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SOME CONSEQUENCES OF PERVERSITY OF VANISHING CYCLES

by **Alexandru DIMCA and Morihiko SAITO**

Introduction.

Let f be a nonconstant holomorphic function on a complex analytic space X. For each $x \in Y := f^{-1}(0)$, we have the vanishing cohomology $\widetilde{H}^j(F_x,\mathbb{Q})$ where F_x denotes the (typical) fiber of the Milnor fibration around x, and $\widetilde{H}$ means the reduced cohomology. It has been observed by many people that there are certain relations between the $\widetilde{H}^j(F_x,\mathbb{Q})$ for $x \in Y$. It is well-known that they form a constructible sheaf on Y (called the vanishing cohomology sheaf). P. Deligne [7] constructed a sheaf complex $\varphi_f\mathbb{Q}_X$ on Y (called the vanishing cycle complex) such that its cohomology sheaves $\mathcal{H}^j\varphi_f\mathbb{Q}_X$ are the vanishing cohomology sheaves.

Let L_x denote the intersection of Y with a sufficiently small sphere around $x \in Y$ (in a smooth ambient space), which is called the *link* of $\{x\}$ in Y. Let T_u, T_s be respectively the unipotent and semisimple part of the monodromy T, and put $N = \log T_u$. Let $\widetilde{H}^{n-1}(F_x,\mathbb{Q})_1$ and $\widetilde{H}^{n-1}(F_x,\mathbb{Q})_{\neq 1}$ denote the unipotent and non unipotent monodromy part, which are defined by $\mathrm{Ker}(T_s-1)$ and $\bigoplus_{\lambda\neq 1}\mathrm{Ker}(T_s-\lambda)$ (after a scalar extension) respectively, and similarly for the cohomology with compact supports.

THEOREM 0.1. — *Assume that $\mathbb{Q}_X[n+1]$ is a perverse sheaf (e.g. X is a locally complete intersection of dimension $n+1$), and $n \geq 1$. Then there are canonical isomorphisms*

$$\widetilde{H}^j(F_x,\mathbb{Q}) = \mathbb{H}^j(L_x,\varphi_f\mathbb{Q}_{X\,|\,L_x}) \quad \textit{for } j < n-1,$$

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and a short exact sequence

$$0 \to \widetilde{H}^{n-1}(F_x,\mathbb{Q}) \longrightarrow \mathbb{H}^{n-1}(L_x,\varphi_f\mathbb{Q}_{X\,|\,L_x}) \longrightarrow K_x \to 0.$$

Here K_x is the kernel of a morphism β_φ which is the direct sum of

$$\beta_{\varphi,1}: H^n_c(F_x,\mathbb{Q})_1(-1) \longrightarrow H^n(F_x,\mathbb{Q})_1,$$

$$\beta_{\varphi,\neq 1}: H^n_c(F_x,\mathbb{Q})_{\neq 1} \longrightarrow H^n(F_x,\mathbb{Q})_{\neq 1},$$

where (-1) denotes the Tate twist, and $\beta_{\varphi,\neq 1}$ coincides with the natural morphism (i.e. corresponds to the natural intersection form if X is a rational homology manifold). If X is a rational homology manifold at x, then $N\beta_{\varphi,1}$ coincides with the natural morphism. These morphisms and the short exact sequence are compatible with mixed Hodge structure.

In the 1-dimensional singular locus case, a similar assertion was obtained in [18], [19], see also [1]. Theorem 0.1 means that $\widetilde{H}^j(F_x,\mathbb{Q})$ for $j < n-1$ (resp. $j = n-1$) is completely (resp. partially) determined by the restriction of $\varphi_f\mathbb{Q}_X$ to the complement of x, and only $\widetilde{H}^n(F_x,\mathbb{Q})$ is essentially interesting if we know well about the restriction of $\varphi_f\mathbb{Q}_X$ to the complement of x. The proof easily follows from the well-known fact that the vanishing cycle complex $\varphi_f\mathbb{Q}_X$ is a (shifted) perverse sheaf. Actually, the first two assertions of Theorem 0.1 are essentially equivalent to the perversity of $\varphi_f\mathbb{Q}_X$, assuming the perversity of its restriction to the complement of x. The hypercohomology $\mathbb{H}^j(L_x,\varphi_f\mathbb{Q}_{X\,|\,L_x})$ can be calculated by using spectral sequences 2.2–2.3. The mixed Hodge structure on $H^j(F_x,\mathbb{Q})$ can be calculated by using the weight spectral sequence 1.5, see also [14] for the unipotent monodromy case, and [20] for the isolated singularity case.

In Theorem 0.1 we can replace the vanishing cycle complex $\varphi_f\mathbb{Q}_X$ with the nearby cycle complex $\psi_f\mathbb{Q}_X$ in [7], and β_φ with $\beta_\psi : H^n_c(F_x,\mathbb{Q}) \to H^n(F_x,\mathbb{Q})$. In this case β_ψ is a natural morphism, and in the isolated singularity case (where X is smooth), we get a well-known relation between the cohomology of the Milnor fiber and the link. Note that the morphism β_φ in Theorem 0.1 for φ in the isolated singularity case is an isomorphism (i.e. the morphism corresponds to a nondegenerate pairing if X is a rational homology manifold), because $\varphi_f\mathbb{Q}_{X\,|\,L_x}$ vanishes, see also 1.3 below.

Let $b^j_\lambda(F_x)$ denote the rank of $H^j(F_x,\mathbb{C})_\lambda (= \mathrm{Ker}(T_s - \lambda))$ for $\lambda \in \mathbb{C}$. Using Theorem 0.1, we can explicitly calculate it for $j \le n-2$ in the case of a divisor with simple normal crossings outside a point as follows (see 4.3 for the proof).

THEOREM 0.2. — *With the notation and the assumption of* 0.1, *assume* $X\setminus\{x\}$ *is smooth,* $Y\setminus\{x\}$ *is a divisor with normal crossings on* $X\setminus\{x\}$, *and the local irreducible components* Y_i $(i=1,\dots,m)$ *of* Y_{red} *at* x *are principal divisors having at most isolated singularities at* x. *Let* a_i *be the multiplicity of* Y *at the generic point of* Y_i, *and* $d=\mathrm{GCD}(a_1,\dots,a_m)$. *Assume* $j\le n-2+\delta_{\lambda,1}$, *where* $\delta_{\lambda,1}=1$ *if* $\lambda=1$, *and* 0 *otherwise. Then* $H^j(F_x,\mathbb{Q})$ *is a pure Hodge structure of type* (j,j); *in particular, the monodromy is semisimple. Furthermore, if* $\lambda^d\ne 1$, *we have* $b^j_\lambda(F_x)=0$, *and if* $\lambda^d=1$, *then*

$$b^j_\lambda(F_x)=\binom{m-1}{j}\quad \text{for } j<n-2+\delta_{\lambda,1},$$

$$b^j_\lambda(F_x)\le\binom{m-1}{j}\quad \text{for } j=n-2+\delta_{\lambda,1}.$$

Here the equality holds also for $j=n-2+\delta_{\lambda,1}$, *if* $Y_I:=\bigcap_{i\in I}Y_i$ *is a rational homology manifold for any subset* I *of* $\{1,\dots,m\}$ *with* $|I|\le n-1$, *where* $Y_\emptyset=X$.

The case $a_i=1$ for any i was studied in [9], see also 4.4 below. In the case where an embedded resolution of (X,Y) can be obtained by one blow-up with a point center (e.g. an equisingular deformation of the affine cone of a divisor with simple normal crossings on a smooth projective variety), we have a more precise statement as follows (see 4.5 for the proof).

THEOREM 0.3. — *With the notation and assumptions of* 0.1, *let* $\pi:\widetilde{X}\to X$ *be the blow-up of* X *with center* x, *and assume that* $\widetilde{X}$ *and the exceptional divisor* $E:=\pi^{-1}(x)$ *are smooth and the total transform* $\widetilde{Y}:=\pi^{-1}(Y)$ *is a divisor with normal crossings. Let* Y' *be the proper transform of* Y, *and put* $U=E\setminus Y'$. *Let* e *be the multiplicity of* $\widetilde{Y}$ *along* U. *Then the monodromy* T *on* $H^j(F_x,\mathbb{Q})$ *is semisimple for any* j, *and* $H^j(F_x,\mathbb{Q})$ *is of type* (j,j) *for* $j<n$. *Furthermore, if* $\lambda^e\ne 1$, *we have* $b^j_\lambda(F_x)=0$ *for any* j, *and if* $\lambda^e=1$, *then*

$$\chi_\lambda(F_x)\ \Big(:=\sum_{0\le j\le n}(-1)^j b^j_\lambda(F_x)\Big)=\chi(U),$$

$$b^j_\lambda(F_x)=\begin{cases}\binom{m-1}{j} & \text{if } j<n \text{ and } \lambda^d=1,\\ 0 & \text{if } j<n \text{ and } \lambda^d\ne 1,\end{cases}$$

$$b^n_\lambda(F_x)=\begin{cases}(-1)^n\chi(U)+\binom{m-2}{n-1} & \text{if } \lambda^d=1,\\ (-1)^n\chi(U) & \text{if } \lambda^d\ne 1.\end{cases}$$

This gives a generalization of formulas in [5], [15] for a generic central arrangement with $a_i = 1$, see 4.6 below. If X is smooth (i.e. if $(X,x) = (\mathbb{C}^{n+1},0)$), then the assumption of 0.3 is equivalent to that the union of the divisors defined by the lowest degree part of a defining equation f_j of Y_j is a reduced divisor with normal crossings on $\mathbb{P}^n$, and we have $e = \sum_j a_j d_j$ where d_j is the degree of the lowest degree part of f_j; in particular, d divides e. We can calculate $\chi(U)$ explicitly in this case, see 4.6.

Let T denote the monodromy of $\varphi_f \mathbb{Q}_X$ with the Jordan decomposition $T = T_u T_s$. For a complex number λ, set $\varphi_{f,\lambda}\mathbb{C}_X = \mathrm{Ker}(T_s - \lambda) \subset \varphi_f \mathbb{C}_X$ (in the abelian category of shifted perverse sheaves), and $N = \log T_u$. As an application of Theorem 0.1, we show

THEOREM 0.4. — *With the notation and the assumption of* 0.1, *let j be a positive integer $< n$. Assume the monodromy of $\widetilde{H}^j(F_x,\mathbb{C})_\lambda$ has a Jordan block of size k. Then the action of N^{k-1} on $\varphi_{f,\lambda}\mathbb{C}_X|_{U\setminus\{x\}}$ is nonzero for any open neighborhood U of x. Furthermore, there exist points y_i $(\neq x)$ sufficiently near x for $i \le j$ such that the monodromy of $\widetilde{H}^i(F_{y_i},\mathbb{C})_\lambda$ has a Jordan block of size k_i and $\sum_{i\le j} k_i \ge k$, where we set $k_i = 0$ if $\widetilde{H}^i(F_y,\mathbb{C})_\lambda = 0$ for $y \neq x$.*

This is a refinement of Corollary 6.1.7 in [8]. There is an example such that the monodromy at degree $n-1$ is not semisimple at x, but is semisimple outside x, see Appendix. Note that the support of the image of N^k in $\psi_f\mathbb{Q}$ (resp. in $\varphi_{f,1}\mathbb{Q}$) as shifted perverse sheaves has dimension $\le n-k$ (resp. $\le n-k-1$), see e.g. [10]. In the case $\dim \mathrm{supp}\, \varphi_{f,\lambda}\mathbb{C}_X = r$, we have $\mathcal{H}^j \varphi_{f,\lambda}\mathbb{C}_X = 0$ for $j < n-r$ (see 2.1.2 below), and the conclusion of Theorem 0.4 for $j = n-r$ means that the monodromy of $\widetilde{H}^{n-r}(F_y,\mathbb{C})_\lambda$ has a Jordan block of size m for any point y of a connected component of $L_x \cap \mathrm{supp}\, \varphi_{f,\lambda}\mathbb{C}_X$ (considering the subsheaf of $\mathcal{H}^{n-r}\varphi_{f,\lambda}\mathbb{C}_X$ defined by the image of N^{k-1} and using 3.5 below). In particular, we get

COROLLARY 0.5. — *If $\dim \mathrm{supp}\, \varphi_{f,\lambda}\mathbb{C}_X = r$ (e.g. if $\dim \mathrm{Sing}\, f = r$) and the monodromy of $\widetilde{H}^{n-r}(F_y,\mathbb{C})_\lambda$ for one point y of each connected component of $L_x \cap \mathrm{supp}\, \varphi_{f,\lambda}\mathbb{C}_X$ is semisimple, then so is that of $\widetilde{H}^{n-r}(F_x,\mathbb{C})_\lambda$.*

For the lowest degree part we have a more precise description of $\mathcal{H}^{n-r}\varphi_{f,\lambda}\mathbb{C}_X$, see 3.5 below.

In Section 1 we review the theory of nearby and vanishing cycles. In Section 2 we calculate the cohomology of some sheaf complexes on the link of a point. In Section 3 we prove Theorems 0.1 and 0.4. In Section 4 we treat the case of simple normal crossings outside a point, and prove Theorems 0.2 and 0.3. In Appendix we give a nontrivial example for Theorem 0.4.

1. Vanishing cycles.

1.1. Nearby and vanishing cycles. — Let f be a nonconstant holomorphic function on a connected complex analytic space X. Assume $\mathbb{Q}_X[n+1]$ is a perverse sheaf in the sense of [2] (in particular, $\dim X = n+1$). This is satisfied if X is a locally complete intersection, see e.g. [8], Theorem 5.1.19. (Indeed, if X is defined locally by a regular sequence $g_1, \dots, g_r$ on a smooth space Z, we can show the acyclicity (except for one degree) of the algebraic local cohomology of $\mathcal{O}_Z$ along X by using the inductive limit of the Koszul complex of $g_1^m, \dots, g_r^m$ for $m \to \infty$, see also 1.6 below.)

Let A be a field of characteristic 0 (e.g. $A = \mathbb{Q}$ or $\mathbb{C}$). We denote by $\psi_f A_X, \varphi_f A_X$ the nearby and vanishing cycle complexes on $Y := f^{-1}(0)$, see [7]. It is well known that $\psi_f A_X[n]$ and $\varphi_f A_X[n]$ are perverse sheaves. (This follows, for example, from [12], [13], see also [3].) We have the action of the semisimple part T_s of the monodromy T on the shifted perverse sheaves. For $\lambda \in A$, let

$$\psi_{f,\lambda} A_X = \operatorname{Ker}(T_s - \lambda) \subset \psi_f A_X \quad \text{(similarly for } \varphi_{f,\lambda} A_X).$$

By definition of vanishing cycles, we have $\psi_{f,\lambda} A_X = \varphi_{f,\lambda} A_X$ for $\lambda \neq 1$.

If A is algebraically closed, we have the decompositions

$$\psi_f A_X = \bigoplus_\lambda \psi_{f,\lambda} A_X, \quad \varphi_f A_X = \bigoplus_\lambda \varphi_{f,\lambda} A_X,$$

In general, we have

$$\psi_f A_X = \psi_{f,1} A_X \oplus \psi_{f,\neq 1} A_X, \quad \varphi_f A_X = \varphi_{f,1} A_X \oplus \varphi_{f,\neq 1} A_X,$$

where $\psi_{f,\neq 1}$, $\varphi_{f,\neq 1}$ denote the non unipotent monodromy part, and $\psi_{f,\neq 1} = \varphi_{f,\neq 1}$.

For $x \in Y$, we have isomorphisms

$$H^j(F_x, A)_\lambda = \mathcal{H}^j(\psi_{f,\lambda} A_X)_x, \quad \widetilde{H}^j(F_x, A)_\lambda = \mathcal{H}^j(\varphi_{f,\lambda} A_X)_x. \tag{1.1.1}$$

Here F_x denotes the Milnor fiber as in the introduction, and $H^j(F_x, A)_\lambda$ is the λ-eigenspace as above. By [16], [17], we have a canonical mixed Hodge structure on these groups (which coincides with the one in [20] for the isolated singularity case), see also [14].

1.2. Cohomology with compact supports. — It is known that there is a proper continuous map $\rho: X_c \to Y$ such that $\psi_f A = \mathbb{R}\rho_* A$, where $X_c = f^{-1}(c)$ for $c \neq 0$ sufficiently small. This can be constructed by using a resolution of singularities. Let $i: \{x\} \to Y$ denote the inclusion morphism. Then for a sufficiently small open ball B_x around x, we have a commutative diagram

$$\begin{array}{ccccc} H_c^k(F_x, A) & = & H_c^k(B_x \cap Y, \psi_f A) & = & H^k i^! \psi_f A \\ \downarrow{\scriptstyle \beta_F} & & \downarrow{\scriptstyle \beta_B} & & \downarrow{\scriptstyle \beta_\psi} \\ H^k(F_x, A) & = & H^k(B_x \cap Y, \psi_f A) & = & H^k i^* \psi_f A, \end{array} \tag{1.2.1}$$

where the horizontal morphisms are canonical isomorphisms, the first two vertical morphisms β_F, β_B are natural morphisms, and β_ψ is induced by the natural morphism $i^! \to i^*$. By (1.2.1), β_F will be identified with β_ψ.

1.3. Unipotent monodromy part. — We have morphisms of perverse sheaves (compatible with mixed Hodge modules [16])

$$\mathrm{can}: \psi_{f,1} A \longrightarrow \varphi_{f,1} A, \quad \mathrm{Var}: \varphi_{f,1} A(1) \longrightarrow \psi_{f,1} A,$$

whose compositions coincide with N on $\psi_{f,1}A, \varphi_{f,1}A$. If $n \geq 1$ and X is a rational homology manifold at x, then they induce isomorphisms

$$\begin{cases} \mathrm{can}: H^n i^* \psi_{f,1} A \xrightarrow{\sim} H^n i^* \varphi_{f,1} A, \\ \mathrm{Var}: H^n i^! \varphi_{f,1} A(1) \xrightarrow{\sim} H^n i^! \psi_{f,1} A, \end{cases} \tag{1.3.1}$$

because the mapping cone of Var is $\mathbb{R}\Gamma_Y A_X(1)[2]$ and $\mathbb{R}\Gamma_{\{x\}} \mathbb{R}\Gamma_Y A_X = \mathbb{R}\Gamma_{\{x\}} A_X$.

By the isomorphisms of (1.3.1), the morphism

$$\beta_{F,1}: H_c^n(F_x, A)_1 \longrightarrow H^n(F_x, A)_1, \tag{1.3.2}$$

which is the restriction of β_F, can be identified with the composition of N and

$$\beta_{\varphi,1}: H^n i^! \varphi_{f,1} A \longrightarrow H^n i^* \varphi_{f,1} A, \tag{1.3.3}$$

which is induced by the natural morphism $i^! \to i^*$. Indeed, using

$\mathrm{can}\circ\mathrm{Var}=N$ together with the commutativity of the natural morphism $i^! \to i^*$ with can, Var, we get a commutative diagram

$$\begin{array}{ccc} H^n i^! \psi_{f,1} & \xrightarrow{\beta_{\psi,1}} & H^n i^* \psi_{f,1} A \\ \uparrow{\scriptstyle \mathrm{Var}} & & \downarrow{\scriptstyle \mathrm{can}} \\ H^n i^! \varphi_{f,1} A(1) & \xrightarrow{N\beta_{\varphi,1}} & H^n i^* \varphi_{f,1} A, \end{array} \tag{1.3.4}$$

where the vertical morphisms are isomorphisms. Note that the morphism $\beta_{\varphi,1}$ in (1.3.3) is an isomorphism in the isolated singularity case, because $\operatorname{supp}\varphi_f A=\{x\}$.

In Theorem 0.1, $\beta_{\varphi,1}$ in (1.3.3) is identified with a morphism $H^n_c(F_x,A)_1(-1) \to H^n(F_x,A)_1$ by using the isomorphisms of (1.2.1) and (1.3.1). For the non unipotent monodromy part, we have $\beta_{\psi,\neq 1}=\beta_{\varphi,\neq 1}$, because $\psi_{f,\neq 1}=\varphi_{f,\neq 1}$.

1.4. Normal crossing case. — Assume that $Y := f^{-1}(0)$ is a divisor with normal crossings on a complex manifold X whose irreducible components $Y_1,\dots,Y_m$ are smooth. Let

$$\mathcal{F}_\lambda=\psi_{f,\lambda}\mathbb{C}_X[n].$$

Since $\psi_{f,\lambda}\mathbb{C}_X\oplus\psi_{f,\bar\lambda}\mathbb{C}_X$ underlies a mixed Hodge Module, $\mathcal{F}_\lambda$ has the weight filtration W which is the monodromy filtration shifted by $n=\dim Y$, i.e.

$$N^k:\operatorname{Gr}^W_{n+k}\mathcal{F}_\lambda \xrightarrow{\sim} \operatorname{Gr}^W_{n-k}\mathcal{F}_\lambda. \tag{1.4.1}$$

Let $\operatorname{PGr}^W_{n+k}\mathcal{F}_\lambda$ denote the N-primitive part, which is defined by $\operatorname{Ker}N^{k+1}\subset\operatorname{Gr}^W_{n+k}\mathcal{F}_\lambda$ for $k\geq 0$, and is zero otherwise. By (1.4.1) we have the primitive decomposition

$$\operatorname{Gr}^W_j\mathcal{F}_\lambda=\bigoplus_{k\geq 0}N^k\operatorname{PGr}^W_{j+2k}\mathcal{F}_\lambda(k). \tag{1.4.2}$$

Let a_j be the multiplicity of f along Y_j, and put $J(\lambda)=\{j:\lambda^{a_j}=1\}$. Let $d=\mathrm{GCD}(a_1,\dots,a_m)$. Then

$$J(\lambda)=\{1,\dots,m\}\quad\text{if and only if}\quad \lambda^d=1. \tag{1.4.3}$$

For $I\subset J(\lambda)$, let $Y_I=\bigcap_{j\in I}Y_j$, $U_I=Y_I\setminus\bigcup_{j\notin J(\lambda)}Y_j$, with the inclusion morphism $j_I:U_I\to Y_I$. By [17], 3.3, we see that the primitive part $\operatorname{PGr}^W_{n+k}\mathcal{F}_\lambda$ is the direct sum of

$$(j_I)_!\mathcal{F}_{\lambda,I}(-k)[n-k]=\mathbb{R}(j_I)_*\mathcal{F}_{\lambda,I}(-k)[n-k] \tag{1.4.4}$$

over $I \subset J(\lambda)$ with $|I| = k+1$, where $\mathcal{F}_{\lambda,I}$ is a local system of rank 1 on U_I. Furthermore, the monodromy of $\mathcal{F}_{\lambda,I}$ around $Y_j\,(j \notin J(\lambda))$ is given by the multiplication by λ^{-a_j} so that (1.4.4) holds.

If each Y_j is a principal divisor defined by a reduced equation f_j and $f = \prod_j f_j^{a_j}$, then the $\mathcal{F}_{\lambda,I}$ are the restrictions of $\mathcal{F}_{\lambda,\emptyset}$ on $U_\emptyset$ which is defined by $\bigotimes_j f_j^* L_j$ where L_j is a local system on $\mathbb{C}^*$ with monodromy λ^{-a_j} for $j \notin J(\lambda)$. This can be verified by reducing to the case where the a_i are independent of i, and using the compatibility of the nearby cycle functor with the direct image under a proper morphism. Indeed, setting $c_j = \mathrm{LCM}(a_1, \dots, a_m)/a_j$, we have a ramified covering of X defined by

$$\big\{(x, t_1, \dots, t_m) \in X \times \mathbb{C}^m :\ f_j(x) = t_j^{c_j} \text{ for any } j\big\}. \tag{1.4.5}$$

For the vanishing cycle $\varphi_{f,1}\mathbb{C}_X[n]$ with $\lambda = 1$, the weight filtration is the monodromy filtration shifted by $n+1$. For the N-primitive part $\mathrm{PGr}^W_{n+1+k}\,\varphi_{f,1}\mathbb{C}_X[n]$, we have

$$\mathrm{PGr}^W_{n+1+k}\,\psi_{f,1}\mathbb{C}_X[n] = \mathrm{PGr}^W_{n+1+k}\,\varphi_{f,1}\mathbb{C}_X[n] \quad \text{for } k \geq 0,$$

because $\varphi_{f,1}\mathbb{C}_X[n]$ can be identified with $\mathrm{Im}\,N \subset \psi_{f,1}\mathbb{C}_X[n]$.

1.5. Weight spectral sequence. — Let $\pi : (X', Y') \to (X, Y)$ be an embedded resolution such that $Y' := \pi^{-1}(Y)$ and $E := \pi^{-1}(x)$ are divisors with simple normal crossings. Let E' be the closure of $Y' \setminus E$, and put $U = E \setminus E'$ with the inclusion $j' : U \to E$. Let $f' = f\pi$. Then by [4], 4.2, the canonical morphism

$$\psi_{f',\lambda}\mathbb{C}_{X\,|\,E'} \longrightarrow \mathbb{R}j'_*(\psi_{f',\lambda}\mathbb{C}_{X\,|\,U}) \tag{1.5.1}$$

is a quasi-isomorphism. (This easily follows from [17], 3.3.) Since the nearby cycle functor commutes with the direct image under a proper morphism, we get canonical isomorphisms (compatible with T)

$$H^i(F_x, \mathbb{C})_\lambda = H^i(E, \psi_{f,\lambda}\mathbb{C}_{X\,|\,E}) = H^i(U, \psi_{f,\lambda}\mathbb{C}_{X\,|\,U}). \tag{1.5.2}$$

Let $Y_1, \dots, Y_m$ denote the irreducible components of Y' (which are assumed to be smooth). We may assume that $Y_1, \dots, Y_r$ are the irreducible components of $E = \pi^{-1}(x)$. Let $Y_I, U_I, \mathcal{F}_{\lambda,I}$ be as in 1.4. For $I \subset \{1, \dots, m\}$, let $s(I) = |I \cap \{1, \dots, r\}| - 1$. Then we have the weight spectral sequence

$$E_1^{-k,j+k} = \bigoplus_{I,a} H^{j-|I|+1}\big(U_I, \mathcal{F}_{\lambda,I}(a+1-|I|)\big) \Longrightarrow H^j(F_x, \mathbb{C})_\lambda, \tag{1.5.3}$$

where the summation is taken over $I\,(\neq\emptyset)\subset J(\lambda)$, $0\le a\le s(I)$ such that $|I|-1-2a=k$. Indeed, $\mathbb{R}j'_*$ is a t-exact functor [2], and $(j_I)_!\mathcal{F}_{\lambda,I}(a+1-|I|)[n+1-|I|]$ comes from the graded pieces of the weight filtration on

$$\mathbb{R}j'_*\big((j_{I'})_!\mathcal{F}_{\lambda,I'}(a+1-|I'|)[n+1-|I'|]_{|U}\big)$$

for $I':=I\cap\{1,\ldots,r\}$. Here we may assume essentially that $\mathcal{F}_{\lambda,I'}$ is a constant sheaf (where the assertion is well-known [6]) because it is of normal crossing type, see [17], 3.1. The range of a comes from the symmetry of the weight filtration (1.4.1) which is related to $\mathcal{F}_{\lambda,I'}$ because we consider it on U.

The spectral sequence (1.5.3) degenerates at E_2, because $E_1^{-k,j+k}$ is pure of weight $j+k$.

Remark 1.6. — If $\mathbb{Q}_X[n+1]$ is a perverse sheaf, then $\mathbb{Q}_Y[n]$ is a perverse sheaf for any locally principal divisor Y on X. Indeed, we have locally a distinguished triangle

$$\mathbb{Q}_Y[n]\longrightarrow\psi_f\mathbb{Q}_X[n]\longrightarrow\varphi_f\mathbb{Q}_X[n]\xrightarrow{+1},\tag{1.6.1}$$

by the definition of φ_f, where f is a local equation of Y. This implies ${}^p\mathcal{H}^j(\mathbb{Q}_Y[n])=0$ except for $j=0,1$, where ${}^p\mathcal{H}^j$ denotes the perverse cohomology functor [2]. Furthermore, the vanishing of ${}^p\mathcal{H}^j(\mathbb{Q}_Y[n])$ for $j>0$ is clear by the definition of semi-perversity. (In general, a sheaf complex $\mathcal{F}$ is called semi-perverse if $\dim\operatorname{supp}\mathcal{H}^{-i}\mathcal{F}\le i$ for any i, see *loc. cit.*)

1.7. Wang sequence. — Let f be a holomorphic function on an analytic space X. Let $L_{X,x}$ be the link of x in X. Then we have the Wang sequence

$$H^j(L_{X,x}\backslash Y,\mathbb{Q})\longrightarrow H^j(F_x,\mathbb{Q})_1\xrightarrow{N}H^j(F_x,\mathbb{Q})_1(-1)\longrightarrow H^{j+1}(L_{X,x}\backslash Y,\mathbb{Q}).$$

In the category of mixed Hodge structures, this follows from

$$i'^*j'_*\mathbb{Q}_X=C(j'_!\mathbb{Q}_X\to j'_*\mathbb{Q}_X)=C\big(N:\psi_{f,1}\mathbb{Q}_X\to\psi_{f,1}\mathbb{Q}_X(-1)\big)[-1],$$

where $i':Y\to X$, $j':X\setminus Y\to X$ are the inclusion morphisms, see e.g. [17], 2.23, for the second isomorphism. (Here $\mathbb{Q}_X$ can be defined locally in the derived category of mixed Hodge Modules, using an embedding into a smooth space.)

2. Cohomology of link with coefficients.

2.1. Localization sequence. — Let $\mathcal{F}$ be a perverse sheaf on Y in the sense of [2]. In particular,

$$\dim \operatorname{supp} \mathcal{H}^{-k}\mathcal{F} \leq k, \tag{2.1.1}$$

$$\mathcal{H}^{-r}\mathcal{F} = 0 \quad \text{for } r > \dim \operatorname{supp} \mathcal{F}. \tag{2.1.2}$$

Let $i:\{x\} \to Y$ and $j:U := Y \setminus \{x\} \to Y$ denote the inclusions. Let L_x be the intersection of a sufficiently small sphere around x with Y. Then

$$\mathbb{H}^k(L_x, \mathcal{F}_{|L_x}) = H^k i^* j_* j^* \mathcal{F}, \tag{2.1.3}$$

and we get a long exact sequence

$$\cdots \to H^k_{\{x\}}\mathcal{F} \longrightarrow H^k\mathcal{F}_x \to \mathbb{H}^k(L_x, \mathcal{F}_{|L_x}) \longrightarrow H^{k+1}_{\{x\}}\mathcal{F} \to \cdots$$

induced by the distinguished triangle

$$\mathbb{R}\Gamma_{\{x\}}\mathcal{F} \longrightarrow \mathcal{F}_x \longrightarrow \mathbb{R}\Gamma(L_x, \mathcal{F}_{|L_x}) \xrightarrow{+1}$$

which is identified with $i^!\mathcal{F} \to i^*\mathcal{F} \to i^*j_*j^*\mathcal{F} \xrightarrow{+1}$ (because $i_*i^! = \mathbb{R}\Gamma_{\{x\}}$).

Let $\mathbb{D}$ denote the functor assigning the dual. Since $\mathbb{D}i^! = i^*\mathbb{D}$, and $\mathbb{D}\mathcal{F}$ is a perverse sheaf, we get

$$H^k_{\{x\}}\mathcal{F} = 0 \quad \text{for } k < 0. \tag{2.1.4}$$

Indeed, (2.1.4) is equivalent to the (dual) semi perversity of $\mathcal{F}$ (see [2]) assuming the perversity of the restriction of $\mathcal{F}$ to the complement of x.

2.2. Leray spectral sequence. — Let $\mathcal{F}$ be a complex of sheaves with constructible cohomology on Y. There is a Leray-type spectral sequence

$$E_2^{p,q} = H^p(L_x, \mathcal{H}^q\mathcal{F}_{|L_x}) \Longrightarrow \mathbb{H}^{p+q}(L_x, \mathcal{F}_{|L_x}) \tag{2.2.1}$$

induced by the filtration τ on $\mathcal{F}$, see [6]. By (2.1.3) this is compatible with mixed Hodge structure (using a t-structure in [17], 4.6) if $\mathcal{F}$ underlies a complex of mixed Hodge modules. The calculation of (2.2.1) is not necessarily easy. One problem is that $\mathcal{H}^q\mathcal{F}$ is a constructible sheaf and not a local system, and some times we have to use the spectral sequence associated to a stratification, which is a special case of (2.3.1) below, to calculate its cohomology. Actually this spectral sequence can be formulated for a complex as below, and we do not have to use spectral sequences twice if we can calculate the E_1-term of (2.3.1). But the calculation of d_r is still nontrivial.

2.3. Spectral sequence associated to a stratification. — Let $\mathcal{F}$ be as above, and let $\{Y_k\}$ be a stratification of Y compatible with $\mathcal{F}$, where the Y_k are locally closed analytic subspaces of Y with pure dimension k such that the restriction of $\mathcal{H}^j\mathcal{F}$ to Y_k is a local system, and $\overline{Y}_k\setminus Y_k$ is the disjoint union of Y_i $(i<k)$. Put $U_k = Y\setminus\overline{Y}_{k-1}$. Then, for each k, there is a subcomplex of $\mathcal{F}$ whose restriction to U_k coincides with $\mathcal{F}_{|U_k}$ and whose restriction to $\overline{Y}_{k-1}$ vanishes (i.e. it is the direct image with proper supports by $U_k\to Y$). Such complexes form a decreasing filtration of $\mathcal{F}$ whose graded pieces are (the direct images with proper supports of) the restrictions of $\mathcal{F}$ to the Y_k. So they induce the spectral sequence associated to the stratification

$$E_1^{p,q} = \mathbb{H}_c^{p+q}(L_x\cap Y_p, \mathcal{F}_{|L_x\cap Y_p}) \Longrightarrow \mathbb{H}^{p+q}(L_x, \mathcal{F}_{|L_x}). \tag{2.3.1}$$

By (2.1.3) this is also compatible with mixed Hodge structure (using the quasi-filtration in [16], 5.2.17).

2.4. Weight spectral sequence. Let $\mathcal{F}$ be a perverse sheaf underlying a mixed Hodge Module, and W be the weight filtration. Then, as in [6], W induces a spectral sequence

$$E_1^{-k,j+k} = \mathbb{H}^j(L_x, \mathrm{Gr}_k^W \mathcal{F}_{|L_x}) \Longrightarrow \mathbb{H}^j(L_x, \mathcal{F}_{|L_x}), \tag{2.4.1}$$

which is called the (generalized) weight spectral sequence. (We can use Verdier's theory of spectral objects, see [2] and also [16], 5.2.18.) By (2.1.3) this is compatible with mixed Hodge structure, but does not necessarily degenerate at E_2, because $E_1^{-k,j+k}$ is not pure of weight $j+k$ in general. It is not easy to calculate this spectral sequence explicitly except for some special cases, see e.g. 4.2 below.

If $X\setminus\{x\}$ is smooth and $Y\setminus\{x\}$ is a divisor with simple normal crossings, then the E_1-complex has a structure of double complex whose differentials are induced by the Čech restriction morphism and the co-Čech Gysin morphism, see e.g. [20]. Indeed, the differential d_1 is induced by the extension class between the graded pieces of the perverse sheaves, and the assertion can be verified by using locally a ramified covering as in (1.4.5) and reducing to the case where the irreducible components of $Y\setminus\{x\}$ have the constant multiplicity.

3. Proofs of Theorems 0.1 and 0.4.

3.1. Proof of Theorem 0.1. — Applying 2.1 to $\mathcal{F}=\varphi_f\mathbb{Q}_X[n]$, the assertion follows from 1.1–1.3 and 2.1.

3.2. Proof of Theorem 0.4. — The first assertion follows from 2.1 applied to $\operatorname{Im} N^{k-1} \subset \varphi_f \mathbb{C}_X$ (defined in the abelian category of shifted perverse sheaves). Indeed, factorizing $N^{k-1} : \varphi_f \mathbb{C}_X \to \varphi_f \mathbb{C}_X(1-k)$ by $\operatorname{Im} N^{k-1}$, we see that $\operatorname{Im} N^{k-1} \neq 0$ on a neighborhood of x. The remaining assertion is clear by (2.2.1). Indeed, if any Jordan block of the monodromy on $\widetilde{H}^i(F_y, \mathbb{C})_\lambda$ has size at most k_i, then $N^{k_i} = 0$ on $H^{j-i}(L_x, \mathcal{H}^i \varphi_{f,\lambda} \mathbb{C}_{X\,|\,L_x})$, and $N^k(\widetilde{H}^j(F_x, \mathbb{C})_\lambda) = 0$ for $k = \sum_{i \leq j} k_i$ by Theorem 0.1 together with (2.2.1), because $N^{k_i}(\operatorname{Gr}_G^{j-i} \widetilde{H}^j(F_x, \mathbb{C})_\lambda) = 0$ where G is the filtration associated to the spectral sequence (2.2.1).

3.3. One-dimensional singular locus case. — If $\Sigma_\lambda := \operatorname{supp} \varphi_{f,\lambda} \mathbb{C}_X$ is 1-dimensional (e.g. if $\operatorname{Sing} f$ is 1-dimensional), let $\Sigma_{\lambda,i}$ be the local irreducible components of Σ_λ at x, and take $x_i \in \Sigma_{\lambda,i} \cap L_x$. Then $H^j \varphi_{f,\lambda} \mathbb{C}_X = 0$ for $j < n-1$, and

$$\mathbb{H}^{n-1}(L_x, \varphi_{f,\lambda} \mathbb{C}_{X\,|\,L_x}) = \bigoplus_i (\widetilde{H}^{n-1}(F_{x_i}, \mathbb{C})_\lambda)^{\tau_i}, \tag{3.3.1}$$

where τ_i denotes the monodromy of the local system on $\Sigma_{\lambda,i} \cap L_x$ (which is called the vertical monodromy in [18], [19]). However, for a given element of $\bigoplus_i (\widetilde{H}^{n-1}(F_{x_i}, \mathbb{C})_\lambda)^{\tau_i}$, it is not easy to determine whether it comes from $\widetilde{H}^{n-1}(F_x, \mathbb{C})_\lambda$ or not. Note that $K_x = \operatorname{Ker} \beta_\varphi$ does not vanish in general. For example, if X is smooth and Y is a reduced divisor with normal crossings it is well-known (see e.g. [20]) that the Milnor fiber is homotopy equivalent to a real torus of dimension $m-1$ where m is the multiplicity of Y at the point. In the case $f = xyz$ and $n = 2$, we have $\dim H^1(F_x, \mathbb{C})_1 = 2$ and $\dim \mathbb{H}^1(L_x, \varphi_{f,1} \mathbb{C}_{X\,|\,L_x}) = 3$, see also [18], [19].

3.4. Remark. — There are examples such that the monodromy of $\widetilde{H}^{n-1}(F_x, \mathbb{Q})$ is semisimple, but that of $\widetilde{H}^{n-1}(F_y, \mathbb{Q})$ for y sufficiently near x has a Jordan block of size n (this implies that the converse of Theorem 0.4 does not hold). For example, consider a germ of $(n-1)$-dimensional hypersurface (Y, x) with isolated singularity whose Milnor monodromy has a Jordan block of size n, take a projective compactification Z of Y in $\mathbb{P}^n$ such that $Z \setminus \{x\}$ is smooth (using finite determinacy of isolated singularity), and then take $f : \mathbb{C}^{n+1} \to \mathbb{C}$ to be a defining equation of Z.

3.5. Lowest degree term. — Assume $\Sigma_\lambda := \operatorname{supp} \varphi_{f,\lambda} \mathbb{C}_X$ is r-dimensional (e.g. $\operatorname{Sing} f$ is r-dimensional). Let Σ_λ^1 be an $(r-1)$-dimensional Zariski-locally closed smooth analytic subspace of Σ_λ such that $\Sigma_\lambda^0 := \Sigma_\lambda \setminus \overline{\Sigma}_\lambda^1$ is smooth (where $\overline{\Sigma}_\lambda^1$ is the closure of Σ_λ^1) and the restrictions of $\mathcal{H}^j\varphi_{f,\lambda}\mathbb{C}_X$ to Σ_λ^0, Σ_λ^1 are local systems for any j. Let $\overline{\Sigma}_\lambda^2 = \overline{\Sigma}_\lambda^1 \setminus \Sigma_\lambda^1$, $U_\lambda = \Sigma_\lambda \setminus \overline{\Sigma}_\lambda^2$ with the inclusions $j' : \Sigma_\lambda^0 \to U_\lambda$, $j'' : U_\lambda \to \Sigma_\lambda$. Then

$$\mathcal{H}^{n-r}\varphi_{f,\lambda}\mathbb{C}_{X\,|\,U_\lambda} \subset j'_*j'^*(\mathcal{H}^{n-r}\varphi_{f,\lambda}\mathbb{C}_{X\,|\,U_\lambda}), \tag{3.5.1}$$

$$\mathcal{H}^{n-r}\varphi_{f,\lambda}\mathbb{C}_X = j''_*(\mathcal{H}^{n-r}\varphi_{f,\lambda}\mathbb{C}_{X\,|\,U_\lambda}) \tag{3.5.2}$$

Indeed, restricting to a subspace transversal to Σ_λ^1, (3.5.1) follows from the 1-dimensional singular locus case, and furthermore, the cokernel of the inclusion in (3.5.1) is given by K_x in Theorem 0.1, see (3.3). Similarly (3.5.2) follows from Theorem 0.1 by induction on strata.

4. Case of simple normal crossings outside a point

4.1. — With the notation of (1.1), assume that $X \setminus \{x\}$ is smooth, and $Y \setminus \{x\}$ is a divisor with *simple* normal crossings on $X \setminus \{x\}$. Here simple means that each irreducible component of $Y \setminus \{x\}$ is smooth. Assume further that the local irreducible components of Y at x are principal divisors. Then, replacing X with a sufficiently small open neighborhood of x if necessary, there exist holomorphic functions $f_i : X \to \mathbb{C}$ and positive integers a_i for $i = 1, \ldots, m$ such that $f = f_1^{a_1} \cdots f_m^{a_m}$ and each $Y_i := f_i^{-1}(0)$ has at most isolated singularity at x, see also [9]. Here we assume $n \geq 2$. Let

$$\mathcal{F}_\lambda = \psi_{f,\lambda}\mathbb{C}_X[n]_{|Y \setminus \{x\}}. \tag{4.1.1}$$

Since $\psi_{f,\lambda}\mathbb{C}_X[n] \oplus \psi_{f,\bar\lambda}\mathbb{C}_X[n]$ underlies a mixed Hodge Module, we have the weight spectral sequence (2.4.1). Here W is the monodromy filtration shifted by $n = \dim Y$, and the N-primitive part $\mathrm{PGr}_{n+k}^W \mathcal{F}_{\lambda\,|\,Y \setminus \{x\}}$ is calculated as in 1.4.

We assume that $\mathbb{Q}_X[n+1]$ is a perverse sheaf. Since the intersection complex of X is given by $\tau_{<0}\mathbb{R}j'_*\mathbb{Q}_{X \setminus \{x\}}[n+1]$ where $j' : X \setminus \{x\} \to X$ denotes the inclusion, this condition is equivalent to

$$\widetilde{H}^j(L_{X,x}, \mathbb{Q}) = 0 \quad \text{for } j < n, \tag{4.1.2}$$

where $L_{X,x}$ is the link of $\{x\}$ in X. This follows from the long exact sequence of perverse sheaves associated to the distinguished triangle

$$\begin{aligned} \mathbb{Q}_X[n+1] &\longrightarrow \tau_{<0}\mathbb{R}j'_*\mathbb{Q}_{X \setminus \{x\}}[n+1] \\ &\longrightarrow (\tau_{<0}\mathbb{R}j'_*\mathbb{Q}_{X \setminus \{x\}}/\mathbb{Q}_X)[n+1] \xrightarrow{+1}, \end{aligned} \tag{4.1.3}$$

because $\mathcal{H}^j(\mathbb{R}j'_*\mathbb{Q}_{X \setminus \{x\}})_x = H^j(L_{X,x}, \mathbb{Q})$.

Proposition 4.2. — *With the above notation and assumptions, let* $\mathcal{F}_{\lambda,I}$, j_I *and* d *be as in* (1.4) *with* Y_j *replaced by* $Y_j\setminus\{x\}$. *Then*

$$\mathbb{H}^i\big(L_x,(j_I)_!\mathcal{F}_{\lambda,I}[n-k]|_{L_x}\big)=0 \quad \text{for} k-n<i<-1, \tag{4.2.1}$$

where $k=|I|-1$. *For* $i=k-n<-1$, *we have*

$$\mathbb{H}^{k-n}\big(L_x,(j_I)_!\mathcal{F}_{\lambda,I}[n-k]|_{L_x}\big)=\begin{cases}\mathbb{C} & \text{if } \lambda^d=1,\\ 0 & \text{if } \lambda^d\neq 1.\end{cases} \tag{4.2.2}$$

Proof. — We prove the assertion by induction on $|I|$. If $I=\emptyset$, we have $\mathcal{F}_{\lambda,\emptyset}$ on $U_\emptyset$ as in 1.4. We may assume $U_\emptyset\neq X$, because the assertion is clear by (4.1.2) if $U_\emptyset=X$. Let B_x be a sufficiently small open ball around x. By the cone theorem, $B_x\cap Y$ is homeomorphic to the topological cone of $\partial B_x\cap Y$ in a compatible way with a given Whitney stratification of Y. (This is proved by using a continuous vector field compatible with the stratification as well-known.) So we have

$$\mathbb{H}^i\big(L_x,\mathbb{R}(j_\emptyset)_*\mathcal{F}_{\bar\lambda,\emptyset}|_{L_x}\big)=\mathbb{H}^i(B_x\cap U_\emptyset,\mathcal{F}_{\bar\lambda,\emptyset}).$$

By duality, (4.2.1) is equivalent to the vanishing of these groups for $n+1<i<2n+1$. (Note that the dual of the λ-eigenspace is the λ^{-1}-eigenspace.) So we get the assertion in this case, using the corresponding de Rham complex and the vanishing of the higher cohomology of coherent sheaves on a smooth Stein space $B_x\cap U_\emptyset$ of dimension $n+1$.

If $I\neq\emptyset$, take $j\in I$, and let $I'=I\setminus\{j\}$. By the exact sequence

$$H^{i-1}\big(L_x,(j_{I'})_!\mathcal{F}_{\lambda,I'}\big)\longrightarrow H^{i-1}\big(L_x,(j_I)_!\mathcal{F}_{\lambda,I}\big)\longrightarrow H^i_c\big(L_x\setminus Y_j,(j_{I'})_!\mathcal{F}_{\lambda,I'}\big),$$

it is enough to show

$$H^i_c\big(L_x\setminus Y_j,(j_{I'})_!\mathcal{F}_{\lambda,I'}\big)=0 \quad \text{for } i<n-k.$$

This is isomorphic to the dual of $H^{2n-2k+1-i}(L_x\setminus Y_j,\mathbb{R}(j_{I'})_*\mathcal{F}_{\bar\lambda,I'})$, because $\dim U_{I'}=n-k+1$. So it is enough to show

$$H^i\big(L_x\cap(Y_{I'}\setminus Y_j),\mathbb{R}(j_{I'})_*\mathcal{F}_{\bar\lambda,I'}\big)=0 \quad \text{for } i>n-k+1.$$

For this, we may replace $L_x\cap(Y_{I'}\setminus Y_j)$ by $B_x\cap(U_{I'}\setminus Y_j)$ (using the cone theorem). Then we get the assertion by using the same argument as above, because $B_x\cap(U_{I'}\setminus Y_j)$ is a smooth Stein space of dimension $n-k+1$.

4.3. Proof of Theorem 0.2. — If $\lambda^d \neq 1$, the assertion follows from 4.2. So we may assume $\lambda^d = 1$, i.e. $J(\lambda) = \{1, \dots, m\}$, see (1.4.3). We define K_λ to be a complex whose j-th component is

$$\bigoplus_{|I|=j} H^0(L_{X,x} \cap Y_I, \mathbb{C}),$$

where $Y_\emptyset = X$, and the differential is given by the Čech restriction morphism. Let σ be the filtration as in [6], II, 1.4.7, and define

$$\widetilde{K}_\lambda = \bigoplus_{i \geq 1} (\sigma_{\geq i} K_\lambda)(1-i)[n+1]. \tag{4.3.1}$$

Let $E_{1,\lambda}$ denote the E_1-complex of the weight spectral sequence (2.4.1) applied to (4.1.1). For $-n \leq j \leq -2$, we see that

$$E_1^{-k,j+k} = K_\lambda^{j+n+1}\Big(\frac{-j-k}{2}\Big), \tag{4.3.2}$$

if $j + k$ is even and $|k| + j + n > 0$, and it is zero otherwise, using 4.2 and the primitive decomposition (1.4.2). So we get

$$\sigma_{\leq -2} \widetilde{K}_\lambda = \sigma_{\leq -2} E_{1,\lambda}, \tag{4.3.3}$$

and $\sigma_{\leq -1} \widetilde{K}_\lambda$ is a quotient complex of $\sigma_{\leq -1} E_{1,\lambda}$. We have the isomorphism for degree ≤ -1 if the last assumption of 0.2 is satisfied, i.e. if for $|I| < n$ we have $H^j(Y_I \cap L_x, \mathbb{C}) = 0$ except for $j = 0$ or $2n + 1 - 2|I|$.

Let $K(\mathbb{C}; v_1, \dots, v_m)$ be the Koszul complex for $v_i = \mathrm{id}: \mathbb{C} \to \mathbb{C}$ $(1 \leq i \leq m)$. Then

$$\sigma_{\leq n-1} K(\mathbb{C}; v_1, \dots, v_m) = \sigma_{\leq n-1} K_\lambda, \tag{4.3.4}$$

and $\sigma_{\leq n} K(\mathbb{C}; v_1, \dots, v_m)$ is a direct factor of $\sigma_{\leq n} K_\lambda$, because Y_I may be reducible if $|I| = n$. So we may replace K_λ with the Koszul complex as long as we calculate the cohomology of degree $\leq n - 1$. Since this Koszul complex is acyclic and the rank of its j-th component is $\binom{m}{j}$, the rank of the nonzero cohomology group of $\sigma_{\geq j} K_\lambda$ (i.e. the image of the differential d^{j-1}) is $\binom{m-1}{j-1}$ for $j \leq n - 1$ by the binomial relation. So the assertion for $\lambda \neq 1$ follows from Theorem 0.1, where the shift of the index j comes from the fact that the complex K_λ is indexed by $|I|$ instead of $k = |I| - 1$.

For $\lambda = 1$, we use a (generalized) weight spectral sequence similar to 2.4:

$$E_1^{-k,j+k} = \bigoplus_{|I|=k} H^{j-k}(L_{X,x} \cap Y_I, \mathbb{Q})(-k) \Longrightarrow H^j(L_{X,x} \setminus Y, \mathbb{Q}).$$

This is induced by the weight filtration W on $(\mathbb{R}j'_*\mathbb{Q}_{X\setminus Y})_{|Y\setminus\{x\}}$ (see [6]) such that

$$\mathrm{Gr}_k^W(\mathbb{R}j'_*\mathbb{Q}_{X\setminus Y})_{|Y\setminus\{x\}} = \bigoplus_{|I|=k} \mathbb{Q}_{Y_I\setminus\{x\}}(-k)[-k],$$

where $j' : X\setminus Y \to X$ is as in (1.7). So $H^j(L_{X,x}\setminus Y, \mathbb{Q})$ is of type (j,j) for $j \le n-1$ because $H^j(L_{X,x} \cap Y_I, \mathbb{Q}) = 0$ for $j \le n-1-|I|$ by (4.1.2) and (1.6).

Since $H^j(F_x, \mathbb{Q})_1$ has weights $\le 2j$ and N is a morphism of type $(-1,-1)$, this assertion implies that $N = 0$ on $H^j(F_x,\mathbb{Q})_1$ for $j \le n-1$ using the Wang sequence 1.7 and considering $\mathrm{Ker}\, N$. The assertion on the rank then follows using the Wang sequence and the binomial relation. This completes the proof of Theorem 0.2.

4.4. Remark. — In [9], the case $a_i = 1$ for any i was treated. The arguments there (e.g. Th. 3.1) imply also the assertion on the rank in 0.2 in this case (see also [5], [15] for the case of a generic central arrangement), and Th. 5.1 corresponds to the vanishing results in (0.2). In Cor. 4.1, it is proved that the monodromy is trivial for $j \le n-1$ in this case.

4.5. Proof of Theorem 0.3. — Let $\mathcal{F}_\lambda = \psi_{f,\lambda}\mathbb{C}_{X|U}$, and let $j_U : U \to E$ denote the inclusion morphism. By (1.5.2) we have canonical isomorphisms (compatible with T)

$$H^i(F_x, \mathbb{C})_\lambda = H^i(E, \psi_{f,\lambda}\mathbb{C}_{X|E}) = H^i(U, \mathcal{F}_\lambda). \tag{4.5.1}$$

By (1.4), $\mathcal{F}_\lambda$ is a local system of rank 1 if $\lambda^e = 1$, and $\mathcal{F}_\lambda = 0$ otherwise. So the action of the monodromy T on $\mathcal{F}_\lambda$ and $H^i(F_x,\mathbb{C})_\lambda$ is the multiplication by λ (i.e. semisimple). The monodromy of $\mathcal{F}_\lambda$ around Y_j is given by the multiplication by λ^{-a_j}. By (4.5.1) we get

$$\chi_\lambda(F_x) = \chi(U) \quad \text{if } \lambda^e = 1, \text{ and 0 otherwise.} \tag{4.5.2}$$

Since we assume that the Y_j are principal, we have

$$H^j(F_x, \mathbb{C})_\lambda = 0 \quad \text{for } j \ne n, \text{ if } \lambda^{a_i} \ne 1 \text{ for some } i, \tag{4.5.3}$$

using the weak Lefschetz theorem, because $E\setminus Y_i'$ is affine where Y_i' is the proper transform of Y_i. Indeed the last assertion can be reduced to the case X smooth, replacing X with an ambient smooth space, because Y_i is principal. So we get

$$b^n_\lambda(F_x) = (-1)^n\chi(U) \quad \text{if } \lambda^e = 1 \text{ and } \lambda^d \neq 1. \tag{4.5.4}$$

If $\lambda^d = 1$, then it is known that $\mathcal{F}_\lambda$ is a constant sheaf on U. (Indeed, $\bigoplus_\lambda \mathcal{F}_\lambda$ is the direct image of a constant sheaf on a finite covering of U which is ramified over $E \cap Y'$, see [20], etc.) Let $D_j := E \cap Y'_i$, and $D^{(k)}$ be the disjoint union of $D_I := \bigcap_{j\in I} D_j$ for $|I| = k$ where $D_\emptyset = E$. Then the cohomology of U is calculated by using the weight spectral sequence [6]

$$E_1^{-k,j+k} = H^{j-k}\big(D^{(k)}, \mathbb{Q}(-k)\big) \Longrightarrow H^j(U,\mathbb{Q}). \tag{4.5.5}$$

By assumption the constant sheaf $\mathbb{Q}_X[n+1]$ is a perverse sheaf, and hence so are $\mathbb{Q}_{Y_I}[n+1-|I|]$ for any I, where $Y_I = \bigcap_{j\in I} Y_j$, see (1.6). On the other hand, it is known that, if there is a blow-up $\pi : X' \to X$ with a point center such that X' and the exceptional divisor E are smooth, then the primitive cohomology of E is isomorphic to the stalk of the intersection cohomology $\mathrm{IC}_X\mathbb{Q}$ of X at x. (Indeed, by the decomposition theorem [2], $\mathbb{R}\pi_*\mathbb{Q}_{X'} = \mathrm{IC}_X\mathbb{Q}[-n-1] \oplus M^\bullet$ with $\operatorname{supp} M^\bullet = \{x\}$, and M is symmetric with center $n+1$, i.e. $\dim M^{n+1-j} = \dim M^{n+1+j}$ by the relative hard Lefschetz theorem for π. On the other hand, $H^\bullet(E,\mathbb{Q}) = (\mathrm{IC}_X\mathbb{Q}[-n-1])_x \oplus M^\bullet$, and it is symmetric with center n by the classical hard Lefschetz theorem. Then the assertion follows from the Lefschetz decomposition because $H^j(E,\mathbb{Q}) = M^j$ for $j > n$.) So the j-th primitive cohomology of the exceptional divisor vanishes for $0 < j < \dim X - 1$, using an exact sequence as in (4.1.3). Similar assertions hold also for any Y_I.

For $0 \le j < n$, the above arguments imply that

$$E_1^{-k,j+k} = \bigoplus_{|I|=k} \mathbb{Q}\big(-\tfrac{1}{2}(j+k)\big), \tag{4.5.6}$$

if $j+k$ is even and $0 \le k \le j$, and it is zero otherwise. For $j = n$, $E_1^{-k,n+k}$ contains $\bigoplus_{|I|=k} \mathbb{Q}(-\frac{1}{2}(n+k))$ if $j+k$ is even and $0 \le k \le n$. Furthermore, the differential is given by the co-Čech Gysin morphism. Thus the cohomology of the E_1-complex of (4.5.5) for $j < n$ is calculated by that of

$$\bigoplus_{j\ge 0} \sigma_{\ge m-j} K(\mathbb{C}; v_1, \ldots, v_m)[m-2j](-j),$$

where the Koszul complex $K(\mathbb{C}; v_1, \ldots, v_m)$ is as in 4.3. So the assertion on the Hodge type and the rank in 0.2 holds for $j \le n-1$ in this case.

Combined with (4.5.4), this implies

$$b^n_\lambda(F_x) = (-1)^n\chi(U) + \binom{m-2}{n-1} \text{ if } \lambda^e = 1 \text{ and } \lambda^d = 1, \tag{4.5.7}$$

because $\binom{m-2}{n-1} = \sum_{0\le k\le n-1}(-1)^k\binom{m-1}{n-1-k}$, see [15], Lemma 2.5. This completes the proof of Theorem 0.3. □

4.6. Remark. — With the assumption of 4.5, assume further X smooth. Then it is known that $\chi(U)$ is explicitly calculated by using d_j. Indeed, we have by (4.5.5)

$$\chi(U) = \bigoplus_{|I|\le n}(-1)^{|I|}\chi(D_I).$$

Furthermore, by the theory of Chern classes (see e.g. [11]), the topological Euler characteristic $\chi(D_I)$ is the coefficient of T^n in

$$(1+T)^{n+1}\prod_{j\in I}\big(d_jT/(1+d_jT)\big) \in \mathbb{Q}[[T]]/(T^{n+1}),$$

because the k-th Chern class of the tangent bundle of D_I gives the topological Euler characteristic for $k = \dim D_I (= n - |I|)$, and the restriction of a cycle on $\mathbb{P}^n$ to D_I is essentially same as the intersection with D_I. Here the truncated formal power series ring is identified with the cohomology ring of $\mathbb{P}^n$ so that $(1+T)^{n+1}$ is the total Chern class of the tangent bundle of $\mathbb{P}^n$, and $1 + d_jT$ is that of the normal bundle of D_j.

Since $1 - d_jT/(1+d_jT) = (1+d_jT)^{-1}$, we see that $\chi(U)$ is the coefficient of T^n in

$$(1+T)^{n+1}\prod_{1\le j\le m}(1+d_jT)^{-1} \in \mathbb{Q}[[T]]/(T^{n+1}).$$

For $m = 1$ and $a_1 = 1$, this is compatible with a well-known formula for the Milnor number of a homogeneous hypersurface isolated singularity (using Theorem 0.3), i.e.

$$1 - d_1\chi(U) = (1-d_1)^{n+1}.$$

In the case of a generic central arrangement (i.e. $d_j = 1$), the above assertion implies

$$\chi(U) = (-1)^n\binom{m-2}{n}. \tag{4.6.1}$$

This is compatible with the formula in [5], [15] using Theorem 0.3.

In general, we can verify that the coefficient of T^k in

$$\prod_{1\le j\le m}(1+d_jT)^{-1}\in\mathbb{Q}[[T]]/(T^{n+1})$$

is a polynomial in $d_1,\dots,d_m$, which is equal to

$$(-1)^k\sum_{1\le i\le m}\Big(d_i^{k+m-1}/\prod_{p\ne i}(d_i-d_p)\Big)$$

in the fraction field $\mathbb{Q}(d_1,\dots,d_m)$. This follows by induction on m, using

$$\sum_{0\le j\le k}d_{m-1}^j d_m^{k-j}=(d_{m-1}^{k+1}-d_m^{k+1})/(d_{m-1}-d_m).$$

Furthermore, the above polynomial vanishes for $1-m\le k<0$, because it is a polynomial, and has negative degree. So we see that $\chi(U)$ for $m>1$ is a polynomial in $d_1,\dots,d_m$, which is equal to

$$\sum_{1\le i\le m}\Big(-d_i^{m-2}(1-d_i)^{n+1}/\prod_{p\ne i}(d_i-d_p)\Big) \tag{4.6.2}$$

in the fraction field. This gives an explicit formula if the d_i are different from each other. In general we have to take a limit (or make some calculation in the fraction field).

Appendix.

We give an example such that the monodromy at degree $n-1$ is not semisimple at the origin, but is semisimple at the other points. This shows that Theorem (0.4) is optimal, and that the extension class between the graded-pieces of the filtration associated to the Leray spectral sequence 2.2 for the nearby cycles is nontrivial as $\mathbb{C}[N]$-modules.

A.1. Embedded resolution of singularities. — We first explain how to get an embedded resolution of a function of type $f=f_d+f_{d+1}$ on the affine cone X of a smooth projective variety E with a very ample line bundle L defining the embeddings $E\to\mathbb{P}^{r-1}$ and $X\to\mathbb{C}^r$ where $r=\dim\Gamma(E,L)$. Here f_j is an element of the j-th symmetric power of $\Gamma(E,L)$, which is identified with a polynomial of degree j in r variables, and defines a function on the affine cone X. We assume that $f_d^{-1}(0)\setminus f_{d+1}^{-1}(0)$

defines a divisor with simple normal crossings on $E\setminus f_{d+1}^{-1}(0)$ (where f_j is viewed as a section of $L^{\otimes j}$).

Let $X^{\vee}$ be the total space of the dual of the line bundle L with the projection $\rho: X^{\vee}\to E$. It is the blow-up of the affine cone X at the origin, and the exceptional divisor is identified with E. Let

$$D_0=f_d^{-1}(0),\quad D_{\infty}=f_{d+1}^{-1}(0)$$

as (not necessarily reduced) divisors on E. Let $Y^{\vee}$ be the proper transform of $f^{-1}(0)$ in $X^{\vee}$. Let $D_{0,\infty}$ be the greatest common divisor of D_0 and D_{∞}, and put

$$D_0^{\mathrm{red}}=D_0-D_{0,\infty},\quad D_{\infty}^{\mathrm{red}}=D_{\infty}-D_{0,\infty}.$$

Then we have a canonical decomposition

$$Y^{\vee}=Y_{\mathrm{hor}}+Y_{\mathrm{ver}},$$

where $Y_{\mathrm{ver}}=\rho^*D_{0,\infty}$ and Y_{hor} corresponds to a rational section σ of the line bundle such that

$$\operatorname{div}\sigma=D_0^{\mathrm{red}}-D_{\infty}^{\mathrm{red}}.$$

Assume there is an embedded resolution $\pi: E'\to E$ of D_0 such that $D_0\cup D_{\infty}$ is a divisor with normal crossings on a neighborhood of D_0. (This is satisfied in the case $n=2$.) Let $\pi: X'\to X^{\vee}$ be the base change of $\pi: E'\to E$ by ρ. We can similarly define $D'_{0,\infty}, D_0'^{\mathrm{red}}, D_{\infty}'^{\mathrm{red}}$, $Y'_{\mathrm{hor}}, Y'_{\mathrm{ver}}$ for $D'_0=\pi^*D_0$, $D'_{\infty}=\pi^*D_{\infty}$, $Y'=\pi^*Y^{\vee}$ so that

$$\operatorname{div}\pi^*\sigma=D_0'^{\mathrm{red}}-D_{\infty}'^{\mathrm{red}}.$$

Blowing up further if necessary, we may assume

$$D_0'^{\mathrm{red}}\cap D_{\infty}'^{\mathrm{red}}=\emptyset. \tag{A.1.1}$$

Then we get an embedded resolution of $f^{-1}(0)$ by iterating blow-ups of X' along the irreducible components of $D_0'^{\mathrm{red}}$. Indeed, Y'_{hor} may be locally defined by $s=\prod_i x_i^{m_i}$ with $x_1,\dots,x_n$ local coordinates of E' and s a local coordinate of the line bundle so that the blow-up along $\{x_i=s=0\}$ corresponds to the substitution of s by sx_i where m_i decreases by 1.

For simplicity, assume $n=2$, D_0 is a reduced divisor with simple normal crossings, and intersects D_{∞} at smooth points of D_0. Since the embedded resolution can be obtained by iterating blowing-ups with point centers, we can verify that $D_0'^{\mathrm{red}}$ may be assumed to be isomorphic to D_0, and does not intersect $D'^{\,\mathrm{red}}_{\infty}$ (calculating the multiplicities of the exceptional divisors). If furthermore D_0 is smooth, then the exceptional divisor of the blow-up along $D'^{\,\mathrm{red}}_0$ is a trivial $\mathbb{P}^1$-bundle over $D'^{\,\mathrm{red}}_0$, because the proper transform of $Y^{\vee}$ gives a trivialization.

For example, if D_0 (resp. D_∞) is defined locally by $y = 0$ (resp. $x = 0$) with multiplicity 1 (resp. m), then the resolution is obtained by iterating m times blow-ups along a point of the proper transform of D_0. Let C_j denote the proper transform of the exceptional divisor of the j-th blow-up for $1 \le j \le m$. Then

$$\pi^* D_0 = \sum_{1\le j\le m} jC_j + D'^{\,\mathrm{red}}_0, \quad \pi^* D_\infty = \sum_{1\le j\le m} mC_j + D''_\infty,$$

where D''_∞ is the proper transform of D_∞ (with multiplicity m), and

$$D'_{0,\infty} = \sum_{1\le j\le m} jC_j, \quad D'^{\,\mathrm{red}}_\infty = \sum_{1\le j\le m-1} (m-j)C_j + D''_\infty.$$

A.2. Conditions for non semisimplicity. — With the notation and the assumptions of 1.5, assume $n = 2$. We consider the conditions for the non semisimplicity of the monodromy on $H^1(F_x, \mathbb{Q})_\lambda$. Define

$$J(\lambda; a, b) = \{I \subset J(\lambda) : |I| - 1 = a,\, s(I) \ge b\},$$
$$J_0(\lambda; a, b) = \{I \in J(\lambda; a, b) : Y_I \cap Y_j = \emptyset \text{ for } j \notin J(\lambda)\}.$$

Let u be an element of $E_1^{-1,2}$ in (1.5.3). It may be viewed as an element of

$$\bigoplus_{I\in J_0(\lambda;1,0)} H^0(Y_I, \mathbb{Q}),$$

because $H^0(U_I, \mathcal{F}_{\lambda,I})$ vanishes for $I \in J(\lambda; 1, 0) \setminus J_0(\lambda; 1, 0)$. Here the Tate twist (-1) is trivialized by choosing $\sqrt{-1}$.

The first condition on u is that it is annihilated by the differential d_1 of the spectral sequence, i.e. its images in

$$\bigoplus_{I\in J_0(\lambda;2,1)} H^0(Y_I, \mathbb{Q}), \qquad \bigoplus_{I\in J(\lambda;0,0)} H^2(U_I, \mathcal{F}_{\lambda,I})(1)$$

vanish. This condition is necessary to assure that it defines an element of $\mathrm{Gr}_2^W H^1(F_x, \mathbb{Q})_\lambda$.

The second condition is that its image in $\bigoplus_{I\in J_0(\lambda;1,1)} H^0(Y_I,\mathbb{Q})$ does not belong to the image of $\bigoplus_{I\in J_0(\lambda;0,0)} H^0(Y_I,\mathbb{Q})$. This condition is necessary to assure that its image by N does not vanish in $\mathrm{Gr}_0^W H^1(F_x,\mathbb{Q})_\lambda$.

A.3. Example. — Let

$$X=\{xw-yz=0\}\subset\mathbb{C}^4,\quad f=(y^2-x^4)(x^2-y^4),$$

where $n=2$. Then $E=\mathbb{P}^1\times\mathbb{P}^1$ with coordinates $(u_0,u_1;v_0,v_1)$ such that

$$x=u_0v_0,\quad y=u_0v_1,\quad z=u_1v_0,\quad w=u_1v_1.$$

We apply the arguments in (A.1) to

$$g=x\pm y^2,\quad h=y\pm x^2,$$

where $g_1=x$, $g_2=y^2$, etc., and L is induced by $\mathcal{O}(1)$ on $\mathbb{P}^3$. Let (u,v) be the affine coordinates on $\{u_1v_1\neq 0\}\subset E$ such that $u=u_0/u_1, v=v_0/v_1$. Let s be the coordinate of the line bundle over $\{w\neq 0\}$, which is induced by w. Then $Y^\vee$ near $(0,1;0,1)$ is locally defined by

$$u^4(v-su)(v+su)=0, \tag{A.3.1}$$

because $g=w(x/w)\pm w^2(y/w)^2$ (and similarly for h), where w is actually s. We have a similar assertion on a neighborhood of $(0,1;1,0)$. So $Y^\vee$ has four reduced components (defined by $v\pm su=0$, etc.) and one multiple component (defined by $u^4=0$).

Let Z_1,Z_2 be the divisors defined by v_0 and v_1 respectively. Then D_0^{red} in A.1 for g (resp. h) is Z_1 (resp. Z_2), and $D_0^{\mathrm{red}}\cap D_\infty^{\mathrm{red}}$ consists of $(0,1;0,1)$ (resp. $(0,1;1,0)$). Let $\pi:E'\to E$ be the blow-up along these two points with exceptional divisors C_1,C_2. This gives a resolution satisfying (A.1.1) by the last argument of A.1 where $m=1$. Let Z_1',Z_2' be the proper transforms of Z_1,Z_2 so that

$$\pi^*Z_1=Z_1'+C_1,\quad \pi^*Z_2=Z_2'+C_2. \tag{A.3.2}$$

Let $\pi:X'\to X^\vee$ be the base change of $\pi:E'\to E$ by ρ. Let $X''\to X'$ be the blow-up along Z_1' and Z_2' with exceptional divisors E_1,E_2. This gives an embedded resolution of $f^{-1}(0)$. We see that E_1 is a trivial $\mathbb{P}^1$-bundle over Z_1', and the intersection of E_1 with the proper transform of $f^{-1}(0)$ consists of two connected components (corresponding to $v-su=0$

and $v + su = 0$) and these are both isomorphic to Z_1' by the projection (and similarly for E_2, Z_2'). Let E_0 be the proper transform of the zero section E' by $X'' \to X'$. For $i = 1, 2$, the proper transform of $\rho^{-1}(C_i)$ will be denoted by E_{i+2}. Let C_i' be the proper transform of C_i, which is equal to $E_0 \cap E_{i+2}$. We will identify Z_i' with $E_0 \cap E_i$ for $i = 1, 2$. Note that the inverse image of the origin is $\bigcup_{0 \le i \le 2} E_i$.

Using this resolution together with the conditions in A.2, we can show that the action of N on $H^1(F_0, \mathbb{Q})_\lambda$ is not semisimple where $\lambda = -1$. We see that the multiplicities of the irreducible components are even except for the proper transforms of the four reduced components of $Y^\vee$. We have to find an appropriate element u as in A.2. We define u by

$$1 \in H^0(Z_1', \mathbb{Q}), \quad 1 \in H^0(C_1', \mathbb{Q}), \quad -1 \in H^0(Z_2', \mathbb{Q}), \quad -1 \in H^0(C_2', \mathbb{Q}).$$

Here we use the natural order of the exceptional divisors E_i for $0 \le i \le 4$ to define these elements, because Čech and co-Čech complexes are involved. We can verify that the two conditions in A.2 are satisfied by using (A.3.2), etc. Note that, if Y_I is E_i with $i = 1$ or 2, then

$$H^j(U_I, \mathcal{F}_{\lambda,I}) = 0 \quad \text{for any } j,$$

because U_I is the product of Z_i' with $\mathbb{P}^1$ minus two points, and the monodromy of $\mathcal{F}_{\lambda,I}$ around the two points are -1 (here we use the Leray spectral sequence for the projection to Z_i'). We can also verify that the Milnor monodromy is semisimple outside the origin, using (A.3.1) and 1.5.

A.4. Remark. — For the moment, we do not know any example as above with X smooth.

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