant R^A_s(x) + R^A_t(x).$$

Hence

$$R^A_{s+t} \geqslant R^A_s + R^A_t$$

and the proof is complete.

THEOREM 3.3. — *For any non-negative hyperharmonic function s and for any two subsets* A, B *of* X *we have*

$$R^{A\cup B}_s + R^{A\cap B}_s \leqslant R^A_s + R^B_s, \qquad \hat{R}^{A\cup B}_s + \hat{R}^{A\cap B}_s \leqslant \hat{R}^A_s + \hat{R}^B_s.$$

The second inequality follows from the first one by corollary 1.1.

Suppose first A, B fine open. We denote

$$s_1 = R^{A\cup B}_s, \qquad s_2 = R^{A\cap B}_s.$$

By theorem 3.1 $a$) we have

$$s_1 = R^{A\cup B}_{s_1}, \qquad s_2 = R^{A\cup B}_{s_2}.$$

By theorem 3.1 $a$)

$$s_1 + s_2 \leqslant R^A_s + R^B_s$$

on $A \cup B$. Hence by the preceding theorem

$$R_s^{A \cup B} + R_s^{A \cap B} = s_1 + s_2 = R_{s_1}^{A \cup B} + R_{s_2}^{A \cup B} = R_{s_1+s_2}^{A \cup B} \leqslant R_s^A + R_s^B.$$

Suppose now A, B arbitrary and let us denote by A′ (resp. B′) the subset of A (resp. B) where $s$ is finite. Let $x$ be a point where $R_s^A + R_s^B$ is finite. Then $R_s^{A \cup B}$, $R_s^{A \cap B}$ are also finite at $x$. Let $t$ be a non-negative hyperharmonic function on X,

$$t \geqslant s$$

on $A \cup B$ and finite at $x$. For any $\varepsilon > 0$ and any non-negative hyperharmonic function $s'$ on X such that

$$s' \geqslant s$$

on $A' \cup B'$ we have

$$s' + \varepsilon t \geqslant s$$

on $A \cup B$. Hence

$$s' + \varepsilon t \geqslant R_s^{A \cup B}.$$

$\varepsilon$ and $s'$ being arbitrary we get

$$R_s^{A' \cup B'}(x) \geqslant R_s^{A \cup B}(x).$$

Similarly we get

$$R_s^{A' \cap B'}(x) \geqslant R_s^{A \cap B}(x).$$

We have further

$$\begin{aligned} R_s^A(x) + R_s^B(x) &\geqslant R_s^{A'}(x) + R_s^{B'}(x) = \inf_{A''} R_s^{A''}(x) + \inf_{B''} R_s^{B''}(x) \\ &= \inf_{A'', B''} (R_s^{A''}(x) + R_s^{B''}(x)) \geqslant \inf_{A'', B''} (R_s^{A'' \cup B''}(x) + R_s^{A'' \cap B''}(x)) \\ &\geqslant R_s^{A' \cup B'}(x) + R_s^{A' \cap B'}(x) \geqslant R_s^{A \cup B}(x) + R_s^{A \cap B}(x), \end{aligned}$$

where A″ (resp. B″) runs through the set of fine open sets containing A′ (resp. B′). We get

$$R_s^A + R_s^B \geqslant R_s^{A \cup B} + R_s^{A \cap B}.$$

Proposition 3.4. — *Let $(A_n)_{n \in N}$ be an increasing sequence of subsets of* X, $A = \bigcup_{n \in N} A_n$ *and $s$ be a non-negative hyperharmonic function on* X *finite on* A. *Then*

$$R_s^{A_n} \uparrow R_s^A.$$

Let $x \in X$. Obviously

$$\lim_{n \to \infty} R_s^{A_n}(x) \leqslant R_s^A(x).$$

In order to prove the converse inequality it is sufficient to assume

$$\lim_{n\to\infty} R_s^{A_n}(x) < +\infty.$$

Let $\varepsilon$ be a positive number. We shall define inductively an increasing sequence of fine open sets $(G_n)_{n\in N}$ such that

$$A_n \subset G_n, \qquad R_s^{G_n}(x) \leqslant R_s^{A_n}(x) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i}.$$

Suppose $G_n$ constructed. By lemma 3.1 there exists a fine open set $G'$ such that

$$A_{n+1} \subset G', \qquad R_s^{G'}(x) \leqslant R_s^{A_{n+1}}(x) + \frac{\varepsilon}{2^{n+1}}.$$

Setting

$$G_{n+1} = G' \cup G_n$$

we get by the preceding theorem

$$R_s^{G_{n+1}}(x) + R_s^{G'\cap G_n}(x) \leqslant R_s^{G'}(x) + R_s^{G_n}(x).$$

Hence

$$\begin{aligned} R_s^{G_{n+1}}(x) &\leqslant R_s^{G'}(x) + R_s^{G_n}(x) - R_s^{G'\cap G_n}(x) \\ &\leqslant R_s^{A_{n+1}}(x) + \frac{\varepsilon}{2^{n+1}} + R_s^{A_n}(x) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} - R_s^{A_n}(x) \\ &\leqslant R_s^{A_{n+1}}(x) + \sum_{i=1}^{n+1} \frac{\varepsilon}{2^i}. \end{aligned}$$

Let us denote

$$G = \bigcup_{n=1}^{\infty} G_n.$$

By theorem 3.1. *c*) we have

$$R_s^{A}(x) \leqslant R_s^{G}(x) = \lim_{n\to\infty} R_s^{G_n}(x) \leqslant \lim_{n\to\infty} R_s^{A_n}(x) + \varepsilon.$$

$\varepsilon$ being arbitrary we get

$$R_s^{A}(x) \leqslant \lim_{n\to\infty} R_s^{A_n}(x).$$

THEOREM 3.4. — *Let $s$ be a non-negative hyperharmonic function on $X$, $A$ be a subset of $X$ and $(f_n)_{n\in N}$ be an increasing sequence of non-negative numerical function on $X$ equal to $0$ on $X - A$ and such that for any $x \in A$*

$$s(x) = \lim_{n\to\infty} f_n(x).$$

*Then*

$$R_{f_n} \uparrow R_s^A, \qquad \hat{R}_{f_n} \uparrow \hat{R}_s^A.$$

The second relation follows from the first one by corollary 1.1. Let $x \in X$. Since the inequality

$$\lim_{n\to\infty} R_{f_n}(x) \leqslant R_s^A(x)$$

is obvious, it is sufficient to prove only the converse one.

Suppose first that $s$ is infinite on A. If for any $n \in N$

$$R_{f_n}(x) = 0,$$

then

$$R_s^A(x) = 0.$$

Indeed for any $\varepsilon > 0$ we may take a sequence $(s_n)_{n\in N}$ of non-negative hyperharmonic functions on X such that

$$s_n(y) \geqslant f_n(y), \qquad y \in A,$$
$$\sum_{n\in N} s_n(x) < \varepsilon.$$

The non-negative hyperharmonic function on X

$$\sum_{n\in N} s_n$$

is infinite on A. Hence

$$R_s^A(x) \leqslant \sum_{n\in N} s_n(x) < \varepsilon.$$

$\varepsilon$ being arbitrary we get

$$R_s^A(x) = 0 \leqslant \lim_{n\to\infty} R_{f_n}(x).$$

We may assume therefore

$$0 < R_{f_k}^A(x) < +\infty$$

for a $k \in N$. Let $t$ be a non-negative hyperharmonic function on X such that

$$t \geqslant f_k \quad \text{on} \quad A,$$
$$t(x) < +\infty.$$

Let $\alpha$ be a positive number. We denote

$$B_n = \{y \in A \mid f_n(y) > \alpha t(y)\},$$
$$B = \bigcup_{n\in N} B_n.$$

Obviously $t$ is finite on B and infinite on A — B. Hence

$$R_{f_k}(x) \leqslant R_t^A(x) \leqslant R_t^B(x) + R_t^{A-B}(x) = R_t^B(x).$$

By the preceding proposition we have

$$\lim_{n\to\infty} R_{f_n}(x) \geqslant \alpha \lim_{n\to\infty} R_t^{B_n}(x) = \alpha R_t^B(x) \geqslant \alpha R_{f_k}(x).$$

$\alpha$ being arbitrary we get

$$\lim_{n\to\infty} R_{f_n}(x) = +\infty \geqslant R_s^A(x).$$

Suppose now $s$ arbitrary. Let $\alpha$ be a real number, $0 < \alpha < 1$, and let us denote

$$C_n = \{y \in A \mid \alpha s(y) < f_n(y)\},$$
$$C = \bigcup_{n\in N} C_n.$$

Obviously

$$C = \{y \in A \mid s(y) < +\infty\}.$$

We have, by the preceding proposition,

$$\lim_{n\to\infty} R_{f_n}(x) \geqslant \alpha \lim_{n\to\infty} R_s^{C_n}(x) = \alpha R_s^C(x).$$

$\alpha$ being arbitrary we get

$$\lim_{n\to\infty} R_{f_n}(x) \geqslant R_s^C(x).$$

Since $s$ is infinite on A — C we have either

$$R_s^{A-C}(x) = 0$$

or

$$R_s^{A-C}(x) = +\infty.$$

In the first case we get

$$R_s^A(x) \leqslant R_s^C(x) + R_s^{A-C}(x) = R_s^C(x) \leqslant \lim_{n\to\infty} R_{f_n}^A(x).$$

In the second case we get, from the first part of the proof,

$$\lim_{n\to\infty} R_{f_n}(x) \geqslant R_s^{A-C}(x) = +\infty \geqslant R_s^A(x).$$

COROLLARY 3.1. — *Let $s$ be a non-negative hyperharmonic function on $X$, $A \subset X$ and $x \in X - A$. If $\{x\}$ is of type $G_\delta$ and there exists a non-negative hyperharmonic function on $X$ finite at $x$ and positive on $\{y \in A \mid s(y) > 0\}$ then $R_s^A(x) = \hat{R}_s^A(x)$.*

## 4. Balayage of measures

A non-negative hyperharmonic function $p$ on X is called *potential on* X if any hyperharmonic function $s$ is non-negative if $s + p$ is non-negative. Obviously the sum of two potentials and a non-negative hyperharmonic minorant of a potential are also potentials.

PROPOSITION 4.1. — *Any non-negative locally bounded hyperharmonic function $s$ on X possesses a unique decomposition*

$$s = p + u,$$

*where $p$ is a potential on X and $u$ is a non-negative harmonic function on X. The function $u$ is a greatest harmonic minorant of $s$. Let $\mathfrak{G}$ be an open covering of X with relatively compact* MP*-sets and $\mathcal{S}$ be the smallest set of non-negative hyperharmonic functions on X containing $s$ and such that for any $t \in \mathcal{S}$ and any $\mathrm{U} \in \mathfrak{G}$ the function $\hat{\mathrm{R}}_t^{\mathrm{X}-\mathrm{U}}$ belongs to $\mathcal{S}$. Then*

$$u = \wedge \mathcal{S}.$$

Let

$$s = p + u,$$

where $p$ is a potentials on X and $u$ a harmonic function on X. If $v$ is a harmonic minorant of $s$ we have

$$p + (u - v) \geqslant 0, \qquad u - v \geqslant 0, \qquad u \geqslant v.$$

Hence $u$ is the greatest harmonic minorant of $s$ and therefore the decomposition of $s$ is unique.

Let us denote now

$$u = \wedge \mathcal{S}$$

and let $\mathrm{U} \in \mathfrak{G}$. Then

$$u = \bigwedge_{t \in \mathcal{S}} \hat{\mathrm{R}}_t^{\mathrm{X}-\mathrm{U}}.$$

Since any $t \in \mathcal{S}$ is locally bounded the function $\hat{\mathrm{R}}_t^{\mathrm{X}-\mathrm{U}}$ is harmonic on U. Hence $u$ is harmonic.

Let $t$ be a hyperharmonic function such that $s + t$ is non-negative. We denote by $\mathcal{S}'$ the set of non-negative hyperharmonic functions $s'$ on X such that $s' + t$ is non-negative. Obviously $\mathcal{S} \subset \mathcal{S}'$ and

$$u + t = \wedge \mathcal{S} + t \geqslant \wedge \mathcal{S}' + t = \bigwedge_{s' \in \mathcal{S}'} (s' + t) \geqslant 0.$$

Let us denote

$$p = s - u.$$

$p$ is non-negative hyperharmonic function and for any hyperharmonic function $t$ on X such that $p + t$ is non-negative we have

$$s - u + t \geqslant 0, \qquad u - u + t \geqslant 0, \qquad t \geqslant 0.$$

Hence $p$ is a potential and

$$s = p + u.$$

Lemma 4.1. — *The following assertions are equivalent:*

*a) For any point of* X *there exists a locally bounded potential on* X *positive at this point;*

*b) For any two different points $x, y \in$ X there exists two locally bounded potentials $p$, $q$ on* X *such that*

$$p(x)q(y) - p(y)q(x) \neq 0;$$

*c) For any two different points $x, y \in$ X there exists two locally bounded non-negative hyperharmonic functions $s$, $t$ on* X *such that*

$$s(x)t(y) - s(y)t(x) \neq 0;$$

*d) For any point $x \in$ X and any regular MP-neighbourhood* V *of $x$ there exists a locally bounded non-negative hyperharmonic function $s$ on* X *such that*

$$s(x) > \int s \, d\omega_x^{\mathrm{V}};$$

*e) For any point $x \in$ X and any regular MP-neighbourhood* V *of $x$ there exists a locally bounded potential $p$ on* X *such that*

$$p(x) > \int p \, d\omega_x^{\mathrm{V}}.$$

*a)* $\Longrightarrow$ *b)*. Let $x, y$ be two different points of X and $p$ be a locally bounded potential, positive at $x$ and $y$, and $\mathfrak{S}$ be the set of regular MP-sets V such that either $x \notin$ V or $y \notin$ V. $\mathfrak{G}$ is a covering of X. Let $\mathcal{S}$ be the smallest set of non-negative hyperharmonic functions on X containing $p$ and such that for any $t \in \mathcal{S}$ and V $\in \mathfrak{G}$ the function $\hat{R}_t^{X-V}$ belongs to $\mathcal{S}$. Since, by the preceding proposition

$$\wedge \mathcal{S} = 0,$$

there exists an $q \in \mathcal{S}$ such that either

$$q(x) = p(x) \qquad \text{and} \qquad q(y) < p(y)$$

or

$$q(x) < p(x) \qquad \text{and} \qquad q(y) = p(y).$$

Since any element of $\mathcal{S}$ is a minorant of $p$ and therefore a potential, $q$ is a potential. Obviously

$$q(x)p(y) - q(y)p(x) \neq 0.$$

*b*) $\Longrightarrow$ *c*) is trivial.

*c*) $\Longrightarrow$ *d*). Let $y$ be a point of the carrier of $\omega_x^V$ and $s, t$ be two non-negative locally bounded hyperharmonic functions on X such that

$$s(x) = t(x), \qquad s(y) < t(y).$$

For an open set U, $x \in$ U, $y \notin \overline{\text{U}}$, we denote

$$s' = R^{U}_{\inf(s,t)}.$$

Obviously

$$s'(y) < t(y), \qquad s'(x) = t(x).$$

Since $s'$ is harmonic on X — $\overline{\text{U}}$,

$$s' < t$$

on a neighbourhood of $y$. Hence

$$s'(x) = t(x) \geqslant \int t \, d\omega_x^V > \int s' \, d\omega_x^V.$$

*d*) $\Longrightarrow$ *e*). Let $s$ be a non-negative locally bounded hyperharmonic function on X such that

$$s(x) > \int s \, d\omega_x^V,$$

and let $p$ (resp. $u$) be a potential (resp. harmonic function) on X such that

$$s = p + u.$$

We have

$$p(x) = s(x) - u(x) > \int s\, d\omega_x^{\mathrm{V}} - \int u\, d\omega_x^{\mathrm{V}} = \int p\, d\omega_x^{\mathrm{V}}.$$

$e) \Longrightarrow a)$ is trivial.

In order to introduce the balayaged of a measure one has to suppose that there exists a large number of potentials on X: *For that purpose we shall assume from now on that one of the equivalent conditions a)-e) is fulfilled.* Obviously Bauer's Trennungsaxiom $\mathrm{T}^+$ implies the condition *c*). Also in Brelot's axiomatic, the existence of a positive potential implies the condition *a*).

The following lemma contains some of the first consequences of this hypothesis.

LEMMA 4.2. —

*a*) X *is an MP-set*;

*b*) *for any real continuous non-negative function* $f$ *on* X, *whose support is compact, the function* $\mathrm{R}_f$ *is a finite continuous potential on* X;

*c*) *any non-negative hyperharmonic function is the least upper bound of an upper directed family of continuous finite potentials.*

*a*) Let $s$ be a hyperharmonic function on X, non-negative outside a compact set K. There exists a potential $p$ on X

$$p \geqslant -s$$

on K. Then

$$p + s \geqslant 0$$

on X and therefore

$$s \geqslant 0.$$

*b*) Since $f$ is a real continuous function with compact support, there exists a locally bounded potential dominating $f$. Hence $\mathrm{R}_f$ is locally bounded. By the remark from the proposition 3.3 it follows that $\mathrm{R}_f$ is a non-negative locally bounded continuous hyperharmonic function. Being dominated by a potential it is itself a potential.

*c*) Let $s$ be a non-negative hyperharmonic function on X. Then

$$s = \sup_f \mathrm{R}_f,$$

where $f$ runs through the set of non-negative real continuous functions on X with compact support and not greater than $s$. The proof is complete.

Let $\mathfrak{L}$ be the set of real continuous functions on X with compact support which may be written in the form

$$p - q,$$

where $p$ and $q$ are finite continuous potentials. Obviously $\mathfrak{L}$ is a real vector space ordered by the relation $\leqslant$. Since

$$\max (p - q, 0) = p - \min (p, q),$$

$\mathfrak{L}$ is a vectorlattice.

Lemma 4.3. — *Let $f$ be a non-negative real continuous function whose support is a compact set* K. *For any neighbourhood* U *of* K *and for any positive number* $\varepsilon$ *there exists a non-negative function $f_0 \in \mathfrak{L}$ whose support lies in* U *such that*

$$|f - f_0| < \varepsilon.$$

Let $x$, $y$ be two different points of X. Let V be a regular MP-neighbourhood of $x$, $y \notin \mathrm{V}$ and $s$ a locally bounded non-negative hyperharmonic function on X such that

$$s(x) > \int s \, d\omega_x^{\mathrm{V}}.$$

By *c*) of the preceding lemma there exists a finite continuous potential $p$ on X such that

$$p(x) > \int p \, d\omega_x^{\mathrm{V}}.$$

Let $q$ be the function on X equal to $p$ on X-V and equal to

$$z \to \int p \, d\omega_z^{\mathrm{V}}$$

on V. $q$ is a finite continuous potential on X. The function $g = p - q$ belongs to $\mathfrak{L}$, is equal to zero at $y$ and is different from zero at $x$. Similarly we may construct a function $g' \in \mathfrak{L}$

equal to zero at $x$ and different from zero at $y$. Hence for any real numbers $\alpha$, $\beta$ there exists an element of $\mathfrak{L}$ equal to $\alpha$ at $x$ and equal to $\beta$ at $y$.

Let U′ be a relatively compact open set,

$$K \subset U' \subset \overline{U}' \subset U.$$

By Stone's theorem there exists an $f' \in \mathfrak{L}$ such that

$$|f' - f| < \varepsilon$$

on U′. Let further $g$ be a real continuous function with compact support on X such that

$$g > \sup_{y \in K} f'(y) + \varepsilon$$

on K,

$$g \leqslant 0$$

on X — U′,

$$g \leqslant -\varepsilon$$

on (X—U′) ∩ Supp $f'$. Again by Stone's theorem there exists a $g' \in \mathfrak{L}$ such that

$$|g' - g| < \varepsilon$$

on $\overline{U}'$ ∪ Supp $f'$. The function

$$f_0 = \max(0, \min(f', g'))$$

belongs to $\mathfrak{L}$ has its support in $\overline{U}'$, and

$$|f_0 - f| < \varepsilon,$$

and the proof is complete.

We denote by $\Lambda$ the set of measures $\mu$ on X such that for any finite continuous potential $p$ on X

$$\int p \, d\mu < +\infty.$$

Obviously any measure with compact carrier belongs to $\Lambda$. If $p$, $q$, $p'$, $q'$ are finite continuous potentials on X such that

$$p - q = p' - q',$$

then by theorem 3.2 for any subset A of X and any $\mu \in \Lambda$,

$$\int \hat{R}_p^A \, d\mu - \int \hat{R}_q^A \, d\mu = \int \hat{R}_{p'}^A \, d\mu - \int \hat{R}_{q'}^A \, d\mu.$$

Hence the map

$$p - q \to \int \hat{R}_p^A \, d\mu - \int \hat{R}_q^A \, d\mu$$

is well defined on $\mathfrak{L}$. It is a linear positive functional. By the preceding lemma there exists a unique measure $\mu^A$, called *the balayaged measure of* $\mu$ *on* A, such that

$$\int (p - q) \, d\mu^A = \int \hat{R}_p^A \, d\mu - \int \hat{R}_q^A \, d\mu$$

for any $p - q \in \mathfrak{L}$. The carrier of $\mu^A$ is contained in $\overline{A}$. Indeed let $p - q \in \mathfrak{L}$ such that

$$p - q = 0$$

on $\overline{A}$. Then

$$\int (p - q) \, d\mu^A = \int \hat{R}_p^A \, d\mu - \int \hat{R}_q^A \, d\mu = 0.$$

LEMMA 4.4. — *For any finite continuous potential* $p$ *on* X, *for any* $A \subset X$ *and for any* $\mu \in \Lambda$ *we have*

$$\int p \, d\mu^A = \int \hat{R}_p^A \, d\mu.$$

Let $\mathcal{G}$ be the smallest set of non-negative hyperharmonic functions on X which contains $p$ and such that for any $q \in \mathcal{G}$ and any regular set V the function $\hat{R}_q^{X-V}$ belongs to $\mathcal{G}$. Since the set of non-negative continuous hyperharmonic functions $s$ on X such that $s \leqslant p$ on X and $s = p$ outside a compact set (depending on $s$) contains $\mathcal{G}$, any element $q$ of $\mathcal{G}$ is a finite continuous potential and $p - q \in \mathfrak{L}$. Hence

$$\int (p - q) \, d\mu^A = \int \hat{R}_p^A \, d\mu - \int \hat{R}_q^A \, d\mu.$$

Since $p$ is a potential

$$\inf_{q \in \mathcal{G}} q = 0.$$

Since $\mu \in \Lambda$ and $\mathcal{G}$ is lower directed

$$0 \leqslant \inf_{q \in \mathcal{G}} \int \hat{R}_q^A \, d\mu \leqslant \inf_{q \in \mathcal{G}} \int q \, d\mu = 0,$$

$$\int p \, d\mu^A = \sup_{q \in \mathcal{G}} \int (p - q) \, d\mu^A = \int \hat{R}_p^A \, d\mu - \inf_{q \in \mathcal{G}} \int \hat{R}_q^A \, d\mu = \int \hat{R}_p^A \, d\mu.$$

Corollary 4.1. — *For any non-negative hyperharmonic function $s$, for any $A \subset X$ and for any $\mu \in \Lambda$ we have*

$$\int^* s\, d\mu^A \leqslant \int^* \hat{R}_s^A\, d\mu.$$

*If A is fine open this inequality becomes an equality.*

By lemma 4.2 *c*) there exists an upper directed family $(p_\iota)_{\iota \in I}$ of finite continuous potentials on X such that

$$\sup_{\iota \in I} p_\iota = s.$$

We have

$$\int^* s\, d\mu^A = \sup_{\iota \in I} \int p_\iota\, d\mu^A = \sup_{\iota \in I} \int \hat{R}_{p_\iota}^A\, d\mu \leqslant \int^* \hat{R}_s^A\, d\mu.$$

Corollary 4.2. — *If A, B are subsets of X such that $A \subset B$, then for any $\mu \in \Lambda$ and any non-negative hyperharmonic function $s$ on X we have*

$$\int^* s\, d\mu^A \leqslant \int^* s\, d\mu^B.$$

By lemma 4.2 *c*) there exists an upper directed family of finite continuous potentials $(p_\iota)_{\iota \in I}$ such that

$$s = \sup_{\iota \in I} p_\iota.$$

We have

$$\int^* s\, d\mu^A = \sup_{\iota \in I} \int p_\iota\, d\mu^A = \sup_{\iota \in I} \int \hat{R}_{p_\iota}^A\, d\mu$$
$$\leqslant \sup_{\iota \in I} \int \hat{R}_{p_\iota}^B\, d\mu = \sup_{\iota \in I} \int p_\iota\, d\mu^B = \int^* s\, d\mu^B.$$

Lemma 4.5. — *Let $(A_n)_{n \in N}$ be an increasing sequence of subsets of X, $A = \bigcup_{n \in N} A_n$ and $\mu \in \Lambda$. Let $(s_n)_{n \in N}$ be a sequence of non-negative hyperharmonic functions on X such that for any $n$*

$$\int^* s_n\, d\mu^{A_n} = \int^* \hat{R}_{s_n}^{A_n}\, d\mu,$$
$$s_n \leqslant s_{n+1}$$

*on $A_n$. If $s$ is a non-negative hyperharmonic function on X such that*

$$s = \lim_{n \to \infty} s_n$$

*on* $\mathrm{A}$ *and* $s \geqslant s_n$ *on* $\overline{\mathrm{A}}$ *for any* $n \in \mathrm{N}$, *then*

$$\int^* s\, d\mu^{\mathrm{A}} = \int^* \hat{\mathrm{R}}_s^{\mathrm{A}}\, d\mu.$$

By corollary 4.1, 4.2 and theorem 3.4 we have

$$\int^* \hat{\mathrm{R}}_s^{\mathrm{A}}\, d\mu \geqslant \int^* s\, d\mu^{\mathrm{A}} \geqslant \lim_{n\to\infty} \int^* s\, d\mu^{\mathrm{A}_n}$$
$$\geqslant \lim_{n\to\infty} \int^* s_n\, d\mu^{\mathrm{A}_n} = \lim_{n\to\infty} \int^* \hat{\mathrm{R}}_{s_n}^{\mathrm{A}_n}\, d\mu = \int^* \hat{\mathrm{R}}_s^{\mathrm{A}}\, d\mu.$$

THEOREM 4.1. — *If s is the limit of an increasing sequence of finite continuous potentials then for any* $\mu \in \Lambda$ *and* $\mathrm{A} \subset \mathrm{X}$

$$\int^* s\, d\mu^{\mathrm{A}} = \int^* \hat{\mathrm{R}}_s^{\mathrm{A}}\, d\mu.$$

The assertion follows from lemma 4.4 and 4.5.

COROLLARY 4.3. — *If* $\mathrm{X}$ *has a countable basis then for any non-negative hyperharmonic function s any* $\mu \in \Lambda$ *and any* $\mathrm{A} \subset \mathrm{X}$ *we have*

$$\int^* s\, d\mu^{\mathrm{A}} = \int^* \hat{\mathrm{R}}_s^{\mathrm{A}}\, d\mu.$$

Let $(f_n)_{n\in\mathrm{N}}$ be an increasing sequence of non-negative real continuous functions with compact support converging to $s$. Then $(\mathrm{R}_{f_n})_{n\in\mathrm{N}}$ is an increasing sequence of finite continuous potentials converging to $s$.

THEOREM 4.2. — *Let* $\mu$ *belong to* $\Lambda$. *If the relation*

$$\int^* s\, d\mu^{\mathrm{A}} = \int^* \hat{\mathrm{R}}_s^{\mathrm{A}}\, d\mu$$

*holds for any relatively compact subset* $\mathrm{A}$ *of* $\mathrm{X}$ *and any locally bounded potential s on* $\mathrm{X}$, *then it holds also for any non-negative hyperharmonic function s on* $\mathrm{X}$ *and any subset* $\mathrm{A}$ *of* $\mathrm{X}$ *which satisfy one of the following conditions:*

*a*) $$\bigwedge_{\mathrm{K}} \hat{\mathrm{R}}_s^{\mathrm{X}-\mathrm{K}} = 0,$$

*where* $\mathrm{K}$ *runs through the set of compact subsets of* $\mathrm{X}$;

*b) There exists a sequence of locally bounded potentials* $(p_n)_{n\in N}$ *such that*

$$\sup_{n\in N} p_n > 0$$

*on* A.

*a)* We have

$$\hat{R}_s^A = \bigvee_K \hat{R}_s^{A\cap K},$$

where K runs through the set of compact subsets of X. Indeed for any K

$$\hat{R}_s^A \leqslant \hat{R}_s^{A\cap K} + \hat{R}_s^{X-K},$$

$$\hat{R}_s^A \leqslant \bigvee_K \hat{R}_s^{A\cap K} + \bigwedge_K \hat{R}_s^{X-K} = \bigvee_K \hat{R}_s^{A\cap K}.$$

Let K be a compact subset of X and $p$ be a locally bounded potential on X positive on K. Since

$$((np)\wedge s)\uparrow s$$

on $\overline{A\cap K}$ and $(np)\wedge s$ is a locally bounded potential we have by lemma 4.5

$$\int^* s\, d\mu^{A\cap K} = \int^* \hat{R}_s^{A\cap K}\, d\mu.$$

From this relations we get

$$\int^* s\, d\mu^A \geqslant \sup_K \int^* s\, d\mu^{A\cap K} = \sup_K \int^* \hat{R}_s^{A\cap K}\, d\mu$$
$$= \int^* \hat{R}_s^A\, d\mu \geqslant \int^* s\, d\mu^A.$$

*b)* Let us denote

$$s_n = s\wedge\left(n\sum_{k=1}^{n} p_k\right)$$

for any $n\in N$. Since $s_n$ is a locally bounded potential we have

$$\bigwedge_K \hat{R}_s^{X-K} = 0,$$

where K runs through the set of compact subsets of X. Hence by *a)* we get

$$\int^* s_n\, d\mu^A = \int^* \hat{R}_{s_n}^A\, d\mu.$$

Since

$$s_n \uparrow s$$

on A and

$$s_n \leqslant s$$

on X we deduce from lemma 4.5

$$\int^* s\, d\mu^A = \int^* \hat{R}_s^A\, d\mu.$$

Let A be a subset of X and $s$ a non-negative hyperharmonic function on X. We denote by $\mathfrak{Q}_s^A$ the set of non-negative hyperharmonic functions $t$ on X such that the restriction of $s$ to

$$\{x \in A | t(x) \leqslant 1\}$$

is continuous. We say that $s$ is *quasicontinuous on* A if

$$\wedge \mathfrak{Q}_s^A = 0.$$

Lemma 4.6. — *Let* A *be a relatively compact subset of* X *and* $s$ *be a non-negative hyperharmonic function on* X *quasicontinuous on* $\bar{A}$. *Then, for any* $\mu \in \Lambda$

$$\int^* s\, d\mu^A = \int^* \hat{R}_s^A\, d\mu.$$

Suppose first that the restriction of $s$ to $\bar{A}$ is continuous. Since A is relatively compact there exists for any $n \in N$ a real continuous function $f_n$ with compact support, not greater than $s$ and equal to min $(n, s)$ on $\bar{A}$. We may suppose

$$f_n \leqslant f_{n+1}.$$

Then by lemma 4.2 *b*) $R_{f_n}$ is a finite continuous potential. Obviously

$$R_{f_n} \uparrow s$$

on A and

$$R_{f_n} \leqslant s$$

on X. Hence by lemma 4.5 we have

$$\int^* s\, d\mu^A = \int^* \hat{R}_s^A\, d\mu.$$

Let now $s$ be quasicontinuous on $\bar{A}$. For any $t \in \mathfrak{Q}_s^{\bar{A}}$ we set

$$A_t = \{x \in A | t(x) \leqslant 1\}.$$

Since $\mathfrak{Q}_s^{\bar{A}}$ is lower directed, the family $(A_t)_{t\in\mathfrak{Q}_s^{\bar{A}}}$ is upper directed. We shall prove that

$$\hat{R}_s^A \leqslant \sup_{t\in\mathfrak{Q}_s^{\bar{A}}} \hat{R}_s^{A_t}.$$

Let $(t_n)_{n\in N}$ be a decreasing sequence of elements of $\mathfrak{Q}_s^{\bar{A}}$ and

$$B = \bigcup_{n\in N} A_{t_n}.$$

We have by theorem 3.4

$$\hat{R}_s^B = \sup_{n\in N} \hat{R}_s^{A_{t_n}} \leqslant \sup_{t\in\mathfrak{Q}_s^{\bar{A}}} \hat{R}_s^{A_t},$$

$$\hat{R}_s^{A-B} \leqslant \sum_{n\in N} t_n,$$

$$\hat{R}_s^A \leqslant \hat{R}_s^B + \hat{R}_s^{A-B} \leqslant \sup_{t\in\mathfrak{Q}_s^{\bar{A}}} \hat{R}_s^{A_t} + \sum_{n\in N} t_n.$$

By lemma 1.3 we have

$$\wedge\left(\sum_{n\in N} t_n\right) = 0,$$

since $\mathfrak{Q}_s^{\bar{A}}$ is lower directed and $s$ quasicontinuous on $\bar{A}$. Hence, by lemma 1.2,

$$\hat{R}_s^A \leqslant \sup_{t\in\mathfrak{Q}_s^{\bar{A}}} \hat{R}_s^{A_t}.$$

We get now, from the first part of the proof,

$$\int^* \hat{R}_s^A \, d\mu \leqslant \sup_{t\in\mathfrak{Q}_s^{\bar{A}}} \int^* \hat{R}_s^{A_t} \, d\mu = \sup_{t\in\mathfrak{Q}_s^{\bar{A}}} \int^* s \, d\mu^{A_t}$$
$$\leqslant \int^* s \, d\mu^A \leqslant \int \hat{R}_s^A \, d\mu,$$

and the proof is complete.

THEOREM 4.3. — *Let $s$ be a non-negative hyperharmonic function and $(K_n)_{n\in N}$ be an increasing sequence of compact subsets of $X$ such that $s$ is quasicontinuous on any $K_n$. Then for any $\mu \in \Lambda$ and for any $A \subset \bigcup_{n\in N} K_n$ we have*

$$\int^* s \, d\mu^A = \int^* \hat{R}_s^A \, d\mu.$$

By the preceding lemma we have

$$\int^* s \, d\mu^{A_n} = \int^* \hat{R}_s^{A_n} \, d\mu,$$

where

$$A_n = A \cap K_n.$$

The assertion follows now from lemma 4.5.

In order to obtain further results, M. Brelot has introduced a supplimentary axiom called *axiom* D. This axiom asserts that *for any non-negative locally bounded hyperharmonic function s and any open relatively compact set* U, *the restriction of* $R_s^U$ *to* U *is the greatest harmonic minorant of s on* U. If this axiom is fulfilled it can be proved like in [2] that any non-negative hyperharmonic function on X is quasicontinuous or any compact subset of X. In this case the hypothesis of the theorem 4.2 is fulfilled and the relation

$$\int^* s \, d\mu^A = \int^* \hat{R}_s^A \, d\mu$$

holds if $s$ and A satify one of the conditions *a*) and *b*) of this theorem. Moreover if Brelot's axiom 3 is fulfilled this relation holds for any non-negative hyperharmonic function $s$ and any subset A of X since in this case there exists a positive locally bounded potential on X.

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