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ON COMPONENT GROUPS OF JACOBIANS OF DRINFELD MODULAR CURVES

by Mihran PAPIKIAN (*)

1. Introduction.

Let N be a positive integer. Consider the modular curve $X_0(N)_{\mathbb{Q}}$ of level N defined over $\mathbb{Q}$. This is a proper smooth geometrically-connected curve over $\mathbb{Q}$. Let $J_0(N)$ be the Jacobian variety of $X_0(N)_{\mathbb{Q}}$, and let $\mathcal{J}$ be the Néron model of $J_0(N)$ over $\mathbb{Z}$. It is known that $\mathcal{J}$ is an abelian scheme over $\mathbb{Z}[1/N]$.

Assume N is prime, so $X_0(N)_{\mathbb{Q}}$ has two cusps; these are labelled 0 and ∞. The two cusps are $\mathbb{Q}$-rational points on $X_0(N)_{\mathbb{Q}}$ and the divisor $(0)-(\infty)$ on $X_0(N)_{\mathbb{Q}}$ generates a finite cyclic subgroup C in $J_0(N)(\mathbb{Q})$ called the *cuspidal divisor group.* Denote by $C_{/\mathbb{Z}}$ the finite flat subgroup scheme of $\mathcal{J}$ generated by $C \subset J_0(N)(\mathbb{Q})$. Let $\overline{C}$ be the $\mathbb{F}_N$-valued points of $C_{\mathbb{Z}}$ ("the specialization" of C in $\mathcal{J}\times\mathbb{F}_N$). It is known that $C\to\overline{C}$ is an isomorphism. Let $\mathcal{J}^0_{\mathbb{F}_N}$ be the connected component of the identity of $\mathcal{J}_{\mathbb{F}_N}$, and let $\Phi_{J_0(N),N} := \mathcal{J}_{\mathbb{F}_N}/\mathcal{J}^0_{\mathbb{F}_N}$ be the group of connected components of $\mathcal{J}_{\mathbb{F}_N}$. This is a finite étale group-scheme over $\mathbb{F}_N$. In [23] one finds the following result:

THEOREM 1.1 (Mazur). — *The canonical maps*

$$C \longrightarrow \Phi_{J_0(N),N} \quad \textit{and} \quad C\longrightarrow J_0(N)(\mathbb{Q})_{\mathrm{tor}}$$

are isomorphisms. In particular, $\Phi_{J_0(N),N}$ *is a constant group-scheme.*

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Motivated by Theorem 1.1 and computer calculations, William Stein [30] made the following conjecture:

Let N be a prime. Let f be a normalized weight-2 newform $f = \sum a_n q^n$ of level N, and let I_f be the kernel of the natural map $\mathbb{T} \to \mathbb{Z}[\ldots, a_n, \ldots]$ that sends a Hecke operator T_n to a_n. Denote A_f the abelian-variety quotient $J_0(N)/I_f J_0(N)$ of $J_0(N)$ by the (connected) abelian subvariety $I_f J_0(N)$; this is $\mathbb{Q}$-simple. Let C_f denote the cyclic subgroup of $A_f(\mathbb{Q})$ generated by the image of the cuspidal divisor group C and $\Phi_{A_f,N}$ the mod-N component group of the Néron model of A_f over $\mathbb{Z}$.

CONJECTURE 1.2 (Stein; refined Eisenstein conjecture). — *The canonical specialization maps*

$$C_f \longrightarrow \Phi_{A_f,N} \quad \text{and} \quad C_f \longrightarrow A_f(\mathbb{Q})_{\mathrm{tor}}$$

are isomorphisms. In particular, $\Phi_{A_f,N}$ *is constant.*

Note that Theorem 1.1 and Conjecture 1.2 imply the following weaker conjecture that can be motivated by Ribet's level-lowering results [28], cf. [30], Conjecture 4.1.

CONJECTURE 1.3. — *The natural map* $\Phi_{J_0(N),N} \to \Phi_{A_f,N}$ *is surjective.*

Shortly after the announcement of these conjectures, Matthew Emerton [8] proved them, using the techniques developed by Mazur and Ribet.

Remark 1.4. — Both conjectures are false without assuming N is prime. The following example is due to Stein. Let $N = 33$. The Jacobian $J_0(33)$ has a 1-dimensional optimal quotient $E = \mathbf{33A}$ corresponding to the newform of level 33

$$f = q + q^2 - q^3 - q^4 - 2q^5 + \cdots$$

with $E(\mathbb{Q}) = \mathbb{Z}/2 \times \mathbb{Z}/2$ but $\Phi_{E,3}(\overline{\mathbb{F}}_3) = \mathbb{Z}/6$. Moreover,

$$\#\,\mathrm{coker}(\Phi_{J_0(33),3} \to \Phi_{E,3}) = 3.$$

The aim of this paper is to discuss versions of Conjectures 1.2–1.3 in the context of function fields. The situation which most closely resembles the classical one is when our function field F is the field of rational functions on $\mathbb{P}^1_{\mathbb{F}_q}$; that is, $F = \mathbb{F}_q(t)$. To get the analogue of $\mathbb{Z}$ one has to choose a point on $\mathbb{P}^1_{\mathbb{F}_q}$, suggestively denoted by ∞. We will choose ∞ to be rational; i.e., $\deg(\infty) = 1$. Without loss of generality, $\infty = 1/t$ and $\mathrm{H}^0(\mathbb{P}^1_{\mathbb{F}_q} - \infty, \mathcal{O}) = \mathbb{F}_q[t]$ is the polynomial ring in one variable over the finite field $\mathbb{F}_q$. Let $\mathfrak{n}$ be an ideal of $\mathbb{F}_q[t]$ and consider the Drinfeld modular curve $X_0(\mathfrak{n})_F$ of level $\mathfrak{n}$. This is a compactified coarse moduli scheme for pairs $(D, Z_\mathfrak{n})$ consisting of a Drinfeld $\mathbb{F}_q[t]$-module D of rank-2 over F and an $\mathfrak{n}$-cyclic subgroup $Z_\mathfrak{n}$ of D. It is a proper smooth geometrically-connected curve over F. Let $J_0(\mathfrak{n})$ be the Jacobian of $X_0(\mathfrak{n})_F$. Denote by $\mathcal{J}$ the Néron model of $J_0(\mathfrak{n})$ over $\mathbb{P}^1_{\mathbb{F}_q}$ and by $\mathcal{J}^0$ the relative connected component of the identity (i.e., the open union of fibral identity components). It is known that $\mathcal{J}$ is an abelian scheme over $\mathbb{F}_q[t][\mathfrak{n}^{-1}]$. On the contrary, the fibres of $\mathcal{J}^0$ at the points in $\mathrm{supp}(\mathfrak{n}\cdot\infty)$ are not abelian varieties over the corresponding residue fields.

Assume $\mathfrak{n}$ is prime. Some effort has been made to transfer Mazur's results on the Eisenstein ideal to the context of Drinfeld modular curves; cf. [11], [31]. One encounters very non-trivial technical difficulties while doing this, so the theory over function fields is not yet as satisfactory as over $\mathbb{Q}$. There is an analogue of the cuspidal divisor group $C \subset J_0(\mathfrak{n})(F)$, and Gekeler proved in [11] that the specialization map $c_\mathfrak{n} : C \to \Phi_{J_0(\mathfrak{n}),\mathfrak{n}}$ is an isomorphism. But as far as I am aware, it is still *unknown* whether $C \to J_0(\mathfrak{n})(F)_{\mathrm{tor}}$ is an isomorphism.(1) Even though one should expect Conjectures 1.2–1.3 to be true in this case, proving these will require additional efforts to develop the techniques of Mazur and Ribet over the function fields.

Since our base $\mathbb{P}^1_{\mathbb{F}_q}$ is a complete curve, aside from the bad fibres of $X_0(\mathfrak{n})$ over the divisors of the level $\mathfrak{n}$ there is one more fibre, namely the fibre over ∞. It is known that $\mathcal{J}^0_{\mathbb{F}_\infty}$ is always a split torus ($\mathfrak{n}$ being arbitrary), from which one deduces that $\Phi_{J_0(\mathfrak{n}),\infty}$ is constant. It is natural to ask about the validity of Stein's conjectures over ∞. The analogue of Conjecture 1.2 fails. Indeed, it turns out [14], §5, that the canonical specialization map $c_\infty : C \to \Phi_{J_0(\mathfrak{n}),\infty}$ need not be surjective or injective, even when $\mathfrak{n}$ is prime. In this paper we will concentrate on the ∞-adic analogue of Conjecture 1.3. We do not impose any restrictions on the level $\mathfrak{n}$.

Let A be a quotient of the Jacobian variety $J_0(\mathfrak{n})$, and assume the kernel B of the corresponding quotient map $\pi : J_0(\mathfrak{n}) \to A$ is an abelian subvariety of $J_0(\mathfrak{n})$. The main result is the following:

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(1) According to the anonymous referee, Ambrus Pal recently made a significant progress in the theory of Eisenstein ideal over function fields.

THEOREM 1.5. — *Assume B is stable under the action of the Hecke algebra $\mathbb{T} \subseteq \mathrm{End}(J_0(\mathfrak{n}))$. For any prime ℓ not dividing $q-1$, the functorial sequence of finite abelian groups*

$$0 \to \Phi_{B,\infty} \longrightarrow \Phi_{J_0(\mathfrak{n}),\infty} \longrightarrow \Phi_{A,\infty} \to 0$$

is short-exact on ℓ-power torsion. In particular, the sequence is always short-exact when $q=2$. For an arbitrary q the sequence is short-exact whenever A or B is an elliptic curve.

The condition on B being $\mathbb{T}$-stable simply means that A arises from the splitting of the $\mathbb{Q}$-vector space of Drinfeld automorphic forms of level $\mathfrak{n}$ into $\mathbb{T}_{\mathbb{Q}}$-invariant subspaces; cf. §3.3. For example, this condition is automatic when $\mathfrak{n}$ is prime. The result we prove is somewhat stronger than as stated in Theorem 1.5. In fact, we determine the possible kernels and cokernels in the sequence of component groups, see Theorem 5.1, and we give a criterion which is sufficient for the exactness of the sequence of component groups. We show that this criterion is always satisfied when either A or B is 1-dimensional; see Corollary 3.15. This is a rather striking fact in view of Remark 1.4.

We should remark that the component groups $\Phi_{J_0(\mathfrak{n}),\infty}$ are much more mysterious that the corresponding groups at the finite places. This is because ∞ does not appear in the formulation of the moduli problem, and hence one cannot use deformation theory to deduce the structure of the special fibre of $X_0(\mathfrak{n})$ at ∞. This complicates the application of Raynaud's results on the specialization of Picard functor to get the structure of the group of connected components, cf. [3], §9.6. Some concrete examples show, cf. [14], §5, that $\Phi_{J_0(\mathfrak{n}),\infty}$ is not determined by the prime decomposition of $\mathfrak{n}$. Rather, it depends on topological properties of $\Gamma_0(\mathfrak{n})$ – namely on the action of $\Gamma_0(\mathfrak{n})$ on the Drinfeld half-plane, cf. §4. Thus, our Theorem 1.5 can also be considered as a modest step toward understanding the structure of $\Phi_{J_0(\mathfrak{n}),\infty}$.

The key for proving Theorem 1.5 is to study the polarization $\pi \circ \pi^\vee : A^\vee \to A$, and to compute the order of the finite group-scheme $\ker(\pi \circ \pi^\vee)$ in two different ways. One is algebraic and relies on Grothendieck's monodromy pairing; see §2. The second is rigid-analytic in nature. To carry out this second calculation, in §3 we give a higher-dimensional generalization of a construction of 1-dimensional quotients of $J_0(\mathfrak{n})$ due to Gekeler and Reversat [16]. In §4, we relate these two expressions for $\#\ker(\pi \circ \pi^\vee)$, and Theorem 1.5 easily follows from this.

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2. Algebraic calculations.

2.1. Grothendieck's orthogonality and a formula for the degree.

Let R be a henselian discrete valuation ring with field of fractions K and residue field k. Let A be an abelian variety over K and let $\mathcal{A}$ be its Néron model over R. Let $\mathcal{A}^0$ be the connected component of identity of $\mathcal{A}$. We let Φ_A denote the group $\mathcal{A}(R)/\mathcal{A}^0(R) = \mathcal{A}_k/\mathcal{A}_k^0$ of geometrically-connected components of the special fibre $\mathcal{A}_k$ of $\mathcal{A}$.

Given an isogeny $\varphi\colon A \to B$ of abelian varieties over K, its *degree* $\deg(\varphi)$ as a finite flat map is equal to the order of the finite flat group-scheme $\ker(\varphi)$. Let $A^\vee$ be the dual abelian variety of A. An isogeny $\lambda\colon A \to A^\vee$ is *symmetric* if the dual $\lambda^\vee : (A^\vee)^\vee \to A^\vee$ is equal to λ via the canonical isomorphism $(A^\vee)^\vee \cong A$. For example, polarizations are symmetric.

We say that the reduction of A over R is (*split*) *purely toric* if $\mathcal{A}_k^0 = T_A$ is a (split) algebraic torus over k; this property is invariant under passage to an isogenous abelian variety. Let $\varphi : A \to B$ be an isogeny, $\varphi^\vee : B^\vee \to A^\vee$ be the dual isogeny, and assume A and B have purely toric reductions. Let $\varphi_t : T_A \to T_B$ be the induced map of the closed fibers of the Néron models. This map is an isogeny, by functoriality, so the kernel $\ker(\varphi_t)$ is a finite multiplicative k-group scheme. We denote by $\varphi_t^\vee : T_{B^\vee} \to T_{A^\vee}$ the analogous map induced by $\varphi^\vee$.

For any finite multiplicative k-group scheme G there is a functorially unique multiplicative finite flat R-group scheme $\widetilde{G}$ with closed fibre G. We have a closed immersion $(\widetilde{\ker\varphi_t})_K \hookrightarrow \ker\varphi$ and likewise we have a natural quotient map $(\ker\varphi^\vee)^\vee \to \widetilde{\ker(\varphi_t^\vee)}{}_K^\vee$ that is dual to the closed immersion using the isogeny $\varphi^\vee$.

By the duality theory for abelian varieties, there is a canonical perfect K-group scheme duality between $\ker(\varphi)$ and $\ker(\varphi^\vee)$ over K. Hence there is a natural quotient map of K-group schemes

$$\ker(\varphi) \cong (\ker\varphi^\vee)^\vee \longrightarrow \widetilde{\ker(\varphi_t^\vee)}{}_K^\vee.$$

The following theorem, whose proof relies on Grothendieck's *Orthogonality Theorem* [20], (2.4), (5.6), will play a key role for our calculations in this section.

THEOREM 2.1. — *The sequence of K-group schemes*

$$0 \to \widetilde{\ker(\varphi_t)}_K \to \ker(\varphi) \to \widetilde{\ker(\varphi_t^\vee)}_K^\vee \to 0$$

is exact.

Proof. — See [5], Theorem 8.6. □

LEMMA 2.2. — *Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian varieties over K. Then B has a purely toric reduction if and only if A and C have purely toric reduction. Moreover, A and C will have split toric reduction when B does.*

Proof. — This can be proven by slightly modifying the argument in the proof of [3], Lemma 7.4/2. □

In this section we assume that all abelian varieties under consideration have toric reduction over R (unless otherwise specified).

For an abelian variety A over K, we let $M_A = \mathrm{Hom}_{\bar{k}}(T_A, \mathbb{G}_m)$ denote the character group of the torus $\mathcal{A}_k^0 = T_A$. This is a free abelian group with continuous action of $\mathrm{Gal}(\bar{k}/k)$ for the discrete topology, and it is contravariantly associated to A – given a homomorphism $\pi: A \to B$ of abelian varieties, we have the induced homomorphism $\pi^*: M_B \to M_A$ of the character groups.

COROLLARY 2.3. — *Let $\varphi: A \to B$ be an isogeny. Then*

$$\deg(\varphi) = \#\,\mathrm{coker}(\varphi^*: M_B \to M_A) \cdot \#\,\mathrm{coker}(\varphi^{\vee *}: M_{A^\vee} \to M_{B^\vee}).$$

Proof. — From Theorem 2.1 the order of $\ker(\varphi)$ as a group-scheme is $\#\ker(\varphi) = \#\ker(\varphi_t) \cdot \#\ker(\varphi_t^\vee)$. From the exact sequence

$$0 \to \ker(\varphi_t) \longrightarrow T_A \longrightarrow T_B \to 0$$

we get the induced exact sequence of character groups

$$0 \to M_B \xrightarrow{\varphi^*} M_A \longrightarrow \mathrm{Hom}_{\bar{k}}\big(\ker(\varphi_t), \mathbb{G}_m\big) \to 0.$$

Since $\#\,\mathrm{Hom}_{\bar{k}}(\ker(\varphi_t), \mathbb{G}_m) = \#\ker(\varphi_t)$, one concludes

$$\#\,\mathrm{coker}(\varphi^* : M_B \to M_A) = \#\ker(\varphi_t).$$

Similarly for $\varphi_t^\vee$. □

COROLLARY 2.4. — *Let $\varphi\colon A^\vee \to A$ be a symmetric isogeny. Then*

$$\deg(\varphi) = \#\operatorname{coker}(\varphi^*\colon M_A \to M_{A^\vee})^2.$$

DEFINITION 2.5. — We will say that the abelian variety A is an *optimal quotient* of the abelian variety B if there is a faithfully flat morphism $\pi\colon B \to A$ whose functorial kernel is represented by an abelian subvariety C of B (that is, $C := \ker\theta$ is connected and smooth).

For an optimal quotient A of B, we have an exact sequence of abelian varieties

$$0 \to C \longrightarrow B \xrightarrow{\pi} A \to 0.$$

Consider the dual exact sequence

$$0 \to A^\vee \xrightarrow{\pi^\vee} B^\vee \xrightarrow{\varphi} C^\vee \to 0.$$

The map $\varphi\colon B^\vee \to C^\vee$ is again an optimal quotient. Let $\lambda\colon B^\vee \to B$ be a symmetric isogeny, and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\varphi^\vee"] & B \arrow[r, "\pi"] & A \arrow[r] & 0 \\ 0 \arrow[r] & A^\vee \arrow[r, "\pi^\vee"] & B^\vee \arrow[u, "\lambda"] \arrow[r, "\varphi"] & C^\vee \arrow[r] & 0. \end{tikzcd} \tag{2.1}$$

The composite $\pi \circ \lambda \circ \pi^\vee : A^\vee \to A$ is a symmetric isogeny. Hence, by Corollary 2.4, we have

$$\deg(\pi \circ \lambda \circ \pi^\vee) = \#\operatorname{coker}\big((\pi \circ \lambda \circ \pi^\vee)^* : M_A \to M_{A^\vee}\big)^2. \tag{2.2}$$

By functoriality there is a commutative diagram

$$\begin{tikzcd} M_A \arrow[r, "\pi^*"] \arrow[d, "(\pi\circ\lambda\circ\pi^\vee)^*"'] & M_B \arrow[d, "\lambda^*"] \\ M_{A^\vee} & M_{B^\vee}. \arrow[l, "(\pi^\vee)^*"'] \end{tikzcd} \tag{2.3}$$

The image of M_A in M_B need not be saturated, i.e., the quotient group $M_B/(\pi^* M_A)$ might have torsion. We denote by

$$\overline{M}_A = (\pi^* M_A \otimes \mathbb{Q}) \cap M_B$$

the *saturation* of $\pi^* M_A$ inside of M_B, and likewise by $\overline{M}_{C^\vee}$ the saturation of $\varphi^* M_{C^\vee}$ inside $M_{B^\vee}$. Consider the sequence

$$M_A \xrightarrow{\pi^*} \overline{M}_A \xrightarrow{(\lambda\circ\pi^\vee)^*} M_{A^\vee}$$

induced from (2.3). The image of M_A in $M_{A^\vee}$ is the same, but now both π^* and $(\lambda \circ \pi^\vee)^*$ are finite-index injections. Hence

$$\begin{aligned}(2.4)\qquad &\#\operatorname{coker}\big((\pi\circ\lambda\circ\pi^\vee)^* : M_A \to M_{A^\vee}\big)\\ &\qquad = [\overline{M}_A : \pi^* M_A]\cdot \#\operatorname{coker}\big((\pi^\vee)^* : \lambda^*\overline{M}_A \to M_{A^\vee}\big).\end{aligned}$$

LEMMA 2.6. — *The sequence*

$$0 \to \overline{M}_{C^\vee} \longrightarrow M_{B^\vee} \xrightarrow{(\pi^\vee)^*} M_{A^\vee} \to 0$$

is exact.

Proof. — Since $\varphi \circ \pi^\vee = 0$, $\varphi^* M_{C^\vee}$ is in the kernel of $(\pi^\vee)^*$. Since $M_{A^\vee}$ is a free abelian group, $\ker(\pi^\vee)^*$ must contain the saturation $\overline{M}_{C^\vee}$. On the other hand, $\operatorname{rank}_{\mathbb{Z}} M_{B^\vee} - \operatorname{rank}_{\mathbb{Z}} \overline{M}_{C^\vee} = \operatorname{rank}_{\mathbb{Z}} M_{A^\vee}$ since $\dim B - \dim C = \dim A$. Hence the sequence is exact on the left. It remains to show that $(\pi^\vee)^*$ is surjective. This follows from $A^\vee \to B^\vee$ being a closed immersion; see [5], Proposition 3.3, Theorem 8.2. □

From the sequence $\overline{M}_A \xrightarrow{\lambda^*} M_{B^\vee} \xrightarrow{(\pi^\vee)^*} M_{A^\vee}$, using Lemma 2.6, we get an exact sequence of abelian groups

$$0 \to \overline{M}_{C^\vee} \longrightarrow M_{B^\vee}/\lambda^*\overline{M}_A \longrightarrow \operatorname{coker}(\lambda\circ\pi^\vee)^* \to 0.$$

Thus,

$$(2.5)\qquad \big[M_{A^\vee} : (\lambda\circ\pi^\vee)^*\overline{M}_A\big] = \big[M_{B^\vee} : \lambda^*\overline{M}_A \oplus \overline{M}_{C^\vee}\big].$$

PROPOSITION 2.7. — *With the morphism as in* (2.1) *and* (2.3) *we have*

$$\deg(\pi\circ\lambda\circ\pi^\vee)^{1/2} = [\overline{M}_A : \pi^* M_A]\cdot[M_{B^\vee} : \lambda^*\overline{M}_A \oplus \overline{M}_{C^\vee}].$$

Proof. — This follows from (2.2), (2.4) and (2.5). □

2.2. Monodromy pairing and component groups.

Let A be an abelian variety over K with *semi-stable reduction.* This means that $\mathcal{A}_k^0$ is an extension of an abelian variety $\mathfrak{A}_k$ over k by a torus T_k:

$$0 \to T_k \longrightarrow \mathcal{A}_k^0 \longrightarrow \mathfrak{A}_k \to 0.$$

We again denote by M_A the character group of T_k.

In [20], §§9–10, Grothendieck defines a canonical bilinear $\mathrm{Gal}(\bar{k}/k)$-equivariant pairing,

$$u_A : M_A \times M_{A^\vee} \longrightarrow \mathbb{Z},$$

which he calls the *monodromy pairing.* This pairing is uniquely characterized by the property that its extension of scalars $u_A \otimes \mathbb{Z}_\ell$, for a prime $\ell \neq \mathrm{char}(k)$, can be expressed in terms of the ℓ-adic Weil pairing on $T_\ell(A) \times T_\ell(A^\vee)$ via a formula given in [20], (9.1.2). The monodromy pairing has the following properties:

THEOREM 2.8 (Grothendieck).

(i) *u_A is non-degenerate.*

(ii) *u_A is bifunctorial in A.*

(iii) *If $\xi\colon A^\vee \to A$ is a polarization, then*

$$u_{A,\xi} : M_A \times M_A \xrightarrow{1\times\xi} M_A \times M_{A^\vee} \xrightarrow{u_A} \mathbb{Z}$$

is symmetric and positive-definite.

(iv) *There is a $\mathrm{Gal}(\bar{k}/k)$-equivariant exact sequence*

$$0 \to M_{A^\vee} \xrightarrow{u_A} \mathrm{Hom}(M_A, \mathbb{Z}) \longrightarrow \Phi_A \to 0.$$

Proof. — Once u_A is constructed compatibly with the Weil pairing (this is quite non-trivial), properties (i)–(iii) follow from the well-known facts about the latter pairing; see [20], Theorem 10.4. For the proof of (iv) see [20], Theorem 11.5. □

To simplify our later notations we give a consequence of Theorem 2.8 (iv). Let $\mathrm{rank}_{\mathbb{Z}} M_A = a$, and consider the pairing induced on $(\wedge^a M_A) \times (\wedge^a M_{A^\vee})$ by u_A. We denote this pairing by the same symbol. Let x and y be the generators of $\wedge^a M_A$ and $\wedge^a M_{A^\vee}$ respectively, then

$$\big|u_A(x,y)\big| = \#\Phi_A. \tag{2.6}$$

Now we return to the situation in (2.1).

LEMMA 2.9. — *With notation as in §2.1 and (2.1), let $(\overline{M}_A)^\perp \subset M_{B^\vee}$ be the orthogonal complement of $\overline{M}_A \subset M_B$ with respect to u_B. Then*

$$\overline{M}_{C^\vee} = (\overline{M}_A)^\perp.$$

Proof. — Let $x \in M_A$ and $y \in M_{C^\vee}$. Then using the bifunctoriality of u_B

$$u_B(\pi^* x, \varphi^* y) = u_A\big(x, (\pi^\vee)^* \circ \varphi^* y\big) = u_A\big(x, (\varphi \circ \pi^\vee)^* y\big).$$

Since $\varphi \circ \pi^\vee = 0$, we conclude $\varphi^* M_{C^\vee} \subset (\pi^* M_A)^\perp$. But u_B is also bilinear, hence the orthogonality extends to the saturations. Now comparing the ranks and using non-degeneracy of u_B, we get the desired equality. □

LEMMA 2.10. — *Let* $\dim A = a$. *Let* x_0 *be a generator of* $\wedge^a \overline{M}_A$. *Then*

$$\deg(\pi \circ \lambda \circ \pi^\vee)^{1/2} \cdot \#\Phi_A = [\overline{M}_A : \pi^* M_A]^2 \cdot \big|u_B(x_0, \lambda^* x_0)\big|.$$

Proof. — Let x and y be generators of $\wedge^a M_A$ and $\wedge^a M_{A^\vee}$ respectively. On the one hand, using bilinearity of u_A and (2.6), we have

$$\begin{aligned}\big|u_A(x, (\pi \circ \lambda \circ \pi^\vee)^* x)\big| &= \#\operatorname{coker}\big((\pi \circ \lambda \circ \pi^\vee)^* : M_A \to M_{A^\vee}\big) \cdot \big|u_A(x, y)\big| \\ &= \#\operatorname{coker}\big((\pi \circ \lambda \circ \pi^\vee)^* : M_A \to M_{A^\vee}\big) \cdot \#\Phi_A.\end{aligned}$$

On the other hand, using bifunctoriality and bilinearity of the monodromy pairing, we have

$$\big|u_A(x, (\pi \circ \lambda \circ \pi^\vee)^* x)\big| = \big|u_B(\pi^* x, \lambda^* \pi^* x)\big| = [\overline{M}_A : \pi^* M_A]^2 \cdot \big|u_B(x_0, \lambda^* x_0)\big|.$$

Using these two equalities along with Corollary 2.4, we get the result. □

COROLLARY 2.11. — *With notations as in Lemma* 2.10, *we have*

$$[M_{B^\vee} : \lambda^* \overline{M}_A \oplus \overline{M}_{C^\vee}] \cdot \#\Phi_A = [\overline{M}_A : \pi^* M_A] \cdot \big|u_B(x_0, \lambda^* x_0)\big|.$$

Proof. — This follows from Proposition 2.7 and Lemma 2.10. □

PROPOSITION 2.12. — *Let* $\pi_* : \Phi_B \to \Phi_A$ *be the morphism induced by* π. *Then*

$$[\overline{M}_A : \pi^* M_A] = \#\operatorname{coker}(\pi_* : \Phi_B \to \Phi_A).$$

Proof. — Consider the commutative diagram induced by π on the sequences in Theorem 2.8 (iv)

$$\begin{tikzcd} 0 \arrow[r] & M_{B^\vee} \arrow[r, "u_B"] \arrow[d] & \operatorname{Hom}(M_B, \mathbb{Z}) \arrow[r] \arrow[d, "\operatorname{Hom}(\pi^*, \mathbb{Z})"] & \Phi_B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_{A^\vee} \arrow[r, "u_A"] & \operatorname{Hom}(M_A, \mathbb{Z}) \arrow[r] & \Phi_A \arrow[r] & 0. \end{tikzcd}$$

Since $\pi^\vee : A^\vee \to B^\vee$ is a closed immersion, the left vertical arrow is surjective by [5], Theorem 8.2. Thus, from the snake lemma

$$\begin{aligned}\#\operatorname{coker}(\pi_* : \Phi_B \to \Phi_A) &= \#\mathrm{Ext}^1_{\mathbb{Z}}(M_B/\pi^* M_A, \mathbb{Z}) \\ &= \#(M_B/\pi^* M_A)_{\mathrm{tor}} = [\overline{M}_A : \pi^* M_A]. \qquad \square\end{aligned}$$

Remark 2.13. — The idea of the proof of Proposition 2.12 is due to K. Ribet [27].

2.3. Some exactness properties.

Let $\lambda_1 : B \to B^\vee$ be a symmetric isogeny and consider a diagram similar to (2.1) along with the induced morphisms

$$\begin{array}{ccccccccc} 0 \to & C & \xrightarrow{\varphi^\vee} & B & \xrightarrow{\pi} & A & \to 0 \\ & & & \big\downarrow{\scriptstyle \lambda_1} & & & \\ 0 \to & A^\vee & \xrightarrow{\pi^\vee} & B^\vee & \xrightarrow{\varphi} & C^\vee & \to 0. \end{array} \tag{2.7}$$

LEMMA 2.14. — *Let* $\dim A = a$ *and* $\dim C = c$. *Let* x_0 *and* y_0 *be generators of* $\wedge^a \overline{M}_A$ *and* $\wedge^c \overline{M}_{C^\vee}$ *respectively. With* λ *and* λ_1 *as in* (2.1) *and* (2.7), *we have*

$$\begin{aligned}&[M_B : \overline{M}_A \oplus \lambda_1^* \overline{M}_{C^\vee}] \cdot [M_{B^\vee} : \overline{M}_{C^\vee} \oplus \lambda^* \overline{M}_A] \cdot \#\Phi_B \\ &\qquad\qquad = \big|u_B(x_0, \lambda^* x_0)\big| \cdot \big|u_{B^\vee}(y_0, \lambda_1^* y_0)\big|.\end{aligned}$$

Proof. — Let V be the $\mathbb{Z}$-linear transformation of M_B into $\overline{M}_A \oplus \lambda_1^* \overline{M}_{C^\vee}$, and let W be the $\mathbb{Z}$-linear transformation of $M_{B^\vee}$ into $\overline{M}_{C^\vee} \oplus \lambda^* \overline{M}_A$. Let $\dim B = b$ and denote by z_0 and $z_0^\vee$ the generators of $\wedge^b M_B$ and $\wedge^b M_{B^\vee}$ respectively. Also denote by $V(z_0)$ and $W(z_0^\vee)$ the generators of $\wedge^b V(M_B)$ and $\wedge^b W(M_{B^\vee})$. Then we have

$$u_B\big(V(z_0), W(z_0^\vee)\big) = \det V \cdot \det W \cdot u_B(z_0, z_0^\vee) = \det V \cdot \det W \cdot \#\Phi_B. \tag{2.8}$$

On the other hand, from Lemma 2.9 we know that $\overline{M}_{C^\vee}$ is orthogonal to $\overline{M}_A$ with respect to monodromy pairing. Hence,

$$u_B\big(V(z_0), W(z_0^\vee)\big) = u_B(x_0, \lambda^* x_0) \cdot u_{B^\vee}(y_0, \lambda_1^* y_0). \tag{2.9}$$

It is clear that

$$\det V = [M_B : \overline{M}_A \oplus \lambda_1^* \overline{M}_{C^\vee}] \quad \text{and} \quad \det W = [M_{B^\vee} : \overline{M}_{C^\vee} \oplus \lambda^* \overline{M}_A].$$

This, combined with (2.8) and (2.9), finishes the proof. $\square$

With notation as in (2.1), consider the functorially-induced homomorphisms on the component groups, $\varphi_* : \Phi_{B^\vee} \to \Phi_{C^\vee}$ and $\pi_* : \Phi_B \to \Phi_A$.

Proposition 2.15. — *There is a canonical exact sequence of abelian groups*

$$0 \to \mathrm{coker}(\varphi_*)^\vee \longrightarrow \Phi_C \longrightarrow \Phi_B \longrightarrow \Phi_A \longrightarrow \mathrm{coker}(\pi_*) \to 0,$$

where $\mathrm{coker}(\varphi_*)^\vee$ *is the Pontrjagin dual of* $\mathrm{coker}(\varphi_*)$.

Proof. — We have the obvious complex

$$0 \to \ker(\varphi_*^\vee) \longrightarrow \Phi_C \longrightarrow \Phi_B \longrightarrow \Phi_A \longrightarrow \mathrm{coker}(\pi_*) \to 0.$$

There is a bifunctorial pairing $\Phi_A \times \Phi_{A^\vee} \to \mathbb{Q}/\mathbb{Z}$, defined by Grothendieck in [20], §1.2, for any abelian variety A over K, which turns out to be perfect when A is semi-stable; see [20], §11.3. Applied to the situation at hand, this pairing induces canonical isomorphisms $\Phi_C \cong (\Phi_{C^\vee})^\vee$ and $\ker(\varphi_*^\vee) \cong \mathrm{coker}(\varphi_*)^\vee$. Hence it is enough to show that

$$\frac{\#\Phi_A \cdot \#\Phi_{C^\vee}}{\#\Phi_B} = \#\,\mathrm{coker}(\pi_*) \cdot \#\,\mathrm{coker}(\varphi_*).$$

Using Corollary 2.11, we have

$$[M_{B^\vee} : \overline{M}_{C^\vee} \oplus \lambda^* \overline{M}_A] \cdot \#\Phi_A = [\overline{M}_A : \pi^* M_A] \cdot \left|u_B(x_0, \lambda^* x_0)\right|,$$
$$[M_B : \overline{M}_A \oplus \lambda_1^* \overline{M}_{C^\vee}] \cdot \#\Phi_{C^\vee} = [\overline{M}_{C^\vee} : \varphi^* M_{C^\vee}] \cdot \left|u_{B^\vee}(y_0, \lambda_1^* y_0)\right|,$$

where x_0 and y_0 are as in Lemma 2.14. Multiplying these equalities and using Lemma 2.14, we get

$$\#\Phi_A \cdot \#\Phi_{C^\vee} = [\overline{M}_A : \pi^* M_A] \cdot [\overline{M}_{C^\vee} : \varphi^* M_{C^\vee}] \cdot \#\Phi_B.$$

Now Proposition 2.12 implies what we want. □

3. Analytic calculations.

Let K be the completion of $F = \mathbb{F}_q(t)$ at $\infty = 1/t$, that is, with respect to the valuation given by the degree on polynomials with coefficients in $\mathbb{F}_q$. Denote by R the ring of integers of K, let ϖ be a uniformizer of K and $k = R/\varpi R$ be the residue field. Note that since $\deg(\infty) = 1$, $q_\infty := \#k$ is equal to q. We will denote by $|.| = q^{-\mathrm{ord}_K(\cdot)}$ the norm attached to the valuation on K, normalized by $\mathrm{ord}_K(\varpi) = 1$.

3.1. Drinfeld modular curves and their Jacobians.

For an ideal $\mathfrak{n}$ in $A = \mathbb{F}_q[t]$, denote by $Y_0(\mathfrak{n})$ the coarse moduli scheme of pairs (D, Z), where D is a rank-2 Drinfeld A-module over K, and Z is a $\mathfrak{n}$-cyclic subgroup of D; for definitions see [7] or [16]. From a theorem of Drinfeld [7], §5, one concludes that $Y_0(\mathfrak{n})$ is a smooth affine curve over K. Moreover, since in our case $\mathrm{Pic}(A) = 1$, $Y_0(\mathfrak{n})$ is geometrically connected; see [16], (2.5). Denote by $X_0(\mathfrak{n})$ the unique smooth compactification of $Y_0(\mathfrak{n})$ over K. Drinfeld modular curves $Y_0(\mathfrak{n})$ have analytic uniformization which we proceed to describe.

Let $\Omega = \mathbb{P}_K^{1,\mathrm{an}} - \mathbb{P}_K^{1,\mathrm{an}}(K)$. Since K has a finite residue field, Ω has a natural structure of a smooth geometrically connected rigid-analytic space [7], §6. It is called the *Drinfeld upper half-plane.* For all $n \in \mathbb{Z}$ set

$$D_n = \{z \in \mathbb{P}_K^{1,\mathrm{an}} - \{\infty\}\,;\ |\varpi|^{n+1} \le |z| \le |\varpi|^n,\ |z - \varpi^n\rho| \ge |\varpi|^n,\\ |z - \varpi^{n+1}\rho| \ge |\varpi|^{n+1},\ \text{for all } \rho \in k^\times\}$$

and for all $x \in F$ let

$$D_{n,x} = x + D_n.$$

Let $A_{n,x}$ be the affinoid algebra of holomorphic functions on $D_{n,x}$. Then we have

$$A_{n,x} = K\Big\langle \frac{z-x}{\varpi^n}, \frac{\varpi^{n+1}}{z-x}, \frac{1}{(z-x)/\varpi^n - \rho},\\ \frac{1}{(z-x)/\varpi^{n+1} - \rho}\ ;\ \rho \in k^\times \Big\rangle; \tag{3.1}$$

this means that $A_{n,x}$ is the set of series of the form

$$f = \sum_{i\ge 0} a_i\Big(\frac{z-x}{\varpi^n}\Big)^i + \sum_{i>0} b_i\Big(\frac{\varpi^{n+1}}{z-x}\Big)^i + \sum_{\rho\in k^\times}\sum_{i>0} c_{i,\rho}\Big(\frac{1}{(z-x)/\varpi^n - \rho}\Big)^i\\ + \sum_{\rho\in k^\times}\sum_{i>0} d_{i,\rho}\Big(\frac{1}{(z-x)/\varpi^{n+1} - \rho}\Big)^i$$

where $a_i, b_i, c_{i,\rho}, d_{i,\rho}$ are in K and satisfy

$$\lim_{i\to\infty} |a_i| = \lim_{i\to\infty} |b_i| = \lim_{i\to\infty} |c_{i,\rho}| = \lim_{i\to\infty} |d_{i,\rho}| = 0.$$

Of course, we have abstract isomorphisms of normed algebras $A_{n,x} \cong A_{n',x'}$, but it's not hard to check that $D_{n,x} = D_{n',x'}$ inside of $\mathbb{P}_K^{1,\mathrm{an}}$ if and only if $n=n'$ and $|x-x'|\le|\varpi|^{n+1}$. The intersection of distinct $D_{n,x}$ and $D_{n',x'}$ is either empty, equal to

$$\operatorname{Sp} K\Big\langle \frac{z-x}{\varpi^n}, \frac{\varpi^n}{z-x-\rho\varpi^n};\ \rho\in k\Big\rangle,$$

or equal to the same set with n replaced by $n+1$. If $D_{n,x}\cap D_{n',x'}\neq\emptyset$ then we glue them along the intersection; this gives the analytic space $D_{n,x}\cup D_{n',x'}$. With this process, one constructs an analytic space $\bigcup D_{n,x}$, which gives the analytic structure of Ω:

THEOREM 3.1. — *The Drinfeld upper half-plane Ω is an admissible open in $\mathbb{P}^{1,\mathrm{an}}_K$, and $\{D_{n,x}\}$ constitutes an admissible cover of Ω.*

Proof. — See [7], Proposition 6.2, and [16], §1.2. □

Let $\|f\|^{\mathrm{sp}}=\sup_{z\in D_{n,x}}|f(z)|$ be the spectral norm on $A_{n,x}$. Let

$$A^0_{n,x}=\{f\in A_{n,x}\,;\ \|f\|^{\mathrm{sp}}\le 1\},\quad A^{00}_{n,x}=\{f\in A_{n,x}\,;\ \|f\|^{\mathrm{sp}}<1\}.$$

As is easy to see [9], §V.1.2, the *analytic reduction* of $D_{n,x}$

$$\overline{D}_{n,x}=\operatorname{Spec}(A^0_{n,x}/A^{00}_{n,x}) \tag{3.2}$$

is k-isomorphic to the union of two projective lines over k meeting transversally in a k-rational point, with the other rational points deleted from both curves. By gluing the reductions $\overline{D}_{n,x}$ along the canonical reductions of $D_{n,x}\cap D_{n',x'}$, we obtain the analytic reduction $\overline{\Omega}$ of Ω with respect to the pure affinoid covering $\{D_{n,x}\}$. (Recall [18], p. 116, that an admissible affinoid covering $\{Z_i\}$ of an analytic space Z is called *pure* if each Z_i meets only finitely many Z_j and for any $Z_i\cap Z_j\neq\emptyset$ there exists an open affine subscheme $U_{i,j}\subset\overline{Z}_i$ in the canonical analytic reduction of Z_i whose preimage in Z_i and in Z_j is $Z_i\cap Z_j$.) The reduction $\overline{\Omega}$ is a scheme over k, locally of finite type, each irreducible component of $\overline{\Omega}$ is isomorphic to $\mathbb{P}^1_k$ and meets exactly $q_\infty+1$ other components. The intersections are ordinary double points which are rational over k.

DEFINITION 3.2. — Let X be a semi-stable curve that is locally of finite type over a field k, and has k-rational singularities. The *dual graph* (or the intersection graph) of X is a graph $\mathcal{G}=\mathcal{G}(X)$ such that the vertices of $\mathcal{G}$ are the irreducible components of X, say $X_1,\dots,X_r,\dots$, and the edges are given by the singular points of X; namely each singular point lying on X_i and X_j defines an edge joining the vertices X_i and X_j ($X_i=X_j$ is allowed). A choice of ordering of the two branches passing through each singular point gives $\mathcal{G}$ an orientation.

The dual graph of $\overline{\Omega}$ is an infinite tree such that each vertex is connected to $q_\infty + 1$ other vertices and no two vertices have double edges between them. Hence we can choose an orientation on $\mathcal{G}(\overline{\Omega})$ to have $\mathcal{G}(\overline{\Omega}) \cong \mathcal{T}$, where $\mathcal{T}$ the *Bruhat-Tits tree* of $\mathrm{PGL}_2(K)$; cf. [16], (1.3).

Let

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)\, ;\; c \equiv 0 \bmod \mathfrak{n} \right\} \tag{3.3}$$

be the *Hecke congruence subgroup* of level $\mathfrak{n}$. This acts on Ω via linear fractional transformations, and the action is *discrete* in the sense of [7], p. 582. Hence we may construct the quotient $\Gamma_0(\mathfrak{n}) \setminus \Omega$ as a 1-dimensional geometrically-connected smooth analytic space over K, and its formation is compatible with the change of the ground field, cf. *loc. cit.* It is a theorem of Drinfeld [7], Proposition 6.6, that there is a canonical isomorphism of analytic spaces

$$\Gamma_0(\mathfrak{n}) \setminus \Omega \xrightarrow{\sim} Y_0(\mathfrak{n})^{\mathrm{an}}. \tag{3.4}$$

One can conclude from this [18], Lemma V.1.5, that $X_0(\mathfrak{n})$ has a model $X_0(\mathfrak{n})_R$ over R which is a Mumford curve:

Definition 3.3. — A scheme X over $\mathrm{Spec}(R)$ is called a *Mumford curve* (or *k-split degenerate curve*) if it has the following properties:

- X is proper and flat over $\mathrm{Spec}(R)$ with one-dimensional geometrically reduced fibres, and geometrically irreducible and smooth generic fibre X_K.

- All of the components of the normalization of the closed fibre X_k are isomorphic to $\mathbb{P}^1_k$, and all the singularities of X_k are ordinary k-rational double points with two k-rational branches.

Let $J_0(\mathfrak{n}) := \mathrm{Pic}^0_{X_0(\mathfrak{n})/K}$ be the Jacobian variety of $X_0(\mathfrak{n})$, and let $\mathcal{J}$ be the Néron model of J over R. By the Néronian property we have a canonical morphism $\mathrm{Pic}^0_{X_0(\mathfrak{n})_R/R} \to \mathcal{J}$, which by [3], Corollary 9.7/2, induces an isomorphism

$$\mathrm{Pic}^0_{X_0(\mathfrak{n})_R/R} \xrightarrow{\sim} \mathcal{J}^0.$$

Hence by Example 8, p. 247 of [3], $\mathcal{J}^0_k \cong \mathbb{G}^g_{m,k}$ is a split torus over k. (Here g is the genus of $X_0(\mathfrak{n})$.) The lifting of tori in [19], Theorem 3.6, implies that the formal completion of $\mathcal{J}^0$ along its closed fibre is canonically isomorphic to a formal split torus $\widehat{\mathbb{G}}^g_m = (\mathrm{Spf}(R\{T, T^{-1}\}))^g$. Consider Raynaud's generic fibre $(\widehat{\mathbb{G}}_m^g)^{\mathrm{rig}} = (\mathrm{Sp}(K\langle T, T^{-1}\rangle))^g$; cf. [2]. There is a canonical open immersion of analytic groups [4], Theorem 5.3.1,

$$i_{\mathcal{J}^0} : (\widehat{\mathbb{G}}_m^g)^{\mathrm{rig}} \hookrightarrow J_0(\mathfrak{n})^{\mathrm{an}}.$$

We also have the analytic split torus $G = (\mathbb{G}_{m,K}^{\mathrm{an}})^g$ associated to $\mathbb{G}_{m,K}^g$, and an open immersion $(\widehat{\mathbb{G}}_m^g)^{\mathrm{rig}} \hookrightarrow G$.

THEOREM 3.4. — *With $J_0(\mathfrak{n})$ and G as above, $i_{\mathcal{J}^0}$ extends uniquely to a rigid-analytic group morphism $\pi : G \to J_0(\mathfrak{n})^{\mathrm{an}}$. The kernel of π is a free lattice $\Lambda \subset G(K)$ of rank g, and we have an isomorphism of rigid-analytic groups*

$$G/\Lambda \cong J_0(\mathfrak{n})^{\mathrm{an}}.$$

Proof. — See [1], Theorem 1.2. □

Recall that a (split) *lattice* of rank $\kappa \leq g$ is a discrete subgroup Λ of $G(K)$, such that under $\log|.| : G(K) \to \mathbb{R}^g$, Λ maps onto a lattice of rank κ in $\mathbb{R}^g$ with finite kernel. In particular, Λ as an abelian group, up to finite torsion, is free of rank κ, and for each affinoid U in G, $U \cap \Lambda$ is finite. If the lattice Λ has no torsion we will say it is *free*. If $\kappa = g$ we will say that the lattice is of *full rank*.

Since $J_0(\mathfrak{n})$ is principally polarized, using [1], Theorem 2.1, it can be shown that the analytic torus G in Theorem 3.4 is canonically isomorphic to $\mathrm{Hom}(\Lambda, \mathbb{G}_{m,K}^{\mathrm{an}})$, where Hom indicates homomorphism of analytic groups. Hence we have a uniformization

$$0 \to \Lambda \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{G}_{m,K}^{\mathrm{an}}) \longrightarrow J_0(\mathfrak{n})^{\mathrm{an}} \to 0. \tag{3.5}$$

Theta functions and the uniformization of $J_0(\mathfrak{n})$.

The uniformization of $J_0(\mathfrak{n})$, which exists due to the nature of the reduction of Drinfeld Jacobians, can be made quite explicit by using rigid-analytic automorphic functions relative to $\Gamma_0(\mathfrak{n})$. This is carefully treated in [16], §§5–7. We recall the main theorem [16], Theorem 7.4.1. To simplify the notation we write $\Gamma := \Gamma_0(\mathfrak{n})$ (assuming $\mathfrak{n}$ is fixed) and $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/(\Gamma^{\mathrm{ab}})_{\mathrm{tor}}$. One can show [16], (3.2), that $\overline{\Gamma}$ is a free abelian group of rank g.

A *holomorphic theta function* for Γ over K is an invertible holomorphic function $u : \Omega \to \mathbb{A}_K^1$ over K such that for each $\gamma \in \Gamma$ there exists

$c_u(\gamma) \in K^\times$ with $u(\gamma z) = c_u(\gamma)u(z)$, and which is holomorphic and non-zero at the cusps of Γ. For $\alpha \in \Gamma$, and $\omega \in \Omega$ an arbitrary base point, consider

$$u_\alpha(z) = \prod_{\gamma \in \widetilde{\Gamma}} \Big(\frac{z - \gamma\omega}{z - \gamma\alpha\omega} \Big),$$

where $\widetilde{\Gamma}$ is the quotient of Γ by its center. The group $\widetilde{\Gamma}$ acts faithfully on Ω.

THEOREM 3.5.

(i) *$u_\alpha(z)$ converges locally uniformly to an invertible function u_α on Ω that does not depend on the choice of $\omega \in \Omega$ (and so is defined over K).*

(ii) *u_α is a holomorphic theta function, whose multiplier $c_\alpha : \gamma \mapsto c_{u_\alpha}(\gamma)$ is a homomorphism $K^\times$ that only depends on the class of α in $\overline{\Gamma}$.*

(iii) *The map $\overline{\Gamma} \times \overline{\Gamma} \to K^\times$ given by $(\alpha,\beta) \mapsto c_\alpha(\beta)$ is symmetric and bilinear.*

(iv) *The symmetric bilinear form $(\alpha,\beta) \mapsto -\operatorname{ord}_K(c_\alpha(\beta))$ on $\overline{\Gamma} \times \overline{\Gamma} \to \mathbb{Z}$ is positive-definite, and*

$$\bar{c} : \overline{\Gamma} \longrightarrow \operatorname{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}}), \quad \alpha \longmapsto c_\alpha$$

is injective.

(v) *We have an exact sequence of analytic groups*

$$0 \to \overline{\Gamma} \xrightarrow{\bar{c}} \operatorname{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}}) \longrightarrow J_0(\mathfrak{n})^{\mathrm{an}} \to 0.$$

Proof. — See [16], §§5.4–5.7, 7.4. □

3.3. Drinfeld's cusp forms and the Hecke algebra.

Let again $\Gamma := \Gamma_0(\mathfrak{n})$ and $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/(\Gamma^{\mathrm{ab}})_{\mathrm{tor}}$. Also let $J := J_0(\mathfrak{n})$. Denote the edges (resp. vertices) of the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(K)$ by $\mathrm{Ed}(\mathcal{T})$ (resp. $\mathrm{Ver}(\mathcal{T})$). Given an edge $e \in \mathrm{Ed}(\mathcal{T})$, we will denote by $\bar{e}$ the edge of $\mathcal{T}$ which corresponds to e with opposite orientation. For any abelian group B, let $\underline{\mathrm{H}}_!(\mathcal{T}, B)^\Gamma$ be the group of maps ϕ: $\mathrm{Ed}(\mathcal{T}) \to B$ subject to

(i) $\phi(\bar{e}) = -\phi(e)$;

(ii) $\sum_{t(e)=v} \phi(e) = 0$ for any $v \in \mathrm{Ver}(\mathcal{T})$, where $t(e)$ is the terminus of e;

(iii) $\phi(\gamma e) = \phi(e)$ for $\gamma \in \Gamma$;

(iv) ϕ has finite support modulo Γ.

We call this group the *group of B-valued cuspidal harmonic cochains* for Γ.

Using the *strong approximation theorem* for function fields, it can be shown that $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{C})^\Gamma$ may be interpreted as a space of automorphic cusp forms on $\mathrm{GL}_2(\mathbb{A}_F)$ which are special at ∞; see [7], Proposition 10.3, and [16], §4. We have the finitely generated $\mathbb{Z}$-algebra of Hecke operators $\mathbb{T} := \{\mathbb{Z}[T_\mathfrak{m}] \mid \mathfrak{m}$ is an ideal of $A\}$ acting on $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{C})^\Gamma$ by the usual formulae; cf. [17], p. 47 or [12], (1.10). Moreover, $\mathbb{T}$ preserves the canonical integral structure $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Z})^\Gamma$. The elements of $\mathbb{T}$ also act on $Y_0(\mathfrak{n})$ as algebraic correspondences – one uses the moduli interpretation of $Y_0(\mathfrak{n})$ to define this action. These correspondences uniquely extend to $X_0(\mathfrak{n})$ and hence $\mathbb{T}$ is naturally enclosed as a subring of $\mathrm{End}_K(J)$. Let ℓ be a prime number not equal to the characteristic of K. Let

$$V_\ell(J) := \big(\varprojlim J[\ell^n](K^{\mathrm{sep}})\big) \otimes \mathbb{Q}_\ell$$

be the ℓ-adic Tate vector space of J. Via the canonical injection

$$\mathrm{End}_K(J) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{End}_{\mathrm{Gal}(K^{\mathrm{sep}}/K)}\big(V_\ell(J)\big),$$

we get $\mathbb{T} \otimes \mathbb{Q}_\ell \subseteq \mathrm{End}_{\mathrm{Gal}(K^{\mathrm{sep}}/K)}(V_\ell(J))$. Let $V_\ell(J)^* = \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(J),\mathbb{Q}_\ell)$ be the linear dual of $V_\ell(J)$. Let $\mathrm{spe}(2)$ be the two-dimensional *special representation* of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, cf. [6], §3.1.2. The following fundamental result is due to Drinfeld [7], Theorem 2.

THEOREM 3.6. — *There is a canonical isomorphism*

$$V_\ell(J)^* \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \underline{\mathrm{H}}_!(\mathcal{T},\overline{\mathbb{Q}}_\ell)^\Gamma \otimes \mathrm{spe}(2).$$

of $\mathbb{T} \times \mathrm{Gal}(K^{\mathrm{sep}}/K)$*-modules.*

One can conclude from this theorem that the abelian subvarieties of J which are stable under the action of $\mathbb{T}$ as a subalgebra of $\mathrm{End}_K(J)$ are in one-to-one correspondence with the stable subspaces of $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Q})^\Gamma$. We recall the decomposition of this latter vector space into $\mathbb{T}$-invariant subspaces, which is the analogue of a well-known result of Atkin and Lehner for the classical weight-2 cusp forms.

Let $\mathfrak{m}$ be an ideal of A. Denote by $\underline{\mathrm{H}}_!^{\mathrm{new}}(\mathcal{T},\mathbb{C})^{\Gamma_0(\mathfrak{m})} \subseteq \underline{\mathrm{H}}_!(\mathcal{T},\mathbb{C})^{\Gamma_0(\mathfrak{m})}$ the orthogonal complement with respect to the Petersson inner product on the cusp forms [16], (4.8), of the different embeddings of $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{C})^{\Gamma_0(\mathfrak{m}')}$

into $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{C})^{\Gamma_0(\mathfrak{m})}$, where $\mathfrak{m}'$ runs through the strict divisors of $\mathfrak{m}$. Let $\phi \in \underline{\mathrm{H}}_!^{\mathrm{new}}(\mathcal{T},\mathbb{C})^{\Gamma_0(\mathfrak{m})}$ be a normalized eigenform for the action of $\mathbb{T}$ – we call ϕ a *newform of level* $\mathfrak{m}$. (There is a more intrinsic definition of newforms given by Casselman; see [17], Proposition 6.17).

Now given a newform ϕ of some level $\mathfrak{n}_\phi \mid \mathfrak{n}$, let $\underline{\mathrm{H}}_\phi$ be the space spanned by the linearly independent forms $\phi\big(\big(\begin{smallmatrix}\mathfrak{a} & 0\\ 0 & 1\end{smallmatrix}\big)e\big)$, where $\mathfrak{a}$ ranges over the monic divisors of $\mathfrak{n}/\mathfrak{n}_\phi$. Then $\underline{\mathrm{H}}_\phi$ is stable under the action of $\mathbb{T}$ and moreover we have

$$\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{C})^{\Gamma} = \bigoplus_{\phi} \underline{\mathrm{H}}_\phi, \tag{3.6}$$

where the sum is taken over all newforms ϕ of some level $\mathfrak{n}_\phi$ dividing $\mathfrak{n}$. Let $[\phi]$ be the Galois orbit (under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) of ϕ. The space

$$\underline{\mathrm{H}}_{[\phi]} = \bigoplus_{\psi\in[\phi]} \underline{\mathrm{H}}_\psi$$

is a finite dimensional $\mathbb{C}$-vector space spanned by $\mathbb{Q}$-valued cusp forms, and we denote by $\underline{\mathrm{H}}_{[\phi]}(\mathbb{Q})$ the $\mathbb{Q}$-subspace spanned by this $\mathbb{Q}$-basis. Then we have $\mathbb{T}_{\mathbb{Q}}$-stable decomposition

$$\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Q})^{\Gamma} = \bigoplus_{[\phi]} \underline{\mathrm{H}}_{[\phi]}(\mathbb{Q}),$$

which reflects the decomposition of J (up to isogeny) into a product of $\mathbb{T}$-stable abelian subvarieties. Let the ideal $I_{[\phi]}$ be the kernel of the action of $\mathbb{T}$ on $\underline{\mathrm{H}}_{[\phi]}(\mathbb{Q})$.

Then $I_{[\phi]}(J)$ is a connected smooth abelian subvariety of J which is stable under $\mathbb{T}$ and is defined over K.

DEFINITION 3.7. — The *optimal abelian quotient* of J associated to $[\phi]$ is the abelian variety $A_\phi = J/I_{[\phi]}J$.

From the definition it is clear that A_ϕ is an abelian variety defined over K, and that there is a natural action of $\mathbb{T}$ on A_ϕ which factors through $\mathbb{T}/I_{[\phi]}$. Moreover, the multiplicity one theorem for cuspidal representations implies that A_ϕ as an optimal quotient of J is uniquely characterized by this property.

The Hecke algebra also acts on $\overline{\Gamma}$. This follows from the analytic uniformization of J in Theorem 3.5. Indeed, by a theorem of van der

Put [32], (3.3), the torus $\mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}})$ is a universal covering space of J^{an}, and hence every endomorphism of J^{an} uniquely lifts to an endomorphism of the torus which preserves the lattice $\bar{c}(\overline{\Gamma})$. This action can be made explicit [16], (9.3), and one has the following important fact

THEOREM 3.8. — *There is a natural* $\mathbb{T}$*-equivariant isomorphism*

$$\overline{\Gamma} \xrightarrow{\sim} \underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Z})^{\Gamma},$$

Proof. — See [16], Lemma 9.3.2, and [15]. □

Another important feature of the action of $\mathbb{T}$ on $\overline{\Gamma}$ is that the map $\bar{c}$ in Theorem 3.5 is $\mathbb{T}$-equivariant.

THEOREM 3.9. — *The action of* $\mathbb{T}$ *is symmetric with respect to the bilinear pairing* $\bar{c}\colon \overline{\Gamma} \times \overline{\Gamma} \to K^{\times}$ *in Theorem* 3.5 (iii). *Moreover, the exact sequence*

$$0 \to \overline{\Gamma} \xrightarrow{\bar{c}} \mathrm{Hom}(\overline{\Gamma},\mathbb{G}_{m,K}^{\mathrm{an}}) \longrightarrow J^{\mathrm{an}} \to 0.$$

is $\mathbb{T}$*-equivariant.*

Proof. — See [16], (9.3)–(9.4). □

Hence we have a decomposition of $\overline{\Gamma}_{\mathbb{Q}}$ into $\mathbb{T}_{\mathbb{Q}}$-stable subspaces similar to (3.6), and to each optimal quotient A of J, such that the kernel of $J \to A$ as an abelian subvariety of J is $\mathbb{T}$-stable, we can associate a unique saturated sublattice Υ of $\overline{\Gamma}$ which is stable under the action of $\mathbb{T}$ (*saturated* means that the quotient $\overline{\Gamma}/\Upsilon$ is torsion free). The sublattice in question is the image under $\bar{c}$ of $\overline{\Gamma} \cap \overline{\Gamma}_{\mathbb{Q},[\phi]}$ using the isomorphism in Theorem 3.8. Conversely, in §3.4, starting with a saturated $\mathbb{T}$-stable sublattice Υ of $\overline{\Gamma}$, we will construct an optimal quotient A_{Υ} of J, which as a quotient of J is uniquely determined by Υ.

3.4. Analytic construction of optimal abelian quotients of $J_0(\mathfrak{n})$.

We will need the following fact

THEOREM 3.10. — *There is a non-degenerate pairing*

$$\mathbb{T} \times \underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Z})^{\Gamma} \longrightarrow \mathbb{Z},$$

which becomes perfect after tensoring with $\mathbb{Z}[p^{-1}]$ *(here* p *is the characteristic of* K*).*

Proof. — The method of the proof is similar to the one used to show that the $\mathbb{Z}$-dual of the lattice of classical modular cusp forms with integral Fourier coefficients forms a free $\mathbb{T}$-module of rank 1, i.e., using the Fourier expansion of cusp forms, one constructs a pairing between $\underline{\mathrm{H}}_!(\mathcal{T},\mathbb{Z})^\Gamma$ and $\mathbb{T}$ which maps (ϕ, T) to the first Fourier coefficient of $T(\phi)$. See [12], Theorem 3.17, for the details. $\square$

PROPOSITION 3.11. — *Let* $s\colon \Upsilon \hookrightarrow \overline{\Gamma}$ *be a saturated* $\mathbb{T}$*-stable subgroup with* $\mathrm{rank}_\mathbb{Z}(\Upsilon) = \kappa$*. Then the image* Λ *of* $\overline{\Gamma}$ *under the composition*

$$\overline{\Gamma} \xrightarrow{\bar{c}} \mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}}) \longrightarrow \mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}}),$$

is a split lattice in $\mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}})$ *of full rank (but not necessarily free).*

Proof. — Let $\mathcal{L}\colon \mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}}) \xrightarrow{\log|\cdot|} \mathbb{R}^\kappa$. By definition, to show that Λ is a lattice we need to show that the restriction of $\mathcal{L}\colon \Lambda \to \mathbb{R}^\kappa$ maps Λ onto a lattice in $\mathbb{R}^\kappa$ with finite kernel. It is enough to show that $\mathrm{rank}_\mathbb{Z}(\Lambda) = \mathrm{rank}_\mathbb{Z}(\mathcal{L}(\Lambda)) = \kappa$ and $\mathcal{L}(\Lambda)$ contains a basis of $\mathbb{R}^\kappa$. From Theorems 3.10 and 3.8 one concludes that $\overline{\Gamma} \otimes \mathbb{Q}$ is a free $\mathbb{T} \otimes \mathbb{Q}$-module of rank 1. Thus the lattice $\overline{\Gamma}$ contains a sublattice $\overline{\Gamma}'$ of full rank which is cyclic under $\mathbb{T}$; i.e., $\overline{\Gamma}' = \mathbb{T}\gamma'$ for some $\gamma' \in \overline{\Gamma}$ and $[\overline{\Gamma} : \overline{\Gamma}']$ is finite. The image of $\overline{\Gamma}'$ in $\mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}})$ is the restriction of $\bar{c}(\overline{\Gamma}', .)$ to Υ. Since by Theorem 3.9 the action of $\mathbb{T}$ on $\overline{\Gamma}$ is symmetric with respect to $\bar{c}$ and Υ is $\mathbb{T}$-stable, the image of $\overline{\Gamma}'$ is just the restriction of $\bar{c}(\gamma', .)$ to Υ. Since $[\overline{\Gamma} : \overline{\Gamma}']$ is finite, we get $\mathrm{rank}_\mathbb{Z}(\Lambda) = \mathrm{rank}_\mathbb{Z}(\Upsilon^*)$, where Υ^* is the linear dual of Υ with respect to $\bar{c}(\gamma', .)$. Hence

$$\mathrm{rank}_\mathbb{Z}(\Lambda) \leq \mathrm{rank}_\mathbb{Z} \Upsilon = \kappa. \tag{3.7}$$

Next, $\mathcal{L}(\Lambda)$ as a subgroup of $\mathrm{Hom}(\Upsilon, \mathbb{R}) \cong \mathbb{R}^\kappa$ contains $-\mathrm{ord}_K\, \bar{c}(\Upsilon, .)_{|\Upsilon}$. By Theorem 3.5 $-\mathrm{ord}(\bar{c})$ is a symmetric positive-definite bilinear pairing on $\overline{\Gamma} \times \overline{\Gamma}$. Hence $\mathcal{L}(\Lambda)$ contains a basis of $\mathbb{R}^\kappa$. In particular, $\mathrm{rank}_\mathbb{Z}(\mathcal{L}(\Lambda)) \geq \kappa$. On the other hand, we obviously have $\mathrm{rank}_\mathbb{Z}(\Lambda) \geq \mathrm{rank}_\mathbb{Z}(\mathcal{L}(\Lambda))$. Combining this with (3.7), we have

$$\kappa \leq \mathrm{rank}_\mathbb{Z}\big(\mathcal{L}(\Lambda)\big) \leq \mathrm{rank}_\mathbb{Z}(\Lambda) \leq \kappa.$$

So equality holds throughout. Finally observe that Λ is split since $\bar{c}(\overline{\Gamma})$ is split. $\square$

PROPOSITION 3.12. — *Let Υ and Λ be as in Proposition 3.11. The quotient $\mathfrak{A} := \mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}})/\Lambda$ is the analytification of an abelian variety A of dimension κ, which is an optimal quotient of J. Moreover, A as an optimal quotient of J is uniquely determined by Υ, and we denote it by A_Υ.*

Proof. — We have a commutative $\mathbb{T}$-equivariant diagram

$$\begin{array}{ccccccc} 0 \to \overline{\Gamma} & \xrightarrow{\bar{c}} & \mathrm{Hom}(\overline{\Gamma}, \mathbb{G}_{m,K}^{\mathrm{an}}) & \longrightarrow & J^{\mathrm{an}} & \to 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & \\ 0 \to \Lambda & \longrightarrow & \mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}}) & \longrightarrow & \mathfrak{A} & \to 0. \end{array} \tag{3.8}$$

By Proposition 3.11 Λ is a split lattice. The quotients of split tori by split lattices always exist as connected proper separated analytic group spaces over K; see [9], Proposition VI.4.5. On the other hand, $\bar{c}$ restricted to Υ is a symmetric bilinear positive-definite form. Hence $\mathfrak{A} = A^{\mathrm{an}}$ for some abelian variety A by [1], Theorem 2.4. The kernel G' of the middle vertical map in (3.8) is isomorphic to $\mathrm{Hom}(\overline{\Gamma}/\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}})$. Since Υ is saturated, $\overline{\Gamma}/\Upsilon$ is free and G' is a split torus. Let $\Lambda' = \bar{c}(\overline{\Gamma}) \cap G'$. This is a full rank sublattice of G'. One way to see this is to observe that $\Upsilon^\perp = \{\gamma \in \overline{\Gamma}\,;\ \mathrm{ord}_K\, \bar{c}(\gamma)_{|\Upsilon} = 0\}$ maps injectively into Λ' with finite index. Hence the kernel of $J^{\mathrm{an}} \to \mathfrak{A}$ is connected. Moreover, it is smooth as the quotient map $G' \to G'/\Lambda'$ is étale. So by GAGA and the definition of optimality 2.5, A is an optimal quotient of J.

As we mentioned previously, by multiplicity-one the optimal quotients A of J for which $\ker(J \to A)$ is $\mathbb{T}$-stable, are uniquely determined by the kernel of the natural homomorphism $\delta_A : \mathbb{T} \to \mathrm{End}_K(A)$. Since (3.8) is $\mathbb{T}$-equivariant, $\ker(\delta_A)$ is isomorphic to the kernel of $\mathbb{T} \to \mathrm{End}(\Upsilon)$. This implies the last part of the claim. □

3.5. Analytic calculation of the degree.

Consider the pairing

$$\langle .,. \rangle : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow \mathbb{Z}, \quad (\gamma_1, \gamma_2) \longmapsto -\mathrm{ord}_K\, \bar{c}(\gamma_1, \gamma_2). \tag{3.9}$$

By Theorem 3.5 this pairing is bilinear, symmetric, and positive-definite. Let Υ be a saturated $\mathbb{T}$-stable sublattice of $\overline{\Gamma}$, and let A_Υ be its corresponding optimal quotient of J, cf. Proposition 3.12. Denote

$$\Upsilon^\perp = \{\gamma \in \overline{\Gamma}\,;\ \langle \gamma, \beta \rangle = 0 \text{ for all } \beta \in \Upsilon\}.$$

It is clear that $\Upsilon^\perp$ is saturated, and since the action of $\mathbb{T}$ is symmetric with respect to $\langle .,. \rangle$, $\Upsilon^\perp$ also $\mathbb{T}$-stable. Moreover, due to positive definiteness of $\langle .,. \rangle$ we have $\Upsilon \cap \Upsilon^\perp = 0$ and $\Upsilon \oplus \Upsilon^\perp$ has finite index in $\overline{\Gamma}$. Consider the dual to the quotient map $\pi\colon J \to A_\Upsilon$. Using the fact that J is principally polarized and π is an optimal quotient, we get a closed immersion $\pi^\vee\colon A_\Upsilon^\vee \hookrightarrow J$, where $A_\Upsilon^\vee$ is the dual abelian variety of A_Υ. The composite $\pi \circ \pi^\vee : A_\Upsilon^\vee \to A_\Upsilon$ is a symmetric isogeny in sense of §2.1; in fact, it is a polarization. Denote $Q = \overline{\Gamma}/(\Upsilon \oplus \Upsilon^\perp)$; this is a finite abelian group, which can be regarded as a commutative étale group-scheme over K. Denote also Λ_{tor} the finite torsion subgroup of Λ in Proposition 3.11. Since $\Lambda_{\mathrm{tor}} \subset (K^\times)^\kappa$, and K has positive characteristic p, in particular $K^\times$ has no non-trivial p-th roots of unity, Λ_{tor} is a commutative étale constant group-scheme of order coprime to the characteristic of K. Let $r = \#Q$ and $d = \#\Lambda_{\mathrm{tor}}$.

PROPOSITION 3.13. — *With previous notation, we have an exact sequence of commutative finite flat group-schemes over K*

$$0 \to H^* \longrightarrow \ker(\pi \circ \pi^\vee) \longrightarrow H \to 0,$$

where $H \in \mathrm{Ext}^1_K(Q, \Lambda_{\mathrm{tor}})$ and $*$ denotes the Cartier dual. In particular,

$$\deg(\pi \circ \pi^\vee)^{1/2} = d \cdot r.$$

Proof. — The subtorus corresponding to the subvariety $A_\Upsilon^\vee \hookrightarrow J$ in the analytic uniformization of J is $\mathrm{Hom}(\overline{\Gamma}/\Upsilon^\perp, \mathbb{G}^{\mathrm{an}}_{m,K})$. Let

$$\Delta = \overline{\Gamma} \cap \mathrm{Hom}(\overline{\Gamma}/\Upsilon^\perp, \mathbb{G}^{\mathrm{an}}_{m,K}) = \left\{\gamma \in \overline{\Gamma}\,;\ \bar{c}(\gamma,\beta) = 1 \text{ for all } \beta \in \Upsilon^\perp\right\}.$$

We have a commutative diagram of rigid-analytic groups with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Delta \arrow[r] \arrow[d] & \mathrm{Hom}(\overline{\Gamma}/\Upsilon^\perp, \mathbb{G}^{\mathrm{an}}_{m,K}) \arrow[r] \arrow[d] & (A_\Upsilon^\vee)^{\mathrm{an}} \arrow[r] \arrow[d, "(\pi\circ\pi^\vee)^{\mathrm{an}}"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & \mathrm{Hom}(\Upsilon, \mathbb{G}^{\mathrm{an}}_{m,K}) \arrow[r] & A_\Upsilon^{\mathrm{an}} \arrow[r] & 0.
\end{tikzcd}$$

The proposition will follow from analyzing this diagram. The middle homomorphism arises from the injective composite $\Upsilon \to \overline{\Gamma} \to \overline{\Gamma}/\Upsilon^\perp$. Since Υ and $\overline{\Gamma}/\Upsilon$ are free, the map of tori is surjective, and its kernel is obviously isomorphic to $\mathrm{Hom}(\overline{\Gamma}/(\Upsilon \oplus \Upsilon^\perp), \mathbb{G}^{\mathrm{an}}_{m,K})$.

By construction of $A^{\mathrm{an}}_{\Upsilon}$, the kernel of the homomorphism $\Delta \to \Lambda$ is

$$\{\gamma \in \bar{\Gamma}\,;\ \bar{c}(\gamma,\beta) = 1 \text{ for all } \beta \in \Upsilon \oplus \Upsilon^{\perp}\}.$$

Since $\Upsilon \oplus \Upsilon^{\perp}$ has finite index in $\bar{\Gamma}$, for any $\gamma \in \bar{\Gamma}$ there exists a natural number m_γ such that $m_\gamma \cdot \gamma \in \Upsilon \oplus \Upsilon^{\perp}$. Hence if $\gamma \in \ker(\Delta \to \Lambda)$ then $\bar{c}(\gamma,\gamma)$ is an m_γ-root of unity. This implies $\langle \gamma,\gamma\rangle = 0$, which contradicts the positive definiteness of $\langle .,.\rangle$ unless $\gamma = 1$. Thus, $\Delta \to \Lambda$ is injective. Applying the snake lemma, we get a short exact sequence

$$0 \to \mathrm{Hom}(Q, \mathbb{G}^{\mathrm{an}}_{m,K}) \longrightarrow \ker\big((\pi \circ \pi^{\vee})^{\mathrm{an}}\big) \longrightarrow \Lambda/\Delta \to 0. \tag{3.10}$$

(If a homomorphism of groups is injective, to simplify the notation, we will denote both the group and its image by the same symbol.) If $\gamma \in \Delta$, then from the definition of Δ we clearly have $\langle \gamma,\beta\rangle = 0$ for all $\beta \in \Upsilon^{\perp}$. Since $(\Upsilon^{\perp})^{\perp} = \Upsilon$ we get $\Delta \subset \Upsilon$.

Next observe that the restriction of the homomorphism $\bar{\Gamma} \to \Lambda$ to Υ is injective. In fact, this amounts to $\Upsilon \xrightarrow{\bar{c}} \mathrm{Hom}(\Upsilon, \mathbb{G}^{\mathrm{an}}_{m,K})$ being injective, which again follows from positive definiteness of $\langle .,.\rangle$. Hence we can naturally decompose Λ/Δ into pieces by applying the kernel-cokernel lemma to $\Delta \to \Upsilon \to \Lambda$:

$$0 \to \Upsilon/\Delta \longrightarrow \Lambda/\Delta \longrightarrow \Lambda/\Upsilon \to 0.$$

For $\gamma \in \bar{\Gamma}$ the image of γ in Λ is in Λ_{tor} if and only if $\langle \gamma,\beta\rangle = 0$ for all $\beta \in \Upsilon$. Hence the image of $\Upsilon^{\perp}$ in Λ is exactly Λ_{tor}. Since $\Upsilon^{\perp}$ is saturated, by comparing the ranks, we have $\bar{\Gamma}/\Upsilon^{\perp} \xrightarrow{\sim} \Lambda_{\mathrm{free}}$, where $\Lambda_{\mathrm{free}} := \Lambda/\Lambda_{\mathrm{tor}}$. We get a commutative diagram

$$\begin{tikzcd} \Upsilon \arrow[r, hook] \arrow[d, hook] & \Lambda \arrow[d] \\ \bar{\Gamma}/\Upsilon^{\perp} \arrow[r, "\sim"] & \Lambda_{\mathrm{free}}. \end{tikzcd}$$

Hence we have an exact sequence of commutative étale group-schemes $0 \to \Lambda_{\mathrm{tor}} \to \Lambda/\Upsilon \to Q \to 0$, and we take $H := \Lambda/\Upsilon$.

As we mentioned, $\bar{c}(\Upsilon^{\perp})_{|\Upsilon} = \Lambda_{\mathrm{tor}}$, so from its definition Δ is the largest sublattice of Υ such that $\bar{c}(\Upsilon^{\perp})_{|\Delta} = 1$. Thus,

$$\Lambda_{\mathrm{tor}} \cong \ker\big(\mathrm{Hom}(\Upsilon, \mathbb{G}^{\mathrm{an}}_{m,K}) \to \mathrm{Hom}(\Delta, \mathbb{G}^{\mathrm{an}}_{m,K})\big).$$

On the other hand, we clearly have

$$\mathrm{Hom}(\Upsilon/\Delta, \mathbb{G}_{m,K}^{\mathrm{an}}) \cong \ker\big(\mathrm{Hom}(\Upsilon, \mathbb{G}_{m,K}^{\mathrm{an}}) \to \mathrm{Hom}(\Delta, \mathbb{G}_{m,K}^{\mathrm{an}})\big).$$

Using the fact that Λ_{tor} is étale and constant of order coprime to p, we get $\Upsilon/\Delta \cong \Lambda_{\mathrm{tor}}$. Combining with the calculation of Λ/Υ,

$$0 \to \Lambda_{\mathrm{tor}} \longrightarrow \Lambda/\Delta \longrightarrow H \to 0. \tag{3.11}$$

Since $\pi \circ \pi^\vee$ is a symmetric isogeny, its kernel is self-dual with respect to Cartier duality. Comparing (3.10) with (3.11), and using the fact that A_Υ has split purely toric reduction (cf. Lemma 2.2) along with an isomorphism $\Lambda_{\mathrm{tor}} \cong \Lambda_{\mathrm{tor}}^*$, we get the desired exact sequence of finite flat group-schemes

$$0 \to H^* \longrightarrow \ker(\pi \circ \pi^\vee) \longrightarrow H \to 0.$$

□

3.6. A criterion for triviality of Λ_{tor}.

Let B be the scheme-theoretic kernel of the quotient map $\pi : J \to A_\Upsilon$. As we proved in Proposition 3.12, this is an abelian subvariety of J. Let $\mathcal{K} = A_\Upsilon^\vee \cap B$, where the scheme-theoretic intersection is taken inside of J. This is a commutative finite flat group-scheme which is canonically isomorphic to its Cartier dual and the order of $\mathcal{K}$ is a perfect square. Indeed, $\mathcal{K}$ is the kernel of the polarization $\pi \circ \pi^\vee : A_\Upsilon^\vee \to A_\Upsilon$ and the statement follows from Proposition 3.13 (see also [24], §§15–16). By a theorem of Deligne any commutative finite flat group-scheme is annihilated by its order. That is, if m is the order of the group-scheme then repeating the group law m times gives a form vanishing identically on the scheme (the identity element). The least natural number with the same property will be called the *exponent* of the group-scheme. In the case of $\mathcal{K} = \ker(\pi \circ \pi^\vee)$ its exponent e divides the square root of its order $\deg(\pi \circ \pi^\vee)^{1/2}$.

The idea of the proof of the following proposition is due to Gekeler, Ribet and Zagier [12], [34].

PROPOSITION 3.14. — *If the exponent of $\mathcal{K}$ is equal to* $\deg(\pi \circ \pi^\vee)^{1/2}$ *then Λ_{tor} is trivial.*

Proof. — We have natural inclusions

$$\mathbb{T} \hookrightarrow \mathrm{End}(J) \hookrightarrow \mathrm{End}(\overline{\Gamma}) \hookrightarrow \mathrm{End}(\overline{\Gamma} \otimes \mathbb{Q}). \tag{3.12}$$

The left and the right injections are obvious. To see the middle injection, observe that due to the semi-stable reduction of J over R we have an injection

$$\mathrm{End}_K(J) \hookrightarrow \mathrm{End}_k(\mathcal{J}_k^0).$$

(Look at the action of the endomorphisms on ℓ-power torsion for $\ell \neq p$.) But $\mathcal{J}_k^0$ is a split torus over k, hence $\mathrm{End}_k(\mathcal{J}_k^0) \cong \mathrm{End}(M)$, where M is the character group of $\mathcal{J}_k^0$. On the other hand, from the construction of analytic uniformization of J in §§3.1–3.2, it follows that M is canonically isomorphic to $\overline{\Gamma}$ as an $\mathrm{End}(J)$-module.

For any finite free $\mathbb{Z}$-module L, the *denominator* of a non-zero $v \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ is the least natural number m such that $mv \in L$. Let $\boldsymbol{e}$ be the idempotent of $\mathrm{End}(\overline{\Gamma} \otimes \mathbb{Q})$ corresponding to $(1,0)$ under the splitting

$$\overline{\Gamma} \otimes \mathbb{Q} = (\Upsilon \otimes \mathbb{Q}) \oplus (\Upsilon^{\perp} \otimes \mathbb{Q}).$$

We claim that the denominator m of $\boldsymbol{e}$ in $\mathrm{End}(\overline{\Gamma})$ is equal to the order r of the finite abelian group $Q := \overline{\Gamma}/(\Upsilon \oplus \Upsilon^{\perp})$. Indeed, it is clear that m divides r. On the other hand, since $m\boldsymbol{e}$ preserves both Υ and $\Upsilon^{\perp}$, it induces an endomorphism of Q. The quotient $\overline{\Gamma}/\Upsilon^{\perp}$ is a free $\mathbb{Z}$-module (as $\Upsilon^{\perp}$ is saturated), and Υ is a full lattice in $(\overline{\Gamma}/\Upsilon^{\perp}) \otimes \mathbb{Q}$. Since $m \cdot \boldsymbol{e}$ acts as multiplication by m on Υ we get that $m \cdot \boldsymbol{e}$ acts as multiplication by m on Q. Applying the same argument with the roles of Υ and $\Upsilon^{\perp}$ reversed, we also get $m \cdot \boldsymbol{e}$ acts as 0 on Q. Thus, the exponent of Q divides m. The assumption of the proposition along with Proposition 3.13 imply that the exponent of Q must be equal the order r of Q, and we conclude $m = r$.

Next, we claim that the denominator of $\boldsymbol{e}$ in $\mathrm{End}(J)$ is equal to the exponent e of $\mathcal{K}$. Indeed, since there are no non-trivial homomorphism between $A_{\Upsilon}^{\vee}$ and B (due to the multiplicity one theorem), and there is an isogeny $\beta \colon B \times A_{\Upsilon}^{\vee} \to J$ given by $\beta(x,y) \mapsto \pi^{\vee}(y) - x$, we have a splitting

$$\mathrm{End}(J)_{\mathbb{Q}} = \mathrm{End}(A_{\Upsilon}^{\vee})_{\mathbb{Q}} \oplus \mathrm{End}(B)_{\mathbb{Q}}.$$

The denominator n of $\boldsymbol{e}$ in $\mathrm{End}(J)$ is the same as the denominator of the idempotent $(1,0)$ in this splitting. As $n \cdot \boldsymbol{e}$ is $n \cdot \mathbf{1}$ on $A_{\Upsilon}^{\vee}$ and 0 on B, in particular, multiplication by n is trivial on $\mathcal{K}$, we have $e \mid n$. Conversely, since multiplication by e kills $\mathcal{K}$, the map $(e,0)$ from $A_{\Upsilon}^{\vee} \times B$ to itself factors through β, so $e \cdot \boldsymbol{e} \in \mathrm{End}(J)$ and $n \mid e$. By the assumption of the proposition, we get $n = \deg(\pi \circ \pi^{\vee})^{1/2}$.

Finally, let s be the denominator of $\boldsymbol{e}$ in $\mathbb{T}$. The inclusions in (3.12) imply $m \mid n \mid s$. From what we proved we get $\deg(\pi \circ \pi^\vee)^{1/2}/r$ divides s/r. But again by Proposition 3.13, $\deg(\pi \circ \pi^\vee)^{1/2}/r = \#\Lambda_{\mathrm{tor}}$. Recall that $\Lambda_{\mathrm{tor}} \subset (K^\times)^\kappa$ and K has positive characteristic p. Hence $K^\times$ has no non-trivial p-th roots of unity, and consequently $d = \#\Lambda_{\mathrm{tor}}$ is coprime to p. From Theorems 3.10 and 3.8 there is a $\mathbb{Z}[p^{-1}]$-perfect pairing $\mathbb{T} \times \overline{\Gamma} \to \mathbb{Z}$. This implies that s/r is a p-power. Hence d must be a p-power, which forces $d = 1$.

To see that s/r is a p-power, let $\mathbb{T}_\Upsilon$ be the subring of $\mathrm{End}_{\mathbb{Z}}(\Upsilon)$ generated by the Hecke operators acting on Υ and $\mathbb{T}_{\Upsilon^\perp}$ be the similar subring of $\mathrm{End}_{\mathbb{Z}}(\Upsilon^\perp)$. Then $\mathbb{T}_\Upsilon$ and $\mathbb{T}_{\Upsilon^\perp}$ are naturally quotients of $\mathbb{T}$, which in turn is a subring of the direct sum $\mathbb{T}_\Upsilon \oplus \mathbb{T}_{\Upsilon^\perp}$ via the injective homomorphism that sends an element T of $\mathbb{T}$ to $(\boldsymbol{e}T, (\mathbf{1} - \boldsymbol{e})T)$. We have an exact sequence

$$0 \to \mathbb{T} \longrightarrow \mathbb{T}_\Upsilon \oplus \mathbb{T}_{\Upsilon^\perp} \longrightarrow S \to 0, \tag{3.13}$$

and the denominator of $\boldsymbol{e}$ in $\mathbb{T}$ is equal to the exponent of the finite abelian group S. Indeed, since $s\boldsymbol{e} \in \mathbb{T}$ the image $s \cdot 1$ of $s\boldsymbol{e}$ in S is 0. So the exponent of S divides s. Conversely, since multiplication by the exponent of S kills the image of $\boldsymbol{e}$ in S, we have $\mathrm{exponent}(S) \cdot \boldsymbol{e} \in \mathbb{T}$. Hence s divides this exponent and we must have an equality. Consider the sequence dual to (3.13):

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}_\Upsilon \oplus \mathbb{T}_{\Upsilon^\perp}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(S, \mathbb{Z}) \to 0.$$

We have an isomorphism $\mathrm{Ext}^1_{\mathbb{Z}}(S, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) = S^\vee$ (which is valid for any finite group S). Hence, after tensoring the above sequence with the flat $\mathbb{Z}$-algebra $\mathbb{Z}[p^{-1}]$ and using the $\mathbb{Z}[p^{-1}]$-perfect pairing between $\mathbb{T}$ and $\overline{\Gamma}$ mentioned before, one concludes that Q and S are isomorphic on their prime-to-p torsion. In particular, s/r is a p-power as we claimed. □

COROLLARY 3.15. — *If either A_Υ or B is an elliptic curve (equivalently, if either rank or corank of Υ in $\overline{\Gamma}$ is* 1) *then* $\Lambda_{\mathrm{tor}} = 1$.

Proof. — Indeed, as one easily verifies, under this assumption $\mathcal{K}$ is the kernel of multiplication by some integer n on an elliptic curve, and hence is isomorphic to an extension of $\mathbb{Z}/n$ by μ_n. Such a group-scheme obviously has exponent n and order n^2. □

4. Monodromy pairing on $J_0(\mathfrak{n})$.

The purpose of this section is essentially to relate §2 and §3. We discuss the monodromy pairing on the Néron model of $J_0(\mathfrak{n})$ over $R = \mathbb{F}_q[[1/t]]$ and relate it to the pairing in (3.9). We follow the notations of §2 and §3.

4.1. The Picard-Lefschetz formula.

In this subsection we can assume that R is any complete discrete valuation ring. Most of what we are about to say applies to a larger class of relative curves over $\mathrm{Spec}(R)$ (the semi-stable curves), but we will restrict ourselves to the special case of Mumford curves as in Definition 3.3, since these are exactly the semi-stable curves which have Jacobians with split toric reduction.

Let X be a Mumford curve over $\mathrm{Spec}(R)$. Let $J := \mathrm{Pic}^0_{X/K}$ be the Jacobian variety of X_K, and let $\mathcal{J}$ be the Néron model of J over R. If we denote by $\mathcal{G}$ the dual graph of X_k then $\mathcal{J}^0_k \approx \mathrm{H}^1(\mathcal{G}, \mathbb{Z}) \otimes \mathbb{G}_{m,k}$; cf. [3], p. 247. Equivalently, we have

$$M \approx \mathrm{H}_1(\mathcal{G}, \mathbb{Z}) \approx \ker(\mathbb{Z}^{\oplus \mathrm{sing}} \to \mathbb{Z}^{\oplus \mathrm{irred}}), \tag{4.1}$$

where $M := M_J$ is the character group of $\mathcal{J}^0_k$, and the map on the right is $[x] \mapsto [\mathcal{X}_x] - [\mathcal{X}'_x]$ with x a singular point of X_k, and $\mathcal{X}_x, \mathcal{X}'_x$ is an ordered choice of irreducible components through x. This depends on the choices of branches, but the composite

$$\varphi_x : M \hookrightarrow \mathbb{Z}^{\oplus \mathrm{sing}} \xrightarrow{\mathrm{pr}_x} \mathbb{Z}$$

is well-defined up to a sign.

Grothendieck's monodromy pairing in §2.2 in the case of Jacobians can be made explicit using the Picard-Lefschetz formula. Choose the θ-polarization on J, i.e., the canonical isomorphism between the Picard and Albanese varieties of X, and hence an isomorphism $M_{J^\vee} \cong M_J$. Then from the monodromy pairing we get a symmetric positive-definite pairing

$$u_{J,\theta} : M \times M \longrightarrow \mathbb{Z}. \tag{4.2}$$

The non-smooth locus in X consists of finitely many rational points $x_1, \dots, x_r \in X_k$. By the étale local theory of ordinary double points

$$\widehat{\mathcal{O}_{X,x_i}} \cong R[[u,v]]/(uv - t_i)$$

for various non-zero non-units $t_1, \dots, t_r \in R$, and each t_i is unique up to a unit multiple, so $n_i = \mathrm{ord}_R(t_i) \in \mathbb{N}$ is intrinsic to x_i. (Note that $n_i = 1$ for all i if and only if X is regular.) Then

$$u_{J,\theta} = \sum_i n_i \varphi_{x_i} \cdot \varphi_{x_i} : M \times M \longrightarrow \mathbb{Z}, \tag{4.3}$$

see [20], 12.10, where this is stated as a fact (the complete proof is given in [22]).

4.2. Model of $X_0(\mathfrak{n})$ over R.

To apply (4.3) we need a semi-stable model of $X_0(\mathfrak{n})$ over $\mathrm{Spec}(R)$. We will construct such a model using the rigid-analytic uniformization of $X_0(\mathfrak{n})$. To simplify the notation we denote $Y = Y_0(\mathfrak{n})$, $X := X_0(\mathfrak{n})$, and $\Gamma = \Gamma_0(\mathfrak{n})$.

The link between analytic and algebraic categories will be given by a formal scheme $\widehat{\Omega}$ over $\mathrm{Spf}(R)$ whose generic fibre $\widehat{\Omega}^{\mathrm{rig}}$ is isomorphic to Ω. We proceed to describe this scheme using Theorem 3.1 and (3.1). With notation as in §3.1, take $\widehat{\Omega}$ to be the formal scheme obtained by gluing formal affines

$$\mathrm{Spf}(A^0_{n,x}) = \mathrm{Spf}\, R\Big\{ \frac{z-x}{\varpi^n}, \frac{\varpi^{n+1}}{z-x}, \frac{1}{(z-x)/\varpi^n - \rho}, \frac{1}{(z-x)/\varpi^{n+1} - \rho};\ \rho \in k^\times \Big\},$$

where $R\{\dots\}$ is defined analogously to (3.1). (That $A^0_{n,x}$ has this form follows from [9], proposition on p. 7.) The underlying topological space $\widehat{\Omega}_k$ of $\widehat{\Omega}$ is isomorphic to $\overline{\Omega}$. Also the local nature of $\widehat{\Omega}$ around the singularities of $\widehat{\Omega}_k$ is apparent from this explicit description. Let, for example, ξ be the double point singularity of $\overline{D}_{n,x}$, where $\overline{D}_{n,x}$ is as in (3.2). Then the completed stalk of $\mathcal{O}_{\widehat{\Omega}}$ at the singular point ξ of the closed fibre $\widehat{\Omega}_k$ is isomorphic to

$$\widehat{\mathcal{O}}_{\widehat{\Omega},\xi} \cong R\Big[\Big[\frac{z-x}{\varpi^n}, \frac{\varpi^{n+1}}{z-x}\Big]\Big] \cong R[[u,v]]/(uv - \varpi), \tag{4.4}$$

where $u = (z-x)/\varpi^n$ and $v = \varpi/u = \varpi^{n+1}/(z-x)$. Hence $\widehat{\Omega}$ is regular. (We should mention that $\widehat{\Omega}$ is constructed directly, without reference to an explicit admissible covering of Ω, in [25]).

Now Drinfeld's theorem (3.4) establishes an isomorphism of analytic spaces $\Gamma \setminus \Omega \cong Y^{\mathrm{an}}$. From $\widehat{\Omega}$ we also get a quotient formal scheme $\mathfrak{Y} \cong \Gamma \setminus \widehat{\Omega}$ such that $\mathfrak{Y}^{\mathrm{rig}} \cong Y^{\mathrm{an}}$. Unfortunately, since $\mathfrak{Y}$ is not proper, we cannot apply Grothendieck's algebraization theorem to get a model for Y over $\mathrm{Spec}(R)$. So we need an explicit affinoid covering of the complete curve X^{an}, or equivalently, a Mumford uniformization of X. This is the subject of a paper by Reversat [26]. We describe the principal result of that paper.

As in §3.1, let $\mathcal{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(K)$. The action of Γ on Ω and $\widehat{\Omega}$ transfers to an action on $\mathcal{T}$, which, as we mentioned, is the dual graph of $\overline{\Omega} \cong \widehat{\Omega}_k$. Consider the quotient graph

$$\Gamma \setminus \mathcal{T} = \mathcal{G} \cup \bigcup_{i=1}^{r} h_i, \tag{4.5}$$

which is the edge-disjoint union of a finite graph $\mathcal{G}$ and a finite number of half-lines h_i, cf. [29], Theorem II.9. An end s of $\mathcal{T}$ gives rise to an end of $\Gamma \setminus \mathcal{T}$ if and only if it is F-rational. Thus we have bijections of finite sets $\{h_1, \dots, h_r\} \cong X - Y \cong \Gamma \setminus \mathbb{P}_K^{1,\mathrm{an}}(F)$; see [16], (2.6).

Let Γ_{tor} be the normal subgroup of Γ generated by the torsion elements, and let $\Gamma' = \Gamma/\Gamma_{\mathrm{tor}}$. Choose $\{s_1, \dots, s_r\}$, $s_i \in \mathbb{P}_K^{1,\mathrm{an}}(F)$, a set of representatives of $\Gamma \setminus \mathbb{P}_K^{1,\mathrm{an}}(F)$. Let h_i be a half-line in $\mathcal{T}$ corresponding to s_i. An infinite half-line is the dual graph of the analytic reduction of a punctured disc, so the preimage of each h_i in Ω consists of a nested sequence of annuli with decreasing radii, whose union is a punctured unit disc which we denote by Ω_{s_i}. One can choose h_i so that if $\gamma(\Omega_{s_i}) \cap \Omega_{s_j} \neq \emptyset$ for $\gamma \in \Gamma$, then $i = j$ and $\gamma \in \mathrm{Stab}_{s_i} \Gamma$, where $\mathrm{Stab}_{s_i} \Gamma$ is the stabilizer of s_i in Γ; see [26], (2.3.1). By [26], Lemma 2.4, for any s_i the quotient $\mathrm{Stab}_{s_i} \Gamma \setminus \Omega_{s_i}$ is analytically isomorphic to a unit disc with the origin removed and there is an open immersion $\mathrm{Stab}_{s_i} \Gamma \setminus \Omega_{s_i} \hookrightarrow \Gamma_{\mathrm{tor}} \setminus \Omega$. Define $\mathbb{B}_{s_i} \cong \mathrm{Sp}\ K\langle t\rangle$ to be $\mathrm{Stab}_{s_i} \Gamma \setminus \Omega_{s_i} \cup \{s_i\}$. Denote by Ξ the analytic space

$$\Xi = (\Gamma_{\mathrm{tor}} \setminus \Omega) \cup \Big(\bigcup_{s_i} \mathbb{B}_{s_i}\Big)$$

obtained by gluing each $\mathbb{B}_{s_i}$ with $\Gamma_{\mathrm{tor}} \setminus \Omega$ along their common admissible open subspace $\mathrm{Stab}_{s_i} \Gamma \setminus \Omega_{s_i}$. For any $s \in \mathbb{P}_K^{1,\mathrm{an}}(F)$ there is $\gamma \in \Gamma$ and $j \in \{1, \dots, r\}$ such that $s = \gamma(s_j)$. Denote $\Omega_s = \gamma(\Omega_{s_j})$.

Proposition 4.1. — *Let $f\colon \Omega \to \Gamma_{\mathrm{tor}} \setminus \Omega \subset \Xi$ be the canonical morphism of analytic spaces. The analytic space Ξ is an open admissible* *subspace of* $\mathbb{P}_K^{1,\mathrm{an}}$ *such that* $\mathbb{P}_K^{1,\mathrm{an}} - \Xi$ *is compact, and*

$$\{f(D_{n,x});\ D_{n,x} \not\subset \Omega_s \text{ for all } s \in \mathbb{P}_K^{1,\mathrm{an}}(F)\} \cup \Big(\bigcup_{s_\imath}\{\mathbb{B}_{s_\imath}\}\Big)$$

is a pure admissible affinoid covering of Ξ*. The group* Γ' *is a free group on* g *generators, where* g *is the genus of* X*. It acts discontinuously on* Ξ*, and this action is compatible with the above admissible covering of* Ξ*. In particular, the quotient* $\Gamma' \setminus \Xi$ *can be endowed with a structure of a* K*-analytic space.*

Proof. — See [26], Lemma 2.6. □

THEOREM 4.2. — *We have a commutative diagram of* K*-analytic spaces*

$$\begin{array}{ccccc}
\Omega & \longrightarrow & \Gamma\setminus\Omega & \xrightarrow{\sim} & Y^{\mathrm{an}} \\
\big\downarrow{\scriptstyle f} & & \big\downarrow & & \big\downarrow \\
\Xi & \longrightarrow & \Gamma'\setminus\Xi & \xrightarrow{\sim} & X^{\mathrm{an}},
\end{array}$$

where the maps in the middle and on the right are open immersions.

Proof. — See [26], Theorem 2.7. □

Let $f(D_{n,x}) = \mathrm{Sp}(A'_{n,x})$. Then the formal scheme $\widehat{\Xi}$ obtained by gluing the formal affines

$$\{\mathrm{Spf}((A'_{n,x})^0)\ ;\ D_{n,x} \not\subset \Omega_s \text{ for all } s \in \mathbb{P}_K^{1,\mathrm{an}}(F)\} \cup \Big(\bigcup_{s_\imath}\widehat{\mathbb{B}}_{s_\imath}\Big),$$

where $\widehat{\mathbb{B}} = \mathrm{Spf}(R\{t\})$, satisfies $\widehat{\Xi}^{\mathrm{rig}} \cong \Xi$ and $(\Gamma' \setminus \widehat{\Xi})^{\mathrm{rig}} \cong X^{\mathrm{an}}$. Since $\Gamma' \setminus \widehat{\Xi}$ is proper R-flat and 1-dimensional, according to *Grothendieck's Algebraization Theorem* [21], (5.1.6), it is the formal completion of some unique proper and flat curve $\mathcal{X}$ over $\mathrm{Spec}(R)$ along its closed fibre $\mathcal{X} \times_R k$. This $\mathcal{X}$ is a model of X over R which we understand fairly well thanks to $\widehat{\Xi}$.

By construction, the closed fibre $\mathcal{X}_k \cong (\Gamma' \setminus \widehat{\Xi})_k$ has dual graph isomorphic to $\mathcal{G}$ in (4.5). The action of Γ on Ω and on $\mathcal{T}$ factors through its image $\widetilde{\Gamma}$ in $\mathrm{PGL}_2(F)$. Also, since Γ acts discontinuously, the stabilizer in Γ of each edge of $\mathcal{T}$ is finite and hence is inside Γ_{tor}. Given an edge $e \in \mathrm{Ed}(\mathcal{G})$, denote by e' some preimage of e in $\mathcal{T}$, and let $n(e) = \#\,\mathrm{Stab}_{e'}\widetilde{\Gamma}$. It is easy to check that this is well-defined. Let ξ be a (normal-crossing) singularity of $\mathcal{X}_k$. Denote by $e_\xi \in \mathrm{Ed}(\mathcal{G})$ the corresponding edge in the dual graph.

PROPOSITION 4.3. — *With notation as above, $\mathcal{X}_K \cong X$ and $\mathcal{X}_k$ is a k-split degenerate curve whose dual graph is isomorphic to $\mathcal{G}$ in* (4.5). *Moreover, the formal completion of the stalk $\mathcal{O}_{\mathcal{X},\xi}$ at a* (*normal-crossing*) *singularity ξ of $\mathcal{X}_k$ is isomorphic to*

$$R[[u,v]]/(uv - \varpi^{n(e_\xi)}).$$

Proof. — We already explained the first part. Let ξ' be a preimage of ξ in $\overline{\Omega}$, and let $\mathrm{Stab}_{\xi'}\widetilde{\Gamma}$ be its stabilizer. The action of Γ on $\overline{\Omega}$ is canonically isomorphic to its action on $\mathcal{T}$. In particular, $\#\,\mathrm{Stab}_{\xi'}\widetilde{\Gamma} = n(e_\xi)$. Next, assume ξ' is the singularity in $\overline{D}_{n,x}$ for some n and x. By (4.4) the étale local neighborhood of the image of ξ' in $\mathrm{Spf}((A'_{n,x})^0)$ is isomorphic to

$$\big(R[[u,v]]/(uv - \varpi)\big)^{\mathrm{Stab}_{\xi'}\widetilde{\Gamma}}.$$

The action of $\mathrm{Stab}_{\xi'}\widetilde{\Gamma}$ does not interchange the two branches through ξ'. Indeed, a choice of ordering of these branches corresponds to a choice of an *oriented* edge $e_{\xi'}$ in $\mathcal{T}$. If some $\gamma \in \mathrm{Stab}_{\xi'}\widetilde{\Gamma}$ interchanges the branches through ξ' then $\gamma(e_{\xi'}) = \bar{e}_{\xi'}$. This latter condition does not occur, as follows from the explicit description of the action of Γ on $\mathcal{T}$ given, for example, in [13]. Hence

$$\widehat{\mathcal{O}}_{\widehat{\Xi},\xi'} \cong R[[u',v']]/(u'v' - \varpi^{\#\,\mathrm{Stab}_{\xi'}\widetilde{\Gamma}}),$$

where $u' = \prod_{g\in \mathrm{Stab}_{\xi'}\widetilde{\Gamma}} gu$ and $v' = \prod_{g\in \mathrm{Stab}_{\xi'}\widetilde{\Gamma}} gv$. Since Γ' acts freely on $\widehat{\Xi}$, we have $\widehat{\mathcal{O}}_{\widehat{\Xi},\xi'} \cong \widehat{\mathcal{O}}_{\mathcal{X},\xi}$, which finishes the proof. □

Consider $\mathcal{G}$ as an oriented graph, that is, keep track of the ordering of the irreducible components through each singular point of $\mathcal{X}_k$. Let α be an oriented cycle in $\mathcal{G}$. One can describe α by a sequence of oriented edges $\{e_1,\dots,e_n\}$ satisfying $t(e_i) = o(e_{i+1})$ for $i = 1,\dots,n-1$ and $t(e_n) = o(e_1)$, where $t(e)$ (resp. $o(e)$) denotes the terminus (resp. origin) of the edge e. Define a pairing $\phi(e_1,e_2)$ between the oriented edges of $\mathcal{G}$ by

$$\phi(e_1,e_2) = \begin{cases} 1 & \text{if } e_1 = e_2, \\ -1 & \text{if } e_1 = \bar{e}_2, \\ 0 & \text{otherwise.} \end{cases}$$

For the cycle α put

$$\phi_\alpha(e) = \sum_{i=1}^{n} \phi(e,e_i).$$

One easily checks that ϕ_α depends only on the class of α in $\mathrm{H}_1(\mathcal{G},\mathbb{Z})$, and $\phi_{\alpha+\beta} = \phi_\alpha + \phi_\beta$.

COROLLARY 4.4. — *Fix an orientation on $\mathcal{G}$. Let M be the character group of $\mathcal{J}_k^0$, where $\mathcal{J}$ is the Néron model of $J = \mathrm{Pic}^0_{X/K}$ over R. Then $M \cong \mathrm{H}_1(\mathcal{G},\mathbb{Z})$ and the θ-polarized monodromy pairing $M \times M \to \mathbb{Z}$ in* (4.2) *is given by*

$$u_{J,\theta}(\alpha,\beta) = \sum_{e \in \mathrm{Ed}(\mathcal{G})} n(e)\phi_\alpha(e)\phi_\beta(e).$$

Proof. — Follows from Proposition 4.3, (4.1), and (4.3). □

Now we return to the pairing $\langle .,. \rangle : \overline{\Gamma} \times \overline{\Gamma} \to \mathbb{Z}$ in (3.9). There is a canonical isomorphism

$$\overline{\Gamma} \xrightarrow{\sim} \mathrm{H}_1(\mathcal{G},\mathbb{Z}), \tag{4.6}$$

where $\mathcal{G}$ as in (4.5); see [29], Corollary 1 to Theorem I.13. Thus $\langle .,. \rangle$ can be regarded as a bilinear symmetric positive-definite pairing on $\mathrm{H}_1(\mathcal{G},\mathbb{Z})$.

PROPOSITION 4.5. — *Via the canonical isomorphisms $\overline{\Gamma} \cong \mathrm{H}_1(\mathcal{G},\mathbb{Z})$ and $M \cong \mathrm{H}_1(\mathcal{G},\mathbb{Z})$ in* (4.6) *and Corollary* 4.4 *respectively, we have $\langle .,. \rangle = u_{J,\theta}$ as $\mathbb{Z}$-valued pairings on $\mathrm{H}_1(\mathcal{G},\mathbb{Z})$.*

Proof. — In the proof of Theorem 5.7.1 in [16] the authors, following an argument of van der Put [33], Theorem 6.4, give an explicit formula for the pairing $\langle .,. \rangle$, which agrees with the formula for $u_{J,\theta}$ in Corollary 4.4. □

5. Main theorems.

We keep the notation of §3. Let A be an optimal quotient of $J := J_0(\mathfrak{n})$ and let B be the scheme-theoretic kernel of the corresponding optimal quotient map $\pi : J \to A$. By the definition of optimality this is an abelian subvariety of J. We further assume that B is preserved under the action of the Hecke algebra $\mathbb{T}$ viewed as a subring of $\mathrm{End}(J)$, cf. §3.3. Let $\pi_* : \Phi_J \to \Phi_A$ be the map induced by π on the component groups of the Néron models of J and A over $\mathrm{Spec}(R)$.

THEOREM 5.1. — *The order of* $\mathrm{coker}(\pi_*)$ *divides* $(q-1)^{\dim(A)}$. *In particular, for any prime ℓ which does not divide $q-1$ we have* $\mathrm{coker}(\pi_*)[\ell^\infty] = 1$.

Proof. — From Proposition 2.7 and Lemma 2.9 applied to $J \xrightarrow{\pi} A$ with $\lambda\colon J^\vee \xrightarrow{\sim} J$ being the θ-polarization, we get

$$\deg(\pi\circ\pi^\vee)^{1/2} = [\overline{M}_A : \pi^* M_A]\cdot[M_J : \overline{M}_A \oplus (\overline{M}_A)^\perp]. \tag{5.1}$$

On the other hand, we have an alternative formula for the degree given by Proposition 3.13

$$\deg(\pi\circ\pi^\vee)^{1/2} = \#\Lambda_{\mathrm{tor}}\cdot[\overline{\Gamma} : \Upsilon\oplus\Upsilon^\perp], \tag{5.2}$$

where Υ is the saturated sublattice of $\overline{\Gamma}$ corresponding to A, and Λ is as in Proposition 3.11. Via the canonical isomorphism $\overline{\Gamma}\cong M_J$ in Proposition 4.5, it is clear that $\Upsilon = \overline{M}_A$. Since by the same proposition the monodromy pairing used to define $(\overline{M}_A)^\perp$ is equal to the pairing $\langle .,. \rangle$ in (3.9) used to define $\Upsilon^\perp$, we also have $(\overline{M}_A)^\perp = \Upsilon^\perp$. Thus (5.1) and (5.2) imply

$$[\overline{M}_A : \pi^* M_A] = \#\Lambda_{\mathrm{tor}}.$$

Recall that $\Lambda_{\mathrm{tor}} \subset (K^\times)^{\dim(A)}$. Since $(K^\times)_{\mathrm{tor}} = \mu_{q-1}$, the order of Λ_{tor} must divide $(q-1)^{\dim(A)}$. But by Proposition 2.12 we also have

$$[\overline{M}_A : \pi^* M_A] = \#\operatorname{coker}(\pi_* : \Phi_J \to \Phi_A),$$

so $\#\operatorname{coker}(\pi_*)$ also must divide $(q-1)^{\dim(A)}$. □

COROLLARY 5.2. — *For any prime ℓ not dividing $q-1$, there is a short exact sequence of finite abelian groups*

$$0 \to \Phi_B[\ell^\infty] \xrightarrow{\varphi_*} \Phi_J[\ell^\infty] \xrightarrow{\pi_*} \Phi_A[\ell^\infty] \to 0.$$

Proof. — The dual to the closed immersion $\varphi\colon B \hookrightarrow J$ is the optimal quotient $\varphi^\vee : J \to B^\vee$ whose kernel $A^\vee$ is $\mathbb{T}$-stable. Theorem 5.1 applied to both $B^\vee$ and A gives $\operatorname{coker}(\varphi_*^\vee)[\ell^\infty] = \operatorname{coker}(\pi_*)[\ell^\infty] = 1$. Now the claim follows from Proposition 2.15. □

THEOREM 5.3. — *Let $\mathcal{K} = B\cap A^\vee$, with the scheme-theoretic intersection taken inside of J. If the exponent of $\mathcal{K}$ is equal to $(\#\mathcal{K})^{1/2}$ then the sequence of component groups*

$$0 \to \Phi_B \xrightarrow{\varphi_*} \Phi_J \xrightarrow{\pi_*} \Phi_A \to 0$$

is exact.

Proof. — Since $\mathcal{K}$ is the kernel of both $\pi \circ \pi^\vee : A^\vee \to A$ and $\varphi^\vee \circ \varphi : B \to B^\vee$, this easily follows from Proposition 3.14 and Proposition 2.15, using the argument in the proof of Theorem 5.1 for the optimal quotients $B^\vee$ and A. □

COROLLARY 5.4. — *If either A or B is an elliptic curve, or $q = 2$, then the sequence*

$$0 \to \Phi_B \xrightarrow{\varphi_*} \Phi_J \xrightarrow{\pi_*} \Phi_A \to 0$$

is short exact.

Proof. — When either A or B is an elliptic curve but q is arbitrary the exactness follows from Theorem 5.3 and Corollary 3.15. When $q = 2$ but A and B are of arbitrary dimension the exactness follows from Corollary 5.2. □

COROLLARY 5.5. — *If one of the conditions of Corollary* 5.4 *holds, then*

$$\deg(\pi \circ \pi^\vee)^{1/2} \cdot \#\Phi_A = \det\big(\langle \gamma_i, \gamma_j \rangle\big)_{i,j},$$

where $\{\gamma_1, \dots, \gamma_\kappa\}$ form a $\mathbb{Z}$-basis of $\overline{M}_A = \Upsilon$ and $\langle .,. \rangle$ is the pairing in (3.9).

Proof. — Indeed, if $\pi_* : \Phi_J \to \Phi_A$ is surjective then Lemma 2.10 and Proposition 2.12 imply

$$\deg(\pi \circ \pi^\vee)^{1/2} \cdot \#\Phi_A = \det\big(u_{J,\theta}(\gamma_i, \gamma_j)\big)_{i,j},$$

where $\{\gamma_1, \dots, \gamma_\kappa\}$ form a $\mathbb{Z}$-basis of $\overline{M}_A$. It remains to use Proposition 4.5. □

This last corollary generalizes a result of Gekeler [12], Corollary 3.20.

Example 5.6. — Let $F = \mathbb{F}_2(t)$ and $\mathfrak{n} = t(t^2 + t + 1)$. Then the genus of the Drinfeld modular curve $X_0(\mathfrak{n})$ is 2. Gekeler showed [12], Example 4.4, that $J = J_0(\mathfrak{n})$ has trivial old subvariety and two optimal quotients E_1 and E_2 of dimension 1, i.e., elliptic curves. The valuations of the j-invariants of these elliptic curves are $\mathrm{ord}_\infty(j_1) = -3$ and $\mathrm{ord}_\infty(j_2) = -5$ respectively. Hence by the *Tate algorithm* $\Phi_{E_1,\infty} \cong \mathbb{Z}/3$ and $\Phi_{E_2,\infty} \cong \mathbb{Z}/5$. Moreover, in [14], Example 5.3.1, Gekeler calculates directly that $\#\Phi_{J,\infty} = 15$. This last fact also follows from our Corollary 5.4.

Example 5.7. — This example is again due to Gekeler. Let $F = \mathbb{F}_7(t)$ and $\mathfrak{n} = t^3 - 2$. In this case the genus of $X_0(\mathfrak{n})$ is 7. As follows from [10], (5.3), and Proposition 4.3, $X_0(\mathfrak{n})$ has a minimal regular model over R whose special fibre has a dual graph consisting of two vertices v_1 and v_2 joined by eight arcs of edges starting at v_1 and ending at v_2. One of the arcs has length 8 and all the others have length 1. Hence by [3], Proposition 9.6/10, the group of components $\Phi_{J,\infty}$ is $\mathbb{Z}/57$. According to [10], Table 10.3 (see also [14], §6), J has two F-simple optimal quotients: one is an elliptic curve with j-invariant of valuation $\mathrm{ord}_\infty(j) = -3$, and the other is a simple abelian variety A of dimension 6. From Corollary 5.4 we conclude $\Phi_{A,\infty} \cong \mathbb{Z}/19$.

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