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SOLUTIONS IN THE LARGE FOR MULTI-DIMENSIONAL, NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER [1]

by **Avron DOUGLIS**

Non-linear first order partial differential equations have solutions in the strict, classical sense only in naturally limited domains, differentiability necessarily failing wherever characteristic curves may happen to collide. Under appropriate hypotheses, strict solutions, however, have unique absolutely continuous extensions satisfying their differential equations at almost all points of a half-space. These absolutely continuous extensions are analogous to the solutions with « shocks » possessed by quasi-linear equations. To study them, it is thus reasonable to turn to shock theory, which suggests at least four approaches to initial value problems. The most traditional of these approaches applies only to solutions that are piecewise smooth and requires an exact accounting of the regions within which the emerging solution admits continuous differentiation. If the boundaries of these regions twist and tangle, this method, however, soon would fail. A second approach consists in moderating, or smoothing, the solution by introducing into the given differential equation a new expression representative of « artificial viscosity » prefixed by a parameter ε; the moderated solutions then must be shown to have a limit, as $\varepsilon \to 0$, that satisfies the original differential equation. Finite difference approximations form the basis of a third possible approach, and mixed difference-differential schemes the basis of a fourth. The second, third, and fourth

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approaches all have been successfully used to obtain solutions under quite general conditions in two dimensions; see [4], [6], [2]. To attack the multi-dimensional case, we have chosen to follow the fourth approach, adapting a difference-differential scheme used in [1] ([2]).

The non-linear equations here considered are required to fulfill a certain condition of definiteness; admitted solutions must conform to a functional restriction generalizing the « entropy condition » of two-dimensional theory. The solutions constructed are of the admitted type, and all such solutions are shown to depend uniquely and continuously upon their initial data.

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1. Statement of problem. Definition of solution.

The differential equations considered are those of the form

$$u_t + F(x, t, u, \text{grad } u) = 0, \tag{1.1}$$

where $u_t = \partial u/\partial t$, grad u denotes the vector with the components $u_{x^r} = \partial u/\partial x^r$, $r = 1, \ldots, n$, and $F(x, t, u, p)$ is a function continuous for $t \geqslant 0$, all values of u, all points

$$x = (x^1, \ldots, x^n),$$

and all vectors $p = (p_1, \ldots, p_n)$, and of class C^2 with respect to all arguments except possibly t. Partial derivatives of F presently to be referred to are written symbolically as follows :

$$F_r = \partial F/\partial p_r, \qquad F_u = \partial F/\partial u, \qquad F_{x^s} = \partial F/\partial x^s,$$
$$F_{rs} = \partial^2 F/\partial p_r \partial p_s, \qquad F_{r,u} = \partial^2 F/\partial p_r \partial u, \qquad F_{r,x^s} = \partial^2 F/\partial p_r \partial x^s$$
$$(r, s = 1, \ldots, n).$$

([2]) Since this paper was submitted, other work on multi-dimensional problems has appeared : S. N. Kruzhkov, The Cauchy Problem in the large for non-linear equations and for certain quasilinear systems of the first order with several variables, *Soviet Math.* 5 (1964), 493-496; E. D. Conway and E. Hopf, Hamilton's theory and generalized solutions of the Hamilton-Jacobi equation, *J. Math. Mech.* 13 (1964), 939-986; E. Hopf, Generalized solutions of non-linear equations of first order, as yet unpublished. Kruzhkov used artificial viscosity, Conway and Hopf an approach through the calculus of variations, and Hopf in his more recent, as yet unpublished, article, a new generalization of the method of envelopes.

The matrix (F_{rs}) always will be assumed to be positive in either of the following senses:

(i_0) Let $T > 0$. For $0 \leqslant t \leqslant T$, all x, u, p, and all (real) $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$(1.2_0) \qquad \sum_{r,s=1}^{n} F_{rs}(x, t, u, p)\lambda_r\lambda_s \geqslant 0.$$

(i_1) Let $T > 0$. A constant $a > 0$ exists such that, for $0 \leqslant t \leqslant T$, all x, u, p, and all $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$(1.2_1) \qquad \sum_{r,s=1}^{n} F_{rs}(x, t, u, p)\lambda_r\lambda_s \geqslant a|\lambda|^2,$$

where $|\lambda| = (\lambda_1^2 + \cdots + \lambda_n^2)^{1/2}$.

Further assumptions, when needed, will be stated below.

The solutions of 1.1 considered are functions u, defined (except in Theorem 3.3) in the half-space

$$S: \quad t \geqslant 0, \qquad -\infty < x^r < \infty, \qquad r = 1, \ldots, n,$$

and uniformly Lipschitz-continuous in any layer

$$S_T: \quad 0 \leqslant t \leqslant T, \qquad -\infty < x^r < \infty, \qquad r = 1, \ldots, n,$$

that satisfy 1.1 at almost all points of S. (In Theorem 3.3, a suitable layer S_T takes the place of S.)

Ordinarily, u also will be subjected to an initial condition of the form

$$(1.3) \qquad u(x, 0) = \varphi(x).$$

In all our problems, the solution u additionally will be required to have the property of being « semi-concave » described below.

A function $f(x)$ is called « semi-concave with constant k » if, for any point $x = (x_1, \ldots, x_n)$ and any vector

$$y = (y_1, \ldots, y_n)$$

of length denoted by $|y|$, the inequality

$$f(x + y) + f(x - y) - 2f(x) \leqslant k|y|^2$$

holds. Any solution $u(x, t)$ we consider will be required, for each positive t, to be semi-concave with a constant of the form $k = A/t + B$, where A and B are non-negative numbers.

A unique semi-concave solution of 1.1 and 1.3 under suitable hypotheses always will exist (Theorems 2.1 and 3.1 to 3.3) but uniqueness may fail when solutions are admitted that are not semi-concave. This failure of uniqueness is illustrated in the following example, adapted from one well-known in the theory of conservation laws (cf. Lax [3], p. 23), concerned with the initial value problem

$$\begin{aligned} u_t + (1/2)(u_x^2 + u_y^2) &= 0 \quad \text{for} \quad t > 0 \\ u(x, y, 0) &= 0 \quad \text{for} \quad x \leqslant 0 \\ &= x \quad \text{for} \quad x \geqslant 0. \end{aligned}$$

Two solutions are offered, namely,

$$\begin{aligned} u_1(x, y, t) &= 0 && \text{for} \quad x \leqslant 0 \\ &= x^2/2t && \text{for} \quad 0 \leqslant x \leqslant t \quad \text{if} \quad t > 0 \\ &= x - t/2 && \text{for} \quad x \geqslant t \end{aligned}$$

and

$$\begin{aligned} u_2(x, y, t) &= 0 && \text{for} \quad x \leqslant t/2 \\ &= x - t/2 && \text{for} \quad x \geqslant t/2, \end{aligned}$$

the latter, however, not semi-concave.

2. The unique, continuous determination of solutions by their initial data. Compactness of solutions.

For any solution $u(x, t)$ considered, let U and P be constants such that $|u| \leqslant U$ and $|u_{x^r}| \leqslant P$ in S_T for $r = 1, \ldots, n$. Than define

$$K = \sup (\Sigma_r(F_r(x, t, u, p))^2)^{1/2},$$

the supremum being taken for $|u| \leqslant U$, $|p_s| \leqslant P$ $(s = 1, \ldots, n)$, $0 \leqslant t \leqslant T$, all x, and $r = 1, \ldots, n$. K is a bound in S_T for the absolute magnitude of the characteristic slope, the vector $(dx_r/dt) = (F_r(x, t, u, \text{grad } u))$.

If $K' \geqslant K$, a semi-concave solution $u(x, t)$ of 1.1 will be seen (Theorem 2.1) to be determined within any cone of the form

$$D_{\xi, T}: \quad \begin{cases} (\Sigma_r(x^r - \xi^r)^2)^{1/2} \leqslant K'(T - t) \\ 0 \leqslant t \leqslant T \end{cases}$$

by the values the solution assumes on its base. Anticipating this, we call $D_{\xi, T}$, when $K' \geqslant K$, a « cone of determinacy » for the solution considered. A horizontal section with altitude t_0 of this cone is denoted by

$$B_{\xi, t_0, T}: \begin{cases} (\Sigma_r(x^r - \xi^r)^2)^{1/2} \leqslant K'(T - t_0) \\ t = t_0. \end{cases}$$

The denomination, « cone of determinacy », is justified under stated hypotheses by the following fact:

THEOREM 2.1 (UNIQUENESS). — *Supposing* F *to satisfy Hypothesis* i_0, *consider, in the sense described, two solutions in* S *of* 1.1, *each semi-concave with constant* $A/t + B$, $A \geqslant 0$, $B \geqslant 0$. *If the solutions coincide on the base of a common cone of determinacy (defined as above), they then coincide within the cone.*

COROLLARY. — *If the solutions are the same on the entire initial plane* $t = 0$, *they are the same in the half-space* S.

Remark 1. — The conclusion of the foregoing theorem is still valid if the solutions exist, and the assumptions hold, only in the cone discussed. Hence, if two *problems* satisfying the hypotheses of Theorem 2.1 are the same in a common cone of determinacy D, their solutions cannot be different within D.

Remark 2. — If k_r and K_r are constants such that

$$k_r \leqslant F_r(x, t, u, p) \leqslant K_r, \qquad r = 1, \ldots, n,$$

for $|u| \leqslant U$, $|p| \leqslant P$, $0 \leqslant t \leqslant T$, and all x, and if ξ_r are arbitrary, we shall call the pyramid

$$K_r(t - T) \leqslant x_r - \xi_r \leqslant k_r(t - T) \quad r = 1, \ldots, n,$$

a « pyramid of determinacy ». Theorems 2.1 and 2.2 and their various consequences are valid for pyramids of determinacy in place of cones of determinacy, a fact we shall later (Section 4) apply when $k_r \geqslant 0$.

Theorem 2.1 is a consequence of the next assertion implying that the solutions of initial value problems depend continuously on their initial data:

THEOREM 2.2 (CONTINUOUS DEPENDENCE). — *Suppose* F *to satisfy Hypothesis* i_0. *In the sense described, let* $u(x, t)$ *and* $v(x, t)$ *be solutions of* (1.1), *each satisfying a uniform Lipschitz* *condition with respect to x and each semi-concave with respect to x with a constant of the form $A/t + B$. Let $D = D_{\xi, T}$ be a common « cone of determinacy » for u and v, $B_t = B_{\xi, t, T}$ the horizontal plane section of D of altitude t. In D, let U denote a common upper bound for $|u|$ and $|v|$, let P be a common Lipschitz constant for u and v, and define*

$$f_0 = \max |F_u(x, t, w, p)| \quad \text{for} \quad (x, t) \text{ in } D, \ |w| \leqslant U, \ |p| \leqslant P.$$

Then for $0 \leqslant t \leqslant T$,

$$e^{-f_0 t} \max_{B_t} |u - v|$$

is a non-increasing function of t.

This theorem will be deduced from an integral inequality for $|u - v|$ in D involving some additional constants we wish now to define. The new constants again depend solely on bounds U and P, supposed to be known, such that

$$|u| \leqslant U, \qquad |v| \leqslant U, \qquad |\text{grad } u| \leqslant P, \qquad |\text{grad } v| \leqslant P \quad \text{in } D$$

and, like f_0, are maxima of functions of x, t, w, p over the domain

$$\mathfrak{D}: \quad \begin{cases} (x, t) & \text{in } D \\ |w| \leqslant U. \\ |p| \leqslant P. \end{cases}$$

Including f_0, which here is repeated, they are:

$$\begin{aligned} f_0 &= \max_{\mathfrak{D}} |F_u(x, t, u, p)| \\ f_1 &= \max_{\mathfrak{D}} |\Sigma_s F_{s, x^s}| + P \Sigma_s \max_{\mathfrak{D}} |F_{s, u}| \\ L &= \max_{\mathfrak{D}} \Sigma_r F_{rr}. \end{aligned}$$

The integral inequality referred to is stated in the following lemma.

LEMMA. — *Under the same assumptions as in Theorem 2.2, for every even integer q,*

$$t^{-LA} e^{-(f_0 q + f_1 + LB)t} \int_{B_t} (u - v)^q \, dx$$

is a non-increasing function of t in the interval $0 < t \leqslant T$.

Proof of Theorem 2.2 — For $0 < \varepsilon < t \leqslant T$, from the lemma we have

$$\int_{B_t} (u - v)^q \, dx \leqslant (t/\varepsilon)^{LA} e^{(f_0 q + f_1 + LB)(t - \varepsilon)} \int_{B_\varepsilon} (u - v)^q \, dx.$$

Raising both sides to the power $1/q$ and letting $q \to \infty$ gives us

$$\max_{B_t} |u - v| \leqslant e^{f_0(t-\varepsilon)} \max_{B_\varepsilon} |u - v|,$$

from which inequality and the continuity of u and v the theorem is clear.

The proof of the lemma depends on the following fact: Any uniformly Lipschitz-continuous, semi-concave function can be uniformly approximated by a sequence of infinitely differentiable functions also semi-concave with the same constant. The approximating functions additionally can be required to have the same absolute bound and the same Lipschitz constant as the original and to be such that their gradients tend to the gradient of the original function almost everywhere. They are obtainable, for instance, by convolving the original function with Friedrichs' « mollifying » kernels.

Proof of Lemma. — Since

$$u_t + F(x, t, u, \text{grad } u) = 0, \qquad v_t + F(x, t, v, \text{grad } v) = 0$$

almost everywhere in D, the difference $w = u - v$ satisfies the differential equation

$$w_t + Gw + \Sigma_s G_s w_{x^s} = 0$$

almost everywhere in D, where

$$\begin{aligned} G &= G(x, t, u, v) \\ &= \int_0^1 F_u(x,\ t,\ v + y(u - v),\ \text{grad } v + y(\text{grad } u - \text{grad } v))\, dy, \\ G_s &= G_s(x, t, u, v) \\ &= \int_0^1 F_s(x,\ t,\ v + y(u - v),\ \text{grad } v + y(\text{grad } u - \text{grad } v))\, dy. \end{aligned}$$

Hence, for

$$W = e^{-ct} w^q, \tag{2.1}$$

where q is an arbitrary even integer and c any positive constant, we have

$$W_t + \Sigma_s G_s W_{x^s} + (qG + c)W = 0 \tag{2.2}$$

at almost all points of D. Eventually, we shall specify

$$c = f_0 q + f_1. \tag{2.3}$$

Let $u'(x, t)$ and $v'(x, t)$ be semi-concave functions of class C_2

with respect to x in D with the same absolute bound U, Lipschitz constant P, and semi-concavity constant $A/t + B$, as u and v. We symbolize the derivatives of these functions as $u'_r = \partial u'/\partial x^r$, $u'_{rs} = \partial^2 u'/\partial x^r \partial x^s$, etc., and we set

$$G'_s = G_s(x, t, u', v'), \qquad s = 1, \ldots, n.$$

Equation (2.2) can be written as

$$W_t + \Sigma_s G'_s W_{x^s} = \Sigma_s (G'_s - G_s) W_{x^s} - (qG + c)W$$

and, hence, as

$$\text{(2.4)} \quad W_t + \Sigma_s (G'_s W)_{x^s} = \Sigma_s (G'_s - G_s) W_{x^s} - (qG + c)W + W\Sigma_s (G'_s)_{x^s}.$$

We shall prove the coefficient in the last term on the right to be of the form

$$\text{(2.5)} \qquad \Sigma_s (G'_s)_{x^s} = H + J,$$

where

$$\text{(2.6)} \qquad |H| \leqslant f_1, \qquad J \leqslant L(A/t + B).$$

To this end, we set

$$H = H_1 + \Sigma_s (J_s u'_s + K_s v'_s), \qquad J = \Sigma_{r,s} (J_{rs} u'_{rs} + K_{rs} v'_{rs}),$$

where

$$H_1 = \int_0^1 \Sigma_s F_{s,x^s}(x, t, v' + y(u' - v'), \text{grad } v' + y\,(\text{grad } u' - \text{grad } v'))\, dy,$$

$$J_s = \int_0^1 y F_{s,u}\, dy, \qquad K_s = \int_0^1 (1 - y) F_{s,u}\, dy,$$

$$J_{rs} = \int_0^1 y F_{rs}\, dy, \qquad K_{rs} = \int_0^1 (1 - y) F_{rs}\, dy,$$

the arguments of the derivative of F in each integrand being the same as in the first. The equality (2.5) and the inequality stipulated for H are obviously valid.

To prove the inequality respecting J, we note first that, since u' is twice continuously differentiable, being semi-concave with constant $k = k(t) = A/t + B$, it must satisfy the following differential condition: For any constants $a_1, \ldots, a_n$ such that $\Sigma_i a_i^2 = 1$,

$$(\Sigma_i a_i (\partial/\partial x^i))^2 u' \leqslant k.$$

Hence, for any real numbers $\lambda_1, \ldots, \lambda_n$,

$$(\Sigma_{r,s} u'_{rs} - k\delta_{rs})\lambda_r\lambda_s \leqslant 0.$$

This is to say that the matrix $U = (U_{rs}) = (u'_{rs} - k\delta_{rs})$ is non-positive.

Because of Hypothesis i_0, the matrices (J_{rs}) and (K_{rs}) are non-negative.

The two sums comprising J are of the same form, and it suffices to consider the first. Letting $M = (J_{rs})$ denote the matrix of coefficients in this sum, we rewrite the latter as

$$\Sigma_{r,s} J_{rs} u'_{rs} = \Sigma_{r,s} J_{rs}(u'_{rs} - k\delta_{rs}) + k\Sigma_r J_{rr} = \Sigma_{r,s} J_{rs} U_{rs} + k\Sigma_r J_{rr} = \operatorname{tr}(MU) + k\Sigma_r J_{rr},$$

tr (MU) signifying the trace of the matrix MU. Since the product of a symmetric non-positive and a symmetric non-negative matrix is non-positive, the stated inequality respecting J immediately follows. (The trace of a product of matrices is independent of their order. Hence, M and U being symmetric, each may be replaced in calculating tr(MU) by the diagonal matrix to which it is similar).

We now integrate the members of equation (2.4) over D_ε^τ, the frustum of D intercepted between B_ε and B_τ, with

$$0 < \varepsilon < \tau \leqslant T.$$

Designating the sloping part of the boundary of D_ε^τ by E_ε^τ and the element of area and unit outward normal at a point of E_ε^τ by dS and $(\nu_1, \ldots, \nu_n, \nu_t)$, respectively, we obtain

$$\int_{B_\tau} W\,dx - \int_{B_\varepsilon} W\,dx + \int_{E_\varepsilon^\tau} W(\nu_t + \Sigma_s G'_s\nu_s)\,dS = \int_{D_\varepsilon^\tau} \Sigma_s(G'_s - G_s)W_{x^s}\,dx\,dt + \int_{D_\varepsilon^\tau} (H - qG - c)W\,dx\,dt + \int_{D_\varepsilon^\tau} JW\,dx\,dt.$$

On E_ε^τ, however, $\nu_t + \Sigma_s G'_s\nu_s \geqslant 0$, and $H - qG - c \leqslant 0$ by 2.3; these facts and (2.6), W not being negative, show that

$$\int_{B_\tau} W\,dx \leqslant \int_{B_\varepsilon} W\,dx + L\int_{D_\varepsilon^\tau} (A/t + B)W\,dx\,dt + \int_{D_\varepsilon^\tau} \Sigma_s(G'_s - G_s)W_{x^s}\,dx\,dt.$$

This relation is to be applied with a succession of choices of u' and v' approximating u and v, respectively, such that grad u' and grad v' tend to grad u and grad v, respectively, at almost all points of D. (It is also required that $|u'|$, $|v'| \leqslant U$ and $|\text{grad } u'|$, $|\text{grad } v'| \leqslant P$.) The last integral on the right will tend to zero as the approximation is made more exact, and we conclude that

$$\int_{B_\tau} W\,dx \leqslant \int_{B_\varepsilon} W\,dx + L\int_{D_\varepsilon^\tau} (A/t + B)W\,dx\,dt.$$

This result, in terms of

$$Z(t) = \int_{B_t} W\,dx,$$

can be written as

$$Z(\tau) \leqslant Z(\varepsilon) + L\int_\varepsilon^\tau (A/t + B)\,Z(t)\,dt.$$

Z is majorized by $z(t) = (t/\varepsilon)^{LA}e^{LB(t-\varepsilon)}Z(\varepsilon)$, i.e.,

$$Z(t) \leqslant z(t) \qquad \text{for} \qquad \varepsilon \leqslant t \leqslant T, \tag{2.7}$$

since $z(t)$ is the solution of the integral relation

$$\zeta(\tau) = Z(\varepsilon) + L\int_\varepsilon^\tau (A/t + B)\zeta(t)\,dt.$$

Inequality (2.7), however, in view of the arbitrariness of t and ε, is equivalent to the property asserted in the lemma to be proved.

Later to lessen the restrictions under which solutions of 1.1 are proved to exist, we shall refer to the result below. In stating this result, it is convenient to designate by a symbol

$$\mathfrak{I}(U, P, A, B)$$

the set of functions $v(x, t)$ satisfying equation (1.1) at almost every point of S_T and subject to the inequalities

$$|v(x, t)| \leqslant U \quad (0 \leqslant t \leqslant T) \tag{2.8a}$$

$$|v(x, t) - v(x', t)| \leqslant P|x - x'| \quad (0 \leqslant t \leqslant T) \tag{2.8b}$$

(2.8c)

$$v(x+y, t) + v(x-y, t) - 2v(x, t) \leqslant (A/t+B)|y|^2 \quad (0 < t \leqslant T),$$

in which U, P, A, and B are uniform constants.

THEOREM 2.3 (THE COMPACTNESS OF SOLUTIONS). — $\mathfrak{J}(U, P, A, B)$ *is compact in the topology of locally uniform convergence. This is to say that any infinite subset of* $\mathfrak{J}(U, P, A, B)$ *contains a sequence of functions converging uniformly on any compact subset of* S_T *to a limit that again is a member of* $\mathfrak{J}(U, P, A, B)$.

In proving this theorem, we need a property of monotonic functions stated as follows:

LEMMA. — *Let* $f_k(\lambda)$, $k = 1, 2, \ldots$, *be a sequence of Lipschitz-continuous functions defined on the real axis with uniformly bounded, monotonic derivatives existing on a subset* Λ_0 *of the real axis whose complement has measure zero. Suppose the limit*

$$f(\lambda) = \lim_{k \to \infty} f_k(\lambda)$$

exists uniformly in any compact interval. Then the first derivative $f'(\lambda)$ *exists at almost every point of the* λ*-axis,* $f'(\lambda)$ *is monotonic, and, at every point* λ_0 *of* Λ_0 *at which* f' *exists,*

$$f'(\lambda_0) = \lim_{k \to \infty} f'_k(\lambda_0).$$

Proof. — By Helly's theorem, a subsequence $(f_{k'})$ of the sequence (f_k) can be selected such that the $f_{k'}$ are monotonic in the same sense and the limit

$$g(\lambda) = \lim_{k' \to \infty} f'_{k'}(\lambda)$$

exists for $\lambda \varepsilon \Lambda_0$. Let us extend g, which is monotonic on Λ_0, monotonically to the entire real axis. Now consider the identities

$$f_{k'}(\lambda) = f_{k'}(\mu) + \int_\mu^\lambda f'_{k'}(\sigma)\, d\sigma.$$

Letting $k' \to \infty$ proves

$$f(\lambda) = f(\mu) + \int_\mu^\lambda g(\sigma)\, d\sigma.$$

It follows that f' exists and equals g at each value λ at which g is continuous and, since the only discontinuities of g are jumps, that f' fails to exist at the points at which g is not continuous.

Hence,

$$\lim_{k'} f'_{k'}(\lambda) = f'(\lambda)$$

at each point of Λ_0 at which f' exists. Therefore,

$$\lim_k f'_k(\lambda) = f'(\lambda)$$

at each point of Λ_0 at which f' exists, as asserted.

We call a function $h(\lambda)$ *semi-decreasing with constant* C if, for $\lambda \neq \mu$,

$$\frac{h(\lambda) - h(\mu)}{\lambda - \mu} \leqslant C.$$

If the functions of a sequence, $h_k(\lambda)$, $k = 1, 2, \ldots$, are semi-decreasing with the same constant C, we say they are *uniformly semi-decreasing.* « Semi-increasing », « uniformly semi-increasing », « semi-monotonic », etc..., are analogously defined. If $h(\lambda)$ is, for instance, semi-decreasing with constant C, $h(\lambda) - C\lambda$ is monotonic decreasing. Hence, the previous lemma applies if the f_k are not monotonic, but uniformly semi-increasing or semi-decreasing.

Proof of Theorem 2.3. — In any infinite subset of

$$\mathfrak{J}(U, P, A, B),$$

Arzela's theorem assures us of the existence of a sequence $u_j(x, t)$, $j = 1, 2, \ldots$, converging uniformly on any compact subset of S_T. The limit of the sequence, which we denote by $u(x, t)$, automatically satisfies the three inequalities (2.8), and it follows from (2.8*c*), in particular, that $u_{x^r}(x, t)$, for $t > 0$, is semi-decreasing with respect to x^r. We prove this in the case $r = 1$. Set $x = (x^1, \xi_1)$, where $\xi_1 = (x^2, \ldots, x^n)$, and define

$$w_h(x^1) \equiv w_h(x^1, \xi_1, t) \equiv h^{-1}(u(x^1 + h, \xi_1, t) - u(x^1, \xi_1, t))$$

for $h \neq 0$. Condition (2.8*c*) implies that

$$h^{-1}(w_h(x^1 + h) - w_h(x^1)) \leqslant C,$$

where $C = A/t + B$, and thus that, if $(\alpha - \beta)/h$ is an integer,

$$\frac{w_h(\alpha) - w_h(\beta)}{\alpha - \beta} \leqslant C.$$

With t and ξ_1 fixed, choose α and β as such values that the derivative $u_{x^1}(x^1, \xi_1, t)$ exists for $x^1 = \alpha$ and $x^1 = \beta$. Then let h

in the previous inequality tend to zero. The result, namely

$$\frac{u_{x^1}(\alpha, \xi_1, t) - u_{x^1}(\beta, \xi_1, t)}{\alpha - \beta} \leqslant C,$$

states that u_{x^1} is semi-decreasing, as asserted. For the same reasons, u_{x^r} and u_{j,x^r}, $j = 1, 2, \ldots$, also are semi-decreasing with respect to x^r, $r = 1, \ldots, n$, with uniform constant $A/t + B$.

Hence, by the foregoing lemma,

$$\lim_j u_{j,x^r} = u_{x^r}$$

at each point at which u_{x^r} and the u_{j,x^r} exist.

Let S′ denote the set of points x_0 such that the derivatives $u_{x^r}(x_0, t)$ and $u_{j,x^r}(x_0, t)$ for $j = 1, 2, \ldots$ exist, and the equations

$$u_{j,t} + F(x_0, t, u_j(x_0, t), \operatorname{grad} u_j(x_0, t)) = 0$$

are satisfied, for almost every value of t in the interval (0, T). The complement of S′ in x-space is a set of measure zero. For x_0 in S′, $0 \leqslant t \leqslant T$, we have by integration

$$u_j(x_0, t) = \varphi(x_0) - \int_0^t F(x_0, t', u_j(x_0, t'), \operatorname{grad} u_j(x_0, t'))\, dt'.$$

Letting $j \to \infty$ gives us

$$u(x_0, t) = \varphi(x_0) - \int_0^t F(x_0, t', u(x_0, t'), \operatorname{grad} u(x_0, t'))\, dt'$$

and, consequently,

$$\int_0^t (u_t + F)\, dt' = 0.$$

Because u is Lipschitz-continuous, u_t and grad u are already known to be measurable in S_T. Hence, we now readily deduce that 1.1 holds at almost all points of S_T and, thus, that u belongs to $\mathfrak{I}(U, P, A, B)$, as asserted.

3. Main results on the existence of solutions.

When one attempts to construct the solution of 1.1, 1.3, the behavior of $F(x, t, u, p)$, $F_u(x, t, u, p)$, and the $F_{x^s}(x, t, u, p)$ for large values of $|u|$ and $|p|$ becomes critical. Three alternative sets of assumptions concerning this type of behavior are given here under which the solution of an arbitrary initial value problem is proved to exist. The simplest of the alternative sets of assumptions is as follows:

$(ii)_1$ $F \equiv F(t, p)$ is independent of x and u.

This is a special case of the second alternative, which is an n-dimensional generalization of a condition first suggested by Vvedenskaya [7] in the case $n = 1$:

(ii_2) Let T be any positive number. For $\rho \geqslant 0$, a positive, non-decreasing function $V(\rho)$, subject to the condition

$$(3.1) \qquad \int_c^\infty \frac{d\rho}{V(\rho)} = \infty \qquad \text{for} \qquad c \geqslant 0,$$

exists such that

$$|F_{x^s}(x, t, u, p) + p_s F_u(x, t, u, p)| \leqslant V(\rho), \qquad s = 1, \ldots, n,$$

in the domain

$$(3.2) \quad 0 \leqslant t \leqslant T, \qquad |p| \leqslant \rho, \; -\infty < u < \infty, \qquad |x| < \infty.$$

Furthermore,

$$K(\rho) \equiv \sup\, (\Sigma_r(F_r(x, t, u, p))^2)^{1/2} < \infty,$$

the supremum being that for the domain (3.2).

It is possible to relax the stringent assumptions above concerning the behavior, for large values of $|u|$, of F and its partial derivatives. This is done, for instance, in the following hypothesis, which, however, is rather extreme and leads to a solution (Theorem 3.3) not necessarily defined in the half-space S, but possibly merely in a suitable layer S_T. (Assumptions of intermediate strength between ii_2 and ii_3 can be given under which the solutions would be defined in all S.) Like its predecessors, the new hypothesis still pertains to solutions permitted to have discontinuous first derivatives.

ii_3) *a*) When t, u, p are held to any finite domains, the functions

$$F,\ F_u,\ F_{x^r},\ F_r, \qquad r = 1, \ldots, n,$$

are bounded uniformly with respect to x.

b) The initial data are uniformly bounded:

$$|\varphi(x)| \leqslant \varphi_0,$$

where φ_0 is a constant.

The theorems below, under varying hypotheses, affirm the existence of a solution of (1.1), (1.3) in S or a layer S_T, the first two theorems also containing estimates for the solution of its constants of Lipschitz continuity and semi-concavity, denoted in such a layer by M(T) and k(T), respectively. The initial values φ of the solution always are taken to be Lipschitz continuous with Lipschitz constant M_0.

THEOREM 3.1. — *Under hypotheses i_1 and ii_1, a unique solution of* 1.1, 1.3 *exists in* S *with* $M(t) = M_0$ *and* $k(t) = 2/at$.

THEOREM 3.2. — *Under hypotheses i_1 and ii_2, a unique solution of* 1.1, 1.3 *exists in* S *with* $M(t) \leqslant P$, *where* P *is determined by the relation*

$$\int_{M_0}^{P} d\nu/V(\nu) = t.$$

Furthermore, $k(t) \leqslant 4/at + B$, *where, in any interval* $0 \leqslant t \leqslant T$, B *can be regarded as a constant depending on* T.

THEOREM 3.3. — *Under hypotheses i_1 and ii_3, a unique solution of* (1.1), (1.3) *exists in a suitable layer* S_T.

4. Preliminaries in the existence problem.

An initial value problem is solvable in the half-space S if solvable in any layer S_T, $T > 0$; it is solvable in S_T if solvable in cones of determinacy within S_T with arbitrary axis. The values of the solution within a given cone of determinacy, moreover, are unresponsive to alterations made outside the cone in F and φ (Theorem 2.1), while such alterations may appreciably simplify the problem. Alterations in F and φ suitable for our difference-differential scheme below are the main subject of this section. (These alterations probably could be dispensed with in an existence proof based on an explicit finite difference scheme, but other details of the proof then would be troublesome.)

Our ability to modify the problem at pleasure in the intended way depends on our foreknowledge of cones of determinacy of arbitrary axis $x = \xi$ and arbitrary altitude T and thus on our foreknowledge of K (Section 2) within S_T. K depends on an absolute bound P for the gradient of the eventual solution and, in the case of the third hypothesis, also on an absolute bound U for the solution itself. We now give, for arbitrary T, determinations of P that will prove to be valid in S_T under Hypothesis ii_1 or ii_2. Under Hypothesis ii_3, we give, for arbitrary, sufficiently large P and U, such T that the P and U selected hold as bounds in S_T. The correctness of all these determinations will be proved in Section 6. Then it will be clear that the cones of determinacy are indeed those that correspond in slope to the K calculated from these determinations of P or of P and U. We shall anticipate this fact in later considerations of this section.

Under Hypothesis ii_1: Select any $T > 0$. Take $P = M_0$.

Under Hypothesis ii_2: Fixing $T > 0$ arbitrarily, determine P such that

$$\int_{M_0}^{P} d\rho / V(\rho) = T.$$

Under Hypothesis ii_3: Select $U > \varphi_0$, $P^* > M_0$, and $T_0 > 0$. Let $V(\rho)$ be a positive, non-decreasing function of its non-negative variable ρ such that

$$|F_{x^r}(x, t, u, p) + p_r F_u(x, t, u, p)| \leqslant V(\rho), \qquad r = 1, \ldots, n,$$

when $0 \leqslant t \leqslant T_0$, $|u| \leqslant U$, and $|p| \leqslant \rho$. With $U' > U$, $P' > P^*$, let Q_0 be a constant such that

$$|F(x, t, u, p)| \leqslant Q_0$$

for all x, $0 \leqslant t \leqslant T_0$, $|u| \leqslant U'$, $|p| \leqslant P'$. With $T_1 = \int_{M_0}^{P^*} d\rho/\nu(\rho)$, define $T = \min(T_0, T_1, (U - \varphi_0)/Q_0)$ and then determine P by the condition

$$\int_{M_0}^{P} d\rho / V(\rho) = T.$$

As a further preliminary, choosing $U_0 > U$ and $P_0 > P$, we normalize F by requiring

$$(4.1) \qquad F_r(x, t, u, p) > 0$$

for all x, $0 \leqslant t \leqslant T$, $|p| \leqslant P_0$, $|u| \leqslant U_0$. (Now and later, the restriction $|u| \leqslant U_0$ is to be disregarded when ii_1 or ii_2 is valid.) To accomplish this normalization, let K_0 denote the value of K (Section 2) corresponding to the indicated T and to the values U_0 and P_0 in place of U and P, respectively, and make the linear change of variables

$$x'^r = x^r + K_0 t, \qquad r = 1, \ldots, n,$$
$$t' = t.$$

Equation (1.1) then becomes

$$u_{t'} + F_0(x, t, u, \text{grad}' u) = 0,$$

where grad$'$ u denotes the vector with the components $u_{x^{r'}}$,

$$r = 1, \ldots, n, \qquad \text{and} \qquad F_0(x, t, u, p) \equiv F(x, t, u, p) + K_0 \Sigma_r p_r.$$

Condition (4.1) is verified for F_0 in place of F, and we now simply drop primes.

Making the hypotheses of Theorem 2.1, and working in the original variables, at this point we select a cone of determinacy (or, rather, an eventual cone of determinacy) $D_{\xi, T}$ in which we desire to know the solution of the given problem. Provided $D_{\xi, T}$ stays a cone of determinacy, changing F and φ outside this cone will not change the values of the solution inside it (this is by Remark 1 after Theorem 2.1), and the same considerations obviously apply to the image D of $D_{\xi, T}$ under the linear change of variables above. Thus justified, adjustments outside D now are made with the object of having $\varphi(x) = \varphi' =$ constant and $F(x, t, \varphi', p) =$ function of p only for $x^r \leqslant \alpha^r$, $r = 1, \ldots, n$, the x^r here being the variables in which 4.1 holds and the α^r suitable constants. The α^r next are made zero by a translation. The original conditions (1.1) and (1.3) herewith are reduced to conditions of the same type for which, however,

(4.2) $F(x, t, \varphi', p) =$ function only of p when any
$$x^r \leqslant 0 \ (r = 1, \ldots, n), \qquad 0 \leqslant t \leqslant T, \qquad |p| \leqslant P_0,$$
and

(4.3) $\varphi(x) = \varphi'$ when any $x^r \leqslant 0, \qquad r = 1, \ldots, n,$

where φ' is a constant.

If the new problem has a solution $u(x, t)$ in S_T, then by Remark 2 after Theorem 2.1

$$(4.4) \quad u(x, t) = \varphi' \quad \text{when any} \quad x_r \leqslant 0, \; r = 1, \ldots, n,$$

and

$$0 \leqslant t \leqslant T.$$

This can be regarded as a boundary condition added to (1.1) and (1.3).

5. A difference-differential scheme for a problem of modified type. Some convergence theorems stated.

The foregoing considerations show that, in proving Theorems 3.1 to 3.3 on the existence of solutions, it suffices to consider problems with the special features described in Section 4. Hence, only such problems from now on will be discussed. Our fundamental result as to the existence of solutions, which implies Theorems 3.1 to 3.3, as noted, is as follows:

THEOREM 5.1 (EXISTENCE). — *Let $\varphi(x)$ be Lipschitz-continuous with Lipschitz constant M_0 and be equal to a constant outside the first « octant »:*

$$(5.1) \quad \varphi(x) = \varphi' \quad \text{when any} \quad x^r \leqslant 0, \quad r = 1, \ldots, n \; (\varphi' = \text{constant}).$$

Let F satisfy Hypothesis i_1. Assume one of the three hypotheses ii_1 to ii_3 to hold and constants T, P and U to be selected or determined as indicated in Section 3. With $P_0 > P$, $U_0 > U$, F also is required to satisfy the additional conditions

$$(5.2) \quad F(x, t, \varphi', p) = \text{function only of } p \text{ when any} \quad x^r \leqslant 0 \; (r = 1, \ldots, n), \quad 0 \leqslant t \leqslant T, \quad |p| \leqslant P_0,$$

and

$$(5.3) \qquad F_r(x, t, u, p) > 0$$

for $0 \leqslant t \leqslant T$, $|p| \leqslant P_0$, and, in the case of Hypothesis ii_3, $|u| \leqslant U_0$. Then a Lipschitz continuous function $u(x, t)$, semi-concave for $0 < t \leqslant T$, exists satisfying the partial differential equation 1.1 at almost all points of S_T and satisfying the initial condition (1.3) and the boundary condition (4.4). The Lipschitz *constant for u in S_T does not exceed* P. *A number* B *independent of φ exists such that $4/at + B$ is a constant of semi-concavity of $u(x, t)$ on any horizontal plane in the layer $0 < T \leqslant T$. Under Hypothesis ii_1, $2/at$ is a constant of semi-concavity on any horizontal plane in the half-space $t > 0$.*

Our proof of Theorem 5.1, which is presented at the end of the section, is based on a difference-differential scheme we now describe.

With any multi-index $i = (i_1, \ldots, i_n)$ and scalar h, let

$$hi = (hi_1, \ldots, hi_n); \tag{5.4}$$

for $h > 0$, let E^h denote the lattice consisting of all the points (5.4) with $i_j = 0, \pm 1, \pm 2, \ldots, j = 1, \ldots, n$. The functions $f(x)$ on this lattice are subject to operations of translation and difference quotient formation defined by

$$T_r f(x) \equiv T_r^h f(x^1, \ldots, x^n) \equiv f(x^1, \ldots, x^{r-1}, x^r + h, x^{r+1}, \ldots, x^n),$$
$$T_r^{-1} f(x) \equiv (T_r^h)^{-1} f(x^1, \ldots, x^n) \equiv f(x^1, \ldots, x^{r-1}, x^r - h, x^{r+1}, \ldots, x^n),$$

and

$$\delta_r \equiv \delta_r^h \equiv h^{-1}(1 - T_r^{-1}).$$

For any $h > 0$, we shall seek to approximate the solution $u(x, t)$ of our problem by a function $v^h(x, t)$ defined for x on E^h and for $0 \leqslant t \leqslant T$. The first derivatives u_{x^r} are to be approximated by $\delta_r^h v^h$. When h is fixed, we set $x_i = hi$ and, for brevity,

$$v_i = v_i(t) = v^h(x_i, t)$$
$$T_r v_i = T_r^h v^h(x_i, t),$$

etc. We also use δv_i or $\delta^h v_i$ to refer to the vector $(\delta_1 v_i, \ldots, \delta_n v_i)$, a presumed approximation to grad u.

The function $v^h(x, t)$ is defined by three conditions, the boundary condition,

(5.5) $v_i(t) = \varphi'$ when any of the indices $i_1, \ldots, i_n$ is zero or negative,

the initial condition,

$$v_i(0) = \varphi(x_i) \quad \text{for all } i, \tag{5.6}$$

and the recursive differential conditions,

$$v_i' + F(x_i, t, v_i, \delta v_i) = 0 \quad \text{for all } i. \tag{5.7}$$

We shall prove that, for a suitable sequence of mesh widths h_k tending to zero, the $v^{h_k}(x, t)$ defined by this scheme converge continuously (see Section 7), as $k \to \infty$, to a solution $v(x, t)$ of (1.1) and (1.3). Unless φ is semi-concave, we do not prove, however, that v is semi-concave. Theorem 5.1 thus is obtained only after further argument, which is given below, the direct outcome of the difference-differential scheme alone being as follows:

THEOREM 5.2 (CONVERGENCE). — *Under the hypotheses of Theorem* 5.1, *a sequence* h_k, $k = 1, 2, \ldots$, *of positive numbers tending to zero exists such that the* $v^{h_k}(x, t)$ *converge continuously, as* $k \to \infty$, *to a Lipschitz-continuous function* $v(x, t)$. *The limit function* $v(x, t)$ *satisfies* 1.1 *at almost all points of* S_T *and satisfies the initial condition* (1.3) *and the boundary condition* (4.4). *With* Δ_r *defined for* $r = 1, \ldots, n$ *by the condition*

$$\Delta_r f(x^1, \ldots, x^n) = h^{-1}(f(x^1, \ldots, x^{r-1}, x^r + h, x^{r+1}, \ldots, x^n) - f(x^1, \ldots, x^n))$$

(δ_r^h *was the restriction of* Δ_r *to the lattice* E_h), v *also satisfies the inequality*

$$(5.8) \quad \Delta_r^2 v(x, t) \leqslant 4/at + B \quad \text{for} \quad 0 < t \leqslant T, \quad r = 1, \ldots, n,$$

where B *is a constant uniform, in particular, for all choices of* φ. *Under Hypotheses* ii_1, *the right side of* 5.8 *can be replaced by* $2/at$.

More can be said if φ is *semi-concave on compact sets* in the following sense: To each positive R corresponds a constant $k(R)$ such that

$$\varphi(x + y) + \varphi(x - y) - 2\varphi(x) \leqslant k(R)|y|^2 \quad \text{for} \quad |x \pm y| \leqslant R.$$

SUPPLEMENT TO THEOREM 5.2. — *If* φ *is semi-concave on compact sets, then, for each* t *in the interval* $0 \leqslant t \leqslant T$, $v(x, t)$, *too, is semi-concave on compact sets.*

Theorem 5.2 and its supplement after some preparation will be proved in Section 8.

COROLLARY TO THEOREM 5.2. — *If all the hypotheses made in Theorem* 5.1 *are in force, and if, in addition,* φ *is semi-concave, then for* $0 < t \leqslant T$ $4/at + B$ *is a constant of semi-concavity for* v.

Proof of Corollary. — Let T here denote any translation in x-space through a distance d: $\mathrm{T}f(x) = f(x + y)$, where $|y| = d$. Let $\Delta = d^{-1}(\mathrm{T} - 1)$. It is necessary to prove that

$$(5.9) \qquad \Delta^2 v(x, t) \leqslant 4/at + \mathrm{B} \qquad \text{for} \qquad 0 < t \leqslant \mathrm{T}.$$

This we shall do in an arbitrary, fixed cone of determinacy C. Since φ is semi-concave, the supplement to Theorem 5.2 shows v to be semi-concave within C and thus (Theorem 2.1) to be uniquely determined by the values of φ on the base of C only. Hence, if we were to rotate the x-axes, renormalize accordingly (Section 4), apply the foregoing difference-differential scheme, but in the new coordinate frame, go to the limit as the mesh width in this scheme tended to zero, and finally return to the original coordinate axes, we would arrive at the same v as before. Since new axes can be chosen one of which is in the direction of the difference operator Δ, inequality (5.9) follows from (5.8).

Proof of Theorem 5.1. — Theorem 5.1, for semi-concave initial data, is contained in Theorem 5.2 and its corollary. Theorem 5.1 with arbitrary data is proved as follows by approximating these data by semi-concave functions and then applying the continuity and compactness properties of Section 2.

By a translation, let us arrange that, for some positive constant ε_0,

$$\varphi = \varphi' \qquad \text{when any} \qquad x^r \leqslant \varepsilon_0.$$

Let $j(x)$ be a function of class C^∞ such that $j(x) \geqslant 0$, $j(x) = 0$ for $|x| > 1$, and $\int j(x)\,dx = 1$, and for $\varepsilon \neq 0$ define

$$\varphi_\varepsilon(x) = \int \varphi(x - \varepsilon y) j(y)\, dy,$$

the domain of integration being the entire n-dimensional x-space. Since φ is Lipschitz-continuous, $\varphi_\varepsilon \to \varphi$ uniformly as $\varepsilon \to 0$.

Since

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \int \varphi(y) j((x - y)/\varepsilon)\, dy,$$

this function is infinitely differentiable, its second derivatives, in particular, being bounded on any compact set. Each φ_ε, therefore, is semi-concave on compact sets in the sense defined after Theorem 5.2. Furthermore, if $0 < \varepsilon < \varepsilon_0$, then $\varphi_\varepsilon = \varphi'$ when any $x^r \leqslant 0$. Hence, the corollary to Theorem 5.2 applies, guaranteeing the existence of a solution $v_\varepsilon(x, t)$ that reduces to φ_ε for $t = 0$ and, for each fixed t in the interval $0 < t \leqslant T$, is semi-concave with uniform constant $4/at + B$ independent of ε. Since $\varphi_\varepsilon \to \varphi$ uniformly, Theorems 2.1 and 2.3 prove v_ε to converge uniformly within any characteristic cone to a Lipschitz-continuous limit $v(x, t)$ that satisfies 1.1 almost everywhere and, for $t > 0$, is semi-concave with constant $4/at + B$. Clearly, $v(x, 0) = \varphi(x)$ and

$$v(x, t) = \varphi' \qquad \text{for} \qquad x^r \leqslant 0, \qquad r = 1, \ldots, n.$$

Thus, Theorem 5.1 is completely proved.

6. Fundamental inequalities for the difference-differential scheme.

Theorem 5.2 is based on four inequalities we prove in this section. The first, arising only in connection with Hypothesis ii_3, is

$$(6.1) \qquad |v_i(t)| \leqslant U \qquad \text{for} \qquad 0 \leqslant t \leqslant T \quad \text{and all } i.$$

The second inequality states that

$$(6.2) \qquad |\delta_r v_i(t)| \leqslant P \quad \text{for} \quad 0 \leqslant t \leqslant T, \quad r = 1, \ldots, n, \text{ and all } i.$$

According to the third, a constant B independent of φ exists such that

$$(6.3) \qquad \delta_r^2 v_i(t) \leqslant 4/at + B \quad \text{for} \quad 0 < t \leqslant T, \quad r = 1, \ldots, n, \quad \text{and all } i;$$

under Hypothesis ii_1, the right side can be replaced by $2/at$.

The fourth inequality relates to problems in which φ is semi-concave on compact sets with constant $\varphi_2(R)$ on the sphere $|x| = R$. To formulate the inequality, let T here symbolize any translation in x-space moving lattice points into lattice points. Thus,

$$Tf(x^1, \ldots, x^n) = f(x^1 + m^1 h, \ldots, x^n + m^n h),$$

where the m^r, $r = 1, \ldots, n$, are integers. With $|m| = \sqrt{\Sigma_r (m^r)^2}$, let

$$\Delta = \frac{T - 1}{|m| h}.$$

The fourth inequality asserts that, to each positive ρ, a constant C depending on ρ and $\varphi_2(\rho)$ exists such that

$$(6.4) \quad \Delta^2 v_i(t) \leqslant C \quad \text{for} \quad 0 \leqslant t \leqslant T \quad \text{and} \quad |i| h \leqslant \rho.$$

Proof of 6.1 *and* 6.2. — The first two of the inequalities must be taken up together. Thus, we begin by deriving equations for the $\delta_r v_i$, to this end noting that

$$\begin{aligned} h\delta_r F(x_i, t, v_i, \delta v_i) &= F(x_i, t, v_i, \delta v_i) - F(T_r^{-1} x_i, t, T_r^{-1} v_i, T_r^{-1} \delta v_i) \\ &= F(x_i, t, v_i, \delta v_i) - F(T_r^{-1} x_i, t, T_r^{-1} v_i, \delta v_i) \\ &+ F(T_r^{-1} x_i, t, T_r^{-1} v_i, \delta v_i) \\ &\qquad - F(T_r^{-1} x_i, t, T_r^{-1} v_i, T_r^{-1} \delta v_i) \\ &= \overline{F}_{x^r} + \overline{F}_u \delta_r v_i + \sum_{s=1}^{n} \overline{\overline{F}}_s \delta_r \delta_s v_i, \end{aligned}$$

the single bar in these equalities referring to the arguments

$$(T_r^{-1} + \overline{\theta}(1 - T_r^{-1}))x_i, \qquad t, \qquad (T_r^{-1} + \overline{\theta}(1 - T_r^{-1}))v_i, \qquad \delta v_i$$

and the double bar to the arguments

$$T_r^{-1} x_i,\ t,\ T_r^{-1} v_i, \qquad (T_r^{-1} + \overline{\overline{\theta}}(1 - T_r^{-1}))\delta v_i,$$

where $0 < \overline{\theta} < 1$, $0 < \overline{\overline{\theta}} < 1$. Using this result, from 5.7 we can immediately write the desired equations. We do so in the form

$$(6.5) \quad (\delta_r v_i - w)' + [w' + \overline{F}_{x^r} + \overline{F}_u \delta_r v_i] + \sum_{s=1}^{n} \overline{\overline{F}}_s \delta_s \delta_r v_i = 0,$$

where $w(t)$ is an arbitrary function chosen according to circumstances. Defining $w_\varepsilon(t)$ $(\varepsilon > 0)$ as below, differently for each hypothesis considered, we shall always choose $w(t) \equiv w_\varepsilon(t)$ or $\equiv - w_\varepsilon(t)$.

Under Hypothesis ii_1, take $w_\varepsilon(t) = M_0 + \varepsilon$.

Under Hypothesis ii_2, take $w_\varepsilon(t)$ as the solution of the initial value problem: $w' = V(w)$, $w(0) = M_0 + \varepsilon$.

Under Hypothesis ii_3, take $w_\varepsilon(t)$ as the solution of the initial value problem: $w' = V(w)$, $w(0) = M_0 + \varepsilon$. ($V(\rho)$ was defined in Section 4.) This solution exists for $0 \leqslant t \leqslant T$.

Under any one of the hypotheses considered, $w_\varepsilon(T) > P$, and $\lim_{\varepsilon \to 0} w_\varepsilon(T) = P$. Choose ε so small that $P < w_\varepsilon(T) \leqslant P_0$. In Hypothesis ii_3, take $U' < U_0$ and $P' < P_0$, and demand additionally of ε that $U + \varepsilon < U'$ and $w_\varepsilon(T) < P'$. To justify 6.2, it obviously suffices to prove that, for such ε,

$$|\delta_r v_i(t)| < w_\varepsilon(t), \tag{6.6}$$

and to justify (6.1) (under Hypothesis ii_3), that

$$|v_i(t)| < U + \varepsilon, \tag{6.7}$$

for $0 \leqslant t \leqslant T$.

The case of Hypothesis ii_1, apart from the special estimate of the semi-concavity constant, which is easily obtained, is included in that of Hypothesis ii_2.

Proof of (6.6) *under Hypothesis* ii_2. — Choosing an arbitrarily large, positive N, we shall keep for the present to the region

$$R: \quad x^r < N, \qquad r = 1, \ldots, n.$$

Let Σ_ε denote the set of values of t in the interval $0 \leqslant t \leqslant T$ such that

$$|\delta_r v_i(\tau) < w_\varepsilon(\tau) \qquad \text{for} \qquad r = 1, \ldots, n, \quad x_i \varepsilon R, \quad 0 \leqslant \tau \leqslant t.$$

Since $w_\varepsilon > M_0$, Σ_ε is not empty; Σ_ε therefore is an interval, which, by continuity, is open in $[0, T]$. Hence, to justify (6.6) in R for $0 \leqslant t \leqslant T$, it suffices to prove that Σ_ε is a closed interval. Consider any value t' such that any smaller (positive) value belongs to Σ_ε; Σ_ε is closed if it must contain such a t'.

If Σ does not contain a value t' of this description, then (6.6) fails in R for $t = t'$: i.e., there are an index s and a lattice point $x_{i'}$ in R such that either

$$\delta_s v_{i'}(t') = w_\varepsilon(t') \tag{6.8'}$$

or

$$\delta_s v_{i'}(t') = - w_\varepsilon(t'); \tag{6.8''}$$

in either case, assume $x_{i'}$ the nearest of such lattice points to the origin. Then in case 6.8′, for instance,

$$\delta_r \delta_s v_{i''}(t') > 0, \qquad r = 1, \ldots, n,$$

and also

$$(\delta_s v_{i''} - w_\varepsilon)'|_{t=t'} \geqslant 0.$$

These inequalities, applied to (6.5) with $w = w_\varepsilon$, imply that the square brackets in that equation are negative, a conclusion, however, that, in view of the definition of Σ_ε, contradicts an assumption of ii_2. Our trial assumption (6.8′), having thus led to a contradiction, is untenable, and the other alternative, (6.8″), similarly is incorrect. It follows that inequality (6.6) holds in R for $t = t'$, as asserted. Hence, Σ_ε is closed, and, from the previous argument, inequality (6.6) is valid in R for $0 \leqslant t \leqslant T$ and x_i in R; R being arbitrary, it is valid as stated. Inequality (6.2) results, finally, by letting $\varepsilon \to 0$.

Proof of 6.6 *and* 6.7 *under Hypothesis* ii_3. — Again keeping x to the region R introduced above, let Σ_ε now denote the set of values of t in the interval $0 \leqslant t \leqslant T$ such that

$$|v_i(\tau)| < U + \varepsilon, \qquad |\delta_r v_i(\tau)| < w_\varepsilon(\tau)$$

for

$$r = 1, \ldots, n, \quad x_i \varepsilon R, \quad 0 \leqslant \tau \leqslant t.$$

Since $U + \varepsilon > \varphi_0$ and $w_\varepsilon > M_0$, Σ_ε is not empty; Σ_ε therefore is an interval that, by continuity, is open in $[0, T]$. Hence, to justify (6.6) and (6.7) in R for $0 \leqslant t \leqslant T$, it suffices to prove that Σ_ε is a closed interval. Consider any value t' such that any smaller positive value belongs to Σ_ε. If Σ_ε excludes t', either (6.6) or (6.7) is excluded (in R) for $t = t'$. Let us first consider (6.7). Since, by the definition of Σ_ε and by continuity,

$$\left.\begin{array}{l} |v_i(t)| \leqslant U + \varepsilon < U' \\ |\delta_r v_i(t) \leqslant w_\varepsilon(t) \leqslant P' \end{array}\right\} \quad \text{for} \quad r = 1, \ldots, n, \quad x_i \varepsilon R, \quad 0 \leqslant t \leqslant t',$$

Hypothesis ii_3 implies

$$|v_i'(t)| = |F(x_i, t, v_i, \delta v_i)| \leqslant Q_0 \qquad \text{for} \qquad 0 \leqslant t \leqslant t'.$$

By integration and by the definition of T, we have from this

$$|v_i(t')| \leqslant \varphi_0 + Q_0 t' \leqslant \varphi_0 + Q_0 T \leqslant \varphi_0 + Q_0((U - \varphi_0)/Q_0) = U.$$

Hence, inequality (6.7) holds for $t = t'$. With this knowledge we now justify (6.6) for $t = t'$ in the same manner as was used in the previous case. It follows that Σ_ε is closed, as well as open, and thus coincides with [0, T], inequalities (6.6) and (6.7) herewith being proved in R and, R being arbitrary, in S_T.

Proof of (6.3) *and* (6.4). — By (5.7), we have

$$\Delta^2 \varphi_i' + \Delta^2 F(x_i, t, \varphi_i, \delta\varphi_i) = 0. \tag{6.10}$$

This and the fact that T and Δ commute with the T_r and the δ_r lead to a differential inequality satisfied by $\Delta^2\varphi_i$.

In the calculations that follow, it is convenient temporarily to change the abbreviations for $\partial F/\partial p_r$ and $\partial^2 F/\partial p_r \partial x^s$ from F_r and F_{r,x^s} to $F_{,r}$ and $F_{,r,x^s}$, respectively, and then to set

$$F_i = F(x_i, t, \varphi_i, \delta\varphi_i)$$

and

$$F_{i,x^r} = F_{x^r}(x_i, t, \varphi_i, \delta\varphi_i), \qquad F_{i,r} = F_{,r}(x_i, t, \varphi_i, \delta\varphi_i),$$
$$F_{i,r,x^s} = F_{,r,x^s}(x_i, t, \varphi_i, \delta\varphi_i),$$

etc. To estimate $\Delta^2 F_i$ appropriately, we write

$$|m|h\Delta^2 F_i = \Delta(T-1)F_i = \Delta T F_i - \Delta F_i$$

and apply Taylor's theorem with a remainder of higher than second order to the two terms on the right. With

$$c_r = m^r/|m| = \Delta x_i^r,$$

we have, first,

$$\begin{aligned}\Delta T F_i &= |m|^{-1}h^{-1}(T-1)TF_i \\ &= \Sigma_r c_r TF_{i,x^r} + \Delta T\varphi_i \cdot TF_{i,u} + \Sigma_r \Delta T\delta_r\varphi_i \cdot TF_{i,r} \\ &\quad + (1/2)\,|m|h\{\Sigma_{r,s} c_r c_s \overline{F}_{i,x^r x^s} + 2\Sigma_r c_r \Delta T\varphi_i \cdot \overline{F}_{i,x^r u} \\ &\quad + \Sigma_{r,s} c_r \Delta T\delta_s\varphi_i \cdot \overline{F}_{i,x^r,s} + 2\Sigma_r \Delta T\varphi_i \cdot \Delta T\delta_r\varphi_i \cdot \overline{F}_{i,u,r} \\ &\quad + (\Delta T\varphi_i)^2 \overline{F}_{i,uu} + \Sigma_{r,s} \Delta T\delta_r\varphi_i \cdot \Delta T\delta_s\varphi_i \cdot \overline{F}_{i,r,s}\},\end{aligned}$$

the bars over the second derivatives of F indicating that these derivatives are to be taken at intermediate arguments. We note that $T\varphi_i$, $\delta T\varphi_i$, $\Delta T\varphi_i$, by (6.1) and (6.2), are bounded in S_T and that the coefficients in the right member being first and second derivatives of F therefore also are bounded. Hence, the last result is of the form

$$\begin{aligned}\Delta T F_i = {} & \Sigma_r c_r TF_{i,x^r} + \Delta T\varphi_i \cdot TF_{i,u} + \Sigma_r \Delta T\delta_r\varphi_i \cdot TF_{i,r} \\ & + |m|h[A_0 + \Sigma_r A_r \Delta T\delta_r\varphi_i + (1/2)\Sigma_{r,s}\Delta T\delta_r\varphi_i \cdot \Delta T\delta_s\varphi_i \cdot \overline{F}_{i,r,s}],\end{aligned}$$

where the A_k are absolutely bounded in S_T. Because of Hypothesis (i_1) (equation 1.2), the combination

$$\Sigma_r A_r \Delta T\delta_r v_i + (1/2)\Sigma_{r,s}\Delta T\delta_r v_i \cdot \Delta T\delta_s v_i \cdot \overline{F}_{i.r,s},$$

however, is not less than

$$\Sigma_r A_r \Delta T\delta_r v_i + (a/2)\Sigma_r(\Delta T\delta_r v_i)^2,$$

and, since for any positive ε

$$|\Delta T\delta_r v_i| \leqslant (\varepsilon/2)(\Delta T\delta_r v_i)^2 + 1/2\varepsilon,$$

is not less than

$$-\Sigma_r|A_r|/2\varepsilon + (1/2)(a - \varepsilon\Sigma_r|A_r|)\Sigma_r(\Delta T\delta_r v_i)^2.$$

Hence,

$$\Delta TF_i \geqslant \Sigma_r c_r TF_{i,x^r} + \Delta T v_i \cdot TF_{i,u} + \Sigma_r \Delta T\delta_r v_i \cdot TF_{i,r} + |m|h[A + (a/4)\Sigma_r(\Delta T\delta_r v_i)^2].$$

A denoting a suitable constant. Similarly,

$$\begin{aligned}-\Delta F_i &= |m|^{-1}h^{-1}(1 - T)F_i \\ &\geqslant -\Sigma_r c_r TF_{i,x^r} - \Delta v_i \cdot TF_{i,u} - \Sigma_r \Delta\delta_r v_i \cdot TF_{i,r} \\ &\quad + |m|h[B + (a/4)\Sigma_r(\Delta\delta_r v_i)^2],\end{aligned}$$

where B is a constant. Adding the preceding two inequalities and dividing by $|m|h$, we obtain

$$\Delta^2 F_i \geqslant \Delta^2 v_i \cdot TF_{i,u} + \Sigma_r \delta_r \Delta^2 v_i \cdot TF_{i,r} + (a/4)\Sigma_r(\Delta\delta_r v_i)^2 - C',$$

C' being a constant we may assume to be zero or positive.

Now we return to (6.10), which in view of the last result leads to the inequality

$$(6.11)\quad (\Delta^2 v_i)' + \Sigma_r TF_{i,r} \cdot \delta_r(\Delta^2 v_i) + TF_{i,u} \cdot \Delta^2 v_i + (a/4)\Sigma_r(\delta_r \Delta v_i)^2 - C' \leqslant 0.$$

To prove (6.4), we drop the squared quantities in (6.11) and make the substitution

$$\Delta^2 v_i = e^{\lambda t}(z_i + C' + \varphi'),$$

where

$$\lambda \geqslant \begin{cases} 0 \\ 1 + \sup F_u \end{cases} \quad \text{and} \quad \varphi' = \max[0, \sup \partial^2\varphi/\partial x^r \partial x^s - C'],$$

the suprema being for all arguments (x, t, u, p) such that

$0 \leqslant t \leqslant T$, $|x| \leqslant \rho$, $|u| \leqslant U$, $|p| \leqslant P$ and, in the second instance, for all r and s, as well. Inequality (6.11) thereby leads to

$$z_i' + \Sigma_r \mathrm{TF}_{i,r} \cdot \delta_r z_i + (\mathrm{TF}_{i,u} + \lambda)(z_i + C' + \varphi') \leqslant C' e^{-\lambda t},$$

which implies a relation of the form

$$z_i' + \Sigma_r E_{ir} \delta_r z_i + E_{i0} z_i \leqslant 0 \tag{6.12}$$

with positive E_{ik}. From this, we shall prove $z_i \leqslant 0$ for $h|i| \leqslant \rho$. First, $z_i(0) \leqslant 0$. Hence, if $z_i(t)$ ever assumes a given positive value ε, it can do so only for positive t. Let t_0 denote the least value of t such that, for any index i with $h|i| \leqslant \rho$, $z_i(t_0) = \varepsilon$; let i_0 denote the index nearest the origin such that $z_i(t_0) = \varepsilon$. Then $z_{i_0}'(t_0) \geqslant 0$ and $\delta_r z_{i_0}(t_0) > 0$ in contradiction to (6.12). Such an ε, therefore, does not really exist, and, hence, $z_i \leqslant 0$ for $h|i| \leqslant \rho$, as asserted. Consequently,

$$\Delta^2 \varphi_i \leqslant e^{\lambda t}(C' + \varphi') \qquad \text{for} \qquad h|i| \leqslant \rho,$$

which is to say that $\varphi^h(x, t)$ is semi-concave on compact sets with respect to x. Inequality (6.4) thereby is proved.

Inequality (6.3) remains, Substituting δ_p for Δ in (6.11), we deduce

$$(\delta_p^2 \varphi_i)' + \Sigma_r \mathrm{TF}_{i,r} \cdot \delta_r(\delta_p^2 \varphi_i) + \mathrm{TF}_{i,u} \cdot \delta_p^2 \varphi_i + \alpha(\delta_p^2 \varphi_i)^2 - C' \leqslant 0,$$

where $\alpha = a/4$. Let us make a substitution in this inequality of the form

$$\delta_p^2 \varphi_i = y_i + 1/\alpha t + \beta,$$

where β is a new constant we shall specify below. We thereby obtain

$$\begin{aligned} y_i' + \Sigma_r \mathrm{TF}_{i,r} \cdot \delta_r y_i + (\mathrm{TF}_{i,u} + 2/t + 2\alpha\beta) y_i \\ + (1/\alpha t)(\mathrm{TF}_{i,u} + 2\alpha\beta) + (\beta \mathrm{TF}_{i,u} + \alpha\beta^2 - C') \leqslant 0. \end{aligned} \tag{6.12}$$

Now we select β in such a way that the parenthesized expressions will be non-negative. With

$$m = \inf F_u(x, t, u, p),$$

the infimum being for $0 \leqslant t \leqslant T$, $|u| \leqslant U$, $|p| \leqslant P$, and all x, it suffices to have

$$2\alpha\beta + m \geqslant 0 \qquad \text{and} \qquad \alpha\beta^2 + m\beta - C' \geqslant 0.$$

These conditions, however, are satisfied if we choose

$$\beta = \frac{-m + \sqrt{m^2 + 4\alpha C'}}{2\alpha},$$

and with this choice inequality (6.12) is reduced to one of the form

$$(6.13) \qquad y_i' + \Sigma_r E_{ir}\delta_r y_i + E y_i \leqslant 0$$

with positive E_{ir}. From this we can prove by an argument similar to one above that $y_i < 0$. In fact, $y_i < 0$ for all sufficiently small values of t, and if $y_i(t) = 0$ for any value of t, this value is positive. If such a positive value of t exists, let i_0 denote the index nearest the origin, and, for this index, t_0 the least value of t, such that $y_{i_0}(t_0) = 0$. Then $\delta_r y_{i_0}(t_0) > 0$ and $y_{i_0}'(t_0) \geqslant 0$. The last relations being incompatible with 6.13, we conclude that no such i_0 and t_0 exist. Hence, $y_i(t) < 0$ for $0 \leqslant t \leqslant T$, as asserted, and thus

$$\delta_p^2 v_i < 1/\alpha t + \beta \qquad \text{for} \qquad 0 \leqslant t \leqslant T.$$

Inequality (6.3) thereby is proved.

The fact that it is possible, when Hypothesis ii_1 is in force, to replace the right side of (6.3) by $2/at$ we see by applying reasoning of the foregoing type to the simpler form

$$(\Delta^2 v_i)' + \Sigma_r TF_{i,r} \cdot \delta_r \Delta^2 v_i + (a/2)\Sigma_r(\Delta\delta_r v_i)^2 \leqslant 0$$

taken by (6.11) in this case.

7. Some lemmas on continuous convergence.

Let $f_k(x)$ be a function defined on the lattice $E_k = E^{h_k}$ (Section 5), where h_k, $k = 1, 2, \ldots$, are positive numbers with integral ratios h_k/h_{k+1}. Let $f(x)$ be defined for all n-dimensional points x. The sequence (f_k) is said to converge continuously to f at a point x_0 if

$$\lim_{k \to \infty} f_k(x_k) = f(x_0)$$

for all sequences of points x_k tending to x_0. (In this and later contexts, x_k, tacitly or explicitly, always is to be understood as belonging to the lattice E_k, the domain of f_k. However, x_0 need not be a lattice point.) This convergence is called uniform if, to each $\varepsilon > 0$, positive δ and N exist such that

$$|f_k(x) - f(x_0)| < \varepsilon$$

whenever $|x - x_0| < \delta$ and $k > N$.

The Theorem of Arzela has been generalized by C. Pucci [5] to continuous convergence. Here we state a special case of Pucci's result of interest to us :

THEOREM 7.1. — *Let f_k be a function defined on a lattice E_k, $k = 1, 2, \ldots$, as above. If a constant K exists such that, for all lattice points x and y,*

$$|f_k(x) - f_k(y)| \leqslant K|x - y|, \qquad k = 1, 2, \ldots,$$

then a subsequence of the f_k converges continuously at every point (of the n-dimensional space in question), and the limit is Lipschitz-continuous with constant K. In any bounded region, the convergence is uniform.

The remaining lemmas are concerned with functions of a single real variable s, $-\infty < s < \infty$. Again requiring that h_k/h_{k+1} be an integer, consider the discrete sets

$$T_k = \{ jh_k | j = 0, \pm 1, \pm 2, \ldots \}$$

partitioning the s-axis into equal subintervals, each partition being a refinement of the preceding.

THEOREM 7.2. — *Let $f_k(s)$ be defined on T_k, and suppose the sequence (f_k) to converge continuously at every point to a continuous limit $f(s)$. Suppose that the difference quotients*

$$g_k(s) = \frac{f_k(s) - f_k(s - h_k)}{h_k},$$

which are defined on T_k, are uniformly bounded in any finite interval and, furthermore, converge continuously almost everywhere to a measurable function $g(s)$. Then

$$g(s) = df/ds \quad \text{almost everywhere.} \tag{7.1}$$

If the only discontinuities of $g(s)$ are jumps, then the points of continuity of $g(s)$ are also the precise points at which df/ds exists, and $g = df/ds$ at all these points.

This is proved as Theorem 6.4 in [2].

The following result concerning semi-monotonic functions (Section 2) comes from Theorem 7.1 of [2]:

THEOREM 7.3. — *Let functions $g_k(s)$, defined on T_k, $k = 1, 2, \ldots,$ be uniformly bounded and uniformly semi-decreasing. Then for a suitable subsequence $(g_{k'})$, a semi-decreasing function $h(s)$ exists such that $g_{k'}$ converges continuously to h at every point s_0 at which h is continuous, and converges continuously at no other point.*

8. The convergence of the difference-differential scheme. Proof of Theorem 5.2.

Select a sequence of partitions E^{h_k} such that h_k/h_{k+1} is an integer > 1, and set

$$v_k(x, t) = v^{h_k}(x, t).$$

In the scheme considered, by (5.7) written for any lattice point z_k belonging to E^{h_k} as

$$(8.1)\quad v_k'(z_k, t) + F(z_k, t, v_k(z_k, t), \delta^{h_k}v_k(z_k, t)) = 0,$$

we have

$$(8.2)\quad v_k(z_k, t) = \varphi(z_k) - \int_0^t F(z_k, t', v_k(z_k, t')\delta^{h_k}v_k(z_k, t'))\, dt'.$$

By (6.1), (6.2), (6.3), the $v_k(z, t)$ and their first differences

$$(8.3)\qquad \delta_r^{h_k}v_k(z, t)$$

are uniformly bounded in S_T; by (8.1), the derivatives $v_k'(z, t)$ are similarly bounded.

We also make the following remarks: 1) If $\varphi(x)$ is semi-concave on a ball $|x| \leqslant \rho$, then by (6.4) for each t in the interval $0 \leqslant t \leqslant T$ the v_k are semi-concave on this ball with a single constant of semi-concavity holding uniformly with respect to k. 2) Let $h > 0$ and, as in Section 5, define Δ_r for $r = 1, \ldots, n$ by the condition

$$\Delta_r f(x^1, \ldots, x^n) = h^{-1}(f(x^1, \ldots, x^{r-1}, x^r + h, x^{r+1}, \ldots, x^n) - f(x^1, \ldots, x^n)).$$

Because

$$(\delta_r^{h_k})^2 v_k(z, t) \leqslant 4/at + \mathrm{B} \quad (0 < t \leqslant \mathrm{T}),$$

where B is a constant, we easily deduce that, when h is an integral multiple of h_k,

$$(\Delta_r)^2 v_k(z, t) \leqslant 4/at + \mathrm{B} \qquad \text{for} \qquad 0 < t \leqslant \mathrm{T}.$$

The uniform boundedness of the t-derivatives and the x-difference quotients of the $v_k(x, t)$, and the foregoing remarks, give us from Theorem 7.1:

THEOREM 8.1. — *The $v_k(x, t)$ converge continuously everywhere to a Lipschitz-continuous limit $u(x, t)$ satisfying the assigned initial conditions* (1.3), *the boundary condition* (4.4), *and n inequalities of the form* (5.8). *The convergence is uniform in any bounded region of the half space $t \geqslant 0$. If φ is semi-concave, u is semi-concave for $0 \leqslant t \leqslant \mathrm{T}$.*

(Strickly speaking, Theorem 8.1 can be justified at this stage only for a subsequence of the v_k, eventual reference to a uniqueness theorem being needed to establish the result as stated. In what follows, we shall identify the convergent subsequence with the sequence itself, i.e., assume that the sequence v_k converges.)

The principal result concerning the convergence of the differences (8.3) is as follows:

THEOREM 8.2. — *For fixed, positive t $(0 < t \leqslant \mathrm{T})$, the differences* (8.3) *converge continuously to u_{x^r} at each point x at which this derivative exists. This is to say that, for any sequence z_k of points of E^{h_k} tending, as $k \to \infty$, to x,*

$$\lim_{k \to \infty} \delta_r^{h_k} v_k(z_k, t) = u_{x^r}(x, t).$$

This result is a corollary of the theorem below.

THEOREM 8.3. — *Fix t in the interval $0 < t \leqslant \mathrm{T}$, fix r as any index from 1 to n, fix arbitrarily the $n - 1$ quantities x^i, $i \neq r$, and let S_r denote the set of values of x^r at which $u_{x^r}(x, t)(x = (x^1, \ldots, x^n))$ then exists.* (S_r *may depend on t and the x^i, $i \neq r$.) Set $\xi^i = x^i$ for $i \neq r$. The function*

$$w_r(x, t) = \limsup_{\substack{\xi^r \to x^r \\ \xi^r \in \mathrm{S}_r}} u_{x^r}(\xi, t) \quad (\xi = (\xi^1, \ldots, \xi^n)),$$

defined for every real value of x^r, *is semi-decreasing with respect to* x^r. *This function is continuous with respect to* x^r *when, and only when,* $x^r \varepsilon S_r$. *For* $x^r \varepsilon S^r$,

$$u_{x^r}(x, t) = w_r(x, t)$$

and also

$$\lim_{k \to \infty} \delta_r^{h_k} v_k(z_k, t) = u_{x^r}(x, t)$$

for all sequences of points $z_k \varepsilon E^{h_k}$ *tending to* x.

Proof. — We shall prove this theorem just for $r = 1$, the $n - 1$ other cases being analogous.

In what follows, z or ζ generally will refer to lattice points or their coordinates, or to sets of their coordinates, and x or ξ, in similar fashion, to arbitrary points. t will be fixed in the interval $0 < t \leqslant T$.

Let $\xi_0 = (x_0^2, \ldots, x_0^n)$ denote an arbitrary point with $n - 1$ coordinates and $\zeta_k = (z_k^2, \ldots, z_k^n)$, $k = 1, 2, \ldots,$ a sequence of points tending to ξ_0:

$$\zeta_k \to \xi_0.$$

The ζ_k, conforming to the foregoing convention, are such that the n-dimensional points $(z_k^1, \zeta_k) = (z_k^1, z_k^2, \ldots, z_k^n)$ belong for suitable choices of their first components z_k^1, to the lattices E^{h_k}, respectively. Regard

$$W_k(z^1) \equiv \delta_1^{h_k} v_k(z^1, \zeta_k, t)$$

for $k = 1, 2, \ldots,$ as a sequence of functions of z^1. These functions are semi-decreasing. Theorem 7.3 thus proves the existence, for all values of x^1, of a semi-decreasing function $\tilde{W}(x^1)$ and of a subsequence $(W_{k'})$ of (W_k) that, as $k' \to \infty$, converges continuously to $\tilde{W}$ at the points of continuity of the latter and just at these points. $\tilde{W}$ can be normalized, without changing any of the aforementioned properties, to be continuous from the left: We merely replace $\tilde{W}(x^1)$ in all future considerations by $W(x^1) \equiv \tilde{W}(x^1 - 0)$. The normalized function being both continuous from the left and semi-decreasing satisfies the condition

$$W(x^1) = \limsup_{\xi^1 \to x^1} W(\xi^1).$$

Regarding

$$V_k(z^1) \equiv v_k(z^1, \zeta_k, t), \qquad k = 1, 2, \ldots,$$

as functions of z^1, and

$$U(x^1) \equiv u(x^1, \xi_0, t)$$

as a function of x^1, the n variables besides x^1 being considered fixed, from Theorem 8.1 we know that the V_k converge continuously to U, as $k \to \infty$, for all values of x^1. Theorem 7.2 thus applies to U, W, the V_k, and the $W_{k'}$ with the following result: The values of x^1 at which $W(x^1)$ is continuous (with respect to x^1), which are the values at which the subsequence $(W_{k'})$ converges to W continuously, are also the values at which $dU/dx^1 = u_{x^1}(x^1, \xi_0, t)$ exists and the equality

$$u_{x^1}(x^1, \xi_0, t) = W(x^1)$$

is satisfied. Hence, we have

$$W(x^1) = \limsup_{\xi^1 \to x^1} W(\xi^1) = w_1(x^1, \xi_0, t),$$

w_1 having been defined above in stating the theorem. Since w_1 is independent of the mode of selection of the subsequence $(W_{k'})$, W thus, too, is independent. Consequently, the entire sequence (W_k) converges continuously to $u_{x^1}(x^1, \xi_0, t)$ at the values of x^1 at which $w_1(x^1, \xi_0, t)$ is continuous with respect to x^1. From the manner of definition of the W_k, $\delta_1^{h_k}\varphi_k(z, t)$ therefore converges continuously to u_{x^1} at all points (x^1, ξ_0, t) at which $w_1(x^1, \xi_0, t)$ is continuous with respect to x^1.

Proof of Theorem 5.2. — Since u is Lipschitz-continuous with respect to x, the (vector) function defined in Section 1 as grad u is bounded and measurable in S_T and, by Fubini's theorem, is bounded and measurable also on almost every line segment

$$L_x: \begin{cases} x = \text{constant} \\ 0 < t \leqslant T. \end{cases}$$

Consider such a segment determined by an appropriate x, and let $z_k \varepsilon E^{h_k}$, $k = 1, 2, \ldots$, be lattice points tending to x. Theorem 8.2 shows that, for almost all t in the interval $0 < t \leqslant T$,

$$\lim_{k \to \infty} \delta_r^{h_k}\varphi_k(z_k, t) = u_{x^r}(x, t), \qquad r = 1, \ldots, n.$$

Hence, the limit of

$$F(z_k, t, \varphi_k(z_k, t), \delta^{h_k}\varphi_k(z_k, t)),$$

as $k \to \infty$, exists and is equal to $F(x, t, u(x, t), \text{grad } u(x, t))$ for almost all t in the interval $0 < t \leqslant T$. Letting $k \to \infty$ in (8.2), we thus obtain the relation

$$u(x, t) = \varphi(x) - \int_0^t F(x, t', u(x, t'), \text{grad } u(x, t'))\, dt',$$

valid for all t and almost all x. It follows that

$$\int_0^t (u_t + F)\, dt' = 0$$

for almost all x and, in consequence, $u_t + F$ being bounded and measurable, that the integral of $u_t + F$ over any rectangular parallelepiped in xt-space is zero. Hence, $u_t + F = 0$ almost everywhere. Herewith, all the contentions of Theorem 5.2 have been proved.

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