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SCHWARZ-TYPE LEMMAS FOR SOLUTIONS OF $\bar{\partial}$-INEQUALITIES AND COMPLETE HYPERBOLICITY OF ALMOST COMPLEX MANIFOLDS

by Sergey IVASHKOVICH and Jean-Pierre ROSAY (*)

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0. Introduction.

In this paper an *almost complex manifold* (X, J) means a smooth $(\mathcal{C}^\infty)$ manifold X with an almost complex structure J of smoothness at least $\mathcal{C}^1$. When more smoothness is required by our proofs, it will be specified in each statement. Through any point in any tangent direction, there are local J-complex discs in X (Theorem III in the pioneering paper [N-W], or see Appendix 1). So, one can define the Kobayashi-Royden pseudo-norm $|\cdot|_{k,J}$ of tangent vectors and then the Kobayashi pseudo-distance $k_J(\cdot,\cdot)$ on (X, J), see § 1. If $k_J(\cdot,\cdot)$ is a distance (separation property), the manifold is said to be *hyperbolic*.

For a compact X k_J is a distance iff X does not contain a J-complex line, i.e., an image of a non-constant J-holomorphic map $u : \mathbb{C} \to X$ (see [K-O], [De-1]) - this is just a Brody reparametrization lemma. The proof originally given by Brody does not use the integrability of the structure. If, for example, X is a product of two Riemann surfaces (S_1, J_1) and (S_2, J_2) each of genus at least two then (X, J) is hyperbolic for any J close to $J_1 \oplus J_2$.

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QUESTION 1. — *Is (X, J) hyperbolic for any J tamed by a standard symplectic form on $X = S_1 \times S_2$?*

If $X = \mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2, \mathbb{T}^{2n}$ equipped with structures tamed by standard symplectic forms then such X always contains a J-complex line (see [Gr] and [Ba]) and therefore is never hyperbolic.

0.1. Hyperbolic distance to a strictly pseudoconvex hypersurface.

In addition to the example of compact complex manifolds without complex lines there is the simple example of bounded domains in $\mathbb{C}^n$ that are also hyperbolic. The Gromov-Schwarz lemma extends this remark to relatively compact domains in symplectic manifolds with exact symplectic forms (the almost complex structure should be tamed by this symplectic form), see [K-O]. The main question that will be studied in this paper is: *for which domains $D \subset (X, J)$ is the almost complex manifold (D, J) complete hyperbolic (i.e. complete for the Kobayashi distance in D)?*

First results were obtained in [D-I]. An important work was done by Gaussier and Sukhov [G-S], who in particular were the first to prove that in any dimension, points always have a basis of hyperbolic complete neighborhoods, for almost complex structures that are smooth enough. See more about their work in Remark 1 below.

Let us state the problem more precisely. Let D be a domain in an almost complex manifold (X, J). We do not assume that D is relatively compact nor that D is hyperbolic. A point $p \in \partial D$ is said to be at finite distance from $q \in D$ if there is a sequence of points $q_j \in D$ converging to p and whose Kobayashi distances to q stay bounded. Distances here are taken in Kobayashi pseudo-metric of (D, J). Otherwise we say that the distance is infinite.

Our first result is the following one (the definition of strict pseudoconvexity is recalled in 2.b):

THEOREM 1. — *Let D be a domain in an almost complex manifold (X, J), J of class $\mathcal{C}^1$. Let $p \in \partial D$. If the boundary of D is strictly J-pseudoconvex at p, the point p is at infinite Kobayashi distance from the points in D.*

The Brody reparametrization lemma immediately gives the following:

COROLLARY 1. — *Let D be a relatively compact strictly pseudoconvex domain of class $\mathcal{C}^2$ in an almost complex manifold (X, J) and assume that J is of class $\mathcal{C}^1$. Then either (D, J) is complete hyperbolic or D contains a J-complex line.*

There is one important case where the absence of J-complex lines in a strictly pseudoconvex D is completely obvious. This is the case when D possesses a global defining strictly plurisubharmonic function i.e. there exists a $\mathcal{C}^2$-function ρ in a neighborhood V of $\bar{D}$ such that $dd^c_J\rho(Y, JY) > 0$, $Y \in T\bar{D} \setminus \{0\}$ and $D = \{x \in V : \rho(x) < 0\}$. Therefore we obtain

COROLLARY 2. — *Let D and (X, J) be as in Corollary 1 and suppose additionally that D possesses a global defining strictly plurisubharmonic function. Then (D, J) is complete hyperbolic.*

Corollary 2 implies the existence of bases of complete hyperbolic neighborhoods of any point on (X, J) provided $J \in \mathcal{C}^1$. Indeed, the problem is local and therefore we can suppose that $X = \mathbb{R}^{2n}$ and $J(0) = J_{st}$. Then $\|\cdot\|^2$ is strongly J-plurisubharmonic near 0. When the real dimension of X is 4 the existence of a basis of hyperbolically complete neighborhoods was proved in [D-I].

It is worth to point out that strictly pseudoconvex domains in almost complex manifolds are far different from those in complex ones, even if the almost complex structure is tamed by some symplectic form. For example, in [McD] a symplectic 4-manifold (X, ω) together with a relatively compact smoothly bounded domain $D \subset X$ is constructed such that ∂D is of contact type (and therefore is strictly pseudoconvex with respect to an appropriate ω-tamed J) but at the same time ∂D is disconnected. In fact it has two connected components. It is not clear however, whether this D contains J-complex lines or not.

It seems to be a proper place to mention a well known problem, which is open even in $\mathbb{C}^2$:

QUESTION 2. — *Let D be a smooth pseudoconvex domain in an almost complex manifold (X, J), e.g. $D = \{x : \rho(x) < 0\}$ for some (not necessarily strictly) plurisubharmonic function ρ. Is D locally complete hyperbolic?*

The reason why we mention this problem here is that by local diffeomorphism D can be put into a nice form (say upper half-space) and then one can try to improve and apply the methods of §§ 1,2 of this paper.

0.2. Distance to a J-complex hypersurface.

We have already mentioned compact complex manifolds and bounded domains in symplectic manifolds with exact symplectic form. Another example of hyperbolic manifold is the manifold $X = \mathbb{CP}^2 \setminus \bigcup_{i=1}^{5} l_i$, where $\{l_1, ..., l_5\}$ are J-complex lines in general position. The structure J should be tamed by the Fubini-Studi form, see [Du].

Therefore we now turn our attention to the hyperbolic distance to complex submanifolds. Let M be a closed submanifold of a domain D, of real codimension 1 or 2 (the case of higher codimension is trivial, see Remark at the end of § 5). For a point $p \in M$, we wish to investigate whether there exist points $q \in D \setminus M$ at finite Kobayashi distance from p, in $D \setminus M$. For codimension 2 our result is the following one:

THEOREM 2. — *Let D be a be a hyperbolic domain in an almost complex manifold (X, J), $J \in \mathcal{C}^2$. Let M be a closed submanifold of D of real codimension 2 and of class $\mathcal{C}^3$.*

(2.A) *If M is a J-complex hypersurface, then for every $p \in M$ and $q \in D \setminus M$ the Kobayashi distance from q to p is infinite.*

(2.B) *Conversely, if $p \in M$ and if the tangent space to M at p is not J-complex, then for any neighborhood D_1 of p in D, there exists $p' \in D_1 \cap M$ that is at finite distance from points in $D \setminus M$.*

It is shown in section § 5 that at least if J is of class $\mathcal{C}^{2,\alpha}$, instead of merely $\mathcal{C}^2$, one can take $p' = p$. This is an immediate consequence of Theorem 3.

Theorems 1 and 2 imply the following

COROLLARY 3. — *Let J be a $\mathcal{C}^2$ almost complex structure in the neighborhood of the origin in $\mathbb{R}^{2n}$. Suppose that there exists a J-complex $\mathcal{C}^3$ hypersurface $M \ni 0$. Then there exists a fundamental system of neighborhoods $\{U_i\}$ of the origin such that (U_i, J) and $(U_i \setminus M, J)$ are complete hyperbolic in the sense of Kobayashi.*

In real dimension 4 this statement was proved in [D-I]. Let us explain our interest for higher dimensions.

QUESTION 3. — *Let $C \subset \mathbb{CP}^2$ be a J-complex curve (not smooth) in general position of sufficiently high degree, J tamed by the Fubini-Studi form. Is it true that $(\mathbb{CP}^2 \setminus C, J)$ is hyperbolic?*

This question has likely a positive answer and therefore the following problem arises in its turn:

QUESTION 4. — *Let J be an almost complex structure in the neighborhood of the origin in $\mathbb{R}^4$ and C a J-complex curve (may be singular) passing through zero. Prove that for a complete hyperbolic neighborhood of zero V the open set $(V \setminus C, J)$ is also complete hyperbolic.*

A proof could possibly go along the following lines. Let $F(z, \bar{z}) = 0$ be the equation of C, $z = (z_1, z_2)$. Consider in $\mathbb{R}^6$ with coordinates z_1, z_2, w the hypersurface $M = \{w - F(z, \bar{z}) = 0\}$. If one could extend J from $\mathbb{R}^4$ to $\mathbb{R}^6$ making M complex then our C will be an intersection of two smooth complex hypersurfaces and we are done due to our Theorem 2.

Of course the existence of J-complex hypersurfaces is totally exceptional unless the real dimension of X is 4. However, Donaldson in [Do] proved that every compact symplectic manifold admits symplectic hypersurfaces in homology classes of sufficiently high degree, therefore giving us almost complex structures with complex hypersurfaces on such manifolds.

0.3. Distance to a Levi flat hypersurface.

The consideration of J-complex hypersurfaces comes naturally in the discussion of real hypersurfaces.

Now we turn to the case when M is a real hypersurface. Let us recall construction of McDuff in more details. In her example $X = T^*Y$ - the total space of the cotangent bundle of a Riemann surface Y of genus $\geqslant 2$ equipped with the standard symplectic form. The domain D in question is defined as $D = \{x : t_1 < \rho(x) < t_2\}$ where $\rho : T^*Y \setminus \{0\} \to \mathbb{R}$ is some smooth function which is strictly pseudoconcave in $D_1 := \{x : t_1 < \rho(x) < t_0\}$ and is strictly pseudoconvex in $D_2 := \{x : t_0 < \rho(x) < t_2\}$ for some $t_1 < t_0 < t_2$. And $dd^c_J\rho = 0$ on Γ_{t_0}, where $\Gamma_{t_i} = \{x : \rho(x) = t_i\}, i = 1, 2, 3$. Therefore we obtain a domain, say D_1 with disconnected boundary $\Gamma_{t_1} \cup \Gamma_{t_0}$ such that one component is strictly pseudoconvex and another is Levi flat. Moreover, this D_1 doesn't contain J-complex lines.

If M is Levi flat, i.e. foliated by J-complex hypersurfaces, then $D\setminus M$ is locally hyperbolically complete as follows immediately from Theorem 2. Therefore the only obstruction for D_1 in the example above to be not (complete) hyperbolic is the eventual presence of a (so called limiting) J-complex line in Γ_{t_0}.

Therefore the following question close to Question 1 seems to be of an interest:

QUESTION 5. — *Let $\Gamma \subset T^*Y$ be a $\mathbb{S}^1$-bundle over a Riemann surface Y of genus $\geqslant 2$. Suppose that Γ is Levi-flat for some ω-tamed J. Can such Γ contain a J-complex line?*

Probably not and if so we would get an example of a complete hyperbolic manifold with disconned boundary.

In the integrable case if the Kobayashi distance to a real hypersurface M is infinite then M is forced to be Levi-flat. This statement is no longer true in case of non integrable J. In § 6 we construct the following:

Example. — There exists a real analytic almost complex structure J on $\mathbb{R}^6$ such that $M = \mathbb{R}^5 \times \{0\}$ is not Levi flat but the Kobayashi pseudo-distance relative to $(\mathbb{R}^6 \setminus M, J)$ of any point in $\mathbb{R}^6 \setminus M$ to M is infinite.

Let a real hypersurface $M \subset D$ be defined by $\rho = 0$ (as usual $\nabla\rho \neq 0$ on M). A tangent vector Y to M, at some point p, is said to be *complex tangent* if $J(p)Y$ is also tangent to M, at p. Recall that M is foliated by J-complex hypersurfaces (i.e. M has a Levi foliation) if and only if for every $p \in M$ and every Y and T, both complex tangent vectors to M at p, $dd^c_J\rho(Y,T) = 0$ (definitions will be recalled later).

THEOREM 3. — *Let D be a domain in an almost complex manifold (X,J). Assume that the closed real hypersurface $M \subset D$ is of class $\mathcal{C}^2$, and J is of class $\mathcal{C}^{3,\alpha}$ (for some $0 < \alpha < 1$). If there exists a complex tangent vector Y to M at a point p, such that $dd^c_J\rho(Y,JY) > 0$, then that point p is at finite distance, in $D \setminus M$ from points in the region defined by $\rho > 0$.*

If $dd^c_J\rho(Y,JY) < 0$, simply replace ρ by $-\rho$.

Theorem 3 implies that if a relatively compact smoothly bounded domain D in almost complex manifold is complete hyperbolic then D should be pseudoconvex (pseudoconvexity being defined by the Levi form).

For non integrable almost complex structures the condition $dd^c_J\rho(Y, JY) = 0$ for all complex tangent vectors Y does not imply the (Frobenius) condition $dd^c_J\rho(Y, T) = 0$ for all complex tangent vectors Y, T. It is illustrated by the above example. But there is a case when the above conditions are equivalent. This is the case of dimension 4. We therefore have the following

COROLLARY 4. — *Let D be a Kobayashi hyperbolic with respect to J domain in X. If X has dimension 4, if the hypersurface $M \subset D$ is of class $\mathcal{C}^3$ and J is of class $\mathcal{C}^{3,\alpha}$, then the following are equivalent:*

(1) *For every point $p \in M$ the Kobayashi distance, in $D \setminus M$, from p to any point in $D \setminus M$ is infinite.*

(2) *M is Levi flat.*

The example in $\mathbb{R}^6$ mentioned above has the interesting feature that, although $M = \mathbb{R}^5 \times \{0\}$ is not Levi flat, through any point in any complex tangential direction, there is a J-complex curve lying entirely in M. It would be interesting to know this in general.

QUESTION 6. — *Let $M = \{\rho = 0\}$ be a real hypersurface in (X, J) such that $dd^c_J\rho(Y, JY) = 0$ for all complex tangent vectors Y. Show that for any $p \in M$ and any $Y \in T^c_pM$ there exists a J-complex disc passing through p, entirely lying in M and tangent to Y at p. Does this condition imply complete hyperbolicity of the complement to M?*

Remarks. — 1. Corollary 2 was first proved by Gaussier and Sukhov ([G-S]) in dimension 4, for almost complex structures that are smooth enough (with partial results for arbitrary dimensions). Although our path is substantially different, their paper has been an inspiration. After our proof of Corollary 1 was written (several months later), Gaussier and Sukhov were able to improve their technique in order to get a proof of Corollary 2 for arbitrary dimension (still for structures that are smooth enough), along the lines of their original attempt.

2. Our Theorem 3 is essentially given by Proposition 6 in the paper [B-M], by Barraud and Mazzilli. But we did not see in [B-M] neither a proof nor an adequate reference for the existence of J-holomorphic discs with appropriate jets (it is the essential point, and a proof is needed even it follows a path known since [N-W]).

3. We give complete proofs including for some well known basic results and the natural extensions of these results that we needed. We aimed for proofs much simpler than the proofs usually given or sketched. We were careful about the needed smoothness assumptions. So we hope that this paper can also be used by non-specialists as an easy introduction to the now basic theory of pseudo-holomorphic discs.

4. Even from the strict point of view of complex analysis, there is an advantage in taking the point of view of almost complex manifolds. This is clearly illustrated by the proof of the upper semi-continuity of the Kobayashi- Royden pseudo-norm. This upper semi-continuity was first proved by Royden [R]. The proof uses only standard tools such as: the existence of Stein neighborhoods for embedded discs (now covered by Siu's Theorem), Grauert's characterization of Stein manifolds in terms of strictly plurtisubharmonic exhaustion functions, the embedding of Stein manifold in $\mathbb{C}^N$, and elementary results on holomorphic vector bundles on the unit disc. But it cannot be considered to be an elementary proof, unlike the proof that one gives in the more general setting of almost complex manifolds (see the Appendix): it is simply basic elliptic theory.

5. Since the beginning of the theory ([N-W]), one has considered non-smooth almost complex structures. In real dimension 2, it has been important to prove that such structures are merely complex structures. (One even considers bounded but non-continuous data, see [C-G] Chapter 1). In real dimension 4, there has been a recent interesting example where non-smooth almost complex structures arise. In [Du], J. Duval had to blow up a point, say the origin. A preliminary work has to be done in order to make the standard complex lines through the origin to be J-holomorphic. It leads to a change of variables that is not smooth.

The structure of the paper is the following:

In § 1, we clarify some notations and recall basic facts from almost complex geometry. It also contains some preliminaries, such as the existence and basic properties of plurisubharmonic functions on almost complex manifolds (including a quick proof of well known basic results that also follow from standard elliptic theory).

In § 2, we prove Theorem 1. The proof uses localization with the help of plurisubharmonic functions and then an appropriate Schwarz-type Lemma 2.3.

In § 3 we give an estimate of a Calderon-Zygmund integral and prove another Schwarz-type Lemma 3.2. In § 4 Lemma 3.2 is applied, and we prove Theorem 2.

Theorem 3 is proved in § 5.The example mentioned above is given in § 6. An appendix gathers some proofs and additional facts.

1. Some notations, definitions, and basic facts.

[McD-S] is a well known reference for almost complex manifolds. An almost complex manifold (X, J) is a manifold X of even real dimension $2n$, with at each point p an endomorphism $J = J(p)$ of the tangent space satisfying $J^2 = -\mathbf{1}$. In this section, and in the next one we shall assume that J is of class $\mathcal{C}^1$.

As usual $\mathcal{C}^{k,\alpha}$ is used to denote spaces of maps whose derivatives of order $\leqslant k$ are Holder continuous of order α, $k \in \mathbb{N}$, $0 < \alpha < 1$. $\mathcal{C}^{k,\alpha}$ regularity of J (i.e. of the map $p \mapsto J(p)$) is preserved by $\mathcal{C}^{k+1,\alpha}$ change of variables.

1.a. J-holomorphic discs and the Kobayashi-Royden pseudo-norm.

We shall study maps u from an open set of $\mathbb{C}$ (always equipped with the standard complex structure - so we will avoid the more complete notation $\overline{\partial}_{J_{st},J}$) into (X, J). We set

$$\overline{\partial}_J u(Y) = \frac{1}{2}[du(Y) + J(u)du(iY)],$$

for any vector Y tangent to $\mathbb{C}$ (at a point where u is defined). The map $u \in \mathcal{C}^0 \cap L^{1,2}$ is J-holomorphic if $\overline{\partial}_J u = 0$ a.e. If $J \in \mathcal{C}^{k,\alpha}$ this implies that $u \in \mathcal{C}^{k+1,\alpha}, k \geqslant 0$. In particular $\mathcal{C}^1$-regularity of J implies $\mathcal{C}^{1,\alpha}$-regularity of u, for any $\alpha \in (0,1)$. We shall see later that more is true: u belongs then to some sub-Lipschitzian class $\mathcal{C}^{1,\phi}$ with $\phi(r) = r \ln \frac{1}{r}$. It will enable us to work under $\mathcal{C}^1$-regularity of the structure.

$\mathbb{D}_R$ will denote the disc of radius R in $\mathbb{C}$ centered at 0, and $\mathbb{D} = \mathbb{D}_1$ will be the open unit disc. Under our regularity assumption on J through each point $p \in X$ in every direction $Y \in T_pX$ there exists a J-complex curve, see § 1.e. More precisely, there exists a J-holomorphic

$u : \mathbb{D}_R \to X$ (for some $R > 0$) such that $u(0) = p$ and $du(0)(\frac{\partial}{\partial x}) = Y$. The Kobayashi-Royden pseudo-norm of the vector $Y \in T_pX$ on the almost complex manifold X is defined as $\|Y\|_K = \inf\{\frac{1}{R} \mid \exists u : \mathbb{D}_R \to X,$ J-holomorphic, $u(0) = p, du(0)(\frac{\partial}{\partial x}) = Y\}$. Another way to say the same is $\|Y\|_K = \inf\left\{\frac{1}{t} : \exists u : \mathbb{D} \to X, J\text{-holomorphic}, u(0) = p, du(0)(\frac{\partial}{\partial x}) = tY\right\}$.

If TX is equipped with some norm $\|\cdot\|$ then $\|Y\|_K = \inf\{\frac{\|Y\|}{\|du(0)(\frac{\partial}{\partial x})\|} \mid u : \mathbb{D} \to X, J-$ holomorphic, $u(0) = p$ and $du(0)(\frac{\partial}{\partial x})$ is parallel to $Y\}$.

The length L of a path $\gamma : [0\ ,\ 1] \to X$ is then defined by $L = \int_0^1 \|\dot\gamma(t)\|_K\, dt$, where the integral is understood as the upper integral, i.e. the infimum of the integrals of the positive measurable majorants.

In case the almost complex structure J is of class $\mathcal{C}^1$ it is not clear that the function $t \mapsto \|\dot\gamma(t)\|_K$ is integrable or even measurable. But, at least if J is of class $\mathcal{C}^{1,\alpha}$, the Kobayashi-Royden pseudo-norm is an upper semi continuous function on TX (see Appendix 2 and [I-P-R]), and the integral makes sense in the ordinary sense and is finite.

The pseudo-distance between two points is of course the infimum of the lengths of the paths joining these two points. Abusively we may say distance instead of pseudo-distance.

1.b. Complete hyperbolicity.

We now state an elementary Lemma that will be used to prove completeness for the Kobayashi metric. It will be applied with $\delta(t) = Ct$ or $Ct \log \frac{1}{t}$ near 0 (C is a constant). Think of χ in the Lemma as a complex coordinate function.

LEMMA 1.1. — *Let D be a domain in an almost complex manifold (X, J). Let $p \in \partial D$. Let χ be either:*

(a) *a $\mathcal{C}^1$ map from $\overline{D}$ into $\mathbb{R}^2$ with $\chi(p) = 0$ and $\chi \neq 0$ on D, or*

(b) *a $\mathcal{C}^1$ map from a neighborhood U of p into $\mathbb{R}^2$, such that $\chi(p) = 0$ and $\chi \neq 0$ on $U \cap \overline{D} \setminus \{p\}$.*

Let δ be a positive function defined on $(0\ ,+\infty)$ satisfying $\int_0^1 \frac{dt}{\delta(t)} = +\infty$. Assume that for every J-holomorphic map u from $\mathbb{D}$ into D, such that $u(0)$ is close to p

$$|\nabla(\chi \circ u)(0)| \leqslant \delta(|(\chi \circ u)(0)|). \tag{1.1}$$

Then, p is at infinite Kobayashi distance from the points in D.

Proof. — We write first the proof for case (a). In the proof, the Kobayashi pseudo-norms are denoted by $|\cdot|_K$, and $|\cdot|$ denotes Euclidean norm in $\mathbb{R}^2$.

Let γ be a $\mathcal{C}^1$ path in D from a point q_0 (fixed) to a point p_1 (to be thought of as close to p). So $\gamma : [0\ ,\ 1] \to D$, $\gamma(0) = q_0$, $\gamma(1) = p_1$. The Kobayashi length of γ is $L = \int_0^1 |\dot\gamma(t)|_K\, dt$. For $t \in [0\ ,\ 1]$, by definition of the Kobayashi metric, there exists a J-holomorphic map $u_t : \mathbb{D} \to X$ with $u_t(0) = \gamma(t)$, and

$$\frac{\partial u_t}{\partial x}(0) = \frac{1}{2|\dot\gamma(t)|_K}\dot\gamma(t).$$

From this we get:

$$\frac{\partial(\chi \circ u_t)}{\partial x}(0) = d\chi_{\gamma(t)}\Big(\frac{\partial u_t}{\partial x}(0)\Big) = \frac{1}{2|\dot\gamma(t)|_K} d\chi_{\gamma(t)}(\dot\gamma(t))) = \frac{1}{2|\dot\gamma(t)|_K} \frac{d}{dt}(\chi \circ \gamma)(t).$$

From (1.1) it follows that

$$|\dot\gamma(t)|_K \geqslant \frac{1}{2}\frac{|\frac{d}{dt}(\chi \circ \gamma)(t)|}{\delta(|(\chi \circ \gamma(t)|)} \geqslant \frac{1}{2}\frac{-\frac{d}{dt}(|(\chi \circ \gamma)(t)|)}{\delta(|(\chi \circ \gamma(t)|)}.$$

Finally, we get

$$L \geqslant \frac{1}{2}\int_{|\chi(p_1)|}^{|\chi(q_0)|} \frac{ds}{\delta(s)},$$

which is arbitrarily large if p_1 is close enough to p (so $|\chi(p_1)| \simeq 0$).

The proof of (b) follows from the above. Shrinking U if needed, we can assume that χ is defined on $\overline{U} \cap \overline{D}$ and $\chi \neq 0$ on $\overline{U} \cap \overline{D} \setminus \{p\}$. We wish to estimate the length of a path from a point q_0 in D to a point p_1 as in the proof of (a). If this path is entirely in U, the proof of (a) applies. Otherwise, let t_0 the largest element in $[0,1]$ such that $\gamma(t_0) \notin U$ (assuming U open). Then simply apply the above estimates to the path from $\gamma(t_0)$ to p_1 obtained by restricting γ to $[t_0, 1]$. It is important to note that the non vanishing of χ on $\overline{U} \cap \overline{D} \setminus \{p\}$, and not only on $\overline{U} \cap D$, insures that there exists $\epsilon > 0$ such that $|\chi| \geqslant \epsilon$ on the boundary of U (in D), and therefore $|\chi(\gamma(t_0))| \geqslant \epsilon$ (ϵ not depending on γ). □

Remark 1. — The Lemma covers the case of maps χ with values in $\mathbb{R}$, such that $\chi|_{D\cap U\setminus\{p\}} < 0$, and $\chi(p) = 0$.

Remark 2. — In a hyperbolic manifold, complete hyperbolicity of a domain is a purely local question. We have more:

An open subset X_0 in an almost complex manifold (X, J) is called *locally complete hyperbolic* if for every $y \in \overline{X}_0$ there exists a neighborhood $V_y \ni y$ such that $V_y \cap X_0$ is complete hyperbolic.

Recall that an open subset X_0 of an almost complex manifold X is called *hyperbolically imbedded* into X if for any two sequences $\{x_n\}$, $\{y_n\}$ in X_0 converging to $x \in X$ and $y \in X$, $x \neq y$, respectively, one has that $\limsup_{n\to\infty} k_{X_0,J}(x_n, y_n) > 0$. Here $k_{X_0,J}$ denotes the Kobayashi pseudo-distance on the manifold (X_0, J).

It is worth observing that if X_0 is a relatively compact domain in X, hyperbolically embedded into (X, J) and if X_0 is locally complete hyperbolic then (X_0, J) is complete hyperbolic, see [Ki].

The result is rather immediate (if u maps the unit disc into X_0, restrict to a smaller disc and rescale), but it is very useful. It reduces the problem of completeness to a purely local problem (in hyperbolic manifolds). We will use it repeatedly without further explanations when switching to local problems.

1.c. Plurisubharmonic functions.

If λ is a function or vector valued map defined on (X, J), $d^c_J\lambda$ is the 1-form (vector valued) defined by

$$d^c_J\lambda(Y) = -d\lambda(JY)$$

for every tangent vector Y. Notice that now the almost complex structure is on the source space X. Then $dd^c_J\lambda$ is defined by usual differentiation. As usual, Δ will denote the Laplacian, ($\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$). The notation d^c is relative to the standard complex structure on $\mathbb{C}$, so for a function h defined on an open set of $\mathbb{C}$: $d^c h = -\frac{\partial h}{\partial y}dx + \frac{\partial h}{\partial x}dy$.

The following formula (1.2), that one finds also in [De-2] and [Ha], is an immediate consequence of the chain rule (with some care needed in case of low regularity). The interesting case of smooth structures but rough functions is studied by Nefton Pali ([P]).

LEMMA 1.2. — *Let J be a $\mathcal{C}^1$ almost complex structure defined on an open set $\Omega \subset \mathbb{R}^{2n}$. Let λ be a $\mathcal{C}^2$ function defined on Ω. If $u : \mathbb{D} \to (\Omega, J)$*

is a J-holomorphic map, then:

$$\Delta(\lambda \circ u) \;=\; [dd^c_J\lambda]_{u(z)}\Big(\frac{\partial u}{\partial x}(z), [J(u(z))]\Big(\frac{\partial u}{\partial x}(z)\Big)\Big). \tag{1.2}$$

Comments. — Since J is only of class $\mathcal{C}^1$, u need not be $\mathcal{C}^2$ (only $\mathcal{C}^{1,\beta}$ for all $\beta < 1$), so the left hand side in the equation above has to be understood in the distributional sense. But the right hand side is different. Since dd^c_J involves only one derivative of J, the right hand side makes sense pointwise (as suggested by our insertion of z's in the right hand side). The statement says that $\Delta(\lambda \circ u)$ is in fact (the distribution defined by) a continuous function, although $\lambda \circ u$ need not be $\mathcal{C}^2$.

Although u is not $\mathcal{C}^2$, the Lemma should not be so surprising. Consider the case of a genuine holomorphic change of variable ψ, one has the formula $\Delta(h \circ \psi) = \Delta h \circ \psi \; |\nabla\psi|^2$, in which the second derivative of ψ plays no role.

Proof. — Since we deal with low regularity, we start with the following simple preliminary remark.

Let U be an open set in $\mathbb{R}^K$ and φ be a $\mathcal{C}^1$ map from U into $\mathbb{R}^N$. Let ω be a $\mathcal{C}^1$ form of class $\mathcal{C}^1$ defined on an open set of $\mathbb{R}^N$ containing $\varphi(U)$. Then $d(\varphi^*\omega) \;=\; \varphi^*(d\omega)$. Note that $\varphi^*\omega$ and $\varphi^*(d\omega)$ are well defined (the case of Lipschitzian data would be more subtle!), and have continous coefficients, but possibly not $\mathcal{C}^1$ coefficients and so $d(\varphi^*\omega)$ may have to be taken in the sense of currents. For smooth φ and ω it is plainly chain rule, and for φ and ω as above it follows by approximation.

Now, we prove the Lemma. We know that u is of class $\mathcal{C}^1$. By the very definition of J-holomorphicity, du commutes with the action of the almost complex structures, so

$$d^c(\lambda \circ u) = u^* \; d^c_J\lambda.$$

Indeed, for any tangent vector Y to the unit disc,

$$\begin{aligned} d^c(\lambda \circ u)(Y) = -d(\lambda \circ u)(iY) = -d\lambda \circ du(iY) &= -d\lambda(J(du(Y))) \\ &= d^c_J\lambda(du(Y)) = u^*d^c_J\lambda(Y). \end{aligned}$$

By the preliminary remark (chain rule) applied with $\varphi = u$ and $\omega = d^c_J\lambda$:

$$dd^c(\lambda \circ u) = d(u^*d^c_J\lambda) = u^*dd^c_J\lambda.$$

Finally,

$$\Delta(\lambda \circ u) = dd^c(\lambda \circ u)\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = dd^c_J\lambda\Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big) = dd^c_J\lambda\Big(\frac{\partial u}{\partial x}, J\frac{\partial u}{\partial x}\Big),$$

the J-holomorphicity of u being again used for the last equality.

The Lemma is thus established.

COROLLARY 1.1. — *Let J be a $\mathcal{C}^1$ almost complex structure defined on an open set $\Omega \subset \mathbb{C}^n$. Let λ be a $\mathcal{C}^2$ real valued function defined on Ω. The following are equivalent:*

(1) *For every tangent vector Y to Ω, $dd^c\lambda(Y, JY) \geqslant 0$.*

(2) *For every J-holomorphic map $u : \mathbb{D} \to \Omega$, $\lambda \circ u$ is subharmonic.*

Let us repeat that since dd^c_J involves only one derivative of J, $dd^c_J\lambda$ makes clear sense. (1) $\Rightarrow$ (2) is an immediate consequence of the Lemma. (2) $\Rightarrow$ (1) is also an immediate consequence, taking into account that for every tangent vector Y, at a point $q \in \Omega$, there exists a J-holomorphic map $u : \mathbb{D} \to \Omega$ such that $u(0) = q$ and $\frac{\partial u}{\partial x}(0) = tY$, for some $t > 0$.

If the equivalent conditions of the Corollary are satisfied by a function λ, that function is said to be *J-plurisubharmonic*. Of course (2) makes sense also for upper-semicontinuous λ giving us the general notion of a plurisubharmonic function. As usual, a function is said to be strictly J-plurisubharmonic if locally any small $\mathcal{C}^2$ perturbation of that function is still J-plurisubharmonic.

There are two important examples of plurisubharmonic functions:

LEMMA 1.3. — *Let (X, J) be an almost complex manifold equipped with an arbitrary smooth Riemannian metric with J of class $\mathcal{C}^1$. Then for any $p \in X$, the function $q \mapsto \big(\mathrm{dist}(q,p)\big)^2$ is strictly plurisubharmonic near p.*

Proof. — If J is of class $\mathcal{C}^{1,\alpha}$ the J-holomorphic discs are $\mathcal{C}^{2,\alpha}$ and the result is trivial. If J is only of class $\mathcal{C}^1$, it follows from (1.2) and from the comments after Lemma 1.2.

The simple fact to be used is the following one: If μ is a continuous function defined near 0 in $\mathbb{R}^2$ such that $\Delta\mu$ (in the sense of distributions) is a continuous function, and such that for some $C > 0$ $\mu(x,y) \geqslant \mu(0) + C(x^2 + y^2)$ (near 0), then $\Delta\mu(0) > 0$.

The next example less trivial is due to Chirka (not published).

LEMMA 1.4. — *Let J be an almost complex structure of class $\mathcal{C}^1$ in the neighborhood of the origin in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. Suppose that $J(0) = J_{st}$.*

Then there exist a neighborhood $V \ni 0$ and a constant $A > 0$ such that $\log|Z| + A|Z|$ *is J-plurisubharmonic in V.*

Consequently, for any point p in a complex manifold (X, J), with J of class $\mathcal{C}^1$, there exists a plurisubharmonic function λ defined near p, continuous and finite except at p such that $\lambda(p) = -\infty$. Indeed we can take local coordinates in which the almost complex structure coincides with the standard one at p.

Proof. — Using dilations the Lemma reduces to the following.

For $Z \in \mathbb{R}^{2n}$ $(= \mathbb{C}^n)$ set $u(Z) = |Z| + \log|Z|$. We consider a continuously differentiable almost complex structure J defined on the unit ball B in $\mathbb{R}^{2n}$. We wish to prove that there exists $\epsilon > 0$ such that if $J(0) = J_{st}$ and $\|J - J_{st}\|_{\mathcal{C}^1} \leqslant \epsilon$, then for $Z \in B$, $Z \neq 0$, and every tangent vector Y at Z:

$$[dd^c_J u(Z)](Y, JY) \geqslant 0\ .$$

As previously dd^c will be used for $dd^c_{J_{st}}$. An elementary computation gives the following result:

$$dd^c u(Y, J_{st}Y) \geqslant \frac{A}{|Z|}\|Y\|^2, \tag{1.4}$$

for some positive A easy to determine. With complex notations: it is equivalent to showing $\sum \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geqslant \frac{A}{4|Z|}|W|^2$.

Note that, using invariance under rotations, it is enough to check the inequality at points Z of the type $Z = (z, 0, \cdots, 0)$. The computations simplify and one immediately gets a better non-isotropic estimate:

$$dd^c u(Y, J_{st}Y) \geqslant \frac{A}{|z|}\|Y\|^2 \ + \frac{A_1}{|z|^2}\|Y'\|^2\ ,$$

with $Y = (Y_1, Y') \in \mathbb{R}^2 \times \mathbb{R}^{2(n-1)}$.

Chirka's Lemma follows from a simple perturbation argument in which we shall use that the first order (resp. second order) derivative of u at Z are of the order of magnitude of $\frac{1}{|Z|}$ (resp. $\frac{1}{|Z|^2}$). We have:

$$dd^c_J u(Y, JY) = [d(d^c_J - d^c)u(Y, JY)] + [dd^c u(Y, (J - J_{st})Y)] + [dd^c u(Y, J_{st}Y)].$$

We now look at each of the terms on the right hand side. For the last one $dd^c u(Y, J_{st}Y)$, we have the estimate (1.4). For the second one: $\|(J(Z) - J_{st})Y\| \leqslant \epsilon|Z|\ \|Y\|$. So, using the estimates on the derivatives of u, if ϵ is small enough: $|dd^c u\ (Y, (J - J_{st})Y)| \leqslant \frac{A}{4|Z|}\|Y\|^2$.

We finally look at the first term. Using $\mathbb{R}^{2n}$ coordinates $(t_1, \dots, t_{2n})$ for $Z \in B$, write

$$d(d^c_J - d^c) = \Big(\sum \alpha^{l,m}_{j,k} \frac{\partial^2}{\partial t_j \partial t_k} + \sum \beta^{l,m}_j \frac{\partial}{\partial t_j}\Big) dt_l \wedge dt_m.$$

Since $J(0) - J_{st} = 0$ and $\|J - J_{st}\|_{C^1} \leqslant \epsilon$, for some universal constant K we get $|\alpha^{l,m}_{j,k}| \leqslant K\epsilon|Z|$ and $|\beta^{l,m}_j| \leqslant K\epsilon$. Using the estimates on the derivatives of u, we again get that if ϵ is small enough $|d(d^c_J - d^c)u(Y, JY)| \leqslant \frac{A}{4|Z|}\|Y\|^2$. Then one has:

$$dd^c_J u(Y, JY) \geqslant \frac{A}{2|Z|}\|Y\|^2.$$

□

1.d. Regularity of J-complex discs.

We want to make a few remarks about the regularity of J-holomorphic discs since the proofs in the literature are not always pleasant. We shall always assume u (as below) and J to be of class $\mathcal{C}^1$ at least, in which case Formula (1.2) allows truly immediate proofs.

But it should be reminded that: First of all if J is merely continuous then J-holomorphic maps $u : \mathbb{D} \to X$ (a priori they are $\mathcal{C}^0 \cap L^{1,2}$) are in $L^{1,p}$ for any p and therefore in $\mathcal{C}^\alpha$ for any $0 < \alpha < 1$ due to Sobolev embedding, see [IS-2] Lemma 2.4.1. Then if J is of class $\mathcal{C}^\alpha$, u is of class $\mathcal{C}^{1,\alpha}$ (Theorem III in [N-W] and [Si]).

Remark 1. — The proof of Lemma 1.2 used only the following:

$$(H) \qquad \begin{cases} \lambda \text{ is a } \mathcal{C}^2 \text{ function,} \\ \text{the almost complex structure } J \text{ is of class } \mathcal{C}^1, \\ u \text{ is a } \mathcal{C}^1\ J\text{-holomorphic map.} \end{cases}$$

It may be worth pointing out that (using the simplest possible results on the regularity of the standard Laplacian Δ) the following is an immediate consequence of the validity of formula (1.2) under the above hypotheses (H) :

If an almost complex structure J on (some open set of) $\mathbb{R}^{2n}$ is of class $\mathcal{C}^{k,\alpha}$, with $k \geqslant 1$ and $0 < \alpha < 1$, then any $\mathcal{C}^1$ J-holomorphic map $u : \mathbb{D} \to (\mathbb{R}^{2n}, J)$ is $\mathcal{C}^{k+1,\alpha}$ regular. If J is only $\mathcal{C}^1$, u is of class $\mathcal{C}^{1+\beta}$ for all $\beta < 1$. More is true (see Remark 3) and will be needed later.

Here, we sketch the argument:

1) Assume that J is of class $\mathcal{C}^1$. Let λ be any smooth function defined on $\mathbb{R}^{2n}$.

By (1.2), $\Delta(\lambda \circ u)$ is a locally bounded function. It immediately follows that $\lambda \circ u$ is of class $\mathcal{C}^{1,\beta}$ for all $0 < \beta < 1$. Taking λ to be the coordinate functions, one sees that u itself is of class $\mathcal{C}^{1,\beta}$.

2) Assume now that J is of class $\mathcal{C}^{k,\alpha}$ ($k \geqslant 1$, $0 < \alpha < 1$). By 1) we already know that u is of class $\mathcal{C}^{1,\alpha}$. Then, by repeating the argument, one sees that if u is of class $\mathcal{C}^{r,\alpha}$ with $1 \leqslant r \leqslant k$, then u is of class $\mathcal{C}^{r+1,\alpha}$.

Indeed, if λ is any smooth function on $\mathbb{R}^{2n}$, (1.2) shows that $\Delta(\lambda \circ u)$ is of class $\mathcal{C}^{r-1,\alpha}$. So by regularity of the Laplacian $\lambda \circ u$ is $\mathcal{C}^{r+1,\alpha}$.

Remark 2. — The other result of basic elliptic theory that is used several times is that if on a bounded open set Ω in $\mathbb{R}^{2n}$, an almost complex structure J is close enough (depending on β below, $0 < \beta < 1$) to J_{st} in $\mathcal{C}^1$ norm (resp. close enough in $\mathcal{C}^{k,\alpha}$ norm, with $k \geqslant 1$), then the J-holomorphic maps u from $\mathbb{D}$ into Ω have uniform $\mathcal{C}^{1,\beta}$ (resp. $\mathcal{C}^{k+1,\alpha}$) bounds on smaller discs. It is at the root of local hyperbolicity, see Lemma 1.5. We shall restrict our discussion to almost complex structures which are at least of class $\mathcal{C}^1$ (see [Si], for lower regularity).

We wish to mention that formula (1.2) can also be used for proving the above result. One first needs an initial regularity result giving L^p bounds for ∇u, to be used with $p > 4$. Then, proceeding as above, a first application of (1.2) gives L^r bounds for Δu with $r = \frac{p}{2} > 2$, therefore Hölder $(1 - \frac{2}{r})$ estimates for ∇u, i.e. $\mathcal{C}^{1,1-\frac{2}{r}}$ bounds for u. After that, (1.2) gives $\mathcal{C}^{k+1,\alpha}$ bounds if J is of class $\mathcal{C}^{k,\alpha}$ and if $\mathcal{C}^{k,\alpha}$ bounds are already known for u.

It happens that L^p bounds for ∇u, for arbitrary $1 < p < \infty$, are extremely easy to get. Provided that ϵ is small enough depending on p (for $p = 2$, $\epsilon < 1$), they follow from the simple differential inequality:

$$\Big|\frac{\partial u}{\partial \overline{z}}\Big| \leqslant \epsilon \Big|\frac{\partial u}{\partial z}\Big|.$$

The argument is well known. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{D})$, satisfying $0 \leqslant \chi \leqslant 1$. We have:

$$\Big|\frac{\partial \chi u}{\partial \overline{z}}\Big| \leqslant \Big|\chi\frac{\partial u}{\partial \overline{z}}\Big| + \Big|\frac{\partial \chi}{\partial \overline{z}}u\Big| \leqslant \epsilon\Big|\frac{\partial \chi u}{\partial z} - \frac{\partial \chi}{\partial z}u\Big| + \Big|\frac{\partial \chi}{\partial \overline{z}}u\Big| \leqslant \epsilon\Big|\frac{\partial \chi u}{\partial z}\Big| + (1+\epsilon)K|u|,$$

with $K = \operatorname{Sup}|\nabla\chi|$. Since $\frac{\partial \chi u}{\partial z} = \frac{-1}{\pi z^2} * \frac{\partial \chi u}{\partial \overline{z}}$, the theory of singular integral gives $\|\frac{\partial \chi u}{\partial z}\|_{L^p} \leqslant C_p \|\frac{\partial \chi u}{\partial \overline{z}}\|_{L^p}$. So, one has

$$\|\frac{\partial \chi u}{\partial \overline{z}}\|_{L^p} \leqslant \epsilon C_p \|\frac{\partial \chi u}{\partial \overline{z}}\|_{L^p} + (1+\epsilon)K\|u\|_{L^p}.$$

If ϵ is chosen small enough so that $\epsilon C_p < 1$, we get:

$$\|\frac{\partial \chi u}{\partial \bar{z}}\|_{L^p} \leqslant \frac{(1+\epsilon)K}{1-\epsilon C_p}\|u\|_{L^p}.$$

Then,

$$\|\frac{\partial \chi u}{\partial z}\|_{L^p} \leqslant C_p \frac{(1+\epsilon)K}{1-\epsilon C_p}\|u\|_{L^p}.$$

Finally,

$$\|\nabla \chi u\|_{L^p} \leqslant (1+C_p)\frac{(1+\epsilon)K}{1-\epsilon C_p}\|u\|_{L^p}.$$

We will need the following that is easily obtained from Remark 2 above.

LEMMA 1.5. — *Let Ω be an open subset of $(\mathbb{R}^{2n}, J)$, J of class $\mathcal{C}^1$. Let K be a compact subset of Ω. There exists $\delta > 0$, such that: for every $r \in [0,1)$ there exists $C > 0$ such that if $u : \mathbb{D} \to \Omega$ is a J-holomorphic disc with $u(\mathbb{D}) \subset K$, then*

$$|\nabla u(z)| \leqslant C \sup_{|t|<1} |u(t) - u(0)|, \tag{1.5}$$

if $\sup_{|t|<1} |u(t) - u(0)| \leqslant \delta$ and $|z| \leqslant r$.

Proof. — Depending on $u(0)$, one can make a linear change of variables such that in the new coordinates $J(u(0)) = J_{st}$. One can choose the linear maps for changing variables so that their norms and the norm of their inverses are uniformly bounded for $u(0) \in K$.

After such a change of variables, set $M = \mathrm{Sup}\,|u(z) - u(0)|$ ($z \in \mathbb{C}$, $|z| < 1$). Set $u_M(z) = \frac{u(z)-u(0)}{M}$. Then u_M is a J_M-holomorphic map from $\mathbb{D}$ into the unit ball in $\mathbb{R}^{2n}$, for the almost complex structure J_M that is the push-forward of J under the map $Z \mapsto \frac{1}{M}(Z - u(0))$. If M is small enough, J_M is close to J_{st} in the $\mathcal{C}^1$ sense and Remark 2 applies. So, for $|z| \leqslant r$ one gets $|\nabla u_M(z)| \leqslant C$ for some absolute constant C, hence $|\nabla u(z)| \leqslant CM$, as desired.

Remark 3. — Let $\mathcal{C}^{1,\phi}$ (see § 3 for an explication of the notation) be the class of continuously differentiable functions or maps that locally satisfy the following sub-Lipschitzian condition

$$|\nabla f(z') - \nabla f(z)| \leqslant C|z' - z| \ \log \frac{1}{|z' - z|},$$

for some constant C. So, the gradients of functions in $\mathcal{C}^{1,\phi}$ are better than Hölder, but not quite Lipschitzian.

LEMMA 1.6. — *Any J-holomorphic disc (that in our proof we assume to be $\mathcal{C}^1$) is in the class $\mathcal{C}^{1,\phi}$ if J is $\mathcal{C}^1$.*

Uniform bounds are obtained as in Remark 2. Lemma 1.6 follows immediately from the observations in Remark 1 and from the elementary Lemma:

LEMMA 1.7. — *Let g be a function defined on some open set in $\mathbb{R}^2 \simeq \mathbb{C}$. If Δg (in the sense of distributions) is a bounded function, then $g \in \mathcal{C}^{1,\phi}$.*

Proof. — Assume that g is defined near 0. Let χ be a smooth cut off function such that $\chi \equiv 1$ near 0. We have $g\chi = \Delta(g\chi) * \frac{1}{2\pi} \log|z|$. So, near 0:

$$\frac{\partial g}{\partial z} = \Delta(g\chi) * \frac{1}{\pi z} \quad \text{and} \quad \frac{\partial g}{\partial \overline{z}} = \Delta(g\chi) * \frac{1}{\pi \overline{z}}.$$

We concentrate on $\frac{\partial g}{\partial z}$, the case of $\frac{\partial g}{\partial \overline{z}}$ being similar.

Write $\Delta(g\chi)$ as the sum of a bounded function v with compact support and of a distribution T with support away from 0. Then, $T * \frac{1}{\pi z}$ is $\mathcal{C}^\infty$ near 0. So we only need to show an estimate of the type:

$$\Big|\Big(v * \frac{1}{\pi z}\Big)(z_0 + t) - \Big(v * \frac{1}{\pi z}\Big)(z_0)\Big| \leqslant C|t| \log \frac{1}{|t|}.$$

Checking that estimate is immediate. Write

$$\Big(v * \frac{1}{\pi z}\Big)(z_0 + t) - \Big(v * \frac{1}{\pi z}\Big)(z_0)$$
$$= \frac{1}{\pi} \int \frac{v(z_0 - \zeta)}{\zeta + t}\, dxdy(\zeta) - \frac{1}{\pi} \int \frac{v(z_0 - \zeta)}{\zeta}\, dxdy(\zeta).$$

Then, simply use the estimates:

$$\int_{|\zeta|<2|t|} \frac{1}{|\zeta|} \quad \text{and} \quad \int_{|\zeta|<2|t|} \frac{1}{|\zeta + t|} = O(|t|),$$

and

$$\int_{|\zeta|>2|t|} \Big|\frac{1}{\zeta + t} - \frac{1}{\zeta}\Big| \leqslant 2|t| \int_{|\zeta|>2|t|} \frac{1}{|\zeta|^2} = O(|t| \log |t|).$$

$\square$

1.e. Jets of J-holomorphic discs.

The case of 1-jets ($k = 1$)is simply the case of discs with prescribed tangent. In the proof of Theorem 3, we shall also need the case of 2-jets, with dependence on parameters. But we state the general case of jets of arbitrary order.

PROPOSITION 1.1. — *Let $k \in \mathbb{N}$, $k \geqslant 1$, and $0 < \alpha < 1$. Let J be a $\mathcal{C}^{k-1,\alpha}$ almost complex structure defined near 0 in $\mathbb{R}^{2n}$. For any $p \in \mathbb{R}^{2n}$ close enough to 0, and every $V = (v_1, \dots, v_k) \in (\mathbb{R}^{2n})^k$ small enough, there exists a $\mathcal{C}^{k,\alpha}$ J-holomorphic map $u_{p,V}$ from $\mathbb{D}$ into $\mathbb{R}^{2n}$ such that $u_{p,V}(0) = p$, and $\frac{\partial^l u_{p,V}}{\partial x^l}(0) = v_l$, for any $1 \leqslant l \leqslant k$. If the structure J is of class $\mathcal{C}^{k,\alpha}$, then $u_{p,V}$ can be chosen with $\mathcal{C}^1$ dependence (in $\mathcal{C}^{k,\alpha}$) on the parameters (p, V) in $\mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k$.*

Here we will give the proof assuming $\mathcal{C}^{k,\alpha}$ regularity of J. It is a rather simple consequence of the implicit function theorem. The proof of the existence of discs under $\mathcal{C}^{k-1,\alpha}$ regularity requires a trick found by [N-W] for one-jets, and the Schauder fixed point Theorem. It is given in the Appendix. We do not know whether continuous dependence on parameters can then be achieved (there are possibly related examples indicating that it is conceivable that it fails).

Before the proof of the Proposition, we start with some preliminaries.

1.e.1. Re-writing of J-holomorphicity.

On $\mathbb{R}^{2n}$, we consider an almost complex structure J and the standard almost complex structure J_{st} (corresponding to multiplication by i in the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$). By definition

$$2\overline{\partial}_J u\Big(\frac{\partial}{\partial x}\Big) = \frac{\partial u}{\partial x} + J(u)\frac{\partial u}{\partial y}. \tag{1.6}$$

With some abuse of notations ($\mathbb{R}^{2n}$ notations on the left hand side and $\mathbb{C}^n$ notations on the right hand side)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \overline{z}}\ ;\ \frac{\partial u}{\partial y} = \frac{1}{i}\Big(- \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \overline{z}}\Big) = -J_{st}\Big(- \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \overline{z}}\Big).$$

By multiplication on the left by $J(u)$, (1.6) gives:

$$2J(u)\overline{\partial}_J u\Big(\frac{\partial}{\partial x}\Big) = [J(u) + J_{st}]\frac{\partial u}{\partial \overline{z}} + [J(u) - J_{st}]\frac{\partial u}{\partial z}.$$

We shall restrict our attention to almost complex structures J such that at each $Z \in X$, $J(Z) + J_{st}$ is invertible, which happens in particular if $J(Z) - J_{st}$ has operator norm < 1. Then, set:

$$Q_J(u) = [J(u) + J_{st}]^{-1}[J(u) - J_{st}].$$

The equation for J holomorphicity of u becomes

$$\frac{\partial u}{\partial \bar{z}} + Q_J(u)\frac{\partial u}{\partial z} = 0. \tag{1.7}$$

Indeed since $\bar\partial_J u$ is simply the $\mathbb{C} - J$ anti-linear part of du, $\bar\partial_J u = 0$ if and only if $\bar\partial_J u(\frac{\partial}{\partial x}) = 0$.

1.e.2. The Cauchy-Green operator T_{CG} and the operator Φ_J.

For a complex valued function g or a map g with values in a complex vector space, continuous on $\overline{\mathbb{D}}$, and $z \in \mathbb{C}$ with $|z| \leqslant 1$, we set:

$$T_{CG}(g)(z) = \Big(g * \frac{1}{\pi\zeta}\Big)(z) = \frac{1}{\pi}\int_D \frac{g(\zeta)}{z-\zeta}\, dxdy(\zeta).$$

We shall need the classical properties of T_{CG} (see Appendix 4):

(a) If $g \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, $k \in \mathbb{N}$,$0 < \alpha < 1$, then $T_{CG}g \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$.

(b) $\frac{\partial}{\partial \bar z}[T_{CG}(g)] = g$. (on $\overline{\mathbb{D}}$.)

Let k be an integer $\geqslant 1$. Assume that J is a $\mathcal{C}^{k,\alpha}$ almost complex structure defined on $\mathbb{R}^{2n}$, such that $J(z) + J_{st}$ is invertible for all $z \in \mathbb{R}^{2n}$. We define the operator Φ_J from $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$ into itself by:

$$\Phi_J(u) = \Big(\mathbf{1} + T_{CG}Q_J(u)\frac{\partial}{\partial z}\Big)u. \tag{1.8}$$

Φ_J is a continuously differentiable map from $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$ into itself, whose derivative at the point $u \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$ is the map:

$$\delta u \mapsto \Big(\mathbf{1} + T_{CG}Q_J(u)\frac{\partial}{\partial z}\Big)(\delta u) + T_{CG}S(\delta u)\frac{\partial u}{\partial z},$$

where S is the operator defined by differentiation of $u \mapsto Q_J(u)$ (as a map from $\mathcal{C}^{k,\alpha}$ into $\mathcal{C}^{k-1,\alpha}$ - T_{CG} re-gaining one derivative), i.e. with obvious notations:

$$S(\delta u) = [J(u)+J_{st}]^{-1}DJ(\delta u) - [J(u)+J_{st}]^{-1}DJ(\delta u)[J(u)+J_{st}]^{-1}[J(u)-J_{st}].$$

It is for this differentiation that it is not enough that J be of class $\mathcal{C}^{k-1,\alpha}$. If $J = J_{st}$, Φ_J is the identity mapping. On any fixed ball in $\mathcal{C}^{k,\alpha}$, Φ_J is a small (non-linear) perturbation of the identity if J is close to J_{st} in $\mathcal{C}^{k,\alpha}$ topology.

Finally note that equation (1.7) and (b) show that u is J-holomorphic if and only if $\Phi_J(u)$ is holomorphic in the ordinary sense.

1.e.3. Proof of Proposition 1.1, assuming $\mathcal{C}^{k,\alpha}$ smoothness of J.

Of course, we can assume that $J(0) = J_{st}$. After linear change of variables (dilations), and cut off of $J - J_{st}$, we can assume that J is defined on $\mathbb{R}^{2n}$, and as close as we wish to J_{st} in $\mathcal{C}^{k,\alpha}$ topology. For $(q, W) \in \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k$ $(W = (w_1, \dots, w_k))$, define the map $h_{q,W}$ on $\overline{D}$ by

$$h_{q,W}(z) = q + \sum_{l=1}^{k} \frac{1}{l!} z^l w_l.$$

Fix $R > 0$ so that for any (q, W) in the ball $\mathbf{B}_R$ of radius R in $\mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k$, $h_{q,W}$ is in the unit ball of $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$. If J is close enough to J_{st}, Φ_J^{-1} is defined as a map from the unit ball of $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$ into $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$. For $(q, W) \in \mathbf{B}_R$, set

$$U_{q,W} = \Phi^{-1} h_{q,W},$$

Since $h_{q,W}$ is holomorphic, $U_{q,W}$ is J-holomorphic. Finally let Ψ_J the map that to $(q, W) \in \mathbf{B}_R$ associates

$$\Big(U_{q,W}(0), \frac{\partial U_{q,W}}{\partial x}(0), \dots, \frac{\partial^k U_{q,W}}{\partial^k x}(0) \Big) \in \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k.$$

If $J = J_{st}$, Ψ_J is the identity mapping. If J is close to J_{st} in $\mathcal{C}^{k,\alpha}$ topology, it is a small continuously differentiable perturbation of the identity, whose image therefore contains a neighborhood of 0, with a $\mathcal{C}^1$ inverse. If $(q, W) = \Psi^{-1}(p, V)$, then $u_{p,V} = U_{q,W}$ is the desired map. □

Proposition 1.1 can be rephrased in terms of matching jets. In Proposition 1.1' we rephrase only the part of proposition 1.1 with $\mathcal{C}^{k,\alpha}$ smoothness assumption.

PROPOSITION 1.1'. — *Let $k \geqslant 1$ and $0 < \alpha < 1$. Let J be a $\mathcal{C}^{k,\alpha}$ almost complex structure defined in a neighborhood of 0 in $\mathbb{R}^{2n}$. If $\varphi : \mathbb{D} \to (\mathbb{R}^{2n}, J)$ is a smooth map such that $\overline{\partial}_J \varphi(z) = o(|z|^{k-1})$, then there exists a J-holomorphic map u from a neighborhood of 0 in $\mathbb{C}$ into $\mathbb{R}^{2n}$ such that $|(u - \varphi)(z)| = o(|z|^k)$.*

If we have a family of maps $\varphi_t : \mathbb{D} \to (\mathbb{R}^{2n}, J)$ as above, with $\mathcal{C}^1$ dependence on t in some neighborhood of 0 in $\mathbb{R}^\ell$, then there are J-holomorphic maps u_t defined for t near 0 on a same neighborhood of 0 *in* $\mathbb{C}$ *and with* $\mathcal{C}^1$ *dependence on* t*, with* $|(u_t - \varphi)(z)| = o(|z|^k)$ *(uniformly in* t*).*

Proof. — Proposition 1.1' follows immediately from Proposition 1.1, due to the following observation. Assume that u and v are $\mathcal{C}^k$ maps from $\mathbb{D}$ into $(\mathbb{R}^{2n}, J)$, where J is an almost complex structure of class $\mathcal{C}^{k-1}$. If

$$\frac{\partial^\ell u}{\partial^\ell x}(0) = \frac{\partial^\ell v}{\partial^\ell x}(0), \text{ for all } 0 \leqslant l \leqslant k,$$
$$\text{and } \overline{\partial}_J v = o(|z|^{k-1}),\ \overline{\partial}_J u = o(|z|^{k-1}),$$

then u and v have same k-jet at 0.

$\overline{\partial}_J u = o(|z|^{k-1})$ implies that

$$\frac{\partial u}{\partial y}(z) = -J(u(z))\frac{\partial u}{\partial x}(z) + o(|z|^{k-1}). \tag{1.9}$$

Similarly for v. Assume that for some $\ell < k$, we have already shown that at 0 all the derivatives of u and v of total order $\leqslant k$ and order $\leqslant \ell$ in y coincide (the case $k = 0$ being given by the hypothesis). We have to show that for derivatives of total order $m + \ell + 1 \leqslant k$ and of order $\ell + 1$ in y, $D = \frac{\partial^{m+\ell+1}}{\partial x^m \partial y^{\ell+1}}$ we have $Du(0) = Dv(0)$.

Differentiation of (1.9) gives

$$Du = -\frac{\partial^{m+\ell}}{\partial x^m \partial y^\ell} J(u)\frac{\partial u}{\partial x} + o(1),$$

and similarly for v. By the induction hypothesis, one gets

$$Du(0) = Dv(0).$$

□

1.e.4. Families of discs.

Without looking for more generality, we state the next Proposition just as we will need it for proving part $2B$ in Theorem 2.

PROPOSITION 1.2. — *Let* Ω *be an open set in* $\mathbb{R}^{2n} \simeq \mathbb{C}^n$*. Let* $\delta > 0$ *and* $\rho > 1$*. Let* ϕ_t *be a family of* (J_{st}) *holomorphic discs* $\phi_t : \mathbb{D}_\rho \to \Omega$*, defined for* $-\delta \leqslant t \leqslant \delta$*, depending continuously on* t*. Let* $\eta > 0$*. For any almost complex structure on* Ω *close enough to* J_{st} *in* $\mathcal{C}^{1,\alpha}$ *topology on* Ω*, there exists a family* ψ_t *of holomorphic discs* $\psi_t : \mathbb{D} \to \Omega$*, continuous in* t*, such that for any* $t \in [-\delta\ ,\delta]$ *and any* $\zeta \in \mathbb{D}$*,* $|\psi_t(\zeta) - \phi_t(\zeta)| \leqslant \eta$*.*

Proof. — The operators Φ_J were introduced and discussed in the two previous sections. $\Phi_{J_{st}}$ is the identity mapping. Given any compact set F in $\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})$, if J is close enough to J_{st}, Φ_J^{-1} can be defined on F and is close to the identity. It maps J_{st} holomorphic discs to J holomorphic discs. □

2. Completeness of strictly pseuconvex domains.

2.a. Localization results.

As a preliminary, we shall start with a first localization (Lemma 2.1) that is not needed when dealing with local problems only, for which Lemma 2.2 can be obtained directly. We shall use localization techniques that are standard in complex manifolds. See in particular Proposition 2.1 in [Be], but we avoid any explicit use of the Sibony metric.

LEMMA 2.1. — *Let D be a domain in an almost complex manifold (X, J), J of class $\mathcal{C}^1$. Let $p \in \partial D$. Assume that there exists a neighborhood U of p in X and a continuous function ρ on $U \cap \overline{D}$ such that*

$$\begin{cases} \rho(p) = 0, \quad \rho < 0 \text{ on } U \cap \overline{D} - \{p\} \\ \rho \text{ is plurisubharmonic on } U \cap D. \end{cases}$$

Then for every $r \in [0, 1)$ and for every neighborhood V of p in X, there exists a neighborhood W of p such that if $u : \mathbb{D} \to D$ is a J-holomorphic disc and $u(0) \in W$, then $u(z) \in V$ for every $z \in \mathbb{C}$ such that $|z| < r$.

Note that there is absolutely no global assumption made on D.

Proof. — We first make the function ρ globally defined on $\overline{D}$ and identical to -1 off a relatively compact subset of U by replacing ρ by $\max(\kappa\rho, -1)$ for $\kappa > 0$ large enough.

According to Lemma 1.4, there exists a plurisubharmonic function λ defined near p, in X, such that λ is finite and continuous except at p and $\lambda(p) = -\infty$.

We then replace λ by a plurisubharmonic function $\lambda^{\#}$ continuous on $\overline{D} - \{p\}$, bounded off any neighborhood of p, plurisubharmonic on D and such that $\lambda^{\#}(q) \to -\infty$ as $z \to p$.

Such a function $\lambda^{\#}$ is obtained as follows.

If $\rho(q) > -\frac{1}{3}$ set $\lambda^{\#}(q) = C\big(\rho(q) + \frac{1}{2}\big) + \lambda(q)$.

If $-\frac{2}{3} \leqslant \rho(q) \leqslant -\frac{1}{3}$ set $\lambda^{\#}(q) = \max\big(\rho(q), C\big(\rho(q) + \frac{1}{2}\big) + \lambda(q)\big)$.

If $\rho(q) \leqslant -\frac{2}{3}$ set $\lambda^{\#}(q) = \rho(q)$.

If C is taken large enough ($C > 0$), the above defines (on D) a global plurisubharmonic function $\lambda^{\#}$ as desired.

Fix $r \in [0,1)$ and the neighborhood U. Let $u_j : \mathbb{D} \to D$ be a sequence of J-holomorphic discs and (ζ_j) a sequence in $\mathbb{C}$, with $|\zeta_j| < r$. Assume that $u_j(0) \to p$. We have to show that $u_j(\zeta_j) \to p$.

Since $u_j(0) \to p$, $\rho \circ u_j(0) \to 0$. By the mean value property

$$\int_0^{2\pi} |\rho \circ u_j(e^{i\theta})| \frac{d\theta}{2\pi} \longrightarrow 0. \tag{2.1}$$

For every $N > 0$ there exists $\varepsilon > 0$ such that $\rho \circ u_j(z) > -\varepsilon$ implies $\lambda^{\#} \circ u_j < -N$. Taking into account that $\lambda^{\#}$ is bounded away from p it follows from (2.1) that

$$\int_0^{2\pi} \lambda^{\#} \circ u_j(e^{i\theta}) \frac{d\theta}{2\pi} \longrightarrow -\infty.$$

From which it follows that $\lambda^{\#} \circ u_j(\zeta_j) \to -\infty$ and so $u_j(\zeta_j) \to p$. □

This first localization is followed now by a more precise localization that will use Lemma 2.1 as a first step.

DEFINITION. — *We say that the boundary of D (as above) is strictly J-pseudoconvex at p, if there exists a $\mathcal{C}^2$ strictly J-plurisubharmonic function ρ defined near p in X such that $\nabla\rho \neq 0$ on ∂D (near p), and near p, D is defined by $\rho < 0$.*

We equip X with an arbitrary smooth Riemannian metric. We still assume J to be of class $\mathcal{C}^1$.

LEMMA 2.2. — *If the boundary of D is strictly J-pseudoconvex at p_0, for any $r \in [0,1)$ there exists $\delta > 0$ and $C > 0$ such that for every J-holomorphic disc $u : \mathbb{D} \to D$ with $\operatorname{dist}(u(0), p_0) < \delta$ then*

$$\operatorname{dist}(u(0), u(z)) \leqslant C\sqrt{\operatorname{dist}(u(0), \partial D)} \tag{2.2}$$

if $|z| < r$.

Proof. — The proof is basically a repetition of the proof of Lemma 2.1 with the function λ replaced below by the functions $q \mapsto |q - p|^2$.

Fix $r < r_1 < 1$. Fix a neighborhood U of p_0 on which D is defined by a strictly plurisubharmonic function ρ, and diffeomorphic to an open set in $\mathbb{R}^{2n}$. On U we will also consider the Riemannian metric obtained by the identification of U_0 with this open set in R^{2n}. The distance from p to q will then be denoted somewhat abusively by $|p-q|$.

There is a neighborhood $V \subset U$ of p_0, and $\varepsilon > 0$ (small enough) such that for any $p \in V$, $q \mapsto \rho_p(q) = \rho(q) - \varepsilon|q-p|^2$ and the functions $q \mapsto |q-p|^2$ are J-plurisubharmonic on V. Also, for appropriate constants A and $B > 0$

$$-B|q-p| \leqslant \rho_p(q) \leqslant -A|q-p|^2.$$

By Lemma 2.1 (applied with ρ_{p_0} instead of ρ), if $u : \mathbb{D} \to D$ be a J-holomorphic map, and if $u(0)$ is close enough to p_0, $u(z) \in V$ whenever $|z| \leqslant r_1$. Take $p \in \partial D$ such that $\operatorname{dist}\big(u(0), \partial D\big) = \operatorname{dist}\big(u(0), p\big)$ (so $p \in V$). For $|z_1| \leqslant r$, by subharmonicity of $|u(z)-p|^2$ there is a constant C such that

$$|u(z)-p|^2 \leqslant C \int_0^{2\pi} |u(r_1 e^{i\theta}) - p|^2 \frac{d\theta}{2\pi}.$$

By the mean value property:

$$-\rho_p \circ u(0) \geqslant \int_0^{2\pi} -\rho_p \circ u(r_1 e^{i\theta}) \frac{d\theta}{2\pi} \geqslant A \int_0^{2\pi} |u(r_1 e^{i\theta}) - p|^2 \frac{d\theta}{2\pi} \geqslant \frac{A}{C} |u(z)-p|^2.$$

It gives

$$\big(\operatorname{dist}(u(z), u(0))\big)^2 \leqslant \big(\operatorname{dist}(u(0), p) + \operatorname{dist}(p, u(z))\big)^2 \leqslant \mathrm{C}\ \operatorname{dist}(u(0), p)$$

for some other constant C, as desired. □

2.b. A Schwarz-type Lemma-I and proof of Theorem 1.

Lemma 2.1 and 2.2 make possible to work locally if we are interested in the distance of a point p of strict pseudoconvexity in the boundary of a domain D, to points in D. Take a C^2-chart in a neighborhood U of p such that $p = 0$, $U \cap \overline{D} \setminus \{0\} \subset \{Z = (z_1, ..., z_n) \in \mathbb{C}^n : ||Z|| < 1, \operatorname{Re} z_1 < 0\}$. Using dilations we can assume that U contains the closed unit ball in $\mathbb{R}^{2n}$ and that J is as close as desired to J_{st} in C^1- norm. For any J-disc $u : \mathbb{D} \to D$ with $u(0)$ close to p, $u(z) \in U$ for $|z| < \frac{1}{2}$. So due to (1.5) and (2.2) we see that

$$|\nabla u(z)| \leqslant C\sqrt{\operatorname{dist}(u(0), \partial D)} \leqslant C\sqrt{-\operatorname{Re} u_1(0)} \quad \text{if} |z| \leqslant 1/4.$$

We state the next Lemma so that, starting from above, it can be used without (the obvious) rescaling (done in the proof).

LEMMA 2.3 (Schwarz-type Lemma-I). — *Let J be of class $\mathcal{C}^1$ on the closure of the unit ball in $\mathbb{R}^{2n}$. Assume that there exists $A > 0$ such that for every J-holomorphic disc $u : \mathbb{D}_{\frac{1}{2}} \to U^- := \{Z = (z_1, ..., z_n) \in \mathbb{C}^n : |Z| < 1, \operatorname{Re} z_1 < 0\}$ such that $u(0)$ is close enough to 0:*

$$(2.3) \qquad |\nabla u(z)| \leqslant A\sqrt{-\operatorname{Re} u_1(0)}$$

if $|z| \leqslant 1/4$. Then there exists $C = C(A) > 0$ such that for every J-holomorphic $u : \mathbb{D} \to U^-$ with $u(0)$ close enough to 0 one has

$$(2.4) \qquad |\nabla \operatorname{Re} u_1(0)| \leqslant C|\operatorname{Re} u_1(0)|.$$

Proof. — Of course, replacing the function $u(z)$ by the function $u(\frac{z}{4})$, we can assume that u is instead defined on $\mathbb{D}$ and that (2.3) holds for $|z| < 1$.

Formula (1.2) applied to the coordinate function $\lambda = x_1 = \operatorname{Re} z_1$ gives

$$(2.5) \qquad \Delta \operatorname{Re} u_1(z) = (dd^c_J x_1)_{u(z)}\Big(\frac{\partial u}{\partial x}(z), J\frac{\partial u}{\partial x}(z)\Big).$$

Using the estimate (2.3) one gets

$$(2.6) \qquad |\Delta \operatorname{Re} u_1(z)| \leqslant C_1|\operatorname{Re} u_1(0)|$$

if $|z| \leqslant 1$, for a constant C_1 depending only on J and A. The estimate (2.4) follows from (2.6) and from the condition $\operatorname{Re} u_1 < 0$.

So, set $f(z) = \operatorname{Re}\, u_1(z)$.

By $\Delta f\tilde{}$, we denote the function equal to Δf on $\mathbb{D}$, and 0 elsewhere. With C_1 as above, set

$$(2.7) \qquad g(z) = f(z) - [\frac{1}{2\pi}\ln|\zeta| * \Delta f\tilde{}\,](z) - C_1|f(0)|.$$

From (2.6), we get $|\frac{1}{2\pi}\ln|\zeta| * \Delta f\tilde{}\,| \leqslant C_1|f(0)|$. So $g < 0$ and $|g(0)| < (2C_1 + 1)|f(0)|$.

The classical Schwarz Lemma for negative harmonic functions gives: $|\nabla g(0)| \leqslant 2|g(0)|$. Therefore (2.7) yields;

$$|\nabla f(0)| \leqslant |\nabla g(0)| + \operatorname{Sup}|\Delta f\tilde{}\,| \leqslant 2|g(0)| + C_1|f(0)| \leqslant (5C_1 + 2)|f(0)|.$$

□

The localization Lemma 2.1 and 2.2, and the above allow us to apply Lemma 1.1 to the function $\chi = \operatorname{Re} z_1$ in order to immediately get our Theorem 1.

Note that there is still no global hypothesis on D. So D may very well not be hyperbolic i.e. the Kobayashi-Royden 'distance' may be only a pseudo-distance in the interior. An easy example is obtained by the blow up of a point. It has been pointed out at the end of 1.c that in an almost complex manifold, every point has a basis of hyperbolic neighborhoods. Lemmas 1.3 and 1.5 show that the small balls (with respect to any given Riemannian metric) are complete hyperbolic, and so are the intersections of small balls with strictly pseudoconvex domains.

2.c. Proof of Corollary 1.

To get Corollary 1, hyperbolicity of the domain, in case D does not contain complex lines, is the only point left.

It follows from the Brody reparametrization lemma. If there is a sequence of points $p_j \in D$ and unit (with respect to some metric) vector $v_j \in T_pX$ such that $|v_j|_k$ tends to 0, then there is a sequence of J-holomorphic maps $u_j : \mathbb{D}_{R_j} \to D$ such that $du_j(\frac{\partial}{\partial x}) = v_j$ and $R_j \to \infty$. After reparametrization we get a sequence $\tilde{u}_j : \mathbb{D}_{R_j} \to D$, still J-holomorphic and such that $\sup \left\{ |d\tilde{u}_j(z) \cdot \frac{R_j^2 - |z|^2}{R_j^2}| : z \in \mathbb{D}_{R_j} \right\} = |d\tilde{u}_j(0)| = 1$. A subsequence converges to a non-constant map from $\mathbb{C}$ into $\overline{D}$, which due to the strict pseudoconvexity of the boundary of D must be a map from $\mathbb{C}$ into $\mathbb{D}$.

Corollary 1 is proved.

3. Estimate of a Calderon-Zygmund Integral and Schwarz-type Lemma II.

Fix the following function $\phi(r) = r \ln \frac{1}{r}, r > 0$. We have already introduced the class $\mathcal{C}^{1,\phi}$. Let us introduce more generally sub-Lipschitzian classes $\mathcal{C}^{k,\phi}$ which are the spaces of functions or maps $f \in \mathcal{C}^k$ that locally satisfy $\frac{|f^{(k)}(z') - f^{(k)}(z)|}{\phi(|z'-z|)} \leqslant C < \infty$ (for some positive C). We also define the

Banach space $\mathcal{C}^{k,\phi}(\overline{\mathbb{D}})$ as the space of complex valued functions on $\mathbb{D}$ that satisfy:

$$\|f\|_{\mathcal{C}^k(\overline{\mathbb{D}})} + \sup_{|z-z|\leqslant\frac{1}{2}} \frac{|f^{(k)}(z') - f^{(k)}(z)|}{\phi(|z'-z|)} < \infty,$$

with the left hand side defining the norm $\|\cdot\|_{k,\phi}$, $k \geqslant 0$. $\mathcal{C}^{0,\phi}$ will be denoted simply by $\mathcal{C}^{\phi}$.

LEMMA 3.1. — *There exists a constant C such that for all complex-valued functions $f \in \mathcal{C}^{\phi}(\mathbb{R}^2)$, and $g \in \mathcal{C}^{1,\phi}(\mathbb{R}^2)$ such that:*

(1) $\|f\|_{\mathcal{C}^{\phi}(\overline{\mathbb{D}})} \leqslant 1$, $\|g\|_{C^{1,\phi}(\overline{\mathbb{D}})} \leqslant 1$;

(2) $\|g\|_{C^0(\overline{\mathbb{D}})} \leqslant \frac{1}{2}$;

(3) $|f(z)| \leqslant |g(z)|$ *and* $g(z) \neq 0$ *for all* $z \in \mathbb{D}$,

one has

$$\left| \int_{\overline{\mathbb{D}}} \frac{1}{z^2} \frac{f(z)}{g(z)} dz d\overline{z} \right| \leqslant C \cdot \log \frac{1}{|g(0)|}. \tag{3.1}$$

Proof. — We can assume $g(0) > 0$. Set $\delta = g(0)$ and split

$$\int_{|z|\leqslant 1} \frac{1}{z^2} \frac{f(z)}{g(z)}\, dxdy = \int_{|z|\leqslant\delta/4} \frac{1}{z^2} \frac{f(z)}{g(z)}\, dxdy + \int_{\delta/4\leqslant|z|\leqslant 1} \frac{1}{z^2} \frac{f(z)}{g(z)}\, dxdy,$$

where clearly

$$\int_{\delta/4\leqslant|z|\leqslant 1} \left|\frac{1}{z^2} \frac{f(z)}{g(z)}\right| dxdy \leqslant C \cdot \log \frac{1}{\delta},$$

since $\delta = g(0) \leqslant \frac{1}{2}$. In order to estimate $\left| \int_{|z|\leqslant\delta/4} \frac{1}{z^2} \frac{f(z)}{g(z)} dxdy \right|$, we shall use the following cancellations (with the first integral in the sense of principal value):

$$\int_{|z|\leqslant\delta/4} \frac{1}{z^2}\, dxdy = \int_{|z|\leqslant\delta/4} \frac{z}{z^2}\, dxdy = \int_{|z|\leqslant\delta/4} \frac{\overline{z}}{z^2}\, dxdy = 0. \tag{3.2}$$

Write $f(z) = f(0) + R_1(z)$, and $g(z) = g(0) + Az + B\overline{z} + Q_2(z)$. Due to the condition of the Lemma that $\|f\|_{\phi}, \|g\|_{1,\phi} \leqslant 1$ we have (with appropriate definitions of the norms)

$$|A|, |B| \leqslant 1,\ |R_1(z)| \leqslant |z| \ln \frac{1}{|z|},\ |Q_2(z)| \leqslant |z|^2 \ln \frac{1}{|z|}.$$

For $|z| \leqslant \frac{\delta}{4}$, we have that $g(z) = \delta\big(1 + \frac{Az+B\overline{z}}{\delta} + \frac{Q_2(z)}{\delta}\big)$ with $\left|\frac{Az+B\overline{z}}{\delta}\right| + \left|\frac{Q_2(z)}{\delta}\right| \leqslant \frac{1}{4} + \frac{1}{4} + \frac{\delta \ln \frac{1}{\delta}}{16} \leqslant \frac{2}{3}$.

For dealing with $\frac{1}{g}$, we shall simply use that if $|a| + |b| \leqslant \frac{2}{3}$, $\frac{1}{1+a+b} = 1 - a + r$ with $r \leqslant C(|a|^2 + |b|)$, for some universal constant C. We will apply it with $a = \frac{Az+B\overline{z}}{\delta}$, and $b = \frac{Q_2(z)}{\delta}$.

So, we can write that

$$\frac{f}{g} = \frac{1}{\delta}[f(0) + R_1(z)]\Big[1 - \Big(\frac{Az + B\overline{z}}{\delta}\Big) + S_2(z)\Big], \tag{3.3}$$

with $|S_2(z)| \leqslant C\big(\big|\frac{Q_2(z)}{\delta}\big| + \big|\frac{Az+B\overline{z}}{\delta}\big|^2\big) \leqslant C\big(\frac{|z|^2}{\delta}\ln\frac{1}{|z|} + \frac{|z|^2}{\delta^2}\big)$. Write (3.3) as

$$\frac{f}{g} = \frac{f(0)}{\delta} - f(0)\frac{Az + B\overline{z}}{\delta^2} + T(z),$$

where $|T(z)| = \frac{f(0)+R_1(z)}{\delta}S_2(z) + \frac{R_1(z)}{\delta}(1 - \frac{Az+B\overline{z}}{\delta})$.

(1) From $|f(0)| \leqslant |g(0| = \delta$, and $|R_1(z)| \leqslant |z|\ln\frac{1}{|z|}$, we get that $\big|\frac{f(0)+R_1(z)}{\delta} S_2(z)\big| \leqslant C(1 + |z|\ln\frac{1}{|z|})(\frac{|z|^2}{\delta^2} + \frac{|z|^2}{\delta}\ln\frac{1}{|z|}) \leqslant C(\frac{|z|^2}{\delta^2} + \frac{|z|^2}{\delta}\ln\frac{1}{|z|} + \frac{|z|^3}{\delta}\ln^2\frac{1}{|z|})$.

(2) $\big|R_1(z)(1 - \frac{Az+B\overline{z}}{\delta})\big| \leqslant 3|z|\ln\frac{1}{|z|}$, so $\frac{1}{\delta}\big|R_1(z)(1 - \frac{Az+B\overline{z}}{\delta})\big| \leqslant \frac{3|z|}{\delta}\ln\frac{1}{|z|}$.

Using cancellation properties of the Calderon-Zygmund operator and the claim we have proved one gets

$$\Big|\int_{|z|\leqslant\frac{\delta}{4}} \frac{1}{z^2}\frac{f(z)}{g(z)}dxdy\Big| = \Big|\int_{|z|\leqslant\frac{\delta}{4}} \frac{T(z)}{z^2}dxdy\Big| \leqslant C\int_{|z|\leqslant\delta/4} \frac{1}{\delta|z|}\ln\frac{1}{|z|}dxdy \leqslant C\ln\frac{1}{\delta}.$$

□

Lemma 3.2 (in which we focus on the behavior near the puncture) will be a generalization of the standard Schwarz Lemma which gives the following estimate for a holomorphic map g from the unit disc into the punctured unit disc:

$$|g'(0)| \leqslant 2|g(0)|\log\Big|\frac{1}{g(0)}\Big|.$$

LEMMA 3.2 (Schwarz-type Lemma-II). — *For A and $B > 0$ there exists $C > 0$ such that for every map g from the unit disc into the punctured disc $\mathbb{D}_{1/2} - \{0\}$ satisfying*

$$\|g\|_{1,\phi} \leqslant A \ \ \textit{and} \ \ \Big|\frac{\partial g(z)}{\partial\overline{z}}\Big| \leqslant B|g(z)|$$

for all $z \in D$, one has

$$|\nabla g(0)| \leqslant C|g(0)| \log \frac{1}{|g(0)|}.$$

Proof. — Extend $\frac{-\frac{\partial g}{\partial \bar{z}}}{g}$ to $\mathbb{C}$ (identified with $\mathbb{R}^2$), by setting it to be equal to 0 outside the unit disc. Set

$$w = \frac{1}{\pi z} * \frac{-\frac{\partial g}{\partial \bar{z}}}{g}.$$

We have $|w| \leqslant 2B$, and on the unit disc,

$$\frac{\partial w}{\partial \bar{z}} = \frac{-\frac{\partial g}{\partial \bar{z}}}{g}.$$

So $h = ge^w$ is holomorphic. The holomorphic function h never vanishes, and it takes values in the disc of radius $\frac{1}{2}e^{2B}$. So, the Schwarz Lemma for holomorphic function gives us a bound:

$$|\nabla h(0)| \leqslant 2|h(0)| \log \left|\frac{C_1}{h(0)}\right|,$$

for some constant C_1 depending only on B. We have

$$\nabla g(0) = e^{-w(0)}\nabla h(0) - h(0)e^{-w(0)}\nabla w(0).$$

Note that we have a bound for $|w(0)|$, so $h(0)$ and $g(0)$ are comparable (bounded ratios), and we can use the above estimate for $\nabla h(0)$. All what is left is to have a correct estimate of $\nabla w(0)$ (by a multiple of $\log \frac{1}{|g(0)|}$). The $\bar{z}$ derivative of w is simply $\frac{-\frac{\partial g}{\partial \bar{z}}}{g}$, which has modulus $\leqslant B$. The needed estimate of $\frac{\partial w}{\partial z}(0)$ which is given by the integral (defined as a principal value):

$$\int_{\mathbb{D}} \frac{1}{z^2} \frac{-\frac{\partial g}{\partial \bar{z}}}{g},$$

is given by Lemma 3.1, applied to the function $g_1 : z \mapsto g(\frac{1}{k}z)$ instead of g, with $k \leqslant \max\,(1, \|g\|_{C^{1,\phi}})$, and $f = c\frac{\partial g_1}{\partial \bar{z}}$, for an appropriate constant c. □

4. Distance to a complex hypersurface.

Proof of (2.A). — Due to hyperbolicity the question can be localized. Therefore we will suppose that X is a neighborhood of 0 in $\mathbb{R}^{2n}$ and that

$M=\{Z=(z_1,\dots,z_n)\in X : z_n=0\}$, where $z_j=x_j+iy_j$ are complex coordinates in $\mathbb{R}^{2n}\cong\mathbb{C}^n$, unrelated to the almost complex structure J. Since M is of class $\mathcal{C}^3$, in the new coordinates we can keep J of class $\mathcal{C}^2$. Without loss of generality we can assume that $J(0)=J_{st}$, the standard complex structure on $\mathbb{C}^n$. We shall write $Z=(z_1,\dots,z_n)=(Z';z_n)$ with $Z'=(z_1,\dots,z_{n-1})$. Because $z_n=0$ is a J-complex hypersurface we already have the almost complex structure J given along $z_n=0$ by

$$J(Z',0)=\begin{pmatrix} A(Z') & \alpha \\ 0 & \beta \end{pmatrix},$$

where α is a $(2n-2)\times 2$ matrix while β is a 2×2 matrix. After shrinking the neighborhood of 0 if needed, consider the $\mathcal{C}^2$ change of variables given by

$$\Phi(z_1,\dots,z_n)=(Z';0)+(0;x_n,0)+y_nJ(Z';0)\,\frac{\partial}{\partial x_n},$$

(using obvious identification of $\mathbb{R}^{2n}$ and $T\mathbb{R}^{2n}$). In the new coordinate system we will have

$$J(Z';0)=\begin{pmatrix} A(Z') & 0 \\ 0 & J_{st}^{(2)} \end{pmatrix} \tag{4.1}$$

where $J_{st}^{(2)}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

From now on we will work with coordinates in which (4.1) holds, with J now of class $\mathcal{C}^1$.

Let $u:\mathbb{D}\to(\mathbb{R}^{2n},J)$ be a J-holomorphic map. Recall the condition for J-holomorphicity in the form (1.7)

$$\frac{\partial u}{\partial\bar z}+Q_J(u)\frac{\partial u}{\partial z}=0 \tag{4.2}$$

with $Q_J(u)=[J(u)+J_{st}]^{-1}[J(u)-J_{st}]$, where $\frac{\partial}{\partial\bar z}$ and $\frac{\partial}{\partial z}$ refer to the standard complex structures on $\mathbb{D}$ and $\mathbb{C}^n$, i.e. in $\mathbb{R}^{2n}$ coordinates

$$\frac{\partial u}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}+J_{st}\frac{\partial u}{\partial y}\right),\quad \frac{\partial u}{\partial z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}-J_{st}\frac{\partial u}{\partial y}\right).$$

Due to (4.1)

$$Q_J(Z',0)=\begin{pmatrix} B(Z') & 0 \\ 0 & 0 \end{pmatrix}.$$

The Cauchy-Riemann equation (4.2) for J-holomorphic maps $u:\mathbb{D}\to(\mathbb{R}^{2n},J)$ gives for the component u_n:

$$\frac{\partial u_n}{\partial\bar z}=\sum_{k=1}^{n}b_k(u_1,\dots,u_n)\frac{\partial u_k}{\partial z}$$

with $b(u_1, \ldots, u_{n-1}, 0) \equiv 0$. For some constant C we therefore get an inequality

$$\left|\frac{\partial u_n}{\partial \bar{z}}\right| \leqslant C|u_n| \sum_{k=1}^{n} \left|\frac{\partial u_k}{\partial z}\right|. \tag{4.3}$$

On the right-hand side $\left|\frac{\partial u_k}{\partial z}\right|$ will be bounded by the $C^{1,\phi}$ regularity of J-holomorphic maps, the factor $|u_n|$ is the important feature.

Since the problem is purely local and since (after the changes of variables), the almost complex structure J is of class $\mathcal{C}^1$, we can use shrinking and rescaling in order to assume that $(X, J) = (\mathbb{D}^n, J)$ with $\mathbb{D}^n$ the unit polydisc in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, equipped with an almost complex structure J sufficiently close to the standard structure in $\mathcal{C}^1$ topology. Then, by Remark 3 in 1.d, J-holomorphic maps from $\mathbb{D}$ into $(\mathbb{D}^n, J)$ have uniformly bounded $C^{1,\phi}$ norm on $\mathbb{D}_{1/2}$.

From some (other) constant C we therefore get:

$$\left|\frac{\partial u_n}{\partial \bar{z}}\right| \leqslant C|u_n| \text{ on } \mathbb{D}_{1/2},$$

and

$$\|u_n\|_{\mathcal{C}^{1,\phi}(\mathbb{D}_{1/2})} \leqslant C.$$

Only now we are going to use the hypothesis that the map u should avoid the hyperplane $M = \{z_n = 0\}$, i.e. $u_n \neq 0$. We apply Lemma 3.2 to the restriction of $\frac{u_n}{2}$ to $\mathbb{D}_{1/2}$ (i.e. to $g(z) = \frac{1}{2}u_n(\frac{z}{2})$), to get for some (other) constant C:

$$|\nabla u_n(0)| \leqslant C|u(_n0)| \log \frac{1}{2|u_n(0)|}.$$

By Lemma 1.1, if $p = (p_1, \ldots, p_n) \in \mathbb{D}^n$ and $p_n \neq 0$, the Kobayashi distance $d_{\mathbb{D}^n \setminus M}(p, M)$ from p to M, relative to $\mathbb{D}^n \setminus M$ and J is infinite.

This achieves the proof of (2.A).

Proof of (2.B). — Since M is not J complex at p and is of class $\mathcal{C}^3$, we find a $\mathcal{C}^3$ change of coordinates so that: $p = 0$, $J(0) = J_{st}$, M coincides with $\mathbb{R}^2 \times \mathbb{C}^{n-2}$ on the ball of radius 6, and that this ball is included in D_1. Note that J stays of class $\mathcal{C}^2$. Consider the following family of J_{st} holomorphic discs defined for $|\zeta| \leqslant 1$, $|t| \leqslant 1$:

$$\phi_t(\zeta) = \left(\zeta, i\left(\frac{t}{8} + \zeta^2\right), 0, \ldots, 0\right).$$

This family has the following properties.

(a) $\phi_t(\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}}) \subset K$ for some compact K not intersecting M.

(b) $\phi_t(\bar{\mathbb{D}}) \cap M = \emptyset$ for $t > 0$;

(c) $\phi_0(\bar{\mathbb{D}}) \cap M = \{0\}$;

(d) $\phi_t(\bar{\mathbb{D}})$ intersects M transversally for $t < 0$.

Using dilations, replacing $J(z)$ by $J(\epsilon z)$, we can assume that J is as close as we need to J_{st}. Note that these dilations leave $M = \mathbb{R}^2 \times \mathbb{C}^{n-2}$ invariant, but that we do not rescale the family (ϕ_t). Applying Proposition 1.2 we then get a family of J holomorphic discs ψ_t with the following properties:

(a') $\psi_t(\{|z| = \frac{3}{4}\}) \subset K$ for some compact K not intersecting M.

(b') $\psi_t(\bar{\mathbb{D}}) \cap M = \emptyset$ for $t > \delta_0$, for some δ_0 close to 0.

(c') $\psi_{\delta_0}(\bar{\mathbb{D}}) \cap M \neq \emptyset$.

Clearly any point $p' = \psi_{\delta_0}(\zeta_0) \in M$ is at finite distance from points in the complement of M. Indeed, for $t > \delta_0$ the distance from $\psi_t(\frac{3}{4})$ to $\psi_t(\zeta_0)$ in $D \setminus M$ stays bounded as $t \to \delta_0$, but $\psi_t(\frac{3}{4})$ tends to a point in the complement of M. □

5. Hyperbolic distance to a real hypersurface.

5.a Construction of the family of discs.

From Proposition 1.1' we will deduce:

PROPOSITION 5.1. — *Let J be a $\mathcal{C}^{2,\alpha}$ almost complex structure defined near 0 in $\mathbb{R}^{2n}$. Let M be a $\mathcal{C}^2$ hypersurface in $(\mathbb{R}^{2n}, J)$ defined by $\rho = 0$, $0 \in M$, $\nabla\rho(0) \neq 0$. If there is a complex tangent vector $Y \in TM(0)$ such that $dd^c_J\rho(Y, J(0)Y) > 0$ then there is a family of J-holomorphic embedded discs $u_t : z \to \varphi_t(z) \in \mathbb{R}^{2n}$ ($z \in \mathbb{D}$) that depend continuously on $t \in [0,1]$ such that*

$$\begin{cases} \text{if } t > 0, \ u_t(\mathbb{D}) \subset \{\rho > 0\} \\ 0 = u_0(0) = u_0(\mathbb{D}) \cap M. \end{cases}$$

Proof. — Let $\psi : \mathbb{D}_R \to (\mathbb{C}^{2n}, J)$ be some J- holomorphic disk with $\psi(0) = 0$ and $\frac{\partial\psi}{\partial x}(0) = Y$. Then $\psi \in \mathcal{C}^{3,\alpha}$ and after a $\mathcal{C}^{3,\alpha}$ change of coordinates we can assume that $\psi : z \mapsto (\operatorname{Re} z, \operatorname{Im} z, 0, \dots, 0)$ $((z, 0, \dots, 0)$, when using complex coordinates) is a J-holomorphic disc, $Y = (1, 0, \dots, 0)$, $J(0)Y = (0, 1, 0, \dots, 0)$ and $J = J_{st}$ along $\mathbb{R}^2 \times \{0\}$. Moreover we can choose coordinates so that the vector $(0, 0, 1, 0, \dots, 0)$ $(\in \mathbb{R}^{2n})$ is a normal vector to M at 0 with $\nabla\rho(0)[(0, 0, 1, 0, \dots, 0)] = 1$. In the new coordinates J is still of class $\mathcal{C}^{2,\alpha}$.

Since ψ is a J-holomorphic disc $d^c(\rho \circ \psi) = \psi_* d^c_J \rho$. By differentiation $dd^c(\rho \circ \psi) = \psi_* dd^c_J \rho$. Therefore the Taylor expansion of $\rho \circ \psi$ at 0 is

$$\rho \circ \psi(z) = \operatorname{Re} az^2 + b|z|^2 + o(|z|^2),$$

with $b > 0$. Using complex $\mathbb{C}^n$ coordinates in the right-hand side, consider $\varphi(z) = (z, -az^2, 0, \dots, 0)$. It need not be a J-holomorphic map. But since $J = J_{st}$ along $\mathbb{C} \times \{0\}$ and since φ is holomorphic for the standard structure, we have $\overline{\partial}_J \varphi = O(|z|^2)$. By Proposition 1.1' there exists a germ of J-holomorphic disc u_0 such that, still using complex coordinates,

$$u_0(z) = (z, -az^2, 0, \dots, 0) + o(|z|^2).$$

It is immediate to check that

$$\rho \circ u_0(z) = b|z|^2 + o(|z|^2).$$

Then one has just to restrict u_0 to a small disc to be identified with $\mathbb{D}$. We then have $u_0(0) = 0$ but $\rho \circ u_0(z) > 0$ if $z \neq 0$.

The construction of the discs u_t is exactly similar replacing $M = \{\rho = 0\}$ by positive level sets of ρ. $\square$

5.b. Proof of Theorem 3.

After change of variables $p = 0$, we are in the situation of Proposition 5.1. Then, the sequence of points $u_{1/n}(0)$ tends to 0, their respective Kobayashi distance to $u_{1/n}(1/2)$ in the complement of M stay bounded, but the points $u_{1/n}(1/2)$ stay in a compact subset of the complement of M. This establishes Theorem 3.

Proof of a stronger version of (2.B). — We now assume that J is of class $\mathcal{C}^{2,\alpha}$, and we wish to show that in $(2.B)$ one can take $p = p'$. Let M be a real codimension 2 submanifold of class $\mathcal{C}^2$ in $(\mathbb{R}^{2n}, J)$, $0 \in M$. Assume that $Y \in TM(0)$ but $J(0)Y \notin TM(0)$. Consider a hypersurface $\widetilde{M}$ defined by $\rho = 0$ containing M such that $J(0)Y$ is tangent to $\widetilde{M}$ at 0. It is an easy exercise to show that by adding a quadratic term to ρ if needed (therefore not changing the tangent space) but keeping $\rho = 0$ on M we can get

$$dd^c_J\rho(Y, J(0)Y) \neq 0.$$

Observe that if g and h are functions defined near 0, with $h(\zeta) = O(|\zeta|^2)$ and $J(0) = J_{st}$ then $dd^c_J(g+h)(0) = dd^c_J g(0) + dd^c h(0)$. So it is enough to take $h = 0$ on M but $dd^c h(Y, J(0)Y) \neq 0$, $(d^c = d^c_{J_{st}})$.

This being done, 0 is at finite distance from points in the complement of $\widetilde{M}$, for the Kobayashi distance relative to the complement of $\widetilde{M}$, and thus a fortiori for the Kobayashi distance relative to the complement of M.

Remark. — Any $\mathcal{C}^2$ submanifold M of codimension > 2 is a submanifold of a submanifold of codimension 2, whose tangent space (at any chosen point of M) is not J-complex. So, as above, points of M are at finite distance from points in the complement of M. But a more direct argument can be given, simply based on the fact that, by simple count of dimensions, J-holomorphic discs generically miss M, and $\mathcal{C}^2$ regularity of M is not needed.

6. An example.

6.a. d^c_J and Levi foliation.

If M is a real hypersurface defined by $\rho = 0$, with $\nabla\rho \neq 0$, a tangent vector Y to X at a point $p \in M$ is a complex tangent vector to M if and only if $d\rho(Y) = d^c_J\rho(Y) = 0$.

The question of foliation of M by J-complex hypersurfaces is much the same as in the complex setting. The question is to know whether when Y and T are complex tangential vector fields to M, the Lie bracket $[Y,T]$ is also complex tangential i.e. if $d^c_J\rho[Y,T] = 0$. By the definition of d, $dd^c_J\rho(Y,T) = Yd^c_J\rho(T) - Td^c_J\rho(Y) - d^c_J\rho([Y,T]) = -d^c_J\rho([Y,T])$. Thus the Frobenius condition $d^c_J\rho([Y,T]) = 0$ is equivalent to $dd^c_J\rho(Y,T) = 0$.

Remark. — It is well known that for complex manifolds, if a hypersurface is not Levi flat, its complement is never complete hyperbolic.

There is a very simple and fundamental difference between the complex case and the almost complex case. Consider the two conditions on a hypersurface

(1) For every complex tangent vector field Y, $[Y, JY]$ is complex tangent (i.e. $dd^c\rho(Y, JY) = 0$).

(2) For every complex tangent vector fields Y and T, $[Y, T]$ is complex tangent.

For almost complex manifolds of real dimension > 4, (1) does not imply (2) (example below). But for complex manifolds it does. The easiest way to see it is by considering the Levi form as a hermitian form on the complex tangent space, or (just a different writing) by using $dd^c = 2i\partial\bar{\partial}$. Here we sketch a direct argument in terms of the Lie brackets. If (1) holds $[Y + JT, JY - T]$ is complex tangent. But $[Y + JT, JY - T] = [Y, JY] + [JT, J(JT)] - [Y, T] + [JT, JY]$. If $J = J_{st}$, $[JY, JT] = [Y, T]$ modulo a complex tangent vector field, as follows from the vanishing of the Nijenhuis tensor (for complex manifolds): $[Y, T] + J[JY, T] + J[Y, JT] - [JY, JT] = 0$. Hence $[Y, T]$ is complex tangent.

For almost complex manifolds of real dimension 4, (1) and (2) are obviously equivalent since if $Y \neq 0$, Y and JY generate the complex tangent space.

6.b.

In an almost complex manifold of real dimension 4, the complement of a hypersurface is (locally) complete hyperbolic if and only if that hypersurface is foliated by J-complex curves. We now present the following example on $\mathbb{R}^6$, (in which it is to be noticed that we do not need to restrict to bounded regions). We use coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$. On $\mathbb{R}^6$ we define the vector fields

$$L_1 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, \quad L_2 = \frac{\partial}{\partial y_2} - y_1 \frac{\partial}{\partial x_3}$$

and we define the almost complex structure J by setting:

$$\begin{aligned} J\Big(\tfrac{\partial}{\partial x_1}\Big) &= \tfrac{\partial}{\partial y_1} && \Big(\text{so } J\Big(\tfrac{\partial}{\partial y_1}\Big) = -\tfrac{\partial}{\partial x_1}\Big) \\ J(L_1) &= L_2 && \big(\text{so } J(L_2) = -L_1\big) \\ J\Big(\tfrac{\partial}{\partial x_3}\Big) &= \tfrac{\partial}{\partial y_3} && \Big(\text{so } J\Big(\tfrac{\partial}{\partial y_3}\Big) = -\tfrac{\partial}{\partial x_3}\Big). \end{aligned}$$

Note that the functions $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are J-holomorphic but that $z_3 = x_3 + iy_3$ is not. Finally we simply consider the hypersurface

$M = \{y_3 = 0\}$. At each point the tangent space to M is generated by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial y_1}$, L_1, L_2 and $\frac{\partial}{\partial x_3}$. The complex tangent space $(TM \cap JTM)$ is therefore spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial y_1}$, L_1, L_2. Since $\left[\frac{\partial}{\partial x_1}, L_1\right] = \frac{\partial}{\partial x_3}$ is not a complex tangent vector, M is not foliated by J-complex hypersurfaces. However:

PROPOSITION 6.1. — *For any point $p \in \mathbb{R}^6 \setminus M$ the Kobayashi pseudo-distance from p to M is infinite.*

It therefore follows from Theorem 2 that for any complex tangential vector field Y on M, $[Y, JY] \in TM \cap JTM$. It can easily be checked directly, but we won't need it.

Proof. — We will need

$$J\Big(\frac{\partial}{\partial x_2}\Big) = J\Big(L_1 - x_1\frac{\partial}{\partial x_3}\Big) = \frac{\partial}{\partial y_2} - y_1\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial y_3}$$
$$J\Big(\frac{\partial}{\partial y_2}\Big) = J\Big(L_2 + y_1\frac{\partial}{\partial x_3}\Big) = -\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + y_1\frac{\partial}{\partial y_3}.$$

We now write in detail the condition in order that a map $u : \mathbb{D} \to (\mathbb{R}^6, J)$ be J-holomorphic. In $\mathbb{D}$ we use coordinate $z = x + iy$, and we write $u = (X_1, Y_1, X_2, Y_2, X_3, Y_3)$.

The condition for J-holomorphicity is $\frac{\partial u}{\partial y} = J\frac{\partial u}{\partial x}$. It gives:

$$\left.\begin{aligned} \frac{\partial X_1}{\partial y} &= -\frac{\partial Y_1}{\partial x} \\ \frac{\partial Y_1}{\partial y} &= \frac{\partial X_1}{\partial x} \end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned} \frac{\partial X_2}{\partial y} &= -\frac{\partial Y_2}{\partial x} \\ \frac{\partial Y_2}{\partial y} &= \frac{\partial X_2}{\partial x} \end{aligned}\right\} \tag{6.2}$$

$$\left.\begin{aligned} \frac{\partial X_3}{\partial y} &= -\frac{\partial Y_3}{\partial x} - Y_1\frac{\partial X_2}{\partial x} - X_1\frac{\partial Y_2}{\partial x} \\ \frac{\partial Y_3}{\partial y} &= \frac{\partial X_3}{\partial x} - X_1\frac{\partial X_2}{\partial x} + Y_1\frac{\partial Y_2}{\partial x} \end{aligned}\right\} \tag{6.3}$$

(6.1) and (6.2) merely say that $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ are holomorphic functions of $z = x + iy$.

We now compute $2\frac{\partial}{\partial \overline{z}} Z_3$:

$$
\begin{aligned}
2\frac{\partial}{\partial \overline{z}}(Z_3) &= \Big(\frac{\partial X_3}{\partial x} - \frac{\partial Y_3}{\partial y}\Big) + i\Big(\frac{\partial X_3}{\partial y} + \frac{\partial Y_3}{\partial x}\Big) \\
&= \Big(X_1\frac{\partial X_2}{\partial x} - Y_1\frac{\partial Y_2}{\partial x}\Big) + i\Big(-Y_1\frac{\partial X_2}{\partial x} - X_1\frac{\partial Y_2}{\partial x}\Big) \\
&= (X_1 - iY_1)\frac{\partial X_2}{\partial x} - (Y_1 + iX_1)\frac{\partial Y_2}{\partial x} \\
&= \overline{Z}_1\frac{\partial X_2}{\partial x} - i\overline{Z}_1\frac{\partial Y_2}{\partial x} = \overline{Z}_1\frac{\partial \overline{Z}_2}{\partial x}.
\end{aligned}
$$

The conclusion of this computation is that $\frac{\partial}{\partial \overline{z}} Z_3$ is an antiholomorphic function of z. Hence Z_3 can be written as the sum of two functions $Z_3 = h_1 + \overline{h}_2$ with h_1 and h_2 both holomorphic.

Consequently if u is a J-holomorphic map from $\mathbb{D}$ into $(\mathbb{R}^6, J)$, Y_3 is a harmonic function of (x, y). If h is a positive harmonic function on $\mathbb{D}$ then we have $|\nabla h(0)| \leqslant 2h(0)$. By applying it to Y_3, or $-Y_3$, we see that if $u : \mathbb{D} \to \mathbb{R}^6 \setminus M$ is a J-holomorphic map $|\nabla Y_3(0)| \leqslant 2|Y_3(0)|$ (remember $M = \{y_3 = 0\}$). Since $\int_0^1 \frac{dt}{t} = +\infty$, Proposition 6.1 follows.

Appendices.

A1. Proof of Proposition 1.1 under the assumption of $\mathcal{C}^{k-1,\alpha}$ regularity of J.

Here we simply indicate how to adapt the proof given in section 1.e, by using the trick already used in [N-W]. We had to solve the equation $\frac{\partial u}{\partial \overline{z}} + Q_J(u)\frac{\partial u}{\partial z} = 0$. Let $\mathcal{B}$ be the closed unit ball in $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$.

In order to avoid any differentiation of J in the first step, we start by fixing a function $\varphi \in \mathcal{B}$. and we solve instead for u the linear equation:

$$
(a.1), \qquad \frac{\partial u}{\partial \overline{z}} + Q_J(\varphi)\frac{\partial u}{\partial z} = 0
$$

with prescribed x-derivatives up to order k at 0. It therefore leads to introducing, in replacement of Φ_J the operator $\Phi_J^{\#}$:

$$
\Phi_J^{\#}(u) = \Big(\mathbf{1} - T_{CG} Q_J(\varphi)\frac{\partial}{\partial z}\Big)u \ ,
$$

which is simply a linear operator. If J is close enough to J_{st} in $\mathcal{C}^{k-1,\alpha}$ topology (independently on φ) one can invert this operator. In order to follow the proof of section 1.e we need to be more precise, we should fix $R > 0$ such that for any (q, W) in the closed ball $\mathbf{B}_R$ of radius R in $\mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k$, $h_{q,W}$ is in the open unit ball of $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$. As in section 1.e, $h_{q,W}(z) = q + \sum_{l=1}^{k} \frac{1}{l!} z^l w_l$.

Then, exactly as in the proof in section 1.e, to each (p, V) in a neighborhood of 0, independent of $\varphi \in \mathcal{B}$, we associate $(q, W) \in B_R$ such that $u_{p,V}^{\varphi} = (\Phi^{\#})^{-1} h_{q,W}$ has the appropriate k-jet at 0. Since the function $\Phi_J^{\#}(u_{p,V}^{\varphi})$ is an ordinary holomorphic function, the function $u = u_{p,V}^{\varphi}$ satisfies (a.1). Set $\chi(\varphi) = u_{p,V}^{\varphi}$. If we had $\varphi = \chi(\varphi)$ then equation (a.1) would give $\frac{\partial u_{p,V}^{\varphi}}{\partial \bar{z}} + Q_J(u_{p,V}^{\varphi}) \frac{\partial u_{p,V}^{\varphi}}{\partial z} = 0$, so $u_{p,V}^{\varphi}$ would be J-holomorphic and would solve the problem. So we simply need to prove that χ has a fixed point in $\mathcal{B}$, provided that (p, V) is close enough to 0.

Note that if $J = J_{st}$, $Q_J = 0$, $\Phi_J^{\#}$ is the identity, $(p, V) = (q, W)$ and $\chi(\varphi) = u_{p,V}^{\varphi} = h_{p,V}$ (of course independent of φ).

Since T_{CG} gains one derivative we have the following. First, if J is close enough to J_{st} in $\mathcal{C}^{k-1,\alpha}$ topology, and (p, V) is close enough to 0, $\chi(\mathcal{B}) \subset \mathcal{B}$. Second, χ has a strong continuity property: it maps continuously $\mathcal{B}$ equipped with the $\mathcal{C}^{k-1,\alpha}$ topology into $\mathcal{B}$ equipped with the $\mathcal{C}^{k,\alpha}$ topology. At any rate, χ defines a continuous map from $\mathcal{B}$ equipped with the $\mathcal{C}^{k-1,\alpha}$ into itself. Since $\mathcal{B}$ is a convex compact set in $\mathcal{C}^{k-1,\alpha}(\overline{\mathbb{D}}, \mathbb{R}^{2n})$, The Schauder Fixed Point Theorem implies that χ has a fixed point, as desired.

A2. Deformation of (big) J-holomorphic discs.

The following Theorem shows the upper semi-continuity of the Kobayashi Royden pseudo-norm for structures of class $\mathcal{C}^{1,\alpha}$ $(\alpha > 0)$. In [I-P-R] a simple example is given to show that this upper semi-continuity fails to be true for structures of class $\mathcal{C}^{\frac{1}{2}}$. Theorem A1 was proved by Kruglikov [K] at least for structures that are smooth enough. Our proof clarifies the smoothness assumption and unlike [K] it does not require a careful and difficult reading of [N-W].

THEOREM A1. — *Let (X, J) be an almost complex manifold with J of Hölder class $\mathcal{C}^{1,\alpha}$ $(\alpha > 0)$. Let u be a J-holomorphic map from a neighborhood of $\overline{\mathbb{D}}$ into X. There exists a neighborhood V of $\big(u(0), \frac{\partial u}{\partial x}(0)\big)$*

in the tangent bundle TX such that for every $(q, Z) \in V$, there exists a J-holomorphic map $v : \mathbb{D} \to X$ with $v(0) = q$ and $\frac{\partial v}{\partial x}(0) = Z$.

Proof. — Let $r > 1$ be such that u is defined on $\mathbb{D}_r$. The map u is $\mathcal{C}^{2,\alpha}$. We can assume that u is an imbedding. Otherwise we add dimensions. We consider the map $z \mapsto \tilde{u}(z) = (z, u(z))$ from $\mathbb{D}_r$ into $\mathbb{R}^2 \times X$, equipped with the product almost complex structure $J_{st}^2 \times J$. After getting a map $\tilde{v}$ from $\mathbb{D}_R$ into $\mathbb{R}^2 \times X$, one simply takes the projection on the X factor.

By simple topological arguments, a neighborhood of $u(\mathbb{D}_r)$ can be identified with an open set in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, and one can find $(n-1)$ smooth vector fields $Y_1, \ldots, Y_{n-1}$ defined on a neighborhood of $u(\mathbb{D}_r)$ such that for every $z \in D_r$ the vectors $\frac{\partial u}{\partial x}(z), Y_1(z), \ldots, Y_{n-1}(z)$ are $J(u(z))$-linearly independent.

It allows to define the $\mathcal{C}^{2,\alpha}$ change of variables

$$(z_1, \ldots, z_n) \mapsto u(z_1) + \sum_{j=1}^{n-1} z_{j+1} Y_j(u(z_1)) \ ,$$

defined for $|z|_1 < r$ and $|z_j|$ small if $j \geqslant 2$.

In that change of variables the structure J is transformed into another almost complex structure still of class $\mathcal{C}^{1,\alpha}$, that coincides with the standard one along $\mathbb{C} \times \{0\} \subset \mathbb{C}^n$, The map u is replaced by the map $z \mapsto (z, 0, \ldots, 0)$. So Theorem A1 reduces to the following Lemma:

LEMMA A1. — *Let J be an almost complex structure on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, of Hölder class $\mathcal{C}^{1,\alpha}$, that coincides with the standard complex structure on $\mathbb{C} \times \{0\}$. Let U be a neighborhood of $\overline{\mathbb{D}} \times \{0\}$. For any $(q, t) \in \mathbb{C}^n \times \mathbb{C}^n$ close enough to $(0, 0)$, there exists a J-holomorphic map $v : \mathbb{D} \to U$ such that $v(0) = q$ and $\frac{\partial v}{\partial x}(0) = (1, 0, \ldots, 0) + t$.*

Proof of Lemma A1. — We will work in a neighborhood of $\overline{\mathbb{D}} \times \{0\}$, on which $J \simeq J_{st}$ and therefore on which as previously the condition for J-holomorphicity can be written:

$$\frac{\partial u}{\partial \overline{z}} + Q_J(u) \frac{\partial u}{\partial z} = 0,$$

and we have $Q_J(z, 0, \cdots, 0) = 0$.

Assume J of class $\mathcal{C}^{1,\alpha}$. Set $\mathcal{E}_0 = \{f : \overline{\mathbb{D}} \to \mathbb{C}^n; f \in \mathcal{C}^{1,\alpha},\ f(0) = 0, \nabla f(0) = 0\}$, $\mathcal{F}_0 = \{g : \overline{\mathbb{D}} \to \mathbb{C}^n; g \in \mathcal{C}^{\alpha}, g(0) = 0\}$, $F(z) = (z, 0, \cdots, 0)$.

Define the map Φ:

$$\begin{aligned}\Phi :&\mathcal{E}_0 \longrightarrow \mathcal{F}_0\\ &f \mapsto \frac{\partial(F+f)}{\partial \overline{z}} + Q_J(F+f)\frac{\partial(F+f)}{\partial z}\end{aligned}$$

Since F is J-holomorphic, $\Phi(0) = 0$. We want to show that the derivative of the map Φ at $f = 0$ is onto. Taking into account that $\frac{\partial F}{\partial \overline{z}} = 0$ and that $Q_J(F) = 0$, one gets:

$$\Phi(f) = \frac{\partial f}{\partial \overline{z}} + A_J(f)\Big(\frac{\partial F}{\partial z}\Big) + o(|f|),$$

where $A_J(f)$ is a $(2n \times 2n)$ matrix with entry that are $\mathcal{C}^\alpha$ in z and $\mathbb{R}$-linear in f.

Denote the derivative at 0 by $D\Phi_0$. With complex instead of real notations, we can write:

$$D\Phi_0(f) = \frac{\partial f}{\partial \overline{z}} + B_1(z)f(z) + B_2(z)\overline{f}(z),$$

where now B_1 and B_2 are $(n \times n)$ matrices with complex coefficients of class $\mathcal{C}^\alpha$.

The surjectivity of $D\Phi_0$ follows therefore from the following theorem:

THEOREM A2. — *If B_1 and B_2 are $(n \times n)$ complex matrices with coefficients in $\mathcal{C}^\alpha(\overline{\mathbb{D}})$, for every $g \in \mathcal{F}_0$, there exists $f \in \mathcal{E}_0$ such that*

$$(*) \qquad \frac{\partial f}{\partial \overline{z}} + B_1(z)f(z) + B_2(z)\overline{f}(z) = g(z).$$

We postpone the proof of Theorem A2, and we now finish the proof of Lemma A1.

We shall apply the following elementary result on maps from a Banach space E to a Banach space F. Let Γ be a $\mathcal{C}^1$ map from $B_E(p,R)$, the ball of radius R in E with center at a point p, into F. Assume that for some $C > 0$, for all $q \in B_E(p,R)$ the equation $D\Gamma_q(x) = y$ can be solved for all $y \in F$ with $\|x\|_E \leqslant C\|y\|_F$. (By the open mapping theorem, the existence of such a constant $C > 0$, for R small enough, is guaranteed as soon as $D\Gamma_0$ is surjective). Assume moreover that for all q and $q' \in B_E(p,R)$, $\|D\Gamma_q - D\Gamma_{q'}\|_{\rm op} \leqslant \frac{1}{2C}$. Then for every $y \in F$, with $\|y - \Gamma(p)\| < \frac{R}{2C}$, there exists $x \in B_E(p,R)$ such that $\Gamma(x) = y$. The proof is standard, by successive approximations.

For (q,t) in $\mathbb{C}^n \times \mathbb{C}^n$, close to $(0,0)$, as in the statement of the Lemma, and $T = (1,0,\cdots,0) + t$, set

$$F^{\#}(z) = q + xT + y(J(q)T) \qquad (z = x+iy).$$

Then, $F^{\#}$ close to F in $\mathcal{C}^{1,\alpha}$ topology with $\overline{\partial}_J F^{\#}(0) = 0$. Define

$$\Phi^{\#} : f \mapsto \frac{\partial(F^{\#} + f)}{\partial \overline{z}} + Q_J(F^{\#} + f)\frac{\partial(F^{\#} + f)}{\partial z}.$$

Since $\overline{\partial}_J F^{\#}(0) = 0$, $\Phi^{\#}$ maps $\mathcal{E}_0$ into $\mathcal{F}_0$. It is a small $\mathcal{C}^1$ perturbation of Φ and hence there exists f (close to 0) such that $\Phi^{\#}(f) = 0$. Define $v(z) = F^{\#}(z) + f(z)$, v has the same first jet at 0 as $F^{\#}$. So $\left(v(0), \frac{\partial v}{\partial x}(0)\right) = (q; (1, \cdots, 0) + t)$ as desired, and $\Phi^{\#}(f) = 0$ means that v is J-holomorphic. To complete the proof of Theorem A1, it only remains to prove Theorem A2.

Before starting the proof of Theorem A2, we need 2 Lemmas.

LEMMA A2. — *Let A_1 and A_2 be continuous matrices with complex coefficients defined on $\overline{\mathbb{D}} - \{0\}$ and bounded (no continuity assumed at 0). For every continuous map $g : \overline{\mathbb{D}} \to \mathbb{C}^n$ there exists a continuous map $f : \overline{D} \to \mathbb{C}^n$ such that (in the sense of distribution)*

$$\frac{\partial f}{\partial \overline{z}} + A_1(z)f + A_2(z)\overline{f} = g.$$

(From the proof: We can add $f(0) = 0$).

All computations are to be thought in $\mathbb{R}^{2n}$. On most lines however we keep complex ($\mathbb{C}^n$) notations that allow a simpler writing of $\overline{\partial}$ and of its solution by mean of the Cauchy Kernel.

Proof of Lemma A2. — Define the operator P:

$$P(f) = \frac{\partial f}{\partial \overline{z}} + A_1(z)f + A_2(z)\overline{f}$$

on $\mathcal{C}(\overline{\mathbb{D}})$ consider the operator:

$$g \mapsto P\Big(g * \frac{1}{\pi z} - g * \frac{1}{\pi z}(0)\Big) \text{ mapping}$$

$\mathcal{C}(\overline{\mathbb{D}})$ into itself. $Q = \mathbb{1} + K$. K a compact operator. Hence the set of $g \in \mathcal{C}(\overline{D})$ such that $g = P(f)$ for some f is a closed subspace ($\mathbb{R}$ subspace not $\mathbb{C}$ subspace) - of finite codimension.

We need to show that it is dense.

Let μ be a $\mathbb{R}^{2n}$-valued measure on $\overline{\mathbb{D}}$ such that for for any $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^{2n})$ such that $\varphi(0) = 0$ (the condition $\varphi(0) = 0$ to make sure that $P\varphi$ is continuous at 0):

$$\int P\varphi \cdot d\mu = 0.$$

We want to prove that $\mu = 0$.

By integration by parts $\int P\varphi \cdot d\mu = 0$ gives $P^*\mu = 0$ on $\mathbb{R}^2 - \{0\}$ where P^* is an operator of the type:

$$P^*\mu = \frac{\partial \mu}{\partial z} + C_1(z)\mu + C_2(z)\overline{\mu}.$$

The computation is elementary but needs to be done by separating real ($\mathbb{R}^n$ valued) and imaginary parts of μ.

Since μ has compact support we first show that $P^*\mu = 0$ on $\mathbb{R}^2 \setminus \{0\}$ implies $\mu = 0$ on $\mathbb{R}^2 \setminus \{0\}$. Note that $P^*\mu = 0$ on $\mathbb{R}^2 - \{0\}$ immediately shows that $\mu \in L^P_{\mathrm{loc}}$ for any $p < 2$ since $\mu = \frac{1}{\pi \overline{z}} * (-c_1(z)\mu + c_2(z)\overline{\mu} + \nu)$, with ν a distribution carried by $\{0\}$. And using the same formulas on sees that μ is a given by a continuous function. Then $P^*\mu = 0$ gives a simple pointwise inequality in $\mathbb{R}^2 - \{0\}$

$$\left|\frac{\partial \mu}{\partial z}\right| < C|\mu|.$$

By well know uniqueness results it follows that $\mu = 0$ on $\mathbb{R}^2 \setminus \{0\}$.

Consequently, if μ annihilates the continuous functions which can be written as $P(f)$ for $f \in \mathcal{C}^\alpha$, μ must be a $2n$-tuple of point masses at 0. Take $f = \overline{z}a$, $a \in \mathbb{C}^n$, one sees that necessarily $\mu = 0$. $\square$

From now on, the notations are the notations in the statement of Theorem A2 (for B_1, B_2 and equation $(*)$).

LEMMA A3. — *For every $\mathcal{C}^\alpha$ map $g : \overline{\mathbb{D}} \to \mathbb{C}^n$ with $g(0) = 0$, there exists a $\mathcal{C}^{1,\alpha}$ map f_0 defined near 0 in $\mathbb{C}$ such that $f_0(0) = 0$, $\nabla f_0(0) = 0$ and solving $(*)$ near 0.*

For r small enough, let $\mathbb{D}_r$ be the disc of radius r. Let $\mathcal{F}_0(r)$ be the space of $\mathcal{C}^\alpha$ maps g from $\overline{\mathbb{D}}$ into $\mathbb{C}^n$, with $g(0) = 0$. To any $g \in \mathcal{F}_0(r)$ associate a $\mathcal{C}^\alpha$ extension $\widetilde{g}$ with compact support in $\mathbb{D}_{2r}$ given by a linear operator and such that

$$\sup |\widetilde{g}| \leqslant 2 \sup_{\mathbb{D}_r} |g|.$$

Now, define P by

$$P(f) = \frac{\partial f}{\partial \overline{z}} + B_1(z)f + B_2(z)\overline{f}.$$

Consider the $\mathbb{R}$-linear map Θ from $\mathcal{F}_0(r)$ into itself defined by:

$$\Theta : g \mapsto P\Big(\tilde{g} * \frac{1}{\pi z} - (za + b)\Big),$$

with: a and $b \in \mathbb{C}^n$ are determined by the condition:

$$\begin{cases} \widetilde{g} * \frac{1}{\pi z}(0) = b \in \mathbb{C}^n \\ \\ \frac{\partial}{\partial z}\left(\widetilde{g} * \frac{1}{\pi z}\right)(0) = a \in \mathbb{C}^n. \end{cases}$$

Due to the choice of b, Θ indeed maps $\mathcal{F}_0(r)$ into itself. It is again a compact perturbation of the identity. But Θ is clearly one to one if r is small enough. Indeed, for r small enough, one has

$$\sup \left|B_1(z)\Big(\widetilde{g} * \frac{1}{\pi z} - (za+b)\Big)\right| + \left|B_2(z)\overline{\Big(\widetilde{g} * \frac{1}{\pi z} - (za+b)\Big)}\right| \leqslant \frac{1}{2} \sup_{\mathbb{D}_r} |g(z)|.$$

Therefore Θ is onto and so one can take $f_0 = \widetilde{g} * \frac{1}{\pi z} - (za+b)$. On $\mathbb{D}_r$ f_0 solves (*), and f_0 is of class $\mathcal{C}^{1,\alpha}$. We have $f_0 = 0$ and due to the choice of a, $\nabla f_0(0) = 0$, as desired.

Proof of Theorem A2. — Extend f_0 obtained in Lemma A3 , and find f by setting $f = f_0 + f_1$. We need to solve $\frac{\partial f_1}{\partial \overline{z}} + B_1(z) f_1(z) + B_2(z)\overline{f}_1(z) = g_1(z)$ with $g_1 \equiv 0$ near 0. It is better to do it on a neighborhood of $\overline{\mathbb{D}}$ after extending the data to a neighborhood of $\overline{\mathbb{D}}$. Applying Lemma A2 to a larger disc, take $f_1(z) = z^2 h(z)$, with h obtained by solving

$$\frac{\partial h}{\partial \overline{z}} + B_1(z)h(z) + \frac{\overline{z}^2}{z^2} B_2(z)\overline{h}(z) = \frac{g_1(z)}{z^2}.$$

Then $f = f_0 + f_1$ solves $(*)$ and satisfies

$$f(z) = o(|z|) \qquad (z \simeq 0).$$

Since f solves $(*)$, f is of class $\mathcal{C}^{1,\alpha}$, by elementary regularity of $\overline{\partial}$. □

A3. Other definition of the Kobayashi distance.

The usual definition of Kobayashi for the Kobayashi distance can also be given based on the following lemma due to Debalme [De-1]

PROPOSITION A1. — *Let J be an almost complex structure of class $C^{1,\alpha}$ in the neighborhood of the origin in $\mathbb{R}^{2n}$. Then for any pair p, q of points sufficiently close to the origin there exists a J-holomorphic disc passing through both of them.*

Proof. — We use again the notations of the proof of Proposition 1.1. But we replace the functions $h_{p,V}$ in the proof of Proposition 1.1 by the functions $h_{p,q}$ defined as follows.

Consider the mapping from Δ to $\mathbb{R}^{2n}$

$$h_{p,q}: \quad z \longmapsto p + 2z(q-p)$$

where $(p,q) \in (\mathbb{R}^{\nvDash\ltimes})^{\nvDash}$ and denote $u_{\varepsilon,p,q} = \Phi_\varepsilon^{-1} h_{p,q}$. We remark that

- $h_{p,q}$ being holomorphic, $\varepsilon u_{\varepsilon,p,q}$ is J-holomorphic.
- $u_{0,p,q} = h_{p,q}$. So it verifies $u_{0,p,q}(0) = p$ and $u_{0,p,q}(\frac{1}{2}) = q$.

Consider the mapping from $[0,\varepsilon] \times (\mathbb{R}^{\nvDash\ltimes})^{\nvDash}$ to $(\mathbb{R}^{\nvDash\ltimes})^{\nvDash}$

$$\Psi: \quad (\varepsilon, p, q) \longmapsto \Big(u_{\varepsilon,p,q}(0), u_{\varepsilon,p,q}\Big(\frac{1}{2}\Big)\Big)$$

Ψ is C^1 and from our last remark $\Psi(0,.,.) = Id_{(\mathbb{R}^{2n})^2}$. So by the implicit function theorem, if ε is sufficiently small , there exists U and U' neighborhoods of zero in $(\mathbb{R}^{2n})^2$ such that $\Psi(\varepsilon,.,.) : U \longrightarrow U'$ is a diffeomorphism. Let p_0 and q_0 two points sufficiently near of zero (i.e. $(\frac{p_0}{\varepsilon}, \frac{q_0}{\varepsilon}) \in U'$). There exists (p,q) such that $\varepsilon u_{\varepsilon,p,q}(0) = p_0$ and $\varepsilon u_{\varepsilon,p,q}(\frac{1}{2}) = q_0$. We have thus made $\varepsilon u_{\varepsilon,p,q}$ a J-holomorphic curve which is going through p_0 and q_0. □

Both definitions of the Kobayashi distance are equivalent, but we worked only with the first one and therefore we will not discuss this matter any further.

A4. Classical Properties of the Cauchy-Green Operator.

For a complex valued function g or a map g with values in a complex vector space, continuous on $\overline{\mathbb{D}}$, and $z \in \mathbb{C}$, we set:

$$T_{CG}(g)(z) = \Big(g * \frac{1}{\pi\zeta}\Big)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{g(\zeta)}{z-\zeta}\, dxdy(\zeta).$$

We have used the following classical properties of T_{CG}:

PROPOSITION A2. — *For g as above:*

(a) $\frac{\partial}{\partial \bar z}[T_{CG}(g)]$ *is the distribution defined by the function equal to g on $\mathbb{D}$, and to 0 on the complement of $\mathbb{D}$;*

(b) *If $g \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, $k \in \mathbb{N}$, $0 < \alpha < 1$, then the restriction of $T_{CG}g$ to $\overline{\mathbb{D}}$ belongs to $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$.*

Proof. — (a) does not need discussion. (b) follows from (a). The $\mathcal{C}^{k+1,\alpha}$ regularity of $T_{CG}(g)$ off the unit circle results from very basic properties of singular integrals (known as Schauder estimates). See e.g. [M] Chapter II, Theorem 1.6 . The regularity up to the boundary in each of the regions $\{|z| \leqslant 1\}$ and $\{|z| \geqslant 1\}$ is an instance of the so-called transmission property in the theory of partial differential equations (Definition 18.2.13 in [Hö]). Since it may be harder to find a satisfactory reference for $\mathcal{C}^{k+1,\alpha}$ regularity, we provide a justification.

Extend g to a function $g_1 \in \mathcal{C}_0^{k,\alpha}(\mathbb{R}^2)$ ([St] Chapter VI). Set $h = g_1 * \frac{1}{\pi z}$. Then $\frac{\partial h}{\partial \overline{z}} = g_1$, and $h \in \mathcal{C}^{k+1,\alpha}(\mathbb{R}^2)$.

Write $h(e^{i\theta}) = h^+(e^{i\theta}) + h^-(e^{i\theta})$, where h^+ is holomorphic on $\{|z| < 1\}$, and $\mathcal{C}^{k+1,\alpha}$ on $\{|z| \leqslant 1\}$; and h^- is holomorphic on $\{|z| > 1\}$, and $\mathcal{C}^{k+1,\alpha}$ on $\{|z| \geqslant 1\}$. Simply take h^+ and h^- to be the Cauchy transform of the function $e^{i\theta} \mapsto h(e^{i\theta})$. (Plemelj's formula, and again Schauder's estimates for singular integrals.)

Consider the function $\tilde{h}$ defined by:

$$\tilde{h} = h - h^+ \text{ on } \overline{D},$$

$$\tilde{h} = h^- \text{ for } |z| \geqslant 1 .$$

Then $\tilde{h}$ satisfies

$$\frac{\partial \tilde{h}}{\partial \overline{z}} = \frac{\partial T_{CG} g}{\partial \overline{z}},$$

on $\mathbb{R}^2$. There is no jump term on the unit circle. So $\tilde{h} - T_{CG}g \in \mathcal{C}^\infty(\mathbb{R}^2)$. Since $\tilde{h}$ is by construction $\mathcal{C}^{k+1,\alpha}$ smooth on $\overline{\mathbb{D}}$, so is $T_{CG}g$. □

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