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ELLIPTIC OPERATORS AND HIGHER SIGNATURES

by Eric LEICHTNAM & Paolo PIAZZA

1. Introduction.

Let M^{4k} an oriented $4k$-dimensional compact manifold. Let g be a Riemannian metric on M. Let us consider the Levi-Civita connection ∇^g and the Hirzebruch L-form $L(M,\nabla^g)$, a *closed* form in $\Omega^{4*}(M)$ with de Rham class $L(M) := [L(M,\nabla^g)]_{\mathrm{dR}} \in H^*(M,\mathbb{R})$ independent of g. Let now M be closed; then
(1.1)
the integral over M of $L(M,\nabla^g)$ is an oriented homotopy invariant of M.

In fact, if $[M] \in H_*(M,\mathbb{R})$ denotes the fundamental class of M then

$$\int_M L(M,\nabla^g) = < L(M),[M] > = \operatorname{ind} D^{\mathrm{sign}}_{(M,g)} = \operatorname{sign}(M),$$

the last term denoting the topological signature of M, an homotopy invariant of M. We shall call the integral $\int_M L(M,\nabla^g)$ the *lower signature of the closed manifold M*.

A second fundamental property of $\int_M L(M,\nabla^g) \equiv < L(M),[M] >$ is its *cut-and-paste invariance*: if Y and Z are two manifolds with diffeomorphic boundaries and if

$$X_\phi := Y \cup_\phi Z^-, \quad X_\psi := Y \cup_\psi Z^-, \quad Z^- := (-Z)$$

with $\phi,\psi : \partial Y \to \partial Z$ oriented diffeomorphisms, then $< L(X_\phi),[X_\phi] > = < L(X_\psi),[X_\psi] >$.

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A third fundamental property will involve a manifold M with boundary. Using Stokes theorem we see easily that the integral of the L-form is now metric dependent; in particular it is not homotopy invariant. However, by the Atiyah-Patodi-Singer index theorem for the signature operator, we know that there exists a boundary correction term $\eta(\partial M, g_{|\partial M})$ such that

$$(1.2)\quad \int_M L(M,\nabla^g) - \frac{1}{2}\eta(\partial M, g_{|\partial M}) \quad \text{is an oriented homotopy invariant.}$$

In fact, this difference equals the topological signature of the manifold with boundary M. We call the difference appearing in (1.2) the *lower signature of the manifold with boundary M*. The term $\eta(\partial M, g_{|\partial M})$, i.e. the term we need to subtract in order to produce a homotopy invariant out of $\int_M L(M,\nabla^g)$, is a spectral invariant of the signature operator $D^{\text{sign}}_{(\partial M, g|_{\partial M})}$ on ∂M; more precisely, this invariant measures the *asymmetry* of the spectrum of this (self-adjoint) operator with respect to $0 \in \mathbb{R}$. We shall review these basic facts in Section 2 and Section 3.

Let now Γ be a finitely generated discrete group. Let $B\Gamma$ be the classifying space for Γ. We shall be interested in the real cohomology groups $H^*(B\Gamma,\mathbb{R})$. Let $\Gamma \to \widetilde{M} \to M$ be a Galois Γ-covering of an oriented manifold M. For example, $\Gamma = \pi_1(M)$ and $\widetilde{M}$ is the universal covering of M. From the classifying theorem for principal bundles we know that $\Gamma \to \widetilde{M} \to M$ is classified by a continuous map $r : M \to B\Gamma$. We shall identify $\Gamma \to \widetilde{M} \to M$ with the pair $(M, r : M \to B\Gamma)$. Assume at this point that M is closed. Fix a class $[c] \in H^*(B\Gamma,\mathbb{R})$; then $r^*[c] \in H^*(M,\mathbb{R})$ and it makes sense to consider the number $< L(M) \cup r^*[c], [M] > \in \mathbb{R}$. The collection of real numbers

$$\{\text{sign}(M, r; [c]) :=< L(M) \cup r^*[c], [M] >, \quad [c] \in H^*(B\Gamma,\mathbb{R})\}$$

are called the *Novikov's higher signatures* associated to the covering $(M, r : M \to B\Gamma)$. It is important to notice that these number are *not* well defined if M has a boundary; in fact, in this case $L(M) \cup r^*[c] \in H^*(M,\mathbb{R})$ whereas $[M] \in H_*(M, \partial M, \mathbb{R})$, and the two classes cannot be paired.

One can give a natural notion of *homotopy equivalence* between Galois Γ-coverings. One can also give the notion of 2 coverings being *cut-and-paste equivalent.* In this paper we shall address the following three questions:

Question 1. Are Novikov's higher signatures *homotopy invariant*?

Question 2. Are Novikov's higher signatures *cut-and-paste invariant*?

Question 3. If $\partial M \neq \emptyset$, *can we define higher signatures and prove their homotopy invariance?* Of course we want these higher signatures on a manifold with boundary M to generalize the lower signature

$$\int_M L(M, \nabla^g) - \frac{1}{2}\eta(\partial M, g_{|\partial}) \,,$$

which is indeed a homotopy invariant.

Question 1 is still open and is known as the *Novikov conjecture.* It has been settled in the affirmative for many classes of groups. In this survey we shall present two methods for attacking the conjecture, both involving in an essential way properties of *elliptic operators.*

The answer to Question 2 is negative: the higher signatures are *not* cut-and-paste invariants (we shall present a counterexample). However, one can give *sufficient conditions* on the group Γ and on the separating hypersurface ensuring that the higher signatures are indeed cut-and-paste invariant.

Finally, under suitable assumption on $(\partial M, r|_{\partial M})$ and on the group Γ one can *define* higher signatures on a manifold with boundary M equipped with a classifying map $r : M \to B\Gamma$ and prove their homotopy invariance. Notice that part of the problem in Question 3 is to give a meaningful definition. Our answers to Question 2 and Question 3 will use in a crucial way properties of elliptic boundary-value problems.

There are several excellent surveys on Novikov's higher signatures; we mention here the very complete historical perspective by Ferry, Ranicki and Rosenberg [37], the stimulating article by Gromov [44], the one by Kasparov [65] and the monograph by Solovyov-Troitsky [116]. The *novelty* in the present work is the *unified treatment* of closed manifolds and manifolds with boundary as well as the *treatment of the cut-and-paste problem* for higher signatures on closed manifolds.

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2. The lower signature and its homotopy invariance.

2.1. The L-differential form

Let (M, g) be an oriented Riemannian manifold of dimension m. We fix a Riemannian connection ∇ on the tangent bundle of M and we consider ∇^2, its curvature. In a fixed trivializing neighborhood U we have $\nabla^2 = R$ with R a $m \times m$-matrix of 2-forms. We consider the L-differential form $L(M, \nabla) \in \Omega^*(M)$ associated to ∇. Recall that $L(M, \nabla)$ is obtained by formally substituting the matrix of 2-forms $\frac{\sqrt{-1}}{2\pi} R$ in the power-series expansion at $A = 0$ of the analytic function

$$L(A) = \det{}^{\frac{1}{2}} \left(\frac{A}{\tanh A} \right), \quad A \in so\,(m).$$

Since $\Omega^*(M) = 0$ if $* > \dim M$, we see that the sum appearing in $L(\frac{\sqrt{-1}}{2\pi} R)$ is in fact finite. More importantly, since $L(\cdot)$ is $SO(m)$-invariant, i.e.

$$L(A) = L(C^{-1} A C), \quad C \in SO(m),$$

one can check easily that $L(M, \nabla)$ is *globally defined*; it is a differential form in $\Omega^{4*}(M, \mathbb{R})$. One can prove the following two fundamental properties of the L-differential form:

$$dL(M, \nabla) = 0, \quad L(M, \nabla) - L(M, \nabla') = dT(\nabla, \nabla') \tag{2.1}$$

where ∇' is any other Riemannian connection and where $T(\nabla, \nabla')$ is the *transgression form* defined by the two connections. Consequently the de Rham class $L(M) = [L(M, \nabla)] \in H^*_{\mathrm{dR}}(M)$ is well defined; it is called the Hirzebruch L-class.

In what follows we shall always choose the Levi-Civita connection associated to g, ∇^g, as our reference connection.

2.2. The lower signature on closed manifolds and its homotopy invariance

Assume now that M is *closed* ($\equiv$ *without boundary*) and that $\dim M = 4k$. Consider

$$\int_M L(M, \nabla^g) \tag{2.2}$$

Because of the properties (2.1), this integral does not depend on the choice of g and is in fact equal to $< L(M), [M] >$, the pairing between the cohomology class $L(M)$ and the fundamental class $[M] \in H_{4k}(M; \mathbb{R})$.

THEOREM 2.3. — *The integral of the L-form*

$$\int_M L(M, \nabla^g)$$

is an integer and is an oriented homotopy invariant.

Proof. — With some of what follows in mind, we give an index-theoretic proof of this theorem, in two steps.

First step: by the Atiyah-Singer index theorem

$$\int_M L(M, \nabla^g) = \operatorname{ind} D^{\mathrm{sign},+}$$

where on the right hand side the index of the signature operator associated to g and our choice of orientation appears [1]. This proves that

$$\int_M L(M, \nabla^g) \in \mathbb{Z}.$$

Second step: using the Hodge theorem one can check that

$$\operatorname{ind} D^{\mathrm{sign},+} = \operatorname{sign}(M) := \text{ signature of } M$$

i.e. the signature of the bilinear form $H^{2k}(M) \times H^{2k}(M) \longrightarrow \mathbb{R}$

$$([\alpha], [\beta]) \longrightarrow \int_M \alpha \wedge \beta.$$

This is clearly an oriented homotopy invariant and the theorem is proved. □

[1] Let us recall the definition of the signature operator on a 2ℓ-dimensional oriented Riemannian manifold. Consider the Hodge star operator

$$\star : \Omega^p(M) \longrightarrow \Omega^{2\ell-p}(M);$$

it depends on g and the fixed orientation. Let $\tau := (\sqrt{-1})^{p(p-1)+\ell}\star$ on $\Omega^p_{\mathbb{C}}(M)$; then $\tau^2 = 1$ and we have a decomposition $\Omega^*_{\mathbb{C}}(M) = \Omega^+(M) \oplus \Omega^-(M)$. The operator $d + d^*$, extended in the obvious way to the complex differential forms $\Omega^*_{\mathbb{C}}(M)$, anticommutes with τ. The signature operator is simply defined as

$$D^{\mathrm{sign}} := \begin{pmatrix} 0 & D^{\mathrm{sign},-} \\ D^{\mathrm{sign},+} & 0 \end{pmatrix}, \quad D^{\mathrm{sign},\pm} = (d + d^*)|_{\Omega^\pm(M)}.$$

If we wish to be precise, we shall denote the signature operator on the Riemannian manifold (M, g) by $D^{\mathrm{sign}}_{(M,g)}$.

We shall also call $\int_M L(M, \nabla^g)$ the *lower signature* of the closed manifold M.

Remark. — The equality

$$\operatorname{sign}(M) = < L(M), [M] >$$

is known as the *Hirzebruch signature theorem*. The original proof of this fundamental result was topological, exploiting the cobordism invariance of both sides of the equation and the structure of the oriented cobordism ring. See, for example, Milnor-Stasheff [96] and Hirzebruch [55].

Remark. — The formulation and the proof of Hirzebruch theorem given here is not historically accurate but has the advantage of introducing the techniques that will be employed later for tackling the homotopy invariance of the higher signatures of a closed manifold. It is important to single out informally the two steps in the proof:

(i) connect the lower signature to an index

(ii) prove that the index is homotopy invariant.

2.3. The lower signature on manifolds with boundary and its homotopy invariance

Assume now that M has a *non-empty boundary*: $\partial M = N \neq \emptyset$. For simplicity, we assume that the metric g is of product-type near the boundary; thus in a collar neighborhood U of ∂M we have $g = dx^2 + g_\partial$ with $x \in C^\infty(M)$ a boundary defining function. We denote the signature operator on (M, g) by $D^{\text{sign}}_{(M,g)}$. We consider once again $\int_M L(M, \nabla^g)$. In contrast with the closed case, this integral does depend now on the choice of the metric g; in particular it is *not* an oriented homotopy invariant. To understand this point we simply observe that if h is a different metric, then, by (2.1), we get

$$\int_M L(M, \nabla^g) - \int_M L(M, \nabla^h) = \int_{\partial M} T(\nabla^g, \nabla^h)|_{\partial M}. \tag{2.4}$$

We ask ourselves if we can add to $\int_M L(M, \nabla^g)$ a correction term making it metric-independent and, hopefully, homotopy-invariant; formula (2.4) shows that it should be possible to add a term that only depends on the metric on ∂M.

In order to state the result we need a few definitions. Consider the boundary ∂M with the induced metric and orientation. Let $D^{\rm sign}_{\partial M, g_\partial}$ the signature operator on the odd dimensional Riemannian manifold $(\partial M, g_\partial)$; this is the so-called *odd signature operator* and it is defined as follows:

$$D^{\rm sign}_{\partial M, g_\partial}\phi := (\sqrt{-1})^{2k}(-1)^{p+1}(\epsilon \star d - d\star)\phi \tag{2.5}$$

with $\epsilon = 1$ if $\phi \in \Omega^{2p}(\partial M)$ and $\epsilon = -1$ if $\phi \in \Omega^{2p-1}(\partial M)$. This is a formally self-adjoint first order elliptic differential operator on the *closed* manifold ∂M. We shall sometime denote the boundary signature operator by $D^{\rm sign}_\partial$. Thanks to the spectral properties of elliptic differential operators on closed manifolds, we know that the following series is absolutely convergent for $\mathrm{Re}(s) \gg 0$:

$$\eta(s) := \sum \lambda |\lambda|^{-(s+1)}, \tag{2.6}$$

with λ running over the non-zero eigenvalues of $D^{\rm sign}_{(\partial M, g_\partial)}$. One can meromorphically continue this function to the all complex plane; the points $s_k = \dim(\partial M) - k$ are poles of the meromorphic continuation. It is a non-trivial result that the point $s = 0$ is regular and one sets

$$\eta(D^{\rm sign}_{(\partial M, g_\partial)}) := \eta(0) \tag{2.7}$$

This is the *eta invariant* associated to $D^{\rm sign}_{(\partial M, g_\partial)}$; it is a spectral invariant measuring the *asymmetry* of the spectrum of $D^{\rm sign}_{(\partial M, g_\partial)}$, a subset of the real line, with respect to the origin. We can now state the main theorem of this subsection:

THEOREM 2.8 (Atiyah-Patodi-Singer). — *The difference*

$$\int_M L(M, \nabla^g) - \frac{1}{2}\eta\Big(D^{\rm sign}_{(\partial M, g_\partial)}\Big) \tag{2.9}$$

is an integer and is an oriented homotopy invariant of the pair $(M, \partial M)$.

We call the difference $\int_M L(M, \nabla^g) - \frac{1}{2}\eta(D^{\rm sign}_{(\partial M, g_\partial)})$ the lower signature of the manifold with boundary M.

Proof. — Following Atiyah-Patodi-Singer [5] we give an index theoretic proof of this theorem, once again in two steps.

There is a well defined restriction map

$$|_{\partial M} : \Omega^+(M) \to \Omega^*(\partial M).$$

Next we observe that to the formally self-adjoint operator $D^{\rm sign}_{(\partial M, g_\partial)}$ we can associate the spectral projection $\Pi_\geqslant$ onto the eigenspaces associated

to its nonnegative eigenvalues. The operator $D^{\mathrm{sign},+}_{(M,g)} \equiv D^{\mathrm{sign},+}$ on M with boundary condition
$$\omega^+|_{\partial M} \in \operatorname{Ker} \Pi_{\geqslant}$$
turns out to be Fredholm when acting on suitable Sobolev completions (more on this in the next subsection). The Atiyah-Patodi-Singer ($\equiv$ APS) index formula computes its index as

(2.10)
$$\operatorname{ind}\left(D^{\mathrm{sign},+}, \Pi_{\geqslant}\right) = \int_M L(M, \nabla^g) - \frac{1}{2}\eta\Big(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\Big) - \frac{1}{2}\dim \operatorname{Ker}\left(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\right).$$

It should be remarked that $\operatorname{Ker}\left(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\right)$ has a natural symplectic structure and it is therefore even dimensional. From (2.10) we infer that

(2.11)
$$\int_M L(M, \nabla^g) - \frac{1}{2}\eta\Big(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\Big) = \operatorname{ind}(D^{\mathrm{sign},+}, \Pi_{\geqslant}) + \frac{1}{2}\dim \operatorname{Ker}\left(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\right).$$

This concludes the first step, connecting the lower signature to an index.[2] Next, using Hodge theory on the complete manifold $\widehat{M}$ obtained by gluing to M a semi-infinite cylinder $(-\infty, 0] \times \partial M$, one can prove that
$$\operatorname{ind}(D^{\mathrm{sign},+}, \Pi_{\geqslant}) + \frac{1}{2}\dim \operatorname{Ker}(D^{\mathrm{sign}}_{(\partial M, g_\partial)}) = \operatorname{sign}(M) := \text{ the signature of } M.$$
Since the latter is an oriented homotopy invariant, the theorem is proved. □

Remark. — In contrast with the closed case, there is no purely topological proof of the homotopy invariance of the difference $\int_M L(M, \nabla^g) - \frac{1}{2}\eta(D^{\mathrm{sign}}_{(\partial M, g_\partial)})$ on manifolds with boundary; in this case *we do need to pass through the Atiyah-Patodi-Singer index theorem.*

Remark. — Part of the motivation for the Atiyah-Patodi-Singer signature index theorem came from the work of Hirzebruch on Hilbert modular varieties. For these singular varieties the Hirzebruch signature formula does not hold; there is a *defect* associated to each cusp. For Hilbert modular surfaces Hirzebruch computed this defect and showed that it was given in terms of the value at $s = 1$ of certain L-series. He then conjectured that a similar result was true for any Hilbert modular variety. The conjecture was established by Atiyah-Donnelly-Singer in [3] [4] and

[2] It could be proved that the right hand side of (2.11) is the index of the boundary value problem corresponding to the projection $\Pi_> + \Pi_L$, where $L \subset \operatorname{Ker}(D^{\mathrm{sign}}_{(\partial M, g_\partial)})$ is the so-called scattering lagrangian.

the proof is based in an essential way on the Atiyah-Patodi-Singer index theorem (with the value of the L-series corresponding to the eta-invariant). Hirzebruch's conjecture was also settled independently and with a different proof by Müller in [101] (see also [102]).

Remark. — Let N be an odd dimensional oriented manifold. The definition of eta-invariant can be given for any formally self-adjoint elliptic pseudodifferential operator. The definition is by meromorphic continuation as in (2.6); the proof that $s = 0$ is a regular point is non-trivial and it is due to Atiyah-Patodi-Singer [7]. Moreover, for the *odd* signature operator D_N^{sign} (in fact, for any Dirac-type operator associated to a unitary Clifford connection) one can give the following formula

$$\eta(D_N^{\mathrm{sign}}) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}\Big(D_N^{\mathrm{sign}}\, e^{-(t\, D_N^{\mathrm{sign}})^2}\Big) dt.$$

Notice that the convergence of this integral near $t = 0$ is non-trivial and its justification requires arguments similar to those involved in the heat-kernel proof of the Atiyah-Singer index theorem, see Bismut-Freed [17], [18].

2.4. More on index theory on manifolds with boundary.

We elaborate further on the analytic features of the above proof. Let M be a manifold with boundary. Simple examples (such as the $\overline{\partial}$-operator on the disc) show that, in general, elliptic operators on M are *not* Fredholm on Sobolev spaces. In order to obtain a finite dimensional kernel and cokernel it is necessary to impose *boundary conditions.* Among the simplest boundary conditions are those of *local type*, Dirichlet, Neumann or more generally Lopatinski boundary conditions. It is not at all clear that these classical local boundary conditions give rise to Fredholm operators. And in fact Atiyah and Bott showed that there exist topological obstructions to the existence of well-posed *local* boundary conditions for an elliptic operator on a manifold with boundary. When these obstructions are zero, Atiyah and Bott do prove an index theorem, see [2]. The Atiyah-Bott index theorem has been greatly extended by Boutet de Monvel in [20]. However, precisely because of their geometric nature, the signature operator is among those operators for which these obstructions are almost always *non-zero.* In trying to prove the signature theorem on manifolds with boundary, Atiyah, Patodi and Singer introduced their celebrated *non-local* boundary condition. This is the boundary condition explained in the proof of Theorem 2.8. In a fundamental series of papers [5] [6] [7] they investigated the index theory of such boundary value problems for general first-order elliptic differential operators; they also gave important applications to geometry and topology. Their theory applies to any Dirac-type operator on an even dimensional manifold with boundary endowed with a Riemannian metric g which is of product-type near the boundary. The Dirac operators acts between the sections of a $\mathbb{Z}_2$-graded Hermitian Clifford module $E = E^+ \oplus E^-$ endowed with a Clifford connection ∇^E and it is odd with respect to the grading of E:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

Classical *examples* of Dirac-type operators are given by the signature operator D^{sign} introduced above, the Gauss-Bonnet operator $d+d^*$, with d equal to the de Rham differential, *the* Dirac operator $\not{D}$ on a spin manifold, the $\overline{\partial}$-operator on a Kaehler manifold. See Berline-Getzler-Vergne [13] for more on Dirac operators.

Near the boundary D can be written (up to a bundle isomorphism) as

$$\begin{pmatrix} 0 & -\partial/\partial u + D_{\partial M} \\ \partial/\partial u + D_{\partial M} & 0 \end{pmatrix}$$

with u equal to the inward normal variable to the boundary and $D_{\partial M}$ the generalized Dirac operator induced on ∂M. For example, in the case of the signature operator D^{sign} the operator induced on the boundary is simply the odd-signature operator. The boundary operator $D_{\partial M}$ is an elliptic and essentially self-adjoint operator on the *closed* compact manifold ∂M. The L^2-spectrum is therefore discrete and real. Let $\{e_\lambda\}$ be an L^2-orthonormal basis of eigenfunctions for $D_{\partial M}$. Let $\Pi_{\geqslant}$ be the spectral projection corresponding to the non-negative eigenvalues of $D_{\partial M}$: thus $\Pi_{\geqslant}(e_\lambda) = e_\lambda$ if $\lambda \geqslant 0$ and $\Pi_{\geqslant}(e_\lambda) = 0$ if $\lambda < 0$. Let

$$\mathcal{C}^\infty(M, E^+, \Pi_{\geqslant}) = \{s \in \mathcal{C}^\infty(M, E^+) | \Pi_{\geqslant}(s|_{\partial M}) = 0\}.$$

Thus a section s belongs to $\mathcal{C}^\infty(M, E^+, \Pi_{\geqslant})$ iff $s|_{\partial M} = \sum_{\lambda<0} s^\lambda_{\partial M} e_\lambda$. The Atiyah-Patodi-Singer theorem, see [5], states that the operator D^+ acting on the Sobolev completion $H^1(M, E^+, \Pi_{\geqslant})$ of $\mathcal{C}^\infty(M, E^+, \Pi_{\geqslant})$, with range $L^2(M, E^-)$, is a Fredholm operator with index

$$\operatorname{ind}(D^+, \Pi_{\geqslant}) = \int_M \mathrm{AS} - \frac{1}{2}(\eta(D_{\partial M}) + \dim \operatorname{Ker} D_{\partial M}).$$

Here $\eta(D_{\partial M})$ is the *eta invariant* of the self-adjoint operator $D_{\partial M}$ as introduced in the previous subsection, whereas the density AS =

$\widehat{A}(M,\nabla^g)\mathrm{ch}'(E,\nabla^E)$ is the local contribution that would appear in the heat-kernel proof of the Atiyah-Singer index theorem for Dirac operators. In the case D is Dirac operator acting on the spinor bundle of a spin manifold, one has AS $= \widehat{A}(M,\nabla^g)$. In the case where D is the signature operator acting on the bundle of differential forms one has AS $= L(M,\nabla^g)$. The $\widehat{A}$-form $\widehat{A}(M,\nabla^g)$ is obtained by substituting X by $\frac{\sqrt{-1}}{2\pi}R$ in the analytic functions

$$\widehat{A}(X) = \det^{\frac{1}{2}}\left(\frac{X/2}{\sinh X/2}\right).$$

There are nowadays many alternative approaches to the Atiyah-Patodi-Singer index formula; we shall mention here the one started by Cheeger, based on conic metrics (see Cheeger [25], Chou [26] and also Lesch [70]) and the one, fully developed by Melrose, based on manifolds with cylindrical ends (see Melrose [92] and also Piazza [105], Melrose-Nistor [93]). For a proof in the spirit of the embedding proof of the Atiyah-Singer index formula on closed manifolds see Dai-Zhang [35].

Remark. — Let $P = P^2 = P^*$ be a finite rank perturbation of the projection $\Pi_{\geqslant}$. Thus, with $\{e_\lambda\}$ still denoting a L^2-orthonormal basis of eigenfunctions for $D_{\partial M}$, we require that for some $R > 0$, $Pe_\lambda = e_\lambda$ if $\lambda > R$ and $Pe_\lambda = 0$ if $\lambda < -R$. Let

$$\mathcal{C}^\infty(M,E^+,P) = \{\, s \in \mathcal{C}^\infty(M,E^+) \mid P(s|_{\partial M}) = 0 \,\}.$$

The operator D^+ with domain $\mathcal{C}^\infty(M,E^+,P)$ extends once again to a Fredholm operator with $\mathrm{ind}(D^+,P) \in \mathbb{Z}$. See, for example Booss-Bavnbek - Wojciechowski [19]. Moreover: let P_1 and P_2 be two such projections and let us consider $H_j = P_j(L^2(\partial M, E|_{\partial M}))$. One can show easily that the operator $P_2 \circ P_1 : H_1 \to H_2$ is Fredholm; its index is called the *relative index* of the two projections and is denoted by $i(P_1,P_2)$. The following formula is known as the *relative index formula* ([19]):

$$\mathrm{ind}(D^+,P_2) - \mathrm{ind}(D^+,P_1) = i(P_1,P_2)\,.$$

For example: $\mathrm{ind}(D^+,\Pi_>) - \mathrm{ind}(D^+,\Pi_\geqslant) = i(\Pi_\geqslant,\Pi_>) = \dim\mathrm{Ker}\, D_{\partial M}$.

3. The cut-and-paste invariance of the lower signature.

Let M and N be two compact $4k-$dimensional oriented manifolds with boundary and let ϕ, ψ : $\partial M \to \partial N$ be two orientation preserving diffeomorphims. Let N^- be N with the reverse orientation. By gluing we obtain two closed oriented $4k$−dimensional manifolds, $M \cup_\phi N^-$ and $M \cup_\psi N^-$. We shall say that

$$M \cup_\phi N^- \quad \text{and} \quad M \cup_\psi N^- \quad \text{are } \textit{cut-and-paste equivalent.}$$

Consider the two integers $< L(M \cup_\phi N^-), [M \cup_\phi N^-] >$ and $< L(M \cup_\psi N^-), [M \cup_\psi N^-] >$.

Proposition 3.1. — *The following equality holds:*

$$< L(M \cup_\phi N^-), [M \cup_\phi N^-] > = < L(M \cup_\psi N^-), [M \cup_\psi N^-] > . \tag{3.2}$$

In words, the integral of the L-class is a cut-and-paste invariant.

In the next three subsections we shall give three different proofs of this proposition.

3.1. The index-theoretic proof.

We set

$$X_\phi := M \cup_\phi N^- \quad \text{and} \quad X_\psi := M \cup_\psi N^- .$$

Using the Atiyah-Patodi-Singer index theorem we shall prove that

$$< L(X_\phi), [X_\phi] > = \operatorname{sign}(M) - \operatorname{sign}(N) = < L(X_\psi), [X_\psi] > . \tag{3.3}$$

Notice that the 2 manifolds X_ϕ and X_ψ are, in general, *distinct*. Fix metrics g_ϕ and g_ψ on X_ϕ and X_ψ respectively. Since the integral of the L-class on closed manifolds in metric-independent, we can assume that these metrics are of product type near the embedded hypersurface $F := \partial M$. Thus we can write

$$X_\phi = M \cup_{\mathrm{Id}} \mathrm{Cyl}_\phi \cup_{\mathrm{Id}} N^-$$

with

$$\mathrm{Cyl}_\phi := ([-1,0] \times (\partial M)^-) \cup_\phi ([0,1] \times \partial N).$$

Denoting generically by ∇^{LC} the Levi-Civita connection associated to the various restrictions of g_ϕ, we can write

$$\begin{aligned}\int_{X_\phi} L(X_\phi, \nabla^{\mathrm{LC}}) &= \int_M L(M, \nabla^{\mathrm{LC}}) + \int_{\mathrm{Cyl}_\phi} L(\mathrm{Cyl}_\phi, \nabla^{\mathrm{LC}}) - \int_N L(N, \nabla^{\mathrm{LC}}) \\ &= \int_M L(M, \nabla^{\mathrm{LC}}) - \frac{1}{2}\eta\Big(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\Big) \\ &\quad + \frac{1}{2}\eta\Big(D^{\mathrm{sign}}_{(\partial M, g_\partial)}\Big) + \int_{\mathrm{Cyl}_\phi} L(\mathrm{Cyl}_\phi, \nabla^{\mathrm{LC}}) - \frac{1}{2}\eta\Big(D^{\mathrm{sign}}_{\partial N}\Big) \\ &\quad - \Big(\int_N L(N, \nabla^{\mathrm{LC}}\Big) - \frac{1}{2}\eta(D^{\mathrm{sign}}_{\partial N})) \\ &= \mathrm{sign}(M) + \mathrm{sign}(\mathrm{Cyl}_\phi) - \mathrm{sign}(N) \\ &= \mathrm{sign}(M) - \mathrm{sign}(N).\end{aligned}$$

We explain why these equalities hold. The first one is obvious; in the second one, we simply added and substracted the same quantities; in the third one, we applied the Atiyah-Patodi-Singer theorem, keeping in mind that the eta invariant is orientation reversing; in the fourth one, we used the topological invariance of sign$(\cdot)$ together with the following two observations:

(i) the diffeomorphism ϕ induces a diffeomeorphism between Cyl_ϕ and $[-1,1] \times \partial M$;

(ii) $\mathrm{sign}([-1,1] \times \partial M) = 0$ (use again the APS-formula).

Since exactly the same argument can be applied to X_ψ, it follows that we have proved (3.3) and thus Proposition 3.1.

3.2. The topological proof.

We start with a simplified situation. Let $X = M \cup N^-$ with $\partial M = \partial N$; in other words $\phi = \mathrm{Id}$. Using Poincaré duality and reasoning in terms of intersection of cycles one can prove in a purely topological way the following *Novikov gluing formula* (Hirsch [54]):

$$\mathrm{sign}(M \cup N^-) = \mathrm{sign}(M) - \mathrm{sign}(N). \tag{3.4}$$

Then, using exactly the same reasoning as in the previous section, one shows that for two different diffeomorphisms ϕ and ψ

$$\mathrm{sign}(M \cup_\phi N^-) = \mathrm{sign}(M) - \mathrm{sign} N = \mathrm{sign}(M \cup_\psi N^-).$$

By the Hirzebruch signature formula this implies

$$< L(M \cup_\phi N^-), [M \cup_\phi N^-] > = < L(M \cup_\psi N^-), [M \cup_\psi N^-] >$$

which is the formula we wanted to prove.

Following a suggestion of W. Lück, we shall now give a more algebraic proof of this equality. This should be considered as a *prélude* to the arguments of Leichtnam-Lück-Kreck [73] that we shall recall in Section 11 below. Since every sub vector space of a real vector space is a direct summand, one can construct a chain homotopy equivalence u between the cellular chain complex of $\mathbb{R}-$vector spaces $C_*(\partial M)$ and a chain complex D_* of finite dimensional $\mathbb{R}-$vector spaces whose $m-$differential $d_m : D_m \to D_{m-1}$ vanishes. With these notations, set $\overline{D}_i = D_i$ for $0 \leqslant i \leqslant m-1$ and $\overline{D}_i = 0$ for $i \geqslant m$. One then gets a so-called Poincaré pair $j_* : D_* \to \overline{D}_*$ whose boundary is D_*. By glueing $j_* : D_* \to \overline{D}_*$ and the Poincaré pair $i_* : C_*(\partial M) \to C_*(M)$ along their boundaries with the help of u one gets a true algebraic Poincaré complex denoted $C_*(M \cup_u \overline{D})$. A reference for these concepts is Ranicki [109], p. 18. Intuitively an algebraic Poincaré pair $j_* : D_* \to \overline{D}_*$ is the algebraic analogue of the injection $i : \partial M \to M$ where M is an oriented manifold with boundary. One can check that the signature $\operatorname{sign}(M \cup_u \overline{D})$ of the non degenerate quadratic form of $C_*(M \cup_u \overline{D})$ does not depend on the choice of u and D. Moreover one can prove that the signature $\operatorname{sign}(\overline{D}_*, D_*)$ of the algebraic Poincaré pair $j_* : D_* \to \overline{D}_*$ is zero.

LEMMA 3.5. — *One has:*

$$\operatorname{sign}(M \cup_\phi N^-) = \operatorname{sign} M - \operatorname{sign} N = \operatorname{sign}(M \cup_\psi N^-).$$

Proof. — Of course the second equality is a consequence of the first one. The algebraic Poincaré complex defined by the cellular chain complex $C_*(M \cup_\phi N^-)$ is (algebraically) cobordant to the following algebraic Poincaré complex:

$$C_*(M \cup_u \overline{D}) + C_*(N^- \cup_{u \circ \phi^{-1}} \overline{D}^-).$$

Hence the signature of $M \cup_\phi N^-$ is the sum of the ones of $C_*(M \cup_u \overline{D})$ and $C_*(N^- \cup_{u \circ \phi^{-1}} \overline{D}^-)$. But one has:

$$\operatorname{sign}(C_*(M \cup_u \overline{D})) = \operatorname{sign} M + \operatorname{sign}(\overline{D}_*, D_*),$$

$$\operatorname{sign}(C_*(N^- \cup_{u \circ \phi^{-1}} \overline{D}^-)) = \operatorname{sign} N^- + \operatorname{sign}(\overline{D}^-_*, D^-_*).$$

Since the signature of $(\overline{D}_*, D_*)$ is zero one gets that $\operatorname{sign} M \cup_\phi N^- = \operatorname{sign} M - \operatorname{sign} N$ which proves the Lemma. □

3.3. The spectral-flow-proof.

Recall that we have set
$$X_\phi := M \cup_\phi N^- \quad \text{and} \quad X_\psi := M \cup_\psi N^- .$$
Fix metrics g_ϕ on X_ϕ and g_ψ on X_ψ. We shall assume that these metrics are of product type near the embedded hypersurface $F := \partial M$. We shall prove, *analytically and without making use of the Atiyah-Patodi-Singer index formula*, that

$$\operatorname{ind}\left(D^{\mathrm{sign},+}_{X_\phi}\right) = \operatorname{ind}\left(D^{\mathrm{sign},+}_{X_\psi}\right). \tag{3.6}$$

By the Atiyah-Singer index theorem for the signature operator on closed manifolds, this will suffice in order to establish $< L(X_\phi), [X_\phi] >=< L(X_\psi), [X_\psi] >$, i.e. Proposition 3.1. The equality of the two indeces will be obtained exploiting two fundamental properties of the Atiyah-Patodi-Singer index: the *variational formula* and the *gluing formula.*

3.3.1. The variational formula for the APS-index. In contrast with the closed case, the APS-index is not stable under perturbations. In Subsection 2.4 we have defined the APS-boundary value problem for any generalized Dirac operator on an even dimensional manifold with boundary, M, endowed with a metric g which is of product type near the boundary. Assume now that $\{D(t)\}_{t\in[0,1]}$ is a smoothly varying family of such operators. As an important example we could consider a family of metrics $\{g(t)\}_{t\in[0,1]}$ on M and the associated family of signature operators $\{D^{\mathrm{sign}}(t)\}_{t\in[0,1]}$. Going back to the general case, consider the family of operators induced on the boundary $\{D_{\partial M}(t)\}_{t\in[0,1]}$; let $\Pi_{\geqslant}(t)$ the corresponding spectral projection associated to the non-negative eigenvalues; then the following variational formula for the APS-indeces holds:

$$\operatorname{ind}(D^+(1), \Pi_{\geqslant}(1)) - \operatorname{ind}(D^+(0), \Pi_{\geqslant}(0)) = \operatorname{sf}(\{D_{\partial M}(t)\}_{t\in[0,1]}) \tag{3.7}$$

where on the right hand side the *spectral flow* of the 1-parameter family of self-adjoint operators $\{D_{\partial M}(t)\}$ appears; this is the net number of eigenvalues changing sign as t varies from 0 to 1 ([7], [92]). Formula (3.7) follows from the APS-index formula, see [7]. It can also be proved analytically, without making use of the APS-index formula. See for example Dai-Zhang [33].

3.3.2. Important remark. If N is odd dimensional and $\{D^{\mathrm{sign}}_N(t)\}_{t\in[0,1]}$ is a one-parameter family of odd signature operators parametrized by a path of metrics $g_N(t)_{t\in[0,1]}$, then

$$\operatorname{sf}(\{D^{\mathrm{sign}}_N(t)\}_{t\in[0,1]}) = 0 \tag{3.8}$$

In fact, the kernel of the odd signature operator is equal to the space of harmonic forms on N; from the Hodge theorem we know that such a vector space is independent of the metric we choose; thus there are not eigenvalues changing sign and the spectral flow is zero.

3.3.3. The gluing formula. We start with a simplified situation: X is a *closed* compact manifold which is the union of two manifolds with boundary. Thus there exists an embedded hypersurface F which separates M into two connected components and such that

$$X = M_+ \cup_F M_-, \quad \text{with} \quad \partial M_+ = \partial M_- = F.$$

We assume that the metric g is of product type near the hypersurface F, i.e. near the boundaries of M_+ and M_-. Let D_X be a Dirac-type operator on X; then we obtain in a natural way two Dirac operators on M_+ and M_-. The following gluing formula holds:

$$\operatorname{ind}(D_X) = \operatorname{ind}(D_{M_+}, \Pi_{\geqslant}) + \operatorname{ind}(D_{M_-}, 1 - \Pi_{\geqslant}). \tag{3.9}$$

The discrepancy in the spectral projections come from the orientation of the normals to the two boundaries (if one is inward pointing, then the other is outward pointing). [3]

Formula (3.9) can be proved directly, in a purely analytical fashion, see Bunke [23], Leichtnam-Piazza [79]. Of course it is also a consequence of the APS-index theorem.

3.3.4. Proof of formula (3.6). The gluing formula (3.9) can be generalized to our more complicated situation, where X_ϕ is a closed manifold obtained by *gluing* two manifolds with boundary *through a diffeomorphism*. Using this gluing formula on X_ϕ (with metric g_ϕ) and on X_ψ (with metric g_ψ), applying then the variational formula for the APS index on M with respect to a path a metrics connecting $g_\psi|_M$ to $g_\phi|_M$ and then doing the same on N (with a path of metrics connecting $g_\psi|_N$ and $g_\phi|_N$), one proves that $\operatorname{ind}(D^{\text{sign},+}_{X_\psi}) - \operatorname{ind}(D^{\text{sign}}_{X_\phi}) = \operatorname{sf}(\{D^{\text{sign}}_{\text{odd}}(\theta)\}_{\theta \in S^1}$. The spectral flow appearing in this formula is associated to a S^1-family of odd signature operators acting on the fibers of the mapping torus $F \to M(\phi^{-1}\psi) \to S^1$ and parametrized by a family of metrics. As remarked in 3.3.2 this spectral flow is zero because of the cohomological

[3] Notice that $1 - \Pi_{\geqslant}$ is not exactly the APS-projection associated to the non-negative eigenvalues of $D_{\partial M_-}$; to be precise $1 - \Pi_{\geqslant} = \Pi^{\partial M_-}_{>}$, the projection onto the *positive* eigenvalues of $D_{\partial M_-}$

significance of the zero eigenvalue for the signature operator. References for this material are, for example, the book [19] and the survey Mazzeo-Piazza [91]. *Summarizing*, the equality of $\operatorname{ind}(D^{\text{sign}}_{X_\phi}) = \operatorname{ind}(D^{\text{sign}}_{X_\phi})$ has been obtained through the following two equalities

$$\operatorname{ind}(D^{\text{sign},+}_{X_\psi}) - \operatorname{ind}(D^{\text{sign}}_{X_\phi}) = \operatorname{sf}(\{D^{\text{sign}}_{\text{odd}}(\theta)\}_{\theta\in S^1}) = 0. \tag{3.10}$$

Remark. — It should be remarked that in this third proof we have not used the APS-index formula; only the analytic properties of the APS boundary value problem were employed. This will be important later, when we shall consider higher signatures.

4. Summary.

Let us summarize what we have seen so far. Let (M, g) be an oriented Riemannian manifold of dimension $4k$ and let $D^{\text{sign}}_{(M,g)}$ be the associated signature operator.

- *If M is closed then $\int_M L(M, \nabla^g)$ is an oriented homotopy invariant.* In fact
$$\int_M L(M,\nabla^g) \; =< L(M), [M] >= \operatorname{ind} D^{\text{sign},+}_{(M,g)} = \operatorname{sign}(M),$$
with $L(M) = [L(M,\nabla^g)] \in H^*_{\text{dR}}(M)$, $[M] \in H_*(M,\mathbb{R})$ and $\operatorname{sign}(M) =$ signature of M.

- If M has a boundary, $\partial M \neq \emptyset$, then we can define a correction term $\eta(D^{\text{sign}}_{(\partial M, g_\partial)})$ such that
$$\int_M L(M,\nabla^g) - \frac{1}{2}\eta(D^{\text{sign}}_{(\partial M, g_\partial)})$$
is an oriented homotopy invariant of the pair $(M, \partial M)$. In fact
$$\int_M L(M,\nabla^g) - \frac{1}{2}\eta(D^{\text{sign}}_{(\partial M, g_\partial)}) = \operatorname{ind}\left(D^{\text{sign}}_{(M,g)}, \Pi_\geqslant\right) \\ + \frac{1}{2}\dim\operatorname{Ker}\left(D^{\text{sign}}_{(\partial M, g_\partial)}\right) = \operatorname{sign}(M).$$

- Let $X_\phi = M\cup_\phi N^-$ and $X_\psi = M \cup_\psi N^-$ be two *cut-and-paste equivalent* closed manifolds. Then
$$< L(X_\phi), [X_\phi] >=< L(X_\psi), [X_\psi] > .$$

5. Novikov higher signatures.

5.1. Galois coverings and classifying maps.

Let Γ be a discrete finitely presented group. Let $\Gamma \longrightarrow \widetilde{M} \longrightarrow M$ be a Galois Γ-covering (the term *normal* is also in common usage). For example $\Gamma := \pi_1(M)$ and $\widetilde{M}=$ universal covering of M. As a particular example to keep in mind, let Σ_g be a closed connected Riemann surface of genus $g \geqslant 2$ and let Γ_g be its fundamental group, then $\Sigma_g \simeq \mathcal{H}/\Gamma_g$ where $\mathcal{H}$ denotes the Poincaré upper halfplane $(\{z \in \mathbb{C}, / \operatorname{Im} z > 0\})$. The projection map $p : \mathcal{H} \to \mathcal{H}/\Gamma_g$ defines the universal covering of Σ_g.

From now on all our Γ-coverings will be Galois. Recall that Γ-coverings are, in particular, principal Γ-bundles. Thus, thanks to the classification theorem for principal bundles, see Lawson-Michelson [71], we know that there exist topological spaces $B\Gamma$, $E\Gamma$, with $E\Gamma$ contractible, and a Γ-covering $E\Gamma \to B\Gamma$ such that the following statement holds:

there is a natural bijection between the set of isomorphism classes of $\Gamma-$coverings on M and the set of homotopy classes of continuous maps $r : M \to B\Gamma$.

The bijection is realized by the map that associates to $(M, r : M \to B\Gamma)$ the Γ-covering $r^* E\Gamma$. The space $B\Gamma$ is uniquely defined up to homotopy equivalences and is called the *classifying space* of Γ. The map r is called the classifying map. In the example above one has $E\Gamma_g = \mathcal{H}, B\Gamma_g = \Sigma_g$ and $r =$ identity. As a different example: $B\mathbb{Z}^k = (S^1)^k$, $E\mathbb{Z}^k = \mathbb{R}^k$ with covering map:

$$(x_1, \ldots, x_k) \in \mathbb{R}^k \to (e^{ix_1}, \ldots, e^{ix_k}).$$

From now on we shall identify a Γ-covering with the corresponding pair $(M, r : M \to B\Gamma)$.

Definition 5.1. — *Let M and M' be closed oriented manifolds. We shall say that two Γ-coverings*

$$(M, r : M \to B\Gamma) \quad \text{and} \quad (M', r' : M' \to B\Gamma)$$

are oriented homotopy equivalent if there exists an oriented homotopy equivalence $h : M' \to M$ such that $r \circ h \simeq r'$, where $\simeq$ means homotopic.

Definition 5.2. — *Let M and N be two oriented compact manifolds with boundary and let $\phi, \psi : \partial M \to \partial N$ be orientation preserving* *diffeomorphisms. Let $r : M \cup_\phi N^- \to B\Gamma$ and $s : M \cup_\psi N^- \to B\Gamma$ be two reference maps. We say that they define cut-and-paste equivalent Γ-coverings if $r|_M \simeq s|_M$ and $r|_N \simeq s|_N$ holds, where $\simeq$ means homotopic.*

Geometrically this means that $r^*E\Gamma \to M \cup_\phi N^-$ and $r^*E\Gamma \to M \cup_\psi N^-$ give rise to isomorphic bundles when restricted to M and N respectively.

5.2. The definition of higher signatures.

Let $\Gamma \to \widetilde{M} \to M$ be a Γ-covering of a closed oriented manifold and let $r : M \to B\Gamma$ be a classifying map for such a covering. Consider the cohomology of $B\Gamma$ with real coefficients $H^*(B\Gamma, \mathbb{R})$. It can be proved that there is a natural isomorphism

$$H^*(B\Gamma, \mathbb{R}) \cong H^*(\Gamma, \mathbb{R})$$

where on the right hand side we have the algebraic cohomology of the group Γ. We recall that $H^*(\Gamma, \mathbb{R})$ is by definition the graded homology group associated to the complex $\{\mathcal{C}^*(\Gamma), d\}$ whose $p-$cochains are functions $c : \Gamma^{p+1} \to \mathbb{R}$ satisfying the invariance condition

$$c(g \cdot g_0, \dots, g \cdot g_p) = c(g_0, \dots, g_p) \quad \forall g, g_0, \dots, g_p \in \Gamma,$$

and with coboundary given by the formula

$$(dc)(g_0, \cdots, g_{p+1}) = \sum_{i=0}^{p+1} (-1)^i c(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{p+1}).$$

Since we deal with real coefficients, the above complex can be replaced by the subcomplex of antisymmetric cochains:

$$\forall \tau \in S_{p+1},\; c(g_{\tau(0)}, \cdots, g_{\tau(p+1)}) = \mathrm{sgn}(\tau)\, c(g_0, \cdots; g_{p+1}).$$

Let us fix a class $[c] \in H^*(B\Gamma, \mathbb{R})$. We take the pull-back $r^*[c] \in H^*(M, \mathbb{R})$ and consider

$$\text{(5.3)} \qquad \mathrm{sign}(M, r; [c]) := < L(M) \cup r^*[c], [M] >$$

This real number is called the *Novikov higher signature* associated to $[c] \in H^*(B\Gamma, \mathbb{R})$ and the classifying map r. Using the de Rham isomorphism we can equivalently write

$$\text{(5.4)} \qquad \mathrm{sign}(M, r; [c]) := \int_M [L(M, \nabla^g)] \wedge r^*[c].$$

If $\dim M = 4k$ and $[c] = 1 \in H^0(B\Gamma, \mathbb{R})$, then

$$\operatorname{sign}(M, r; 1) = \int_M L(M) (= \operatorname{sign}(M))$$

and we reobtain the lower signature.

Remark. — We have defined the Hirzebruch L-class as the de Rham class of the L-form $L(M, \nabla^g)$. In fact, using a more topological approach to characteristic classes, one can define the L-class in $H^*(M, \mathbb{Q})$; consequently the higher signatures $\operatorname{sign}(M, r; [c])$ can be defined for each $[c] \in H^*(B\Gamma, \mathbb{Q})$.

For motivation and historical remarks concerning Novikov higher signatures the reader is referred to the survey by Ferry-Ranicki-Rosenberg [37].

6. Three fundamental questions.

Having defined the higher signatures

$$\{\operatorname{sign}(M, r; [c]) \,, \quad [c] \in H^*(B\Gamma, \mathbb{R})\}$$

and keeping in mind the properties of the lower signature, we can ask the following three fundamental questions.

Question 1. Are the higher signatures *homotopy invariant*?

Question 2. Are the higher signatures *cut-and-paste invariant*?

Question 3. If $\partial M \neq \emptyset$, *can we define higher signatures and prove their homotopy invariance?* Of course we want these higher signatures on a manifold with boundary M to generalize the lower signature

$$\int_M L(M, \nabla^g) - \frac{1}{2}\eta\big(D^{\text{sign}}_{(\partial M, g_\partial)}\big),$$

which is indeed a homotopy invariant by Theorem 2.8.

We anticipate our answers: Question 1 is still open and is known as the *Novikov conjecture*. It has been settled in the affirmative for many classes of groups. For instance, the following groups satisfy the Novikov conjecture: virtually nilpotent groups and more generally amenable groups, any discrete subgroup of $GL_n(F)$ where F is a field of characteristic zero, Artin's braid groups B_n, one-relator groups, the discrete subgroups of Lie groups with finitely many path components, $\pi_1(M)$ for a complete

Riemanniann manifold with non-positive sectional curvature. The Novikov conjecture has also been proved for hyperbolic groups and, more generally, for groups acting properly on bolic spaces (see the recent work of Kasparov and Skandalis). A few relevant references are Mishchenko [98], Kasparov [63], [64], [65], Weinberger [121], Connes-Moscovici [29], Connes-Gromov-Moscovici [30], [31], Ferry-Ranicki-Rosenberg [37], Gromov [44], Higson-Kasparov [49],Kasparov-Skandalis [67] (see also Kasparov-Skandalis [66], Solov'ev [115]), Guentner-Higson-Weinberger [46]. For related material see Lafforgue [69], Cuntz [32], Mathai [90], Lück-Reich [88], Schick [113].

The answer to Question 2 is negative: the higher signatures are *not* cut-and-paste invariants (we shall present a counterexample below). However, one can give sufficient conditions on the separating hypersurface F and on the group Γ ensuring that the higher signatures are indeed cut-and-paste invariant.

Finally, under suitable assumption on $(\partial M, r|_{\partial M})$ and on the group Γ one can *define* higher signatures on a manifold with boundary M equipped with a classifying map $r : M \to B\Gamma$ and *prove their homotopy invariance.*

Relevant references for the solution to the last 2 questions will be given along the way.

7. The Novikov conjecture on closed manifolds: the K-theory approach.

In this section we shall describe one of the approaches that have been developed in order to attack, and sometime solve, the Novikov conjecture. We begin by introducing important mathematical objects associated to M, Γ and $r : M \to B\Gamma$.

7.1. The reduced group C^*-algebra $C^*_r\Gamma$.

We consider the group ring $\mathbb{C}\Gamma$. It can be identified with the complex-valued functions on Γ of compact support. Any element $f \in \mathbb{C}\Gamma$ acts on $\ell^2(\Gamma)$ by left convolution. The action is bounded in the ℓ^2 operator norm $\|\cdot\|_{\ell^2(\Gamma)\to\ell^2(\Gamma)}$. The reduced group C^*-algebra, denoted $C^*_r\Gamma$, is defined as the completion of $\mathbb{C}\Gamma$ in $B(\ell^2(\Gamma))$. Let us give an example: if $\Gamma = \mathbb{Z}^k$ then using Fourier transform one can prove that there is a natural isomorphism of C^*algebras:

$$C_r^*\mathbb{Z}^k \longleftrightarrow C^0(T^k)$$

with $T^k = \mathrm{Hom}(\mathbb{Z}^k, U(1))$ the dual group associated to $\mathbb{Z}^k$ (a k-dimensional torus).

7.2. K-Theory.

Let A be a unital C^*-algebra, such as $C_r^*\Gamma$. We recall that $K_0(A)$ is defined as the group generated by the *stable* isomorphism classes of finitely generated projective left $A-$modules; more precisely such a module is the range of a projection p in a matrix algebra $M_n(A)$ and one identifies two pairs of projections $(p,q) \in M_n(A)^2$ and $(p',q') \in M_{n'}(A)^2$ if for suitable $k, k' \in \mathbb{N}$,

$$p \oplus q' \oplus \mathrm{Id}_k \oplus 0_{k'} \text{ is conjugate to } p' \oplus q \oplus \mathrm{Id}_k \oplus 0_{k'} \text{ in } M_{n+n'+k+k'}(A).$$

One then denotes by $[p-q]$ ($=[p'-q']$) the class of (p,q); similarly, if E and F are finitely generated projective left A-modules, then we denote by $[E-F]$ the associated class in $K_0(A)$. $K_0(A)$ is an additive group. When A is a non unital C^*-algebra one introduces the unital C^*-algebra $\widetilde{A} = A \oplus \mathbb{C}$ obtained by adding the unit element $0 \oplus 1$ to A; one considers the morphism $\epsilon : \widetilde{A} \to \mathbb{C}$ defined by $\epsilon(a \oplus \lambda) = \lambda$. One then defines $K_0(A)$ to be equal to the kernel of the map $\epsilon_* : K_0(\widetilde{A}) \to K_0(\mathbb{C})$ induced by ϵ. Observe that $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = \mathbb{Z}$. We also define $K_1(A)$ to be equal to $K_0(A \otimes C_0(\mathbb{R}))$ where $A \otimes C_0(\mathbb{R})$ is the *suspension of* A. For instance $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = 0$. Alternatively, $K_1(A)$ can be identified with the set of connected components of $GL_\infty(A)$. We recall that for any compact Hausdorff space M, the $K-$theory group $K^0(M)$ is defined as the set of formal differences of isomorphism classes of complex vector bundles over M. Then Swann's theorem states that $K_0(C^0(M))$ is isomorphic to $K^0(M)$. Thus, from the previous sub-section one gets an isomorphism: $K_0(C_r^*\mathbb{Z}^k) \simeq K^0(T^k)$.

7.3. The index class of the signature operator in $K_*(C_r^*\Gamma)$.

7.3.1. $C_r^*\Gamma$-linear operators. Let (M,g) be a closed, compact and oriented Riemannian manifold. Let $\pi : \widetilde{M} \to M$ be a Galois Γ-covering.

Let $r : M \to B\Gamma$ be a classifying map for this covering. We consider a Hermitian Clifford module $E \to M$, endowed with a Clifford connection ∇^E, and let D be the associated Dirac-type operator acting on $C^\infty(M, E)$. For example we could consider the signature operator associated to g and our choice of orientation. Notice that we can lift the operator D to a Γ-invariant differential operator $\widetilde{D}$ on $\widetilde{M}$; $\widetilde{D}$ acts on the section of the Γ-equivariant bundle $\widetilde{E} := \pi^* E$. Consider now $C^*_r\Gamma$. The group Γ acts in a natural way on $C^*_r\Gamma$ by right translation. It also act on $\widetilde{M}$ (on the left) by deck transformations: we can therefore consider the associated bundle

$$\mathcal{V} := C^*_r\Gamma \times_\Gamma \widetilde{M}$$

which is a vector bundle with typical fiber $C^*_r\Gamma$. We shall be interested in the space of sections $C^\infty(M, E \otimes \mathcal{V})$. If $\operatorname{rank} E = N$ and $s \in C^\infty(M, E \otimes \mathcal{V})$, then in a trivializing neighborhood U we can identify $s|_U$ with a N-tuple of $C^*_r\Gamma$-valued functions $(s^1, \dots, s^N)$. This shows that $C^\infty(M, E \otimes \mathcal{V})$ is in a natural way a left $C^*_r\Gamma$-module. Moreover, using the Hermitian metric $h(\cdot,\cdot)$ on E we can define a $C^*_r\Gamma$-valued inner product $< \cdot, \cdot >$: if $s, t \in C^\infty_0(U, (E \otimes \mathcal{V})|_U)$ then

$$< s, t > := \int_U \sum h_{ij} s^i t^j \,\mathrm{dvol}_g \in C^*_r\Gamma$$

The general case is obtained by using a partition of unity. $C^\infty(M, E \otimes \mathcal{V})$ equipped with the above $C^*_r\Gamma$-valued inner product is a pre-Hilbert $C^*_r\Gamma$-module, in the sense that it satisfies the following properties: $\forall a \in C^*_r\Gamma, \forall s, t, u \in C^\infty(M, E \otimes \mathcal{V})$:

$$< s, t + u > = < s, t > + < s, u >, \; < a \cdot s, t > = a \cdot < s, t >, \; < s, a \cdot t > = a^* < s, t > .$$

The completion of $C^\infty(M, E \otimes \mathcal{V})$ with respect to the norm

$$\|s\| = \sqrt{\| < s, s > \|_{C^*_r\Gamma}}$$

is denoted by $L^2_{C^*_r\Gamma}(M, E \otimes \mathcal{V})$; it is a *Hilbert* $C^*_r\Gamma$*-module.*

The product bundle $C^*_r\Gamma \times \widetilde{M} \to \widetilde{M}$ is endowed with the trivial flat connection. It induces a (non trivial) flat connection $\nabla^{\mathcal{V}}$ on the $C^*_r\Gamma$−bundle $\mathcal{V}$. Then the bundle $E \otimes \mathcal{V} \to M$ is endowed with the connection $\nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathcal{V}}$. We denote by $\mathcal{D}_{(M,r)}$ the associated twisted Dirac type operator. Directly from the definition we see that:

$$\mathcal{D}_{(M,r)} : C^\infty(M, E \otimes \mathcal{V}) \to C^\infty(M, E \otimes \mathcal{V}) \text{ is } C^*_r\Gamma - \text{linear.} \tag{7.1}$$

A good reference for seeing the details of this approach is Schick [114]. We also remark that it is possible to introduce Sobolev $C^*_r\Gamma$-modules

$H^m_{C^*_r\Gamma}(M, E\otimes\mathcal{V})$ and $\mathcal{D}_{(M,r)}$ extends to a bounded $C^*_r\Gamma$-linear operator from $H^1_{C^*_r\Gamma}(M, E\otimes\mathcal{V})$ to $L^2_{C^*_r\Gamma}(M, E\otimes\mathcal{V})$.

If M is even dimensional, then E is $\mathbb{Z}_2$-graded, $E = E^+\oplus E^-$; thus

$$\mathcal{D}_{(M,r)} = \begin{pmatrix} 0 & \mathcal{D}^-_{(M,r)} \\ \mathcal{D}^+_{(M,r)} & 0 \end{pmatrix},$$

with $\mathcal{D}^+_{(M,r)}$ and $\mathcal{D}^-_{(M,r)}$ both $C^*_r\Gamma$-linear.

7.3.2. The index class in $K_*(C^*_r\Gamma)$. From (7.1) we infer that $\operatorname{Ker}\mathcal{D}^+_{(M,r)}$ and $\operatorname{coker}\mathcal{D}^+_{(M,r)}$ are both $C^*_r\Gamma$-modules. In general, the modules $\operatorname{Ker}\mathcal{D}^+_{(M,r)}$ and $\operatorname{coker}\mathcal{D}^+_{(M,r)}$ are not finitely generated and projective so they cannot be used directly to define the index class $\operatorname{Ind}(\mathcal{D}^+_{(M,r)}) \in K_0(C^*_r\Gamma)$ as $[\operatorname{Ker}\mathcal{D}^+_{(M,r)}] - [\operatorname{coker}\mathcal{D}^+_{(M,r)}]$. [4] However this is true up to a smoothing perturbation $\mathcal{R}$; one defines the index class as

$$\operatorname{Ind}(\mathcal{D}^+_{(M,r)}) := [\operatorname{Ker}\mathcal{D}^+_{(M,r)} + \mathcal{R}] - [\operatorname{coker}\mathcal{D}^+_{(M,r)} + \mathcal{R}] \in K_0(C^*_r\Gamma)$$

(and the definition does not depend on the choice of $\mathcal{R}$). Let us see the details.

The Mishchenko-Fomenko pseudodifferential calculus. One can define a space of $C^*_r\Gamma$-linear *differential* operators $\operatorname{Diff}^*_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$; these are simply operators locally given by a $N\times N$-matrix A_{ij}, $N = \operatorname{rk}E$, with

$$A_{ij} = \sum_{|\alpha|\leqslant k} a(ij)_\alpha \frac{\partial^\alpha}{\partial^{\alpha_1}_{x_1}\cdots\partial^{\alpha_n}_{x_n}}, \quad \text{with} \quad a(ij)_\alpha \in C^\infty(U, C^*_r\Gamma).$$

In a very natural way we can give the notion of ellipticity in $\operatorname{Diff}^*_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$. From the definitions, we discover first of all that $\mathcal{D}_{(M,r)} \in \operatorname{Diff}^1_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$; moreover the ellipticity of D implies that $\mathcal{D}_{(M,r)}$ is elliptic in $\operatorname{Diff}^1_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$. Mishchenko and Fomenko have developed a pseudodifferential calculus for $C^*_r\Gamma$-linear operators

$$\Psi^*_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V}) \supset \operatorname{Diff}^*_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V}).$$

Using this calculus one can prove that given an elliptic operator $\mathcal{P} \in \operatorname{Diff}^k_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$, it is possible to find an inverse $\mathcal{Q} \in \Psi^{-k}_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$ modulo elements in $\Psi^{-\infty}_{C^*_r\Gamma}(M; E\otimes\mathcal{V}, E\otimes\mathcal{V})$. Notice that the smoothing operators in the Mishchenko-Fomenko calculus are simply the

[4] This is similar to the problem one encounters in defining the index class of a family $\mathcal{F} := (F_\theta)_{\theta\in T}$ of Fredholm operators parametrized by a space T: the kernel-bundle and the cokernel-bundle do not in general vary continuously.

integral operators with a Schwartz kernel on $M \times M$ locally given by a smooth function with values in $M_{N\times N}(C^*_r\Gamma)$.

Let in particular M be even dimensional and let $E = E^+ \oplus E^-$ be the $\mathbb{Z}_2$-graded bundle appearing in the definition of our Dirac operator. The operator

$$\mathcal{D}^+_{(M,r)} \in \mathrm{Diff}^*_{C^*_r\Gamma}(M; E^+ \otimes \mathcal{V}, E^- \otimes \mathcal{V})$$

is elliptic and there exists therefore a parametrix $\mathcal{Q} \in \Psi^{-1}_{C^*_r\Gamma}(M; E^- \otimes \mathcal{V}, E^+ \otimes \mathcal{V})$ such that

$$\mathcal{D}^+_{(M,r)} \circ \mathcal{Q} = \mathrm{Id} - \mathcal{R}_-, \;\; \mathcal{Q} \circ \mathcal{D}^+_{(M,r)} = \mathrm{Id} - \mathcal{R}_+ \tag{7.2}$$

with $R_\pm \in \Psi^{-\infty}_{C^*_r\Gamma}(M; E^\pm \otimes \mathcal{V}, E^\pm \otimes \mathcal{V})$. This part of the theory runs quite parallel to the usual case, when the C^*-algebra is equal to $\mathbb{C}$; the main differences arise in the functional analytic consequences of (7.2). The point is that doing functional analysis on a Hilbert A-module, with A a C^*-algebra, is a more delicate matter than doing functional analysis on a Hilbert space (see Wegge-Olsen [120] and Higson [47] for more on this delicate point).

The Mishchenko-Fomenko decomposition theorem. On a Hilbert A-module there exists a natural notion of A-compact operator: using (7.2), elliptic regularity and the fact that elements in $\Psi^{-\infty}_{C^*_r\Gamma}$ are $C^*_r(\Gamma)$-compact on $L^2_{C^*_r\Gamma}$, one can prove a decomposition of the space of sections of $E \otimes \mathcal{V}$ with respect to $\mathcal{D}^+_{(M,r)}$, i.e.

$$C^\infty(M, E^+ \otimes \mathcal{V}) = \mathcal{I}_+ \oplus \mathcal{I}_+^\perp, \quad C^\infty(M, E^- \otimes \mathcal{V}) = \mathcal{I}_- \oplus \mathcal{D}^+_{(M,r)}(\mathcal{I}_+^\perp), \tag{7.3}$$

with $\mathcal{I}_+$ and $\mathcal{I}_-$ *finitely generated projective* $C^*_r\Gamma$-modules. Notice that the second decomposition is not, a priori, orthogonal. However, $\mathcal{D}^+_{(M,r)}$ induces an isomorphism (in the Fréchet topology) between $\mathcal{I}_+^\perp$ and $\mathcal{D}^+_{(M,r)}(\mathcal{I}_+^\perp)$. Intuitively $\mathcal{I}_+$ should be thought of as the kernel of $\mathcal{D}^+_{(M,r)}$ and $\mathcal{I}_-$ as the cokernel.

The index class. The index class of $\mathcal{D}^+_{(M,r)}$, à la Mishchenko-Fomenko, is precisely given by

$$\mathrm{Ind}(\mathcal{D}^+_{(M,r)}) = [\mathcal{I}_+] - [\mathcal{I}_-] \;\in\; K_0(C^*_r\Gamma). \tag{7.4}$$

Although the decomposition (7.3) is not unique, the index class is uniquely defined in $K_0(C^*_r\Gamma)$. The main reference for this material is the original article of Mishchenko and Fomenko [100]; see also [74], Appendix A. Working a little bit more one can show that the orthogonal projection

Π_+ onto $\mathcal{I}_+$ and the projection Π_- onto $\mathcal{I}_-$ along $\mathcal{D}^+(\mathcal{I}_+^\perp)$ are elements in $\Psi^{-\infty}_{C^*_r\Gamma}$ (see [74], Appendix A) . Thus

$$\mathcal{D}^+_{(M,r)} - \Pi_-\mathcal{D}^+_{(M,r)}\Pi_+$$

is a *smoothing* perturbation of $\mathcal{D}^+_{(M,r)}$ with the property that its kernel and cokernel are finitely generated and projective.

Summarizing: there exists a smoothing perturbation $\mathcal{R}$ of $\mathcal{D}^+_{(M,r)}$ such that $\operatorname{Ker}(\mathcal{D}^+_{(M,r)}+\mathcal{R})$ and $\operatorname{coker}(\mathcal{D}^+_{(M,r)}+\mathcal{R})$ are finitely generated projective $C^*_r\Gamma$-modules; the index class can be defined as

$$\operatorname{Ind}\left(\mathcal{D}^+_{(M,r)}\right) := \left[\operatorname{Ker}\left(\mathcal{D}^+_{(M,r)}+\mathcal{R}\right)\right] - \left[\operatorname{coker}\left(\mathcal{D}^+_{(M,r)}+\mathcal{R}\right)\right] \in K_0(C^*_r\Gamma)$$

and it is not difficult to prove that it does not depend on the choice of $R \in \Psi^{-\infty}_{C^*_r\Gamma}$.

If M is odd dimensional, then the Clifford module E will be ungraded; we obtain in this case an index class $\operatorname{Ind}\mathcal{D}_{(M,r)} \in K_1(C^*_r\Gamma)$. We shall not give the details here.

7.3.3. The example $\Gamma = \mathbb{Z}^k$. Let N be a closed oriented manifold with $\pi_1(N) = \mathbb{Z}^k$. Let r be the classifying map. In this case the higher index class $\operatorname{Ind}\left(\mathcal{D}^{\rm sign}_{(N,r)}\right)$ has, thanks to Lustzig [89], a geometric description Details for the material that follows can be found in [89] and Lott [83]. As already remarked the space $B\mathbb{Z}^k$ is a k-dimensional torus; more precisely, it is the dual torus $(T^k)^*$ to $T^k = \widehat{\mathbb{Z}^k} = \operatorname{Hom}(\mathbb{Z}^k, U(1))$. Using the duality between the two tori it is easy to see that on the product $(T^k)^* \times T^k$ there is a canonical Hermitian line bundle H with a canonical Hermitian connection ∇^H. The bundle H is flat when restricted to any fibre of the projection $(T^k)^* \times T^k \to T^k$. Using the map $r \times \mathrm{id} : N \times T^k \to (T^k)^* \times T^k$ we obtain a line bundle F on $N \times T^k$ with a natural Hermitian (pulled-back) connection ∇^F. In this way we have obtained a fibration of closed manifolds $\phi : N \times T^k \to T^k$ and a Hermitian line bundle F over the total space with a flat structure in the fibre directions. Let $\theta \in T^k$ and let F_θ be the restriction of F to $N \times \{\theta\}$. Since F_θ is flat, the de Rham differential can be extended to act on $\Lambda^*(M) \otimes F_\theta$; we obtain a twisted de Rham differential d_θ. Let $D^{\rm sign}_\theta$ be the corresponding twisted signature operator on N. As θ varies in T^k, we obtain a smoothly varying family of twisted signature operators. Thus, according to Atiyah and Singer [8], we obtain an index class $\operatorname{Ind}(\{D^{\rm sign}_\theta\}_{\theta\in T^k}) \in K^*(T^k)$, with $* = \dim N$. It can be proved that

$$\operatorname{Ind}(\mathcal{D}^{\rm sign}_{(N,r)}) \in K_*(C^*_r(\mathbb{Z}^k)) \quad\text{and}\quad \operatorname{Ind}(\{D^{\rm sign}_\theta\}_{\theta\in T^k}) \in K^*(T^k)$$

corresponds under the isomorphisms $K_*(C_r^*(\mathbb{Z}^k)) \simeq K_*(C^0(T^k)) \simeq K^*(T^k)$.

7.4. The symmetric signature of Mishchenko.

Let A be an involutive algebra and let us introduce $L^0(A)$, the Witt group of non singular Hermitian forms on A: it classifies Hermitian forms Q on finitely generated left projective modules on A. Given E a finitely generated left projective module over A, a Hermitian form Q on E is a sesquilinear form $E \times E \to A$ such that:

$$\forall \xi, \eta \in E,\ \forall a, b \in A,\ Q(a \cdot \xi, b \cdot \eta) = a \cdot Q(\xi, \eta) \cdot b^*,\ Q(\xi, \eta)^* = Q(\eta, \xi).$$

The form Q is said to be invertible when the map from E to $\mathrm{Hom}_A(E, A)$ given by $\xi \to Q(\cdot, \xi)$ is invertible. *The Witt group $L^0(A)$ is the group generated by the isomorphism classes of invertible Hermitian forms with the relations:* $[Q_1 \oplus Q_2] = [Q_1] + [Q_2]$, $0 = [Q] + [-Q]$. When A is a C^*-algebra with unit then each finitely generated left projective module over A admits an invertible Hermitian form Q satisfying the positivity condition $Q(\xi, \xi) \geqslant 0$ for any $\xi \in E$. (Recall that an element x of the C^*-algebra A is positive if and only if it is of the form $x = yy^*$, or equivalently, x is self-adjoint and its spectrum lies in $[0, +\infty[$). Moreover, on E all such positive Hermitian forms are pairwise isomorphic so that there is a well defined map $K_0(A) \to L^0(A)$ sending E to (E, Q) with Q an invertible positive Hermitian form on E; it turns out that this map is an isomorphism.

Let M be an oriented $2n-$dimensional manifold and let $r : M \to B\Gamma$ be a (continuous) reference map. We are going to recall, following Mishchenko [98], [99] (see also Kasparov [65], [63]), the construction of the Mishchenko symmetric signature $\sigma_{\mathbb{C}\Gamma}(M, r) \in L^0(\mathbb{C}\Gamma)$.

Denote by $\widetilde{M} \to M$ the associated Galois $\Gamma-$covering. Take a (suitably nice) triangulation of M and pull it back to $\widetilde{M}$ to a $\Gamma-$invariant triangulation of $\widetilde{M}$. Let (C_*, ∂_*) and (C^*, δ_*) denote the associated simplicial chain complex and cochain complex: $\delta_j : C^j \to C^{j+1}$, $\partial_j : C_{j+1} \to C_j$, for $0 \leqslant j \leqslant 2n$. The C^j, C_j are finitely generated free left $\mathbb{C}[\Gamma]-$modules. There is a chain map $\xi_j : C^j \to C_{2n-j}$, $0 \leqslant j \leqslant 2n$, defining Poincaré duality, which satisfies $\partial_{2n-j}\xi_j = (-1)^j \xi_{j+1}\delta_j$ and induces a chain homotopy equivalence. It can be arranged that $\xi_j^* = (-1)^j \xi_{2n-j}$ where ξ_j^* denotes the adjoint of the left $\mathbb{C}[\Gamma]-$linear map ξ_j^*. We can add a $(\mathbb{C}[\Gamma])^k$, for a suitable $k \in \mathbb{N}$, to both C^{2n-1} and C_1 to make $\delta_{2n-1} : C^{2n-1} \to C^{2n}$

surjective. Then we add $((\mathbb{C}[\Gamma])^k)^*$ to both C^1 and C_{2n-1} in order to preserve Poincaré duality and modify accordingly δ_0 and ∂_{2n}. Now we may split off ξ_0 and ξ_{2n} and repeat this algebraic surgery process so as to come down to a complex concentrated in middle dimension: $\xi_n : C^{2n} \to C_{2n}$, $(i^n\xi_n)^* = i^n\xi_n$. Since $i^n\xi_n$ defines a non degenerate Hermitian form one gets an element, denoted $\sigma_{\mathbb{C}\Gamma}(M,r)$, of $L^0(\mathbb{C}[\Gamma])$. *Mishchenko has shown that $\sigma_{\mathbb{C}\Gamma}(M,r)$ depends only on the oriented homotopy type of (M,r).*

Consider now the natural homomorphism $L^0(\mathbb{C}\Gamma) \to L^0(C_r^*\Gamma)$ induced by the inclusion $\mathbb{C}\Gamma \hookrightarrow C_r^*\Gamma$. We compose it with the inverse isomorphism $K_0(C_r^*\Gamma) \leftarrow L^0(C_r^*\Gamma)$ and get a well defined homomorphism

$$J : L^0(\mathbb{C}\Gamma) \to K_0(C_r^*\Gamma)\,.$$

Let

$$\sigma(M,r) := J(\sigma_{\mathbb{C}\Gamma}(M,r)) \in K_0(C_r^*\Gamma);$$

$\sigma(M,r)$ is the *$C_r^*\Gamma$-valued Mishchenko symmetric signature.* It is a homotopy invariant of the pair $(M, r : M \to B\Gamma)$.

7.5. Homotopy invariance of the index class.

The following theorem will play a crucial role both in the treatment of the Novikov conjecture and of the cut-and-paste invariance of higher signatures. It is due to Mishchenko and Kasparov, [99], [63]:

THEOREM 7.5. — *Let $(M, r : M \to B\Gamma)$ be an oriented manifold with classifying map r. Then the index class $\mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}_{(M,r)}) \in K_*(C_r^*\Gamma)$, $* = \dim M$, is equal to $\sigma(M,r)$, the $C_r^*\Gamma$-valued Mishchenko symmetric signature:*

$$\mathrm{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(M,r)}\right) = \sigma(M,r) \text{ in } K_*(C_r^*\Gamma). \tag{7.6}$$

As a corollary we then get the following fundamental information:

COROLLARY 7.7. — *The index class $\mathrm{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(M,r)}\right) \in K_*(C_r^*\Gamma)$ is an oriented homotopy invariant.*

Remark. — It is possible to give a purely analytic proof of Corollary 7.7. This important result is due to Hilsum and Skandalis [53]. See also the work of Kaminker-Miller [60].

Remark. — When $\Gamma = \mathbb{Z}^k$, Lusztig was the first to establish the homotopy invariance of $\mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}_{(M,r)}) \in K_*(C^*_r\mathbb{Z}^k)$. The proof of Kaminker-Miller [60] cited above is an extension of Lusztig's proof to the noncommutative context. For a very direct and short proof of the homotopy invariance of the signature index class see the recent paper [106].

Important remark. Although Theorem 7.5 and Corollary 7.7 are extremely interesting results, they still do not settle in anyway the Novikov conjecture. In fact, these results should be viewed as the higher analogue of *only one* out of the two steps we used in order to prove that $\int_M L(M)$ is an homotopy invariant. This step is, more precisely, the homotopy invariance of the signature and its equality with the index. What we are still missing in the present higher case is the first step, the one relating $\int_M L(M)$ to the index. The problem we face now is therefore quite clear:

Fundamental Problem: *how can one use the homotopy invariance of the index class* $\mathrm{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(M,r)}\right)$ *in order to prove the homotopy invariance of the higher signatures* $< L(M) \cup r^*[c], [M] >, [c] \in H^*(B\Gamma, \mathbb{R})$ *?*

Alternatively:

how can we connect the index class and its homotopy invariance to the higher signatures?

We shall present below two answers to this question. The first one, due to Kasparov, employs the K-homology of $B\Gamma$, $K_*(B\Gamma)$, and a natural map $\mu : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$; the second one, due to Connes and Moscovici, employs cyclic cohomology.

7.6. The assembly map and the Strong Novikov Conjecture.

We are considering a closed oriented manifold M and a classifying map $r : M \to B\Gamma$. Let $L(M) \cap [M]$ be the Poincaré dual to $L(M)$ and consider $r_*(L(M) \cap [M]) \in H_*(B\Gamma, \mathbb{R})$. One can check, using some basic algebraic topology, that

$$\mathrm{sign}(M, r; [c]) = < [c], r_*(L(M) \cap [M]) > .$$

Thus the homotopy invariance of the real homology class $r_*(L(M) \cap [M])$ implies the homotopy invariance of all the higher signatures

$$\{\mathrm{sign}(M, r; [c]), [c] \in H^*(B\Gamma, \mathbb{R})\}.$$

It is well known that K-theory is a generalized cohomology theory; it thus admits a dual theory, *K-homology*, and there is a homological Chern character map $\mathrm{Ch} : K_*(\ ,\mathbb{Z}) \to H_*(\ ,\mathbb{Z})$ which is an isomorphism modulo torsion. Summarizing: the K-homology of $B\Gamma$ is well defined and there is an isomorphism $\mathrm{Ch}^{-1} : H_*(B\Gamma,\mathbb{R}) \to K_*(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$. Thus we are led to consider the following K-homology class

$$\mathrm{Ch}^{-1}(r_*(L(M) \cap [M]) \in K_*(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Clearly: *if this class is homotopy invariant, then the Novikov conjecture is true.*

In order to understand why we wish to pass from homology to K-homology we shall simply mention that besides the abstract definition (a dual theory), there are other characterizations of K-homology, directly connected to elliptic operators. Historically, Atiyah was the first to realize that cycles in $K_*(X)$ should be thought of as "abstract elliptic operators" [1]. His ideas were further pursued by Kasparov [62] and Brown-Douglas-Fillmore [21]. At the same time, Baum and Douglas [12] proposed a purely topological definition of K-homology and showed that it was compatible with the analytic one of Atiyah. We shall present this topological definition, since it is the easiest to explain and leads directly to the map $\mu : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ that was mentioned at the end of the previous section. We shall concentrate on the even dimensional case and pretend that $B\Gamma$ is compact (the general case is obtained by taking an inductive limit).

Cycles in the (topological) K-homology groups $K_0(X)$ of a compact topological Hausdorff space X are given by triples $(M, r : M \to X, E)$ where M is an even dimensional oriented manifold, $r : M \to X$ is continuous, and E is a $\mathbb{Z}_2$-graded vector bundle over M which can be given the structure of graded Clifford module. [5] One then introduces an equivalence relation on this triples given by *bordism*, *direct sum* and *vector bundle modification*. We do not enter into the details here. The quotient turns out to be the K_0-homology group of X. For example $[M, \mathrm{id}, \Lambda^{\mathrm{sign}}_{\mathbb{C}}(M)]$, with $\Lambda^{\mathrm{sign}}_{\mathbb{C}}(M) = \Lambda^+_{\mathbb{C}}(M) \oplus \Lambda^-_{\mathbb{C}}(M)$ the Clifford module defining the signature operator, is a class in $K_0(M)$. Similarly, if $r : M \to B\Gamma$ is a classifying map, then $[M, r : M \to B\Gamma, \Lambda^{\mathrm{sign}}_{\mathbb{C}}(M)]$ defines an element in $K_0(B\Gamma)$.

Let now $[M, r : M \to B\Gamma, E^+ \oplus E^-]$ be an element in $K_0(B\Gamma)$: we define a map

$$\mu : K_0(B\Gamma) \longrightarrow K_0(C^*_r\Gamma) \tag{7.8}$$

[5] The original definition of Baum-Douglas was slightly different: it assumed M to be spin_c but left E arbitrary; Keswani has proved, see [68], that the two definitions are equivalent.

by sending $[M, r : M \to B\Gamma, E^+ \oplus E^-]$ to the index class, in $K_0(C_r^*\Gamma)$, associated to the $C_r^*\Gamma$-linear Dirac operator associated to the Clifford module E and the classifying map $r : M \to B\Gamma$. Thus if D^E is the Dirac operator associated to E on M and if, as usual, we denote by $\mathcal{D}^E_{(M,r)}$ the operator D^E twisted by the flat bundle $\mathcal{V} = r^*E\Gamma \times_\Gamma C_r^*\Gamma$, then the map (7.8) is given by

$$K_0(B\Gamma) \ni [M, r : M \to B\Gamma, E] \xrightarrow{\mu} \operatorname{Ind} \mathcal{D}^{E,+}_{(M,r)} \in K_0(C_r^*\Gamma).$$

As a fundamental example we have:

$$\mu\,[M, r : M \to B\Gamma, \Lambda^{\text{sign}}_{\mathbb{C}}(M)] = \operatorname{Ind} \mathcal{D}^{\text{sign},+}_{(M,r)} \in K_0(C_r^*\Gamma).$$

A similar map, from $K_1(B\Gamma)$ to $K_1(C_r^*\Gamma)$, can be defined in the odd case, considering odd dimensional manifolds and ungraded Clifford modules in the definition of the cycles of $K_1(B\Gamma)$. We shall denote by $\mu_{\mathbb{R}}$ the map induced from $K_*(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$ to $K_*(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$. The map μ is called the *assembly map*; it is also referred to as the *Kasparov map*. If Γ is torsion free then it also known as the *Baum-Connes map*. One can check, unwinding the definitions, that

$$\operatorname{Ch}^{-1}(r_*(L(M) \cap [M])) = [M, r : M \to B\Gamma, \Lambda^{\text{sign}}_{\mathbb{C}}(M)] \in K_*(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Hence

$$\mu_{\mathbb{R}}(\operatorname{Ch}^{-1}(r_*(L(M) \cap [M]))) = \operatorname{Ind} \mathcal{D}^{\text{sign}}_{(M,r)} \in K_*(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{7.9}$$

We thus arrive at the following fundamental conclusion:

THEOREM 7.10. — *If the map $\mu_{\mathbb{R}}$ is injective then the Novikov conjecture is true.*

Proof. — If $(M, r : M \to B\Gamma)$ and $(N, s : N \to B\Gamma)$ are homotopy equivalent, then by Corollary 7.7 we have $\operatorname{Ind} \mathcal{D}^{\text{sign}}_{(M,r)} = \operatorname{Ind} \mathcal{D}^{\text{sign}}_{(N,s)}$. Using (7.9), the injectivity of $\mu_{\mathbb{R}}$ and the bijectivity of Ch we get $r_*(L(M) \cap [M]) = s_*(L(N) \cap [N])$, which implies the equality of all the higher signatures. □

For later use we notice that the conclusion we can draw is slightly more general:

PROPOSITION 7.11. — *If $\mu_{\mathbb{R}}$ is injective then the equality of the index classes $\operatorname{Ind} \mathcal{D}^{\text{sign}}_{(M,r)} = \operatorname{Ind} \mathcal{D}^{\text{sign}}_{(N,s)}$ in $K_*(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$ implies the equality of all the higher signatures:*

$$< L(M) \cup r^*(c), [M] > = < L(N) \cup s^*(c), [N] >, \quad \forall c \in H^*(B\Gamma, \mathbb{R}).$$

This remark will be important in treating the cut-and-paste problem for higher signatures. Of course, since $\operatorname{Ind}\mathcal{D}^{\text{sign}}_{(M,r)} = \sigma(M,r)$ (the $C^*_r\Gamma$-valued symmetric signature of Mishchenko), we can also state the following

PROPOSITION 7.12. — *If $\mu_{\mathbb{R}}$ is injective, then the equality of the $C^*_r\Gamma$-valued symmetric signatures, $\sigma(M,r) = \sigma(N,s)$, implies the equality of all the higher signatures.*

The injectivity of $\mu_{\mathbb{R}}$ (in fact, of $\mu_{\mathbb{Q}}$) is known as the *Strong Novikov Conjecture* ($\equiv$ SNC); it is still open. Most of the groups for which the Novikov conjecture has been verified, satisfy the SNC as well.

We refer to the nice survey of Kasparov [65] for seeing, informally, the Dirac-dual Dirac method for constructing a left inverse of $\mu_{\mathbb{R}}$. See also [64].

For the connection between the Strong Novikov Conjecture and the existence of metrics of positive scalar curvature (an important topic that will be left out of the present survey) we refer, for example, to Rosenberg [110], [111], [112], Stolz [117], Joachim-Schick [57], [113].

7.7. The Baum-Connes conjecture.

The *Strong Novikov Conjecture* states that the rational assembly map

$$\mu_{\mathbb{Q}} := \mu \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}} : K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(C^*_r\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. We have just seen that the Strong Novikov conjecture implies the Novikov conjecture.

If the discrete group Γ is *torsion free* then the *Baum-Connes conjecture* states that the assembly map $\mu : K_0(B\Gamma) \to K_0(C^*_r\Gamma)$ is *bijective.*

When the group Γ has torsion then, in general, the map μ is not an isomorphism. For instance, if $\Gamma = \frac{\mathbb{Z}}{2\mathbb{Z}}$ then $K_0\big(B\frac{\mathbb{Z}}{2\mathbb{Z}}\big) \otimes \mathbb{Q} = \mathbb{Q}$ whereas $K_0\big(C^*_r(\frac{\mathbb{Z}}{2\mathbb{Z}})\big) = \mathbb{Z} \oplus \mathbb{Z}$ so that $\mu \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}$ (and thus μ) cannot be surjective.

In the general non-torsion-free case, Baum, Connes and Higson have introduced the space $\underline{E}\Gamma$, classifying the proper actions of Γ. Such a space $\underline{E}\Gamma$ is uniquely defined up to $\Gamma-$equivariant homotopy (see [11]). Baum, Connes and Higson have then constructed an assembly map:

$$\widehat{\mu} : K^{\Gamma}_0(\underline{E}\Gamma) \to K_0(C^*_r\Gamma).$$

The *Baum-Connes conjecture* states that $\widehat{\mu}$ is an *isomorphism*. Notice that there is a natural map $\sigma : E\Gamma \to \underline{E}\Gamma$ which induces a map

$$\sigma_* : K_0(B\Gamma) \simeq K_0^\Gamma(E\Gamma) \longrightarrow K_0^\Gamma(\underline{E}\Gamma).$$

The map μ is given by $\mu = \widehat{\mu} \circ \sigma_*$. One can prove [11], Section 7 that the injectivity of $\widehat{\mu}$ implies the rational injectivity of μ. In other words, the Baum-Connes conjecture implies the Strong Novikov Conjecture.

For more on the Baum-Connes conjecture we also refer the reader to Valette [119], Lafforgue [69], Lück-Reich [88], Schick [113].

8. The cyclic-cohomology approach to the Novikov conjecture.

Let $M \xrightarrow{r} B\Gamma$ be a closed oriented manifold with classifying map r. In the previous subsection we have explained one way to link the index class $\mathrm{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(M,r)}\right) \in K_*(C^*_r\Gamma)$ (and its homotopy invariance), to the higher signatures $< L(M) \cup r^*[c], [M] >$, $[c] \in H^*(B\Gamma, \mathbb{R})$. This link is provided by the assembly map $\mu : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$. In this subsection we shall explain a different approach for establishing such a link; this method, due to Connes and Moscovici [29], will use cyclic cohomology. Our presentation will heavily employ results by Lott [82]. In order to understand the main ideas, we begin by the abelian case, $\Gamma = \mathbb{Z}^k$, thus explaining the *seminal work of Lusztig.*

8.1. The abelian case: family index theory.

Let us assume $\Gamma = \mathbb{Z}^k$. In subsection 8.1 we recalled the construction of the index class $\mathrm{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(M,r)}\right) \in K_*(C^*_r\mathbb{Z}^k)$ in terms of the *index bundle* associated to the Lusztig's family $\{D^{\mathrm{sign}}_\theta\}_{\theta\in T^k}$, $T^k = \mathrm{Hom}(\mathbb{Z}^k, U(1))$. We briefly denote this family by $\{D^{\mathrm{sign}}_\theta\}$. The index bundle lives in $K^*(T^k)$ and we can therefore consider its Chern character $\mathrm{Ch}(\mathrm{Ind}\,\{D^{\mathrm{sign}}_\theta\}\,) \in H^*_{\mathrm{dR}}(T^k)$. An application of the Atiyah-Singer *family index theorem* gives

$$\mathrm{Ch}(\mathrm{Ind}\{D^{\mathrm{sign}}_\theta\}) = \left[\int_M L(M) \wedge \omega\right] \in H^*(T^k), \tag{8.1}$$

with ω an explicit closed form in $\Omega^*(M \times T^k)$. Let now $[c] \in H^\ell(\mathbb{Z}^k, \mathbb{R}) = H^\ell(B\mathbb{Z}^k, \mathbb{R}) = H^\ell((T^k)^*, \mathbb{R})$; starting from $[c]$, Lusztig defines in a natural way $[\tau_c] \in H_\ell(T^k, \mathbb{R})$ so that

$$r^*[c] = \frac{1}{C(\ell)} < \omega, [\tau_c] >, \quad C(\ell) \in \mathbb{R} \setminus \{0\}. \tag{8.2}$$

Consequently

$$\int_M L(M) \cup r^*[c] = \frac{1}{C(\ell)} < \operatorname{Ch} \operatorname{Ind}(\{D_\theta^{\text{sign}}\}_{\theta \in T^k}), [\tau_c] >, \quad C(\ell) \neq 0.$$

Lusztig settled the Novikov conjecture in the abelian case by using the last formula and showing furthermore that the index bundle $\operatorname{Ind}\{D_\theta^{\text{sign}}\}$ is a *homotopy invariant.*

8.2. Bismut's proof of the family index theorem.

It is clear from what has been just explained that Lusztig's treatment of the Novikov conjecture is heavily based on the Atiyah-Singer *family index formula.* Besides the original K-theoretic proof of Atiyah and Singer, see [8], there is a *heat kernel proof* of the family index theorem, due to Bismut [14]. Bismut's theorem applies to any family of Dirac operators along the fibers of a fiber bundle $X \to B$; notice that in the present case this fiber bundle is nothing but $M \times T^k \to T^k$. We briefly explain Bismut's approach, as we shall need it later.

8.2.1. The superconnection heat-kernel. Consider the bundle $\mathcal{E}$ on T^k whose fiber $\mathcal{E}_\theta$ at $\theta \in T^k$ is $C^\infty(M, \Lambda_{\mathbb{C}}^{\text{sign}}(M) \otimes F_\theta)$. The Levi-Civita connection on $M \times T^k$ and the connection ∇^F on the vector bundle F on $M \times T^k$ (see subsection 7.3.3) define together a connection on $\mathcal{E}$:

$$\nabla^{\mathcal{E}} : C^\infty(T^k, \mathcal{E}) \longrightarrow \Omega^1(T^k, \mathcal{E}) := C^\infty(T^k, \Lambda^1(T^k) \otimes \mathcal{E}). \tag{8.3}$$

The sum

$$\mathbb{A} := \{D_\theta^{\text{sign}}\} + \nabla^{\mathcal{E}}$$

is called a *superconnection*; its curvature, $\mathbb{A}^2$, turns out to be a T^k-family of differential operators on M with coefficients in $\Omega^*(T^k)$. Thus $\exp(-\mathbb{A}^2)$ is a T^k-family $\{K(\theta)\}_{\theta \in T^k}$ of *smoothing operators* on M with coefficients differential forms on T^k.

Remark. — The concept of superconnection is due to D. Quillen [107] who moreover suggested that it could be used in a heat kernel proof of the family index theorem. Quillen's heuristic arguments were rigorously developed by Bismut in [14].

8.2.2. The fiber-supertrace. Let $\Lambda_\theta^*(T^k)$ the Grassmann algebra of the cotangent space to T^k in θ. One can see more precisely that the

Schwartz kernel of $K(\theta)$ restricts to the diagonal $\Delta \leftrightarrow M$ in $M \times M$ as a section of the bundle $\Lambda^*_\theta(T^k) \otimes \mathrm{End}(\Lambda^{\mathrm{sign}}_{\mathbb{C}}(M) \otimes F_\theta)$ over M. Let str_θ denote the natural supertrace on $\mathrm{End}(\Lambda^{\mathrm{sign}}_{\mathbb{C}}(M) \otimes F_\theta)$; we can extend this supertrace to $\Lambda^*_\theta(T^k) \otimes \mathrm{End}(\Lambda^{\mathrm{sign}}_{\mathbb{C}}(M) \otimes F_\theta)$ by letting it act on the first factor as the identity. Thus

$$\mathrm{str}_\theta(K(\theta)|_\Delta) \in \Lambda^*_\theta(T^k) \otimes C^\infty(M).$$

We conclude that if $\{K(\theta)\}_{\theta \in T^k}$ denotes the family of Schwartz kernels associated to $\exp(-\mathbb{A}^2)$, then

$$\int_M \mathrm{str}_\theta K(\theta)|_\Delta \in \Lambda^*_\theta(T^k)$$

and as θ varies in T^k we obtain a differential form. *Summarizing* we can give the following

DEFINITION 8.4. — *The functional analytic fiber-supertrace* $\mathrm{STR}(\exp(-\mathbb{A}^2))$ *is the differential form on* T^k *defined by the equality*

$$\mathrm{STR}(\exp(-\mathbb{A}^2))(\theta) = \int_M \mathrm{str}_\theta K(\theta)|_\Delta.$$

8.2.3. Bismut's theorem. Consider the so-called rescaled superconnection $\mathbb{A}_s := s\{D^{\mathrm{sign}}_\theta\} + \nabla^{\mathcal{E}}$ for $s > 0$. Bismut's theorem, in this special case, states that

- for each $s > 0$ the differential form $\mathrm{STR}(\exp(-\mathbb{A}_s^2))$ is closed in $\Omega^*(T^k)$;
- for each $s > 0$ it represents the Chern character of the index bundle:
$$\mathrm{Ch}(\mathrm{Ind}\{D^{\mathrm{sign}}_\theta\}) = [\mathrm{STR}(\exp(-\mathbb{A}_s^2))] \quad \text{in} \quad H^*_{\mathrm{dR}}(T^k);$$
- the short-time limit can be computed, giving
$$\lim_{s \downarrow 0} \mathrm{STR}(\exp(-\mathbb{A}_s^2)) = \int_M L(M, \nabla^g) \wedge \omega.$$

The notion of superconnection can be given for any family of Dirac operators $\{D_b\}_{b \in B}$ acting on the sections of a vertical Clifford module E on a non-trivial fiber bundles $Z \to M \xrightarrow{\pi} B$. [6]

Besides the original article of Bismut, [14], the reader is also referred to [13], Chapter 9 and 10.

[6] It is an operator $\mathbb{A} : C^\infty(B, \mathcal{E}) \to C^\infty(B, \Lambda^*(B) \otimes \mathcal{E})$, with $\mathcal{E}_b = C^\infty(\pi^{-1}(b), E|_{\pi^{-1}(b)})$, which is odd with respect to the total grading defined by E and $\Lambda^*(B)$, satisfies Leibnitz

It is clear that if we wish to generalize Lustzig's approach to a noncommutative group Γ then we need to bring to the noncommutative context the notion of Chern character, defined on $K_*(C_r^*\Gamma)$, and prove a *noncommutative family index theorem*. In order to do so we need the notion of cyclic (co)homology.

8.3. Cyclic (co)homology.

Let A be a unital k-algebra over $k = \mathbb{R}$ or $k = \mathbb{C}$. The cyclic cohomology groups $HC^*(A)$ Connes [27] (see also Tsygan [118]) are the cohomology groups of the complex (C^n_λ, b) where C^n_λ denotes the space of $(n+1)$-linear functionals φ on A satisfying the condition:

$$\varphi(a^1, a^2, \dots, a^n, a^0) = (-1)^n \varphi(a^0, \dots, a^{n+1}), \quad \forall a^i \in A$$

and where b is the Hochschild coboundary map given by

$$(b\varphi)(a^0, \dots, a^{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, \dots, a^n).$$

Set $\overline{C^0_\lambda} = C^0_\lambda$ and, for any $n \in \mathbb{N}^*$, denote by $\overline{C^n_\lambda}$ the sub vector space of C^n_λ formed by the $(n+1)$-linear functionals φ such that $\varphi(a^0, a^1, \dots, a^n) = 0$ if $a^i = 1$ for some $i \in \{1, \dots, n\}$. $(\overline{C^n_\lambda}, b)$ is then a subcomplex of (C^n_λ, b) whose cohomology groups are called the *reduced cyclic cohomology groups* $\overline{HC^*}(A)$.

Of particular importance to us will be the cyclic cohomoly group $HC^*(\mathbb{C}\Gamma)$. Let $c \in H^k(\Gamma, \mathbb{C})$ be a group cocycle. Connes has associated to c

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rule and can be written as

$$\mathbb{A} = \{D_b\} + \nabla^{\mathcal{E}} + \sum_{j=2}^{k} \mathbb{A}_{[k]} \text{ with } \mathbb{A}_{[k]} : C^\infty(B, \mathcal{E}) \longrightarrow C^\infty(B, \Lambda^k(B) \otimes \mathcal{E}).$$

The first two results in Bismut's theorem are true for any superconnection; the short-time limit, on the other hand, only holds for a specific superconnection, nowadays called the *Bismut's superconnection*; its rescaled version can be written as

$$\mathbb{A}_s^{\text{Bismut}} = s\{D_b\} + \nabla^{\mathcal{E}} + \frac{1}{s}\mathbb{A}_{[2]} \tag{8.5}$$

with $\mathbb{A}_{[2]}$ an additional term involving the curvature of the fiber bundle $Z \to M \xrightarrow{\pi} B$. In particular, if the fiber bundle is trivial, as in the Lustzig's family, this additional term is zero.

a cyclic cocycle τ_c and thus a cyclic class $[\tau_c] \in HC^k(\mathbb{C}\Gamma)$: if $\gamma_0, \dots, \gamma_k \in \Gamma$ then set

$$\tau_c(\gamma_0, \dots, \gamma_k) = c(1_\Gamma, \gamma_0, \dots, \gamma_0 \cdots \gamma_{k-1}) \quad \text{if } \gamma_0 \cdots \gamma_k = 1_\Gamma,$$
$$\tau_c(\gamma_0, \dots, \gamma_k) = 0 \quad \text{if } \gamma_0 \cdots \gamma_k \neq 1_\Gamma.$$

If $k \geqslant 1$, then, using the fact that c is antisymmetric, one checks that τ_c is a reduced cyclic cocycle.

We can also introduce cyclic *homology*. Denote by $A^{\otimes, n+1}$ the tensor product over k of $n+1$ copies of A and consider the endomorphism t of $A^{\otimes, n+1}$ defined by:

$$t(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n t(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}).$$

Consider also the map $b : A^{\otimes, n+1} \to A^{\otimes, n}$ defined by:

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Then set $C_n^\lambda(A) = \frac{A^{\otimes, n+1}}{\operatorname{Im} \operatorname{Id} - t}$. The cyclic homology groups $HC_*(A)$ are then the homology groups of the complex $(C_n^\lambda(A), b)$. Next, denote by $\overline{C_n^\lambda}(A)$ the quotient of $C_n^\lambda(A)$ by the sub k-module generated by the tensor products $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ where $a_i = 0$ for some $i \in \{1, \dots, n\}$. Then the reduced cyclic homology groups $\overline{HC_*}(A)$ are defined to be the homology groups of the complex $(\overline{C_n^\lambda}(A), b)$.

8.4. Noncommutative de Rham homology and the Chern character.

We follow Karoubi [59]. Recall that k is $\mathbb{R}$ or $\mathbb{C}$. Let A be a unital k-algebra and consider a graded algebra

$$\Omega_*(A) = \Omega_0(A) \oplus \Omega_1(A) \oplus \Omega_2(A) \dots$$

with $\Omega_0(A) = A$, endowed with a k-linear derivation of degree 1, $d = d_j : \Omega_j(A) \to \Omega_{j+1}(A)$ satisfying $d^2 = 0$ and

$$d(\omega_j \cdot \omega_l) = d\omega_j \cdot \omega_l + (-1)^j \omega_j \cdot d\omega_l, \ \forall \omega_j \in \Omega_j(A), \ \omega_l \in \Omega_l(A).$$

Denote by $[\Omega_*(A), \Omega_*(A)]_t$ the k-module generated by the graded commutators $[\omega_j, \omega_l] = \omega_j \cdot \omega_l - (-1)^{jl} \omega_l \cdot \omega_j$ where $j + l = t$ and $\omega_j \in \Omega_j(A)$, $\omega_l \in \Omega_l(A)$. We then set:

$$\overline{\Omega_t(A)} = \frac{\Omega_t(A)}{[\Omega_*(A), \Omega_*(A)]_t}.$$

It is clear that the derivation d induces a $k-$linear differential, still denoted d, on the graded $k-$vector space $\overline{\Omega_t(A)}$, we then denote by $\overline{H}_*(A)$ the homology of this quotient complex and call it the non commutative de Rham homology of $\Omega_*(A)$.

Now let E be a finitely generated projective left $A-$module, a connection D on E is a $k-$linear homomorphism

$$D : E \to \Omega_1(A) \otimes_A E$$

satisfying Leibniz's rule

$$\forall (a, s) \in A \times E,\ D(a \cdot s) = da \otimes s + a \otimes D(s).$$

Set $\Omega_{\text{even}}(A) = \oplus_{k\in\mathbb{N}}\Omega_{2k}(A)$. One then checks that D^2 extends a left linear $\Omega_{\text{even}}(A)-$endomorphism of $\Omega_{\text{even}}(A) \otimes_A E$ sending $\Omega_{2k}(A) \otimes_A E$ into $\Omega_{2k+2}(A) \otimes_A E$ for each $k \in \mathbb{N}$. Since E is assumed to be finitely generated and projective, there is a natural trace map:

$$\mathrm{TR} : \Omega_{\text{even}}(A) \otimes_A E \to \frac{\Omega_{\text{even}}(A)}{[\Omega_{\text{even}}(A), \Omega_{\text{even}}(A)]} \to \overline{\Omega_{\text{even}}}(A)$$

where the last $\to$ is the obvious one. The Chern character is then defined by

$$\mathrm{Ch} : K_0(A) \to \overline{H}_{\text{even}}(A)$$
$$E \to \mathrm{TR}e^{-D^2}.$$

It is indeed a theorem (see Section 1 of [59]) that $\mathrm{TR}e^{-D^2}$ defines a cycle in $\overline{H}_{\text{even}}(A)$ which does not depend on the choice of D.

8.5. Cyclic (co)homology and noncommutative de Rham homology.

We recall that Connes has constructed an operator B from $\overline{HC_*}(A)$ (resp. $HC_*(A)$) to the Hochschild homology group $H_{*+1}(A, A)$ where B is a non commutative analogue of the de Rham exterior derivative. In Section 2 of [59] the following is proved. For $* > 0$, $\overline{H}_*(A)$ is isomorphic to the kernel of B acting on $\overline{HC_*}(A)$, while $\overline{H}_0(A)$ is isomorphic to the kernel of B acting on $HC_0(A)$. We shall not give the details here but only retain the information that, for $* > 0$, there is a pairing between noncommutative de Rham *homology* $\overline{H}_*(A)$ and the reduced cyclic *cohomology* group $\overline{HC^*}(A)$. For $* = 0$ there is is a pairing between noncommutative de Rham *homology* $\overline{H}_0(A)$ and cyclic *cohomology* $HC^0(A)$.

8.6. Topological cyclic (co)homology.

Now let A be a unital Fréchet locally convex k-algebra; i.e. a Fréchet locally convex topological vector space for which the product is continuous. The *topological* cyclic cohomology groups $HC^n(A)$ are defined as above but by considering only continuous linear $(n+1)-$linear functionals. Similarly, the topological cyclic homology groups $HC_n(A)$ are defined as above but considering completed projective tensor products. Moreover, one can define a completion $\widehat{\Omega}_*(A)$ of $\Omega_*(A)$ which is a Fréchet differential graded algebra. The noncommutative topological de Rham homology $\widehat{H_*}(A)$ is defined as the homology of the complex

$$\left(\widehat{\Omega}_*(A)/\overline{[\widehat{\Omega}_*(A),\widehat{\Omega}_*(A)]},\ d\right);$$

it pairs with the topological cyclic cohomology $HC^*(A)$. In fact, if $* > 0$, it pairs with the reduced topological cyclic cohomology.

8.7. Smooth subalgebras of C^*-algebras.

In general the *topological* cyclic homology of a C^*-algebra is too poor. For instance on a smooth manifold M

$$HC^{2p}(C^0(M)) \simeq HC^0(C^0(M)) \text{ and } HC^{2p+1}(C^0(M)) = 0 \ \ \forall p \in \mathbb{N}.$$

In fact the right algebra to consider in order to recover the (co)homology of a smooth manifold M is the algebra of *smooth functions* on M, as there are many more interesting cyclic cocycles on $C^\infty(M)$ than on $C^0(M)$ [7]. In order to further clarify this point let us recall that Connes has defined a periodicity operator

$$S : HC^k(A) \to HC^{k+2}(A),$$

and introduced the two periodic cyclic cohomology groups

$$PHC^{\text{even}}(A) = \lim_{+\infty \leftarrow S} HC^{2k}(A), \quad PHC^{\text{odd}}(A) = \lim_{+\infty \leftarrow S} HC^{2k+1}(A).$$

The relationship between the homology of M and cyclic cohomology is then the following:

$$PHC^{\text{even}}(C^\infty(M)) = \oplus_{k\in\mathbb{N}} H_{2k}(M;\mathbb{C}),$$
$$PHC^{\text{odd}}(C^\infty(M)) = \oplus_{k\in\mathbb{N}} H_{2k+1}(M;\mathbb{C}).$$

[7] For example, the following interesting $2-$cyclic cocycle on $C^\infty(S^2)$, $(a^0,a^1,a^2) \to \int_{S^2} a^0 da^1 \wedge da^2$ does not extend to $C^0(S^2)$.

Notice now that $C^\infty(M)$ is a *dense subalgebra* of $C^0(M)$ which is furthermore *closed under holomorphic functional calculus.* In general, if A is a C^*-algebra and $\mathcal{B} \subset A$ is a (Fréchet locally convex) dense subalgebra closed under holomorphic functional calculus, then $K_*(A) \simeq K_*(\mathcal{B})$; such a subalgebra is usually referred to as a *smooth subalgebra.* Thus, for example, $K_*((C^\infty(M)) \simeq K_*(C^0(M))$. So, considering a *smooth subalgebra* $\mathcal{B}$ of a C^*-algebra A allows us on the one hand to leave the K-theory unchanged and, on the other hand, to consider an interesting *topological* cyclic cohomology and thus, from the previous subsection, an *interesting Chern character homomorphism*:

$$\mathrm{Ch} : K_0(\mathcal{B}) \to \widehat{H_*}(\mathcal{B}).$$

8.8. The smoothing of the index class.

On the basis of our discussion so far, it is clear that in order to apply an interesting Chern character to our index class $\mathrm{Ind}(D^{\mathrm{sign}}_{(M,r)})$, we need to *fix a subalgebra $\mathcal{B}^\infty$ of $C^*_r\Gamma$ which is dense and closed under holomorphic functional calculus.* As we have explained, it is only by fixing such a subalgebra that we can hope to land, via the Chern character, into an interesting noncommutative de Rham homology.

Such an algebra does exist and it is called the Connes-Moscovici algebra. Let us see the definition. Fix a word metric $\|\cdot\|$ on Γ. Define an unbounded operator D on $\ell^2(\Gamma)$ by setting $D(e_\gamma) = \|\gamma\| e_\gamma$ where $(e_\gamma)_{\gamma\in\Gamma}$ denotes the standard orthonormal basis of $\ell^2(\Gamma)$. Then consider the unbounded derivation $\delta(T) = [D, T]$ on $B(\ell^2(\Gamma))$ and set

$$\mathcal{B}^\infty = \{T \in C^*_r(\Gamma) / \, \forall k \in \mathbb{N},\ \delta^k(T) \in B(\ell^2(\Gamma))\}.$$

One can prove that $\mathcal{B}^\infty$ is dense in $C^*_r\Gamma$ and closed under holomorphic functional calculus. Thus $K_*(C^*_r\Gamma) \simeq K_*(\mathcal{B}^\infty)$; the image of $\mathrm{Ind}(D^{\mathrm{sign}}_{(M,r)}) \in K_*(C^*_r\Gamma))$ in $K_*(\mathcal{B}^\infty)$ under this isomorphism should be thought of as a "smoothing" of the index class, since in the commutative context it is nothing but the passage from a *continuous* index bundle for the Lustzig's family to a *smooth* index bundle. Since $\mathcal{B}^\infty$ is a smooth subalgebra one may define $\widehat{\Omega}_*(\mathcal{B}^\infty)$ and $\widehat{H}_*(\mathcal{B}^\infty)$ as above.

The smoothing of the index class can in fact be achieved directly and explicitly. We wish to explain this point, for it will be important in the next subsection. We do it directly for the signature operator but it is clear that what we explain will hold for any Dirac-type operator. Let $\mathcal{B}^\infty$, $\mathbb{C}\Gamma \subset \mathcal{B}^\infty \subset C^*_r\Gamma$, be *any* smooth subalgebra of $C^*_r\Gamma$. Thus $\mathcal{B}^\infty$ is dense and holomorphically closed in $C^*_r\Gamma$. Consider the flat $\mathcal{B}^\infty$-bundle $\mathcal{V}^\infty = \mathcal{B}^\infty \times_\Gamma \widetilde{M} \to M$ and set

$$\mathcal{E}^\infty := \mathcal{V}^\infty \otimes_{\mathbb{C}} \Lambda^{\mathrm{sign}}_{\mathbb{C}}(M), \quad \mathcal{E}^{\infty,\pm} := \mathcal{V}^\infty \otimes_{\mathbb{C}} \Lambda^{\mathrm{sign},\pm}_{\mathbb{C}}(M).$$

Proceeding as in subsection 7.3 we see that the signature operator on M defines in a natural way an odd $\mathcal{B}^\infty$-linear signature operator

$$D^{\mathrm{sign},\infty}_{(M,r)} : C^\infty(M,\mathcal{E}^\infty) \to C^\infty(M,\mathcal{E}^\infty).$$

For simplicity, we keep the notation $D^{\mathrm{sign}}_{(M,r)}$ for this operator. It is possible to develop a $\mathcal{B}^\infty$-pseudodifferential calculus $\Psi^*_{\mathcal{B}^\infty}(M,\mathcal{E}^\infty)$ and construct a parametrix for $\mathcal{D}^{\mathrm{sign}}_{(M,r)}$ with rests $\Psi^{-\infty}_{\mathcal{B}^\infty}(M,\mathcal{E}^\infty)$. Starting from a $\mathcal{B}^\infty$-parametrix one can prove a decomposition theorem analogous to the one appearing in (7.3); thus

(8.6)

$$C^\infty(M,\mathcal{E}^{\infty,+}) = \mathcal{I}_+(\infty)\oplus\mathcal{I}_+^\perp(\infty),\ C^\infty(M,\mathcal{E}^{\infty,-}) = \mathcal{I}_-(\infty)\oplus\mathcal{D}^+_{(M,r)}(\mathcal{I}_+^\perp(\infty)),$$

with $\mathcal{I}_+(\infty)$ and $\mathcal{I}_-(\infty)$ *finitely generated projective* $\mathcal{B}^\infty$-modules and $\mathcal{D}^{\mathrm{sign},+}_{(M,r)}$ inducing an isomorphism (in the Fréchet topology) between $\mathcal{I}_+^\perp(\infty)$ and $\mathcal{D}^+_{(M,r)}(\mathcal{I}_+^\perp(\infty))$. The proof of this $\mathcal{B}^\infty$-decomposition theorem rests ultimately on the fact that $\mathcal{B}^\infty$ is dense and closed under holomorphic functional calculus in $C^*_r\Gamma$. For the proof see Leichtnam-Piazza [74], Appendix A and also Lott [84] Section 6. Summarizing, the index class can be defined directly in $\mathcal{B}^\infty$:

$$\mathrm{Ind}(\mathcal{D}^+_{(M,r)}) = [\mathcal{I}_+(\infty)] - [\mathcal{I}_-(\infty)] \in K_0(\mathcal{B}^\infty). \tag{8.7}$$

8.9. The higher index theorem of Connes-Moscovici (following Lott).

One can prove that the heat operator associated to the Dirac laplacian on $\widetilde{M}$ defines a heat operator $\exp\big(-\big(s\mathcal{D}^{\mathrm{sign}}_{(M,r)}\big)^2\big)$ which is a $\mathcal{B}^\infty$-smoothing operator, i.e. $\exp\big(-\big(s\mathcal{D}^{\mathrm{sign}}_{(M,r)}\big)^2\big) \in \Psi^{-\infty}_{\mathcal{B}^\infty}$. Inspired by Bismut's heat-kernel proof of the family index theorem, Lott has defined in [82] a certain noncommutative connection on $\mathcal{E}^\infty \equiv \mathcal{V}^\infty \otimes_{\mathbb{C}} \Lambda^{\mathrm{sign}}_{\mathbb{C}}(M)$:

$$\nabla : C^\infty(M,\mathcal{E}^\infty) \to C^\infty(M,\widehat{\Omega}_1(\mathcal{B}^\infty)\otimes_{\mathcal{B}^\infty}\mathcal{E}^\infty). \tag{8.8}$$

This is the analogue, in the nonabelian case, of (8.3). He has then considered the rescaled superconnection $\mathbb{A}_s := s\mathcal{D}^{\text{sign}}_{(M,r)} + \nabla$ and, using Duhamel expansion, the heat operator

$$e^{-\mathbb{A}_s^2} : C^\infty(M,\mathcal{E}^\infty) \to C^\infty(M, \widehat{\Omega}_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \mathcal{E}^\infty).$$

For any real $s > 0$, this is, in a sense that can be made precise, a $\mathcal{B}^\infty$-smoothing operator with coefficients in $\widehat{\Omega}_*(\mathcal{B}^\infty)$. The restriction of the superconnection heat kernel $\mathcal{K}(e^{-\mathbb{A}_s^2})$ to the diagonal $\Delta \leftrightarrow M$ in $M \times M$ is an element in

$$\widehat{\Omega}_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} C^\infty(M, \mathcal{V}^\infty \otimes_{\mathbb{C}} \text{End}E);$$

taking the vector bundle supertrace str_E we get a supertrace

$$\text{STR}(e^{-\mathbb{A}_s^2}) := \int_M \text{str}_E \mathcal{K}(e^{-\mathbb{A}_s^2})|_\Delta d\text{vol}_g$$

with values in $\widehat{\Omega}(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}(\mathcal{B}^\infty), \widehat{\Omega}(\mathcal{B}^\infty)]}$.

Notice that since the algebra of non commutative differential forms $\widehat{\Omega}_*(\mathcal{B}^\infty)$ is not commutative, the super trace STR must take values in the quotient space

$$\widehat{\Omega}_*(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}_*(\mathcal{B}^\infty)]}$$

(i.e. modulo the closure of the space of graded commutators; we take the closure so as to ensure that the quotient space is Fréchet). Using Getzler's rescaling [38] and adapting to the noncommutative context Bismut's proof of the family index theorem, Lott proves in [82] that

- the noncommutative differential form $\text{STR}(e^{-\mathbb{A}_s^2})$ is closed
- its homology class is equal to the Chern character of the index:
$$\text{Ch}\,\text{Ind}(D^{\text{sign}}_{(M,r)}) = [\text{STR}e^{-\mathbb{A}_s^2}] \quad \text{in} \quad \widehat{H}_*(\mathcal{B}^\infty).$$
- there exists a certain closed biform $\omega_{(M,r)} \in \Omega^*(M) \otimes \widehat{\Omega}_*(\mathcal{B}^\infty)$ such that
$$\lim_{s\downarrow 0} \text{STR}e^{-\mathbb{A}_s^2} = \int_M L(M,\nabla^g) \wedge \omega_{(M,r)}$$
with the limit taking place in $\widehat{\Omega}(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}(\mathcal{B}^\infty), \widehat{\Omega}(\mathcal{B}^\infty)]}$.

In this way, we have explained how Lott has proved *the higher index theorem on Galois covering*:

$$\text{(8.9)} \qquad \text{Ch}\,\text{Ind}(\mathcal{D}^{\text{sign}}_{(M,r)}) = \left[\int_M L(M,\nabla^g) \wedge \omega_{(M,r)}\right] \in \widehat{H}_*(\mathcal{B}^\infty)$$

In fact, one can prove that $\omega_{(M,r)}$ is an element in $\Omega^*(M) \otimes \Omega_*(\mathbb{C}\Gamma)$; however, we do point out that the equality (8.9) only makes sense in $\mathcal{B}^\infty$.

8.10. The Novikov conjecture for hyperbolic groups.

Let $[c] \in H^{\ell}(B\Gamma, \mathbb{C}) \equiv H^{\ell}(\Gamma, \mathbb{C})$ and let $[\tau_c] \in HC^{\ell}(\mathbb{C}\Gamma)$ the corresponding cyclic cocycle. Lott has also proved [82] that, in general, there exists a nonzero constant $C(\ell)$ such that

$$\frac{1}{C(\ell)} < \Big[\int_M L(M) \wedge \omega_{(M,r)}\Big]; [\tau_c] >= \int_M L(M) \wedge r^*(c)$$

where on the left-hand-side the pairing between noncommutative de Rham *homology* and cyclic *cohomology* has been used.

By formula (8.9), this means that if $[\tau_c] \in HC^{\ell}(\mathbb{C}\Gamma)$ *extends* to $HC^{\ell}(\mathcal{B}^{\infty})$ then

$$\begin{aligned} \frac{1}{C(\ell)} < \operatorname{Ch}\operatorname{Ind}(\mathcal{D}^{\text{sign}}_{(M,r)}), [\tau_c] > &= \frac{1}{C(\ell)} < \Big[\int_M L(M, \nabla^g) \wedge \omega_{(M,r)}\Big], [\tau_c] > \\ &= \int_M L(M) \wedge r^*(c). \end{aligned} \tag{8.10}$$

The equality of the first and last term is due to Connes and Moscovici and it is known as the *Connes-Moscovici higher index theorem on Galois coverings*. We anticipate that the extra information given by Lott's heat-kernel proof will be crucial on manifolds with boundary. Thus, for cyclic cocycles that *extends from* $HC^*(\mathbb{C}\Gamma)$ *to* $HC^*(\mathcal{B}^{\infty})$ we have expressed the higher signatures in terms of the index class:

$$\int_M L(M) \wedge r^*(c) = \frac{1}{C(\ell)} < \operatorname{Ch}\operatorname{Ind}(\mathcal{D}^{\text{sign}}_{(M,r)}), [\tau_c] > .$$

Since the index class is a homotopy invariant, we conclude that the Novikov conjecture is established for all those groups having the extension property for all the cocycles τ_c. Connes and Moscovici have shown that Gromov hyperbolic groups do satisfy this fundamental property; their proof exploits results by Haagerup, de la Harpe and Jolissaint. We shall not give here the definition of Gromov hyperbolic group but refer the reader instead to Gromov [43], Ghys [39], Connes-Moscovici [29] and Connes [28]. Basic examples of hyperbolic groups are provided by fundamental groups of a compact connected Riemann surfaces of genus $g > 1$ or more generally by fundamental groups of compact, negatively curved manifolds. Summarizing:

THEOREM 8.11 (Connes-Moscovici [29]). — *If Γ is Gromov hyperbolic, then the Novikov conjecture is true.*

In fact, Connes and Moscovici even proved that the Strong Novikov conjecture holds for Gromov hyperbolic groups. It should be remarked that there are now K-theoretic proofs of this result: after the work of Connes-Moscovici appeared, Ogle has proved [104] by K-theoretic methods that $\mu_{\mathbb{R}} : K_*(B\Gamma) \otimes \mathbb{R} \to K_*(C^*_r\Gamma) \otimes \mathbb{R}$ is *injective* for Gromov hyperbolic groups. In fact recently Mineyev and Yu have proved that the Baum-Connes conjecture holds for Gromov-hyperbolic groups, see [97].

8.11. Groups having the extension property.

We can slightly generalize the content of the previous subsection as follows. Let Γ be a finitely generated group. We shall say that Γ *has the extension property* if there exists a subalgebra $\mathcal{B}^\infty$ of $C^*_r\Gamma$, $\mathbb{C}\Gamma \subset \mathcal{B}^\infty \subset C^*_r\Gamma$, with the following 2 properties:

- $\mathcal{B}^\infty$ is dense and holomorphically closed in $C^*_r\Gamma$.
- Each class $[c] \in H^*(\Gamma;\mathbb{C})$ has a cocycle representative whose corresponding cyclic cocycle $\tau_c \in ZC^*(\mathbb{C}\Gamma)$ extends to a continuous cyclic cocycle on $\mathcal{B}^\infty$.

Examples of groups satisfying the extension property are Gromov hyperbolic groups and also virtually nilpotent groups, see [58]. For this latter example it suffices to recall that by a result of Gromov a group Γ is virtually nilpotent if and only if is of polynomial growth with respect to a (and thus any) word metric; the smooth subalgebra for such a group is simply given by

$$\mathcal{B}^\infty := \left\{ f : \Gamma \to \mathbb{C} \mid \forall N \in \mathbb{N},\ \sup_{\gamma\in\Gamma}(1 + \|\gamma\|)^N |f(\gamma)| < \infty \right\}.$$

The following theorem, again due to Connes and Moscovici, is the main result of this entire section 8:

THEOREM 8.12. — *If Γ has the extension property, then the Strong Novikov conjecture is true.*

9. The cut-and-paste problem for higher signatures.

Let M and N be two oriented compact manifolds with boundary and let $\phi, \psi : \partial M \to \partial N$ be orientation preserving diffeomorphisms. We consider the closed oriented manifolds

$$M \cup_\phi N^- \quad \text{and} \quad M \cup_\psi N^-;$$

we shall sometime use the notation $X_\phi := M \cup_\phi N^-$ and $X_\psi := M \cup_\psi N^-$. Let $r : M \cup_\phi N^- \to B\Gamma$, $s : M \cup_\psi N^- \to B\Gamma$ be reference maps and assume that the two coverings are cut-and-paste equivalent, see definition 5.2.

The cut-and-paste problem for higher signatures can be then stated as follows: *for any $c \in H^*(B\Gamma, \mathbb{Q})$, compare the two higher signatures:*

$$\int_{M \cup_\phi N^-} L(M \cup_\phi N^-) \cup r^*(c), \quad \int_{M \cup_\psi N^-} L(M \cup_\psi N^-) \cup s^*(c).$$

The problem (raised by J. Lott and S. Weinberger, see [86], Section 4.1 and [122]) is then to determine which higher signatures of closed manifolds are cut and paste invariant; we refer to [86], Section 4.1 for further discussion.

As remarked by Lott in [86], Section 4.1, it is implicitly established in [61], [103], that, in general, higher signatures of closed manifolds are not cut and paste invariant. We shall describe below a recent counterexample constructed in [73], Example 1.10 to which we refer for the details.

Example. — Let $s : \mathbb{C}P^2 \times S^1 \to B\mathbb{Z} = S^1$ be the reference map given by $s(z, e^{i\theta}) = e^{i\theta}$. Then there exists a compact oriented 4−dimensional manifold F endowed with an orientation preserving diffeomorphism h such that $(\mathbb{C}P^2 \times S^1, s)$ is cobordant to $M((F,h),T)$ where $M(F,h)$ denotes the mapping torus obtained from $[0,1] \times F$ by identifying $(0,x)$ with $(1, h(x)$. It is shown by M. Kreck in [73] that F may be choosen of the form $(S^1 \times S^3)\#(\mathbb{C}P^2 \times \overline{\mathbb{C}P}^2)\#m(S^2 \times S^2)$ for a suitable $m \in \mathbb{N}$. The reference map $T : M(F,h) \to B\mathbb{Z}$ induces a map $r : F \to B\mathbb{Z}$ such that r and $r \circ h$ are homotopic as (continuous) maps from F to $B\mathbb{Z}$. M. Kreck has shown that one may assume that $r : F \to B\mathbb{Z}$ is two-connected. Moreover there exists a manifold with boundary W such that $\partial W = F$ and there are two maps R, R' from W to $B\mathbb{Z}$ such that $r = R_{|\partial W}$ and $r \circ h = R'_{\partial W}$. Therefore, $(M(F,h),T)$ (and thus $(\mathbb{C}P^2 \times S^1, s)$) is cobordant to:

$$(W \cup_{\mathrm{id}} W, R \cup R) - (W \cup_h W, R \cup R').$$

Thus, $(\mathbb{C}P^2 \times S^1 \times S^1, s \times \mathrm{id}_{S^1})$ is cobordant to

$$((W \cup_{\mathrm{id}} W) \times S^1, (R \cup R) \times \mathrm{id}_{S^1}) - ((W \cup_h W) \times S^1, (R \cup R') \times \mathrm{id}_{S^1})$$

where $s \times \mathrm{id}_{S^1} : \mathbb{C}P^2 \times S^1 \times S^1 \to B\mathbb{Z} \times B\mathbb{Z}$.

Now, let ω_1 denote the fundamental class of S^1. Then, since the signature of of $\mathbb{C}P^2$ is not zero, one checks immediately that

$$\int_{\mathbb{C}P^2\times S^1\times S^1} L(\mathbb{C}P^2 \times S^1 \times S^1) \wedge (s \times \mathrm{id}_{S^1})^*(\omega_1 \times \omega_1) \neq 0.$$

Then, by cobordism invariance, it is clear that $((W \cup_{id} W) \times S^1, (R \cup R) \times \mathrm{id}_{S^1})$ and $((W \cup_h W) \times S^1, (R \cup R') \times \mathrm{id}_{S^1})$ do not have the same higher signatures. □

Despite the negative result explained in the previous example, we can ask whether we can give sufficient conditions (on Γ and on the two coverings defined by $r|_{\partial M}$ and $s|_{\partial M}$) ensuring that the higher signatures are indeed cut-and-paste invariant. This would answer, at least partially, Question 2 in subsection 6. Now, for the lower signature $\int_M L(M)$ we have explained 3 different ways for treating the cut-and-paste problem; the first method makes use of the Atiyah-Patodi-Singer index formula for the signature of a manifold with boundary , the second method employs a purely topological argument, whereas the third method uses a spectral flow argument (based, ultimately, on a gluing formula for the index and a variational formula for the Atiyah-Patodi-Singer index).

As we shall now see, these 3 methods can be pursued in the higher case too. We shall begin by the first method and in fact explain a general theory of *higher signatures on manifolds with boundary*, thereby answering simultaneously to Question 2 and Question 3 of section 6.

10. Higher signatures on manifolds with boundary.

10.1. Introduction and main strategy for the definition.

10.1.1. Introduction. Let M be an oriented manifold *with boundary* and let $r : M \to B\Gamma$ be a classifying map. Let $[c] \in H^*(B\Gamma, \mathbb{R})$. Since the expression $\int_M L(M, \nabla^g) \cup r^*[c]$ depends on the choice of the metric g, it is not obvious how to define the higher signature $\mathrm{sign}(M, r; [c])$ associated to r and $[c] \in H^*(B\Gamma, \mathbb{R})$. Still, Theorem 2.8 shows that the difference

$$\int_M L(M, \nabla^g) - \frac{1}{2}\eta(D^{\mathrm{sign}}_{(\partial M, g_\partial)})$$

is an oriented homotopy invariant of the pair $(M, \partial M)$; in other words by *subtracting* a suitable *boundary correction term* to the metric-dependent integral $\int_M L(M, \nabla^g)$ we have produced a homotopy invariant of the pair $(M, \partial M)$. In the higher case, having observed that on a manifold with boundary M the higher analogue of $\int_M L(M, \nabla^g)$ is the metric-dependent integral

$$\int_M L(M, \nabla^g) \wedge \omega_{(M,r)} \in \widehat{\Omega}(\mathcal{B}^\infty) \tag{10.1}$$

appearing in (8.10), we ask ourselves the following:

Fundamental question: *which boundary correction term should we subtract to (10.1) in order to obtain a homotopy invariant noncommutative de Rham class in $\widehat{H}_*(\mathcal{B}^\infty)$?*

To have a feeling on the strategy we shall follow, let us recall how Lustzig managed to prove the Novikov conjecture for $\Gamma = \mathbb{Z}^k$ in the closed case. The proof was in four steps:

(i) Define a suitable family of twisted signature operators $\{D^{\rm sign}_\theta\}_{\theta \in T^k}$, $T^k := \mathrm{Hom}(\mathbb{Z}^k, U(1))$, and its index class in $K^0(T^k)$.

(ii) Prove the homotopy invariance of the index class $\mathrm{Ind}(\{D^{\rm sign}_\theta\}_{\theta \in T^k}) \in K^0(T^k)$.

(iii) Apply the family index formula, thus computing the Chern character of the index class as $[\int_M L(M) \wedge \omega] \in H^*(T^k, \mathbb{R})$, $\omega \in \Omega^*(M \times T^k)$.

(iv) Express the higher signatures in terms of the pairing between this cohomology class and a homology class $[\tau_c] \in H_*(T^k, \mathbb{R})$ naturally defined by $[c] \in H^*(B\mathbb{Z}^k, \mathbb{R}) \equiv H^*((T^k)^*, \mathbb{R})$.

10.1.2. Strategy in the commutative case. Let now M have a boundary, $\partial M \neq \emptyset$. We assume again $\pi_1(M) = \mathbb{Z}^k$. The Lustzig's family $\{D^{\rm sign}_\theta\}_{\theta \in T^k}$ is still perfectly defined and is a family of Dirac-type operators on the manifold with boundary M. Keeping in mind Lustzig's approach and our discussion in the case of a single manifold (Theorem 2.8), we would like to

(i) define a Atiyah-Patodi-Singer ($\equiv$ APS) *index class*, in $K^*(T^k)$, for the Lustzig's family.

(ii) establish the homotopy invariance of this index class.

(iii) prove a *family index formula* for its Chern character in $H^*(T^k, \mathbb{R})$; this formula will involve the *boundary correction term* we alluded to in the fundamental question raised above

(iv) *define* the higher signatures by coupling the Chern character with $\tau_c \in H_*(T^k, \mathbb{R})$.

10.1.3. Strategy in the noncommutative case. Let us pass to the noncommutative case and consider $(M, r : M \to B\Gamma)$. Keeping in mind the analogy between higher index theory and family index theory, we would like to

(i) define a APS *index class* associated to $D^{\rm sign}_{(M,r)}$; this class will live in $K_*(\mathcal{B}^\infty) = K_*(C^*_\Gamma)$.

(ii) establish the homotopy invariance of this index class.

(iii) prove a *higher index formula* for its Chern character in $\widehat{H}_*(\mathcal{B}^\infty)$; this formula will have to involve the *boundary correction term* we alluded to in the fundamental question.

(iv) *define* the higher signatures $\mathrm{sign}(M, r; [c])$ *for a group satisfying the extension property*, by coupling the Chern character in $\widehat{H}_*(\mathcal{B}^\infty)$ with the *extended* cyclic cocycle $\tau_c \in HC^*(\mathcal{B}^\infty)$ defined by $[c] \in H^*(\Gamma, \mathbb{C}) \equiv H^*(B\Gamma, \mathbb{C})$.

The details of this program, which was conceived by Lott in [83], shall now be explained. We begin once again by the commutative case.

10.2. The Bismut-Cheeger eta form.

Let M be an even dimensional oriented manifold with boundary with $\pi_1(M) = \mathbb{Z}^k$, as in the previous subsection. Consider an odd Dirac-type operator $D : C^\infty(M, E) \to C^\infty(M, E)$ acting on the sections of a $\mathbb{Z}_2$-graded Clifford bundle. For each $\theta \in T^k$, one has a twisted operator D_θ acting on $C^\infty(M; E \otimes F_\theta)$ where F_θ is the flat complex line bundle of M associated with $\theta \in T^k := \mathrm{Hom}(\mathbb{Z}^k, U(1))$ (see Section 7.3.3). Let us consider the family $\mathcal{D} := \{D_\theta\}_{\theta \in T^k}$ on M parametrized by the torus T^k . From the variational formula for the Atiyah-Patodi-Singer index, see (3.7), one realizes immediately that the family of Atiyah-Patodi-Singer boundary value problems associated to the family of Dirac-type operators $\mathcal{D} := \{D_\theta\}_{\theta \in T^k}$ *is not continuous* in $\theta \in T^k$, unless the boundary family $\mathcal{D}_\partial := \{D_{\theta, \partial M}\}_{\theta \in T^k}$ is invertible (notice that in the latter case there would not be any spectral flow). Under this additional assumption Bismut and Cheeger defined an index class $\mathrm{Ind}(\mathcal{D}, \Pi_\geqslant) \in K^0(T^k)$ and proved a family index formula for its Chern character in $H^{\mathrm{even}}(T^k,\mathbb{R})$:

(10.2)

$$\mathrm{Ch}(\mathrm{Ind}(\mathcal{D},\Pi_{\geqslant}))=\Big[\int_M \widehat{A}(M,\nabla^g)\mathrm{Ch}'(E,\nabla^E)\wedge\omega-\frac{1}{2}\widetilde{\eta}(\mathcal{D}_\partial)\Big]\in H^{\mathrm{even}}(T^k,\mathbb{R}).$$

In this formula ω is the bi-form in $\Omega^*(M\times T^k)$ we met in subsection 8.1, whereas $\widetilde{\eta}(\mathcal{D}_\partial)\in\Omega^*(T^k)$ is the Bismut-Cheeger *eta form* associated to $\mathcal{D}_\partial$. This is our *boundary correction term.* The eta form is defined as

$$\widetilde{\eta}(\mathcal{D}_\partial)=\frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{STR}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}_s}{ds}e^{-\mathbb{B}_s^2}\right)ds\in\Omega^{\mathrm{even}}(T^k)\tag{10.3}$$

with $\mathbb{B}_s$ the superconnection induced on the boundary by the rescaled Bismut superconnection $\mathbb{A}_s$. The supertrace appearing in this formula is the *odd* fiber-supertrace on the odd-dimensional boundary; it is defined using the isomorphism $E|_{\partial M}\simeq E^+|_{\partial M}\otimes\mathrm{Cl}(1)$, with $\mathrm{Cl}(1)$ denoting the complex Clifford algebra generated by 1 and σ.

As an example, the 0-degree part of this differential form, computed at $\theta\in T^k$, is simply the eta invariant of $D_{\theta,\partial}$:

$$\widetilde{\eta}(\mathcal{D}_\partial)_{[0]}(\theta)=\eta(D_{\theta,\partial}).$$

Notice that the operator $\frac{d\mathbb{B}_s}{ds}e^{-\mathbb{B}_s^2}$ is again a smoothing operator with differential form coefficients. The convergence of the s-integral in (10.3) near zero is non-trivial and requires Bismut's local index theorem for families. The convergence at ∞ depends heavily on the assumption that the family is invertible. The Bismut-Cheeger eta form can be defined for any *invertible* family of Dirac-type operators, not necessarily arising as a boundary family. It is more generally defined for any *invertible* family $\{D_b\}_{b\in B}$ acting on the sections of a vertical Clifford bundle on a fiber bundle $Z\to X\to B$ with odd dimensional fiber.

10.3. Lott's higher eta invariant in the invertible case.

We now pass to the noncommutative case. Let $(N, r: N\to B\Gamma)$ be closed and odd dimensional (for example the boundary of an even dimensional manifold with boundary). We fix a Riemannian metric g on N and consider a Dirac-type operator D on N acting between the sections of an ungraded Clifford module E. We consider $E\otimes\mathrm{Cl}(1)\simeq E\oplus E$, with $\mathrm{Cl}(1)$ the complex Clifford algebra generated by 1 and σ. Let $\mathcal{D}_{(N,r)}: C^\infty(N,E\otimes\mathcal{V})\to C^\infty(N,E\otimes\mathcal{V})$ be the associated $C^*_r(\Gamma)$-linear operator, with $\mathcal{V}=C^*_r\Gamma\times_\Gamma r^*E\Gamma$. Fix now a *smooth subalgebra*

$\mathcal{B}^\infty \subset C^*_r\Gamma$; for example the Connes-Moscovici algebra. We still denote by $\mathcal{D}_{(N,r)}$ the operator acting on $C^\infty(N, E \otimes \mathcal{V}^\infty)$, with $\mathcal{V}^\infty = \mathcal{B}^\infty \times_\Gamma r^*E\Gamma$. The rescaled Lott superconnection in this odd dimentional context has the form $\mathbb{B}_s = s\sigma\mathcal{D}_{(N,r)} + \nabla$. The Schwartz kernel $\mathcal{K}(t)$ of the operator $\sigma\mathcal{D}_{(N,r)} \exp(-(\nabla + t\sigma\mathcal{D}_{(N,r)})^2$, which is a smoothing operator with coefficients in $\widehat{\Omega}_*(\mathcal{B}^\infty)$, can be restricted to the diagonal in $N \times N$, giving $\mathcal{K}(t)|_\Delta$, an element in

$$\widehat{\Omega}_*(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} C^\infty(N, \mathcal{V}^\infty \otimes_{\mathbb{C}} \mathrm{End}(E \otimes \mathrm{Cl}(1)))$$

where we identify $N \leftrightarrow \Delta$. As in the previous section there is an odd-supertrace $\mathrm{Str}_{\mathrm{Cl}(1)}$ acting on the endomorphisms of $E \otimes \mathrm{Cl}(1)$; using this vector-bundle odd supertrace we can define the odd supertrace $\mathrm{STR}_{\mathrm{Cl}(1)}$ of the smoothing operator $\sigma\mathcal{D}_{(N,r)} \exp(-(\nabla + t\sigma\mathcal{D}_{(N,r)})^2$; this is the non-commutative differential form defined by

$$\mathrm{STR}_{\mathrm{Cl}(1)}\left[\sigma\mathcal{D}_{(N,r)} \exp(-(\nabla + t\sigma\mathcal{D}_{(N,r)})^2)\right] := \int_N \mathrm{Str}_{\mathrm{Cl}(1)}\mathcal{K}(t)|_\Delta \, d\mathrm{vol}_g.$$

Once again, since $\widehat{\Omega}_*(\mathcal{B}^\infty)$ is not commutative, the odd super trace $\mathrm{STR}_{\mathrm{Cl}(1)}$ must take values in the quotient space

$$\widehat{\Omega}_*(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}_*(\mathcal{B}^\infty)]}$$

(i.e. modulo the closure of the space of graded commutators). Summarizing, for each $t \in (0,\infty)$ we can consider the following noncommutative differential form

$$\text{(10.4)} \qquad \widetilde{\eta}(\mathcal{D}_{(N,r)})(t) := \frac{2}{\sqrt{\pi}} \mathrm{STR}_{\mathrm{Cl}(1)}[\sigma\mathcal{D}_{(N,r)} \exp(-(\nabla + t\sigma\mathcal{D}_{(N,r)})^2)]$$

This is the non commutative analogue of the integrand in formula (10.3). The following theorem is due to J. Lott:

THEOREM 10.5. — *Assume that $\mathcal{D}_{(N,r)}$ is invertible in the Mishchenko-Fomenko calculus. Then*

$$\text{(10.6)} \qquad \widetilde{\eta}(\mathcal{D}_{(N,r)}) = \int_0^\infty \widetilde{\eta}(\mathcal{D}_{(N,r)})(t)dt$$

converges in

$$\widehat{\Omega}_*(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}_*(\mathcal{B}^\infty)]}.$$

The resulting form is called the higher eta invariant of $\mathcal{D}_{(N,r)}$.

Remarks. — (i) The invertibility of $\mathcal{D}_{(N,r)}$ in the Mishchenko-Fomenko calculus is equivalent to the existence of a full gap at $\lambda = 0$

in the L^2-spectrum of the operator $\widetilde{D}$ on $\widetilde{N}$, i.e. to the L^2-invertibility of $\widetilde{D}$.

(ii) Theorem 10.5 is proved by Lott in [83] for virtually nilpotent groups and implicitly in [84] in the general case. [8] For additional details on the general case see also [77], Theorem 4.1.

(iii) The higher eta invariant of Lott is the noncommutative analogue of the Bismut-Cheeger eta form; the convergence of the integral near $t = 0$ follows from the local index theory developed by Lott, in the same way that the convergence of the eta-form for families is due to Bismut's local index theory. On the other hand, the convergence for $t \to +\infty$ is much more delicate. Once again, *the proof depends heavily on the invertibility of* $\mathcal{D}_{(N,r)}$.

10.4. Higher Atiyah-Patodi-Singer index theory in the invertible case.

Let (M, g) be a compact even-dimensional Riemannian manifold with boundary. We assume g to be of product type near ∂M and we let D be a generalized Dirac operator acting on the sections of a $\mathbb{Z}_2$−graded Hermitian Clifford module E. As a fundamental example we could consider the signature operator D^{sign}. Let Γ be a finitely generated discrete group and let $\mathcal{B}^\infty \subset C^*_r\Gamma$ be a smooth subalgebra. Let $r : M \to B\Gamma$ be a continuous map defining a Γ−covering $\widetilde{M} \to M$. We denote by $\widetilde{D}$ the lift of D to $\widetilde{M}$. We denote by $\mathcal{D}_{(M,r)} : C^\infty(M, E \otimes \mathcal{V}^\infty) \to C^\infty(M, E \otimes \mathcal{V}^\infty)$ the $\mathcal{B}^\infty$−left linear operator induced by $\widetilde{D}$. The boundary operator associated to D will be denoted, as usual, by D_∂. Making use of $\widetilde{D}_\partial$ we also get an operator $\mathcal{D}_{(\partial M, r|_{\partial M})}$ which is nothing but the boundary operator of $\mathcal{D}_{(M,r)}$. We set $r|_{\partial M} := r_\partial$. Assume now that $\mathcal{D}_{(\partial M, r_\partial)}$ is invertible in the Mishchenko-Fomenko calculus; equivalently, we assume that $\widetilde{D}_\partial$ is L^2−invertible. Let

$$\Pi_{\geqslant} = \frac{1}{2}\left(1 + \frac{\mathcal{D}_{(\partial M, r_\partial)}}{|\mathcal{D}_{(\partial M, r_\partial)}|}\right);$$

[8] Charlotte Wahl pointed out that in the proof of Proposition 19 in [84] it is implicitly used that each Banach algebra B_j is closed under the holomorphic functional calculus. This is an extra assumption that should be added to the hypotheses of the Proposition. In the case treated here, where the algebra $\mathcal{B}$ is the Connes-Moscovici algebra of a discrete group, this assumption is indeed satisfied, as can be seen using the arguments of [56], Proof of Theorem 1.2.

this is a 0th order $\mathcal{B}^\infty$-pseudodifferential operator and we can consider the domain

$$C^\infty(M, E^+ \otimes \mathcal{V}^\infty, \Pi_\geqslant) = \{s \in C^\infty(M, E^+ \otimes \mathcal{V}^\infty) \,|\, \Pi_\geqslant(s|_{\partial M}) = 0\}.$$

The following Theorem is conjectured in [83] and proved in [74], [77], Appendix.

THEOREM 10.7. — *Assume that $\mathcal{D}_{(\partial M, r_\partial)}$ is invertible in the Mishchenko-Fomenko calculus. Then the operator $\mathcal{D}_{(M,r)}$ with domain $C^\infty(M, E^+ \otimes \mathcal{V}^\infty, \Pi_\geqslant)$ gives rise to a well defined APS-index class $\operatorname{Ind}(\mathcal{D}_{(M,r)}, \Pi_\geqslant)$ in $K_0(\mathcal{B}^\infty) \simeq K_0(C^*_r(\Gamma))$. The following formula holds in the non commutative topological de Rham homology of $\mathcal{B}^\infty$:*

$$\operatorname{Ch}\operatorname{Ind}(\mathcal{D}_{(M,r)}, \Pi_\geqslant) = \left[\int_M \mathrm{AS} \wedge \omega - \frac{1}{2}\widetilde{\eta}(\mathcal{D}_{(\partial M, r_\partial)})\right] \in \widehat{H}_*(\mathcal{B}^\infty) \tag{10.8}$$

with $\mathrm{AS} = \widehat{A}(M, \nabla^g) \wedge \operatorname{Ch}'(E, \nabla^E)$.

In particular: *under the invertibility assumption we have proved that Lott's higher eta invariant is the boundary correction term we have been looking for.*

The proof of the theorem rests ultimately on the heat-kernel proof of the higher index theorem given by Lott and on an extension to Galois coverings of Melrose's b-pseudodifferential calculus on manifolds with boundary. For the latter, the reader is referred to the book by Melrose [92] and also to the surveys [91] Grieser [41].

10.5. Higher signatures on manifolds with L^2-invisible boundary.

Let (M, g) be a Riemannian manifold with boundary; we assume the metric to be of product type near the boundary. Let $\widetilde{M} \to M$ be a Galois Γ-covering; let $r : M \to B\Gamma$ be a classifying map. We shall assume that the operator $\mathcal{D}^{\mathrm{sign}}_{(\partial M, r_\partial)}$ is invertible in the Mishchenko-Fomenko calculus. Equivalently, the operator $\widetilde{D}^{\mathrm{sign}}_\partial$ is L^2-invertible, or, again equivalently, the differential-form Laplacian $\Delta_{\partial\widetilde{M}}$ is L^2-invertible in each degree. *We shall say that the boundary $\partial\widetilde{M}$ is L^2-invisible.* Recent results of Farber and Weinberger show that there do exist coverings having a L^2-invisible boundary, see [36]. See also the subsequent paper [51] from which the term L^2-invisible is borrowed. Since $(\partial M, r_\partial)$ is L^2-invisible, the higher eta invariant of Lott, $\widetilde{\eta}(\mathcal{D}^{\mathrm{sign}}_{(\partial N, r_\partial)})$, is well defined. We set

$$\widetilde{\eta}(\mathcal{D}^{\mathrm{sign}}_{(\partial N, r_\partial)}) := \widetilde{\eta}_{(\partial N, r_\partial)} \tag{10.9}$$

DEFINITION 10.10. — *We define the higher signature class in $\widehat{H}_*(\mathcal{B}^\infty)$ of a covering $(M, r : M \to B\Gamma)$ with L^2-invisible boundary as*

$$\widehat{\sigma}(M,r) = \left[\int_M L(M,\nabla^g)\wedge\omega_{(M,r)} - \frac{1}{2}\widetilde{\eta}_{(\partial N, r_\partial)}\right] \in \widehat{H}_*(\mathcal{B}^\infty). \tag{10.11}$$

Notice that in this formula $\mathcal{B}^\infty$ is any smooth subalgebra of $C^*_r\Gamma$, for example the Connes-Moscovici algebra.

Let now N be a manifold with boundary and let $(N, s : N \to B\Gamma)$ be a Galois covering. Let $h : N \to M$, with $h(\partial N) \subset \partial M$, a homotopy equivalence between $(N, s : N \to B\Gamma)$ and $(M, r : M \to B\Gamma)$. A fundamental result of Gromov and Shubin [45] states that $(\partial N, s|_{\partial N} : \partial N \to B\Gamma)$ is then also L^2-invisible. The following result is conjectured in Lott [83] and proved in Leichtnam-Piazza [77]:

THEOREM 10.12. — *Let M be an oriented manifold with boundary, let $r : M \to B\Gamma$ be a classifying map and assume that $(\partial M, r_\partial : \partial M \to B\Gamma)$ is L^2-invisible. Then the higher signature class $\widehat{\sigma}(M,r)$ is a oriented homotopy invariant of the pair $(M, \partial M)$ and of the map $r : M \to B\Gamma$.*

Proof. — Following techniques of Kaminker-Miller [60], one proves that the APS-index class introduced in Theorem 10.7, $\mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}_{(M,r)}, \Pi_\geqslant) \in K_0(\mathcal{B}^\infty)$, is a homotopy invariant. The Theorem follows at once from the higher index formula (10.8) applied to the signature operator. □

DEFINITION 10.13. — *Let $[c] \in H^\ell(\Gamma, \mathbb{C})$. Let Γ have the extension property, see subsection 8.11, and let $\tau_c \in ZC^\ell(\mathcal{B}^\infty)$ be the extended cyclic cocycle associated to c. We define higher signatures* $\mathrm{sign}(M, r, [c]) \in \mathbb{C}$ *on a manifold with L^2-invisible boundary by setting*

$$\begin{aligned} \mathrm{sign}(M,r,[c]) &:=< \widehat{\sigma}(M,r), [\tau_c] > \\ &\equiv< \left[\int_M L(M,\nabla^g)\wedge\omega_{(M,r)} - \frac{1}{2}\widetilde{\eta}_{(\partial N, r_\partial)}\right], [\tau_c] > . \end{aligned} \tag{10.14}$$

If the boundary is empty then, up to the constant $C(\ell)$ appearing in (8.10), we reobtain the Novikov higher signatures.

COROLLARY 10.15. — *Let Γ be a finitely generated discrete group having the extension property. On manifolds $(M, r : M \to B\Gamma)$ with L^2-invisible boundary the higher signatures (10.14) are oriented homotopy invariants for each $[c] \in H^*(\Gamma, \mathbb{C})$.*

The result hold more generally for certain *twisted* higher signatures, manufactured out of the index class of twisted signature operators. See [77].

Remark. — Corollary 10.15 should be seen as a topological application of the *higher* Atiyah-Patodi-Singer index theorem 10.7. For applications in the realm of positive scalar curvature metrics see Leichtnam-Piazza [78].

10.6. Non-invertibility, perturbations and index classes.

Let $(M, r : M \to B\Gamma)$ be a covering with non-empty boundary. We set, as usual, $\widetilde{M} = r^*E\Gamma$, $r_\partial := r|_{\partial M}$. The invertibility assumption on $\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial)}$, or, equivalently, the L^2-invertibility assumption on $\Delta_{\partial\widetilde{M}}$, is very strong. In fact, until the recent work of Farber-Weinberger [36], it was an open question whether for a Galois covering $\Gamma \to \widetilde{N} \to N$ it is always the case that the operator $\Delta_{\widetilde{N}}$ is not L^2-invertible (see [85]).

For example, when $\Gamma = \mathbb{Z}^k$ the invertibility condition requires the cohomology groups of ∂M with coefficients in the flat bundle F_θ to vanish *for all* $\theta \in T^k$. Although this is indeed a strong hypothesis, there is no way to avoid it if one wants to set up a *continuous* family of APS-boundary value problems for the Lustzig's family or if one wants to prove the large time convergence of the integral defining the eta form. Similarly, in the noncommutative context, we do need the invertibility of $\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial)}$ for the projection

$$\Pi_\geqslant = \frac{1}{2}\left(1 + \frac{\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial)}}{|\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial}|}\right)$$

to make sense as $C^*_r\Gamma$-linear operator [9]. The invertibility is also necessary in order to prove the convergence of the higher eta invariant. The question then arises as to whether it is possible to lift the invertibility assumption on the boundary operator and still develop a family index theory or a higher index theory on manifolds with boundary. This problem was tackled for the first time by Melrose and Piazza in [94] [95] and subsequently extended to the noncommutative context in Leichtnam-Piazza [75], [79]. We shall now

[9] Notice that in the context of C^*-algebras Hilbert modules we only have a *continuous* functional calculus; in particular the operator $\chi_{[0,\infty)}(\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial)})$ does not make sense as $\mathcal{B}^\infty$-linear or $C^*_r\Gamma$-linear operator. It is only by going to a Von Neumann context that one can make sense of the operator $\chi_{[0,\infty)}(\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial)})$.

give a very short account of this theory, concentrating on the results leading to the definition of a (generalized) Atiyah-Patodi-Singer index class.

10.6.1. Spectral sections. Let D be a Dirac-type operator acting between the sections of a Hermitian Clifford module E. Let $(M, r : M \to B\Gamma)$ be a Galois covering of a manifold with boundary M. We shall concentrate on the even-dimensional case; thus ∂M is odd-dimensional. Let $\mathcal{D}_{(M,r)}$ be the $C_r^*\Gamma$-linear operator associated to D and $(M, r : M \to B\Gamma)$. The starting point in Melrose-Piazza [94] is the observation that although the boundary operator $\mathcal{D}_{(\partial M, r_\partial)}$ is not invertible, its index class in $K_1(C_r^*\Gamma)$ is equal to zero (by cobordism invariance). In order to define a higher APS-index class in $K_0(C_r^*\Gamma)$ we need a projection $\mathcal{P}$ playing the role of the non-existing projection $\Pi_\geqslant$. Of course, we need somewhat special projections; these are nowadays called spectral sections. Let $(N, s : N \to B\Gamma)$ be a odd-dimensional closed Galois covering (we shall eventually choose $(N, s : N \to B\Gamma) = (\partial M, r_\partial : \partial M \to B\Gamma)$). A *spectral section* associated to $\mathcal{D} \equiv \mathcal{D}_{(N,s)}$ is a self-adjoint $C_r^*\Gamma$-linear projection $\mathcal{P}$ with the additional property that there exists smooth functions

$$\chi_j : \mathbb{R} \to [0,1] \text{ with } \chi_j(t) = 0 \text{ for } t \ll 0\ ,\quad \chi_j(t) = 1 \text{ for } t \gg 0,$$

$\chi_2 \equiv 1$ on a neighborhood of the support of χ_1, and such that

$$\operatorname{Im} \chi_1(\mathcal{D}_{(N,s)}) \subset \operatorname{Im} \mathcal{P} \subset \operatorname{Im} \chi_2(\mathcal{D}_{(N,s)}).$$

Intuitively, $\mathcal{P}$ is equal to 1 on the large positive part of the spectrum and equal to 0 on the large negative part of the spectrum, precisely as $\Pi_\geqslant$ when the latter is defined. In fact, we have already encountered spectral sections in this paper; see the Remark at the end of subsection 2.4. The main result is then the following:

THEOREM 10.16 ([94] [124] [79]). — *A spectral section for $\mathcal{D}_{(N,s)}$ exists if and only if* $\operatorname{Ind}(\mathcal{D}_{(N,s)}) = 0$ *in* $K_1(C_r^*\Gamma)$.

10.6.2. Index classes and relative index theorem. The cobordism invariance of the numeric index can be extended to index classes Rosenberg [112] [75], Proposition 2.3. Thus $\operatorname{Ind}(\mathcal{D}_{(\partial M, r_\partial)}) = 0$ in $K_1(C_r^*\Gamma)$; hence there exists a spectral section $\mathcal{P}$ for $\mathcal{D}_{(\partial M, r_\partial)}$. We can use this $C_r^*\Gamma$-linear projection in order to define the domain

$$C^\infty(M, E^+ \otimes \mathcal{V}, \mathcal{P}) = \{s \in C^\infty(M, E^+ \otimes \mathcal{V}) \mid \mathcal{P}(s|_{\partial M}) = 0\}.$$

One can prove that $\mathcal{D}_{(M,r)}$ with domain $C^\infty(M, E^+ \otimes \mathcal{V}, \mathcal{P})$ gives rise to an index class $\operatorname{Ind}(\mathcal{D}_{(M,r)}, \mathcal{P}) \in K_0(C_r^*\Gamma)$ à la Atiyah-Patodi-Singer, see

Wu [125] [79] for the proof. Different choices of spectral sections produces different index classes; however there is a relative index theorem describing how these index classes are related: given spectral sections $\mathcal{P}$, $\mathcal{Q}$ there is a difference class $[\mathcal{P}-\mathcal{Q}]\in K_0(C^*_r\Gamma)$ such that

$$\operatorname{Ind}(\mathcal{D}_{(M,r)},\mathcal{Q})-\operatorname{Ind}(\mathcal{D}_{(M,r)},\mathcal{P})=[\mathcal{P}-\mathcal{Q}]\ \text{ in }\ K_0(C^*_r\Gamma). \tag{10.17}$$

This relative index theorem, first proved in [94] and then extended in [76] [81], is the higher analogue of the last formula of subsection 2.4.

10.6.3. Perturbations. Let $(N,s:N\to B\Gamma)$ odd dimensional and D a Dirac type operator. Assume that $\operatorname{Ind}(\mathcal{D}_{(N,s)})=0$. Fix a spectral section $\mathcal{P}$. Using $\mathcal{P}$ one can construct a smoothing operator $\mathcal{C}_{N,\mathcal{P}}\in\Psi^{-\infty}_{C^*_r\Gamma}$ such that $\mathcal{D}_{(N,s)}+\mathcal{C}_{N,\mathcal{P}}$ is invertible in the Mishchenko-Fomenko calculus. Moreover

$$\mathcal{P}=\frac{1}{2}\left(\operatorname{Id}+\frac{\mathcal{D}_{(N,s)}+\mathcal{C}_{N,\mathcal{P}}}{|\mathcal{D}_{(N,s)}+\mathcal{C}_{N,\mathcal{P}}|}\right).$$

In words: $\mathcal{P}$ is the positive spectral projection for the perturbed operator $\mathcal{D}_{(N,s)}+\mathcal{C}_{N,\mathcal{P}}$. We call such an operator $\mathcal{C}_{N,\mathcal{P}}$ a *trivializing perturbation*. Let us go back to the case where $(N,s)=(\partial M,r_\partial)$, with $(M,r:M\to B\Gamma)$ an even dimensional Galois covering with boundary. Fix a spectral section $\mathcal{P}$ for $\mathcal{D}_{(\partial M,r_\partial)}$; fix a trivializing perturbation $\mathcal{C}_{\partial M,\mathcal{P}}$. One can extend the operator $\mathcal{C}_{\partial M,\mathcal{P}}$ to the whole manifold with boundary M. The resulting operator $\mathcal{C}_{M,\mathcal{P}}$ gives *a perturbation $\mathcal{D}_{(M,r)}+\mathcal{C}_{M,\mathcal{P}}$ which has, by construction, an invertible boundary operator.* It turns out that the index class $\operatorname{Ind}(\mathcal{D}_{(M,r)},\mathcal{P})$ à la Atiyah-Patodi-Singer can also be described as an L^2-index class for the perturbed operator $\mathcal{D}_{(M,r)}+\mathcal{C}_{M,\mathcal{P}}$ extended to the manifold obtained by adding a cylindrical end to M. Cylindrical index theory is also referred to as b-index theory, because of the exhaustive treatment given by Melrose using the b-pesudodifferential calculus. See [92]. *Summarizing*: the index class $\operatorname{Ind}(\mathcal{D}_{(M,r)},\mathcal{P})$ is equal to the b-index class $\operatorname{Ind}_b(\mathcal{D}_{(M,r)}+\mathcal{C}_{M,\mathcal{P}})$. The advantage in considering the latter index class comes from the invertibility of the boundary operator: this allows us to consider the higher eta invariant of the boundary operator, $\widetilde{\eta}((\mathcal{D}_{(\partial M,r_\partial)}+\mathcal{C}_{\partial M,\mathcal{P}}))$ and prove a higher index formula similar to 10.7. The higher eta invariant, denoted

$$\widetilde{\eta}_{(\partial M,r_\partial),\mathcal{P}}:=\widetilde{\eta}((\mathcal{D}_{(\partial M,r_\partial)}+\mathcal{C}_{\partial M,\mathcal{P}}))$$

only depends on $\mathcal{P}$ (and not on the particulat choice of perturbation) *modulo exact forms.* This program is achieved in [94] [95] in the family case and in [75] in the Galois covering case. Recent topological applications of this general theory are given in Piazza-Schick [106].

10.7. Middle-degree invertibility and a perturbation of the signature complex.

10.7.1. The middle-degree assumption. Let us now go back to the signature operator $\mathcal{D}^{\text{sign}}_{(M,r)}$ on a covering with boundary $(M, r : M \to B\Gamma)$ and to the problem of defining higher signatures when the operator $\mathcal{D}^{\text{sign}}_{(\partial M, r_\partial)}$ is *not* invertible. The above subsection shows how to extend the Atiyah-Patodi-Singer index theory developed in the invertible case to this general case: crucial to this extension is the notion of *trivializing perturbation*. Unfortunately, the relative index formula (10.17) shows very clearly that the resulting index classes will depend on the choice of the trivializing perturbation. This is not very encouraging if our goal is to produce a *homotopy invariant APS index class*. In his fundamental paper [83] Lott points out an heuristic cancellation mechanism indicating why the following assumption might be sufficient for defining a *canonical* signature class.

Let $(N, s : N \to B\Gamma)$ be an odd dimensional Galois covering of a closed oriented manifold. For example $(N, s : N \to B\Gamma) = (\partial M, r_\partial : \partial M \to B\Gamma)$. Let $2m - 1 = \dim N$. Let d denote the de Rham differential on $\widetilde{N}$. Endow $\widetilde{N}$ with a $\Gamma-$invariant Riemannian metric.

ASSUMPTION 10.19. — *The differential form Laplacian acting on $L^2(\widetilde{N}, \Lambda^{m-1}(\widetilde{N}))/(\ker d)$ has a strictly positive spectrum.*

If $\mathcal{V} = C^*_r\Gamma \times_\Gamma s^*E\Gamma$ and if $d_\mathcal{V}$ denotes the twisted de Rham differential, then it is proved in [72] that Assumption (10.19) for (N, s) is equivalent to the following:

ASSUMPTION 10.20. — *Let $\Omega^\ell_{(2)}(N, \mathcal{V})$ denote the $L^2_{C^*_r\Gamma}$-completion of $\Omega^\ell(N, \mathcal{V})$. The operator*

$$d_\mathcal{V} : \Omega^{m-1}_{(2)}(N, \mathcal{V}) \to \Omega^m_{(2)}(N, \mathcal{V}),$$

with domain equal to the $C^*_r\Gamma$-Sobolev space $H^1_{C^*_r\Gamma}$, has closed image.

These equivalent assumptions are for example satisfied when N has a cellular decomposition without any cell of dimension m. Thanks to a deep result of Gromov-Shubin [45] we know that these are *homotopy invariant conditions*. Notice that if Assumption 10.19 is satisfied, then necessarily the index class of the signature operator in $K_1(C^*_r\Gamma)$ is equal to zero.

Since the index class of the signature operator is concentrated in middle degree, Assumption 10.19 makes us guess that it should be possible to find a set of *symmetric* trivializing perturbations of the boundary operator producing first of all a well defined higher eta invariant and, secondly, a well defined index class, both independent of the perturbation chosen. This is indeed the case. There are in fact two equivalent ways to proceed: one, proposed by Lott in [86] and fully developed in [72] contructs perturbations of the *signature complexes*, on ∂M and on M, with the right symmetry property for making the eta invariant and the index class well defined. This is the approach we shall explain below. The other approach, developed in Leichtnam-Piazza [76], makes use of a special set of spectral sections for the boundary signature operator; these spectral sections have a certain symmetry property with respect to forms of degree $(m-1)$. We mention the approach through *symmetric spectral sections* because we shall use it later, in conjunction with the cut-and-paste problem for higher signatures. Now, following John Lott [86] and Leichtnam-Lott-Piazza [72] we shall explain how it is possible to *add* a finitely $\mathcal{B}^\infty$–generated perturbation to the *complex* of $\mathcal{B}^\infty$–differential forms on ∂M and consequently perturb $\mathcal{D}^{\rm sign}_{(\partial M, r_\partial)}$ into an *invertible* (generalized) signature operator. To this aim we have to recall, in the next sub-section, how to express $\mathcal{D}^{\rm sign}_{(\partial M, r)}$ in terms of the $\mathcal{B}^\infty$–flat exterior derivative d and the Hodge duality operator τ acting on the Hermitian complex of differential forms.

10.7.2. More on the signature complex on closed manifolds. First of all we recall the following

DEFINITION 10.21. — *A graded regular n-dimensional Hermitian complex consists of*

1. *A $\mathbb{Z}$-graded cochain complex $(\mathcal{E}^*, D)$ of finitely-generated projective left $\mathcal{B}^\infty$-modules,*

2. *A nondegenerate quadratic form $Q : \mathcal{E}^* \times \mathcal{E}^{n-*} \to \mathcal{B}^\infty$ and*

3. *An operator $\tau \in \mathrm{Hom}_{\mathcal{B}^\infty}(\mathcal{E}^*, \mathcal{E}^{n-*})$ such that*

   1. $Q(bx, y) = bQ(x, y)$.
   2. $Q(x, y)^* = Q(y, x)$.
   3. $Q(Dx, y) + Q(x, Dy) = 0$.
   4. $\tau^2 = I$.
   5. $< x, y > \equiv Q(x, \tau y)$ *defines a Hermitian metric on* $\mathcal{E}$.

Let M be a closed oriented n-dimensional Riemannian manifold and let $r : M \to B\Gamma$ be a reference map. We set $\mathcal{V}^\infty = \mathcal{B}^\infty \times_\Gamma r^* E\Gamma$. Let $\Omega^*(M;\mathcal{V}^\infty)$ denote the vector space of smooth differential forms with coefficients in $\mathcal{V}^\infty$. The twisted de Rham differential will be still denoted by d. If $n = \dim(M) > 0$ then $\Omega^*(M;\mathcal{V}^\infty)$ is not finitely-generated over $\mathcal{B}^\infty$, but we wish to show that it still has all of the formal properties of a graded regular n-dimensional Hermitian complex. If $\alpha \in \Omega^*(M;\mathcal{V}^\infty)$ is homogeneous, denote its degree by $|\alpha|$. In what follows, α and β will sometimes implicitly denote homogeneous elements of $\Omega^*(M;\mathcal{V}^\infty)$. Given $y \in M$ and $(\lambda_1 \otimes e_1), (\lambda_2 \otimes e_2) \in \Lambda^*(T_y^*M) \otimes \mathcal{V}_y^\infty$, we define $(\lambda_1 \otimes e_1) \wedge (\lambda_2 \otimes e_2)^* \in \Lambda^*(T_y^*M) \otimes \mathcal{B}^\infty$ by

$$(\lambda_1 \otimes e_1) \wedge (\lambda_2 \otimes e_2)^* = (\lambda_1 \wedge \overline{\lambda_2}) \otimes < e_1, e_2 > .$$

Extending by linearity (and antilinearity), given $\omega_1, \omega_2 \in \Lambda^*(T_y^*M) \otimes \mathcal{V}_y^\infty$, we can define $\omega_1 \wedge \omega_2^* \in \Lambda^*(T_y^*M) \otimes \mathcal{B}^\infty$. Define a $\mathcal{B}^\infty$-valued quadratic form Q on $\Omega^*(M;\mathcal{V}^\infty)$ by

$$Q(\alpha,\beta) = i^{-|\alpha|(n-|\alpha|)} \int_M \alpha(y) \wedge \beta(y)^*.$$

It satisfies $Q(\beta,\alpha) = Q(\alpha,\beta)^*$. Using the Hodge duality operator $*$, define $\tau : \Omega^p(M;\mathcal{V}^\infty) \to \Omega^{n-p}(M;\mathcal{V}^\infty)$ by $\tau(\alpha) = i^{-|\alpha|(n-|\alpha|)} * \alpha$. Then $\tau^2 = 1$ and the inner product $< \cdot,\cdot >$ on $\Omega^*(M;\mathcal{V}^\infty)$ is given by $< \alpha, \beta >= Q(\alpha, \tau\beta)$. Define $D : \Omega^*(M;\mathcal{V}^\infty) \to \Omega^{*+1}(M;\mathcal{V}^\infty)$ by

$$D\alpha = i^{|\alpha|} d\alpha. \tag{10.22}$$

Warning: in this subsection the differential D should not be confused with a Dirac-type operator.

It satisfies $D^2 = 0$. Its dual D' with respect to Q, i.e., the operator D' such that $Q(\alpha, D\beta) = Q(D'\alpha, \beta)$, is given by $D' = -D$. The formal adjoint of D with respect to $< \cdot,\cdot >$ is $D^* = \tau D' \tau = -\tau D \tau$.

DEFINITION 10.23. — *If n is even, the signature operator is*

$$\mathcal{D}^{\text{sign}}_{(M,r)} = D + D^* = D - \tau D \tau. \tag{10.24}$$

It is formally self-adjoint and anticommutes with the $\mathbb{Z}_2$-grading operator τ. If n is odd, the signature operator is

$$\mathcal{D}^{\text{sign}}_{(M,r)} = -i(D\tau + \tau D). \tag{10.25}$$

It is formally self-adjoint.

10.7.3. More on the signature complex on manifolds with boundary. Now suppose that M is a compact oriented manifold-with-boundary of dimension $n = 2m$. Let $r : M \to B\Gamma$ be a reference map and let ∂M denote the boundary of M. We fix a Riemannian metric on M which is isometrically a product in an (open) collar neighbourhood $\mathcal{U} \equiv (0,2)_x \times \partial M$ of ∂M. Let $\mathcal{V}_0^\infty$ denote the pullback of $\mathcal{V}^\infty$ from M to ∂M; there is a natural isomorphism

$$\mathcal{V}^\infty|_{\mathcal{U}} \cong (0,2) \times \mathcal{V}_0^\infty .$$

One can show that, up to explicit isomorphisms, the signature operator can be written near the boundary as $\mathcal{D}^{\mathrm{sign},+}_{(M,r)} = \partial_x + \mathcal{D}^{\mathrm{sign}}_{(\partial M,r)}$.

10.7.4. The perturbed signature complex. Recall that $\mathcal{B}^\infty$ denotes the Connes-Moscovici sub-algebra of $C^*_r(\Gamma)$ and that on ∂M we have the bundles:

$$\mathcal{V} = C^*_r(\Gamma) \times_\Gamma \partial\widetilde{M}, \qquad \mathcal{V}^\infty = \mathcal{B}^\infty \times_\Gamma \partial\widetilde{M}.$$

The following Proposition, from [86], is proved using Assumption 10.19 in a crucial way. See [72].

Proposition 10.26. — *There exists a cochain complex $C^* = \bigoplus_{k=-1}^{2m} C^k$ where $C^k = \Omega^k(\partial M;\mathcal{V}^\infty) \oplus \widehat{W}^k$ and $\widehat{W}^*$ is a complex made by finitely generated left projective $\mathcal{B}^\infty$−modules. There are two maps $\widehat{f} : \Omega^*(\partial M;\mathcal{V}^\infty) \to \widehat{W}^*$ and $\widehat{g} : \widehat{W}^* \to \Omega^*(\partial M;\mathcal{V}^\infty)$ such that the following property is satisfied. For any real $\epsilon > 0$ the differential D_C on C^* defined by*

(10.27)

$$D_C = \begin{pmatrix} D_{\partial M} & \epsilon\widehat{g} \\ 0 & -D_{\widehat{W}} \end{pmatrix} \text{ if} * < m - \frac{1}{2}, \quad D_C = \begin{pmatrix} D_{\partial M} & 0 \\ -\epsilon\widehat{f} & -D_{\widehat{W}} \end{pmatrix} \text{ if} * > m - \frac{1}{2}.$$

is such that $D_C^2 = 0$ and the complex (C^*, D_C) has vanishing cohomology.

Define a duality operator τ_C on C^* by

$$\tau_C = \begin{pmatrix} \tau_{\partial M} & 0 \\ 0 & \tau_{\widehat{W}} \end{pmatrix}. \tag{10.28}$$

The signature operator associated to the perturbed complex (C^*, D_C) is defined to be

$$\mathcal{D}^{\mathrm{sign}}_{C,\partial}(\epsilon) = -i(\tau_C D_C + D_C \tau_C)$$

If $\epsilon > 0$, it follows from the vanishing of the cohomology of C^* that

(10.29) $\mathcal{D}^{\mathrm{sign}}_{C,\partial}(\epsilon)$ is an invertible self-adjoint $\mathcal{B}^\infty$-operator.

We shall set $\mathcal{D}^{\mathrm{sign}}_{C,\partial} := \mathcal{D}^{\mathrm{sign}}_{C,\partial}(1)$.

10.8. Lott's higher eta invariant in the non-invertible case.

We are now in a position to recall the definition of the higher eta invariant of Lott for a closed $(2m-1)$-dimensional covering $(N, s : N \to B\Gamma)$ satisfying Assumption 10.19. This material comes from [86] and [72]. We shall concentrate directly on the case $(N, s : N \to B\Gamma) = (\partial M, r_\partial)$.

Let

$$\nabla : \Omega^*(\partial M; \mathcal{V}^\infty) \to \Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \Omega^*(\partial M; \mathcal{V}^\infty)$$

be Lott's connection for the bundle $E = \Lambda^*(\partial M)$, see (8.8). As in [86], (3.28), let

$$\nabla^{\widehat{W}^*} : \widehat{W}^* \to \Omega_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \widehat{W}^* \tag{10.30}$$

be a connection on $\widehat{W}^*$ which is invariant under the grading operator and preserves the quadratic form of $\widehat{W}^*$. Set $\nabla^C = \nabla \oplus \nabla^{\widehat{W}^*}$; thus

$$\nabla^C : \Omega^*(\partial M; \mathcal{V}^\infty) \oplus \widehat{W}^* \to \widehat{\Omega}_1(\mathcal{B}^\infty) \otimes_{\mathcal{B}^\infty} \left(\Omega^*(\partial M; \mathcal{V}^\infty) \oplus \widehat{W}^*\right).$$

Let $\mathrm{Cl}(1)$ be the complex Clifford algebra of $\mathbb{C}$ generated by 1 and σ, with $\sigma^2 = 1$. Let $\epsilon \in C^\infty(0,\infty)$ now be a nondecreasing function such that $\epsilon(s) = 0$ for $s \in (0,1]$ and $\epsilon(s) = 1$ for $s \in [2,+\infty)$. Consider the element in $\widehat{\Omega}_{\mathrm{even}}(\mathcal{B}^\infty)/\overline{[\widehat{\Omega}_*(\mathcal{B}^\infty), \widehat{\Omega}(\mathcal{B}^\infty)]}$

$$\widetilde{\eta}_{(\partial M, r_\partial)}(s) = \frac{2}{\sqrt{\pi}} \mathrm{STR}_{\mathrm{Cl}(1)} \left(\left(\frac{d}{ds}\left[\sigma s \mathcal{D}^{\mathrm{sign}}_{C,\partial}(\epsilon(s))\right]\right) \exp\left[-(\sigma s \mathcal{D}^{\mathrm{sign}}_{C,\partial}(\epsilon(s)) + \nabla^C)^2\right]\right); \tag{10.31}$$

here $\mathrm{STR}_{\mathrm{Cl}(1)}$ is defined as in subsection 10.2. The higher eta invariant of $(\partial M, r_\partial)$ is, by definition,

$$\widetilde{\eta}_{\partial M} = \int_0^\infty \widetilde{\eta}_{\partial M}(s) ds. \tag{10.32}$$

Since $\epsilon(s) = 0$ for $s \in (0,1]$, it follows that the integral is convergent for $s \downarrow 0$ (in fact, the integrand near $s = 0$ is the same as the one for the unperturbed operator and for the latter we know that convergence is implied by Lott's heat-kernel proof of the higher index theorem). Since $\epsilon(s) = 1$ for $s > 2$ and *since the perturbed signature operator $\mathcal{D}^{\mathrm{sign}}_{C,\partial}$ is invertible*, it follows that the integral is also convergent as $s \uparrow \infty$. It is shown in [86], Proposition 14 that, *modulo exact forms*, the higher eta invariant $\widetilde{\eta}_{(\partial M, r_\partial)}$ is independent of the particular choices of the function ϵ, the perturbing complex $\widehat{W}^*$ and the self-dual connection $\nabla^{\widehat{W}}$.

10.9. Homotopy invariant higher signatures on a manifold with boundary

10.9.1. Conic and cylindrical higher index classes. Having defined the higher eta invariant under the more general hypothesis of middle-degree invertibility, we would like to show that it enters as a *boundary correction term* in a higher index theorem for a homotopy-invariant index class on our covering with boundary. We begin [72] by recalling the construction of a perturbed conic signature operator $\mathcal{D}^{\text{sign,cone}}_{C,M}$, with boundary operator equal to the *invertible* perturbed signature operator $\mathcal{D}^{\text{sign}}_{C,\partial}$ introduced in the previous subsection, see (10.29).

We take an (open) collar neighborhood of ∂M which is diffeomorphic to $(0,2)\times\partial M$. Let $\varphi\in C^\infty(0,2)$ be a nondecreasing function such that $\varphi(x)=x$ if $x\leqslant 1/2$ and $\varphi(x)=1$ if $x\geqslant 3/2$. Given $t>0$, consider a Riemannian metric on $\operatorname{int}(M)$ whose restriction to $(0,2)\times\partial M$ is

$$g_M = t^{-2}dx^2+\varphi^2(x)g_{\partial M}. \tag{10.33}$$

Consider the complex $\Omega^*_c(0,2)\widehat{\otimes}\widehat{W}^*$. It is endowed with a natural differential D_{alg}. Then set:

$$C^* = \Omega^*_c(M;\mathcal{V}^\infty)\oplus\left(\Omega^*_c(0,2)\widehat{\otimes}\widehat{W}^*\right),$$

C^* is endowed with a natural direct sum duality operator τ_C.

Let $\phi\in C^\infty(0,2)$ be a nonincreasing function satisfying $\phi(x)=1$ for $0<x\leqslant\frac14$ and $\phi(x)=0$ for $\frac12\leqslant x<2$. We extend $\widehat{f}$ and $\widehat{g}$ to act on $\Omega^*_c(0,2)\widehat{\otimes}\Omega^*(\partial M;\mathcal{V}^\infty_0)$ and $\Omega^*_c(0,2)\widehat{\otimes}\widehat{W}^*$, respectively, by

$$\widehat{f}(\omega_0+dx\wedge\omega_1)=\widehat{f}(\omega_0)-idx\wedge\widehat{f}(\omega_1)$$

and

$$\widehat{g}(w_0+dx\wedge w_1)=\widehat{g}(w_0)-idx\wedge\widehat{g}(w_1).$$

Using the cutoff function ϕ, it makes sense to define an operator on C^* by

$$D^{\text{cone}}_C=\begin{cases}\left(\begin{pmatrix}D_M & \phi\widehat{g}\\ 0 & D_{\text{alg}}\end{pmatrix}\right) & \text{if } *\leqslant m-1,\\ \left(\begin{pmatrix}D_M & 0\\ 0 & D_{\text{alg}}\end{pmatrix}\right) & \text{if } *=m,\\ \left(\begin{pmatrix}D_M & 0\\ -\phi\widehat{f} & D_{\text{alg}}\end{pmatrix}\right) & \text{if } *\geqslant m+1.\end{cases}$$

Note that $(D_C^{\rm cone})^2 \neq 0$, as ϕ is nonconstant. We have thus defined an "almost" differential $D_C^{\rm cone}$ on the conic complex

$$C^* = \Omega_c^*(M; \mathcal{V}^\infty) \oplus \left(\Omega_c^*(0,2)\widehat{\otimes}\widehat{W}^*\right).$$

The perturbed conic signature operator $\mathcal{D}_C^{\rm sign,cone} = D_C^{\rm cone} + (D_C^{\rm cone})^*$ satisfies

$$\mathcal{D}_C^{\rm sign,cone} = D_C^{\rm cone} - \tau D_C^{\rm cone}\tau.$$

By construction, the boundary signature operator associated to $\mathcal{D}_C^{\rm sign,cone}$ is precisely $\mathcal{D}_{C,\partial}^{\rm sign}$, the perturbed signature operator constructed in the previous section.

Summarizing: we have defined a *perturbed signature complex* on $(M, r : M \to B\Gamma)$ with the property that the associated signature *operator* has an invertible boundary operator.

Using this fundamental fact one can prove that $\mathcal{D}_C^{\rm sign,cone,+}$ defines an index class

$$\text{(10.35)} \qquad \mathrm{Ind}\mathcal{D}_C^{\rm sign,cone,+} \in K_0(\mathcal{B}^\infty).$$

The proof, see [72], employs in a crucial way elliptic analysis on conic manifolds, see [25], Brüning-Seeley [22]. We shall see in a moment that the conic index class is homotopy invariant. This is a fundamental step in our strategy for defining homotopy-invariant higher signatures. The last step will consist in proving an index theorem. However, to do so it turns out that the cylindrical, or b, picture is more convenient. Thus we sketch briefly the construction of a b-signature operator $\mathcal{D}_C^{\rm sign,b}$ in an extended version of Melrose b-calculus; the boundary operator will be once again $\mathcal{D}_{C,\partial}^{\rm sign}$.

Thus, we consider a b-metric g which is product like near the boundary:

$$g = \frac{dx^2}{x^2} + g_{\partial M},$$

for $0 < x \leqslant \frac{1}{2}$. Recall that a b-differential form is locally of the form $a(x,y)\frac{dx}{x} \wedge dy^I$. The space of b-differential forms is usually denoted by ${}^b\Omega^*$.

We consider a new differential D_C on the perturbed complex $C^* = {}^b\Omega^*(M; \mathcal{V}^\infty) \oplus ({}^b\Omega^*[0,2)\widehat{\otimes}\widehat{W})$; on the degree j-subspace we put

$$D_C \equiv \left(\begin{pmatrix} D_M & 0 \\ 0 & D_{\rm alg} \end{pmatrix}\right) + \begin{cases} \left(\begin{pmatrix} 0 & \widehat{g}_b \\ 0 & 0 \end{pmatrix}\right) & \text{if } j < m \\ \left(\begin{pmatrix} 0 & 0 \\ -\widehat{f}_b & 0 \end{pmatrix}\right) & \text{if } j > m \end{cases}$$

where $\widehat{g}_b$ and $\widehat{f}_b$ are b−operators associated in a natural way to $\phi\widehat{g}$ and $\phi\widehat{f}$ respectively.

Let $\mathcal{D}_C^{\mathrm{sign},b} = D_C + (D_C)^*$ be the b-signature operator associated to the b-complex (C^*, D_C). Then $\mathcal{D}_C^{\mathrm{sign},b} = D_C - \tau_C D_C \tau_C$ is odd with respect to the $\mathbb{Z}_2$-grading defined by the Hodge duality operator τ_C on C^*. Since the boundary operator is equal to $\mathcal{D}_{C,\partial}^{\mathrm{sign}}$ and is therefore invertible, one can prove that the *perturbed* b-signature operator $\mathcal{D}_C^{\mathrm{sign},b,+}$ is $C_r^*\Gamma$-Fredholm, i.e. invertible modulo $C_r^*\Gamma$-compact operators. Thus there is a well defined index class $\mathrm{Ind}\mathcal{D}_C^{\mathrm{sign},b,+} \in K_0(\mathcal{B}^\infty)$. To prove these statements an extended version of Melrose's b-calculus must be used, see [72].

The following theorem is proved in [72].

THEOREM 10.37. — *The following equality holds in* $K_0(\mathcal{B}^\infty) = K_0(C_r^*\Gamma)$:

$$\mathrm{Ind}\mathcal{D}_C^{\mathrm{sign,cone},+} = \mathrm{Ind}\,\mathcal{D}_C^{\mathrm{sign},b,+}.$$

Proof. — There is also a perturbed signature operator $\mathcal{D}_C^{\mathrm{sign}}$ with respect to an ordinary product-like metric on M (meaning, of type $dx^2 + g_{\partial M}$ near the boundary). Since the associated boundary operator is still $\mathcal{D}_{C,\partial}^{\mathrm{sign}}$, hence invertible, we can define the projection

$$\Pi_{\geqslant} = \frac{1}{2}\left(\mathrm{Id} + \frac{\mathcal{D}_{C,\partial}^{\mathrm{sign}}}{|\mathcal{D}_{C,\partial}^{\mathrm{sign}}|}\right)$$

and a higher index class $\mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},+}, \Pi_{\geqslant})$ à la Atiyah-Patodi-Singer. One proves that the following two equalities hold in $K_0(\mathcal{B}^\infty) = K_0(C_r^*\Gamma)$:

$$\mathrm{Ind}\,\mathcal{D}_C^{\mathrm{sign,cone},+} = \mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},+}, \Pi_{\geqslant}), \quad \mathrm{Ind}(\mathcal{D}_C^{\mathrm{sign},+}, \Pi_{\geqslant}) = \mathrm{Ind}\mathcal{D}_C^{\mathrm{sign},b,+}.$$

□

10.9.2. Homotopy invariance of the index class. We can finally state the first crucial result toward a definition of homotopy invariant higher signatures:

THEOREM 10.38. — *Let* $(M, r : M \to B\Gamma)$ *be such that* $(\partial M, r_\partial)$ *satisfy the middle-degree assumption 10.19. The index class* $\mathrm{Ind}\mathcal{D}_C^{\mathrm{sign,cone},+} \in K_0(\mathcal{B}^\infty)$ *is a homotopy invariant of the pair* $(M, \partial M)$ *and the classifying map* $r : M \to B\Gamma$. *Consequently, the b-index class* $\mathrm{Ind}\mathcal{D}_C^{\mathrm{sign},b,+}$ *is also a homotopy invariant.*

Proof. — (Sketch) One observes that the resolvent of $\mathcal{D}_C^{\text{sign,cone}}$ is $C_r^*(\Gamma)$-compact and that $(D_C^{\text{cone}})^2$ is small, provided that the real $t > 0$ is small (i.e. the length of the cone is large). Then one can extend fundamental results of Hilsum-Skandalis [53] for $t > 0$ small enough, proving the homotopy invariance of the index class. (We recall that Hilsum and Skandalis have proved the homotopy invariance of the index class for a signature operator with coefficients in an almost flat bundle of C^*−algebras). The details are somewhat of a technical nature and can be found in [72]. □

10.9.3. The index theorem and the higher signature class $\widehat{\sigma}(M,r) \in \widehat{H}_*(\mathcal{B}^\infty)$. Now we can state the following theorem, proved in [72].

THEOREM 10.39. — *Under Assumption 10.19 the following formula holds:*

$$\operatorname{ch}\operatorname{Ind}\mathcal{D}_C^{\text{sign},b,+} = \left[\int_M L(M) \wedge \omega - \frac{1}{2}\widetilde{\eta}_{(\partial M, r_\partial)}\right] \quad in \ \widehat{H}_*(\mathcal{B}^\infty)$$

where $\omega_{(M,r)}$ is, once again, the bi-form appearing in Lott's heat-kernel proof of the higher index theorem and $\widetilde{\eta}_{(\partial M, r_\partial)}$ is the higher eta invariant for the perturbed signature operator $\mathcal{D}_{C,\partial}^{\text{sign}}$.

Thus, under the middle-degree assumption 10.19 on the boundary covering $(\partial M, r_\partial : \partial M \to B\Gamma)$ we are finally in the position of extending the definition of higher signature class given in subsection 10.10

$$\widehat{\sigma}(M,r) := \left[\int_M L(M) \wedge \omega_{(M,r)} - \frac{1}{2}\widetilde{\eta}_{(\partial M, r_\partial)}\right] \in \widehat{H}_*(\mathcal{B}^\infty) \tag{10.40}$$

Using 10.38 and 10.39 we can finally state one of the main results of [72]:

THEOREM 10.41. — *The class $\widehat{\sigma}(M,r)$ in $\widehat{H}_*(\mathcal{B}^\infty)$ is a homotopy invariant of the pair $(M, \partial M)$ and the classifying map $r : M \to B\Gamma$.*

10.9.4. Homotopy invariant higher signatures in the non-invertible case. We are approaching the end of our journey. Let Γ be a group with the extension property. For example, Γ is Gromov hyperbolic or virtually nilpotent. Let $c \in H^\ell(\Gamma, \mathbb{C})$ be a group cycle and let $\tau_c \in ZC^*(\mathbb{C}\Gamma)$ be the associated cyclic cocycle. We can assume τ_c to be extendable and we still let $\tau_c \in ZC^*(\mathcal{B}^\infty)$ be the extended cocycle.

DEFINITION 10.42. — *The complex number*

$$\operatorname{sign}(M,r;[c]) =< \widehat{\sigma}(M,r), [\tau_c] >\in \mathbb{C}$$

is called the higher signature associated to (M,r) *and* $[c]$.

The following theorem gives an answer to **Question 3** in section 6:

THEOREM 10.43 [72]). — *Let* $(M,r : M \to B\Gamma)$ *be a Galois covering with boundary* $(\partial M, r_\partial : \partial M \to B\Gamma)$ *satisfying the middle-degree assumption 10.19. Let* Γ *be a finitely generated group with the extension property. The higher signatures*

$$\operatorname{sign}(M,r;[c]) =< \widehat{\sigma}(M,r), [\tau_c] >,\ \widehat{\sigma}(M,r) := \Big[\int_M L(M)\wedge\omega_{(M,r)} - \frac{1}{2}\widetilde{\eta}_{(\partial M, r_\partial)}\Big]$$

are homotopy invariants for each $[c] \in H^*(\Gamma,\mathbb{C})$.

10.10. Cut-and-paste invariance of higher signatures: the index theoretic approach.

We now go back to the cut-and-paste invariance of Novikov's higher signatures on a *closed* manifold. We are looking for sufficient conditions ensuring that the higher signatures are indeed cut-and-paste invariant. Recall that for the lower signature we explained 3 approaches to the problem:

(i) index theoretic,

(ii) topological,

(iii) via a spectral-flow argument.

The following theorem, from [72], extends to the higher case the first of these approaches. We shall only treat the even-dimensional case, the odd-dimensional case being more complicated to state and to treat.

Let M and N be two compact oriented $2m$-dimensional manifolds with boundary. Let ϕ and ψ two orientation preserving diffeomorphisms from ∂M onto ∂N. Consider the closed manifolds $X_\phi := M \cup_\phi N^-$ and $X_\psi := M \cup_\psi N^-$. Let $r : M \cup_\phi N^- \to B\Gamma$ and $s : M \cup_\psi N^- \to B\Gamma$ be two reference maps; we assume that these two coverings are *cut-and-paste equivalent.*

THEOREM 10.44 ([72]). — *Assume that* Γ *has the extension property and that* $(\partial M, r_\partial : \partial M \to B\Gamma)$ *satisfies Assumption 10.19. Then, for* *every* $[c] \in H^*(\Gamma, \mathbb{C}) = H^*(B\Gamma, \mathbb{C})$ *one has:*

$$(10.45)\quad \begin{aligned} &< L(M \cup_\phi N^-) \cup r^*[c], [M \cup_\phi N^-] > = < \widehat{\sigma}(M, r_{|M}), [\tau_c] > \\ &\quad + < \widehat{\sigma}(N^-, r_{|N^-}), [\tau_c] > \end{aligned}$$

(10.46)

$$< L(M \cup_\phi N^-) \cup r^*[c], [M \cup_\phi N^-] > = < L(M \cup_\psi N^-) \cup s^*[c], [M \cup_\psi N^-] > .$$

In particular under the stated assumptions the higher signatures are cut-and-paste invariant.

Remark. — Notice that $(\partial M, r_\partial : \partial M \to B\Gamma)$ satisfies Assumption 10.19 iff $(\partial M, s_\partial : \partial M \to B\Gamma)$ satisfies it.

Proof. — We begin by (10.45). As in subsection 3.1 we write:

$$M \cup_\phi N^- = M \cup_{\mathrm{Id}} \mathrm{Cyl}_\phi \cup_{\mathrm{Id}} N^-$$

where $\mathrm{Cyl}_\phi = ([-1, 0] \times (\partial M)^-) \cup_\phi ([0, 1] \times \partial N)$ is isomorphic to $\mathrm{Cyl} := [-1, 1] \times \partial M$ via ϕ. Moreover

(10.47)

$$\widehat{\sigma}(\mathrm{Cyl}_\phi, r_{|\mathrm{Cyl}_\phi}) = \int_{\mathrm{Cyl}_\phi} L(\mathrm{Cyl}_\phi) \wedge \omega + \frac{1}{2}\widetilde{\eta}_{(\partial M, r_{|\partial M})} - \frac{1}{2}\widetilde{\eta}_{(\partial N, r_{|\partial N})} = 0.$$

since by the established homotopy invariance $\widehat{\sigma}(\mathrm{Cyl}_\phi, r_{|\mathrm{Cyl}_\phi}) = \widehat{\sigma}(\mathrm{Cyl}, r_{|\partial M} \times \mathrm{Id})$ and the latter is zero for the usual orientation argument concerning the eta invariant. By Lott's higher index theorem on closed manifolds

$$< L(M \cup_\phi N^-) \cup r^*[c], [M \cup_\phi N^-] > = < [\tau_c]; \int_{M \cup_\phi N^-} L(M \cup_\phi N^-) \wedge \omega >$$

We can rewrite the left hand side of (10.45) as

$$< [\tau_c]; \int L(M) \wedge \omega - \frac{1}{2}\widetilde{\eta}_{(\partial M, r_{|\partial M})} > + < [\tau_c]; \int_{\mathrm{Cyl}_\phi} L(\mathrm{Cyl}_\phi) \wedge \omega$$

$$+ \frac{1}{2}\widetilde{\eta}_{(\partial M, r_{|\partial M})} + \frac{1}{2}\widetilde{\eta}_{(\partial N^-, r_{|\partial N^-})} > + < [\tau_c]; \int_{N^-} L(N^-) \wedge \omega - \frac{1}{2}\widetilde{\eta}_{(\partial N^-, r_{|\partial N^-})} > .$$

From (10.47) we immediately obtain (10.45). Moreover, (10.46) is an immediate consequence of (10.45). □

11. The topological approach to the cut-and-paste problem for higher signatures.

In this section we shall describe a topological approach to the study of cut and paste properties of higher signatures. This material comes from

Leichtnam-Lück-Kreck [73] and should be seen as the higher analogue of what we presented in subsection 3.2. Namely, assuming that $(\partial M, r_{\partial M})$ satisfies Assumption 10.19, we shall define a symmetric signature $\sigma(M,r) \in K_0(C^*_r(\Gamma))$ which is *both* a higher generalization of the lower topological signature of $(M, \partial M)$ *and* a generalization of the Mishchenko symmetric signature when the boundary is empty. The properties of $\sigma(M,r)$, namely *additivity* and *homotopy invariance*, will allow us to extend Theorem 10.44 to the discrete finitely presently groups Γ satisfying the Strong Novikov Conjecture.

11.1. The symmetric signature on manifolds with boundary.

We shall follow the notation in [73]; in particular we denote by $\overline{M} \to M$ a Galois covering with base M.

Let $n = 2m$ be an even integer and M be an oriented compact n-dimensional manifold possibly with boundary. Let $(M, r : M \to B\Gamma)$ a Galois covering. Let $\overline{\partial M} \to \partial M$ and $\overline{M} \to M$ be the Γ-coverings associated to the maps $r|_{\partial M} : \partial M \to B\Gamma$ and $r : M \to B\Gamma$. Following Lott [83], Section 4.7 and [72], Assumption 1 and Lemma 2.3, we make the following assumption about $(\partial M, r|_{\partial M})$.

ASSUMPTION 11.1. — *Recall that $n = 2m$. Let $C_*(\overline{\partial M})$ be the cellular $\mathbb{Z}\Gamma$-chain complex. Then we assume that the $C^*_r(\Gamma)$-chain complex $C_*(\overline{\partial M}) \otimes_{\mathbb{Z}\Gamma} C^*_r(\Gamma)$ is $C^*_r(\Gamma)$-chain homotopy equivalent to a $C^*_r(\Gamma)$-chain complex D_* whose m-th differential $d_m : D_m \to D_{m-1}$ vanishes.*

Lemma 2.3 in Leichtnam-Lott-Piazza [72] shows that this assumption is equivalent to Assumption 10.19. Notice that Assumption 11.1 is equivalent to the assertion that the m-th Novikov-Shubin invariant of $\overline{\partial M}$ is ∞^+ in the sense of Lott-Lück [87], Definition 1.8, 2.1 and 3.1.

Under Assumption 11.1 we shall now assign to (M, r) an element

$$\sigma(M,r) \in K_0(C^*_r(\Gamma)), \tag{11.2}$$

Fix a chain homotopy equivalence $u : C_*(\overline{\partial M}) \otimes_{\mathbb{Z}\Gamma} C^*_r(\Gamma) \to D_*$ as in Assumption 11.1. Define $\overline{D}_*$ as the quotient chain complex of D_* such that $\overline{D}_i = D_i$ if $0 \leqslant i \leqslant m-1$ and $\overline{D}_i = 0$ for $i \geqslant m$. One then gets a Poincaré pair $j_* : D_* \to \overline{D}_*$ whose boundary is D_*. By glueing [109] $j_* : D_* \to \overline{D}_*$ with the Poincaré pair

$$i_* : C_*(\overline{\partial M}) \otimes_{\mathbb{Z}\Gamma} C^*_r(\Gamma) \to C_*(\overline{M}) \otimes_{\mathbb{Z}\Gamma} C^*_r(\Gamma)$$

with the help of u (along the boundary $C_*(\overline{\partial M})$) one gets a true Poincaré complex whose signature in $L^0(\mathbb{C}\Gamma)$ is denoted $\sigma_{\mathbb{C}\Gamma}(M,r)$. Our symmetric signature $\sigma(M,r) \in K_0(C_r^*\Gamma)$ is the image of this class under the composition

$$L^0(\mathbb{C}\Gamma) \to L^0(C_r^*\Gamma) \leftrightarrow K_0(C_r^*\Gamma).$$

This construction of the invariant $\sigma(M,r)$ by glueing algebraic Poincaré bordisms is motivated by and extends the one of Weinberger [122] (see also [86], Appendix A) who uses the more restrictive assumption that $C_*(\overline{\partial M}) \otimes_{\mathbb{Z}\Gamma} C_r^*(\Gamma)$ is $C_r^*(\Gamma)$-chain homotopy equivalent to a $C_r^*(\Gamma)$-chain complex D_* with $D_m = 0$. In fact, when $D_m = 0$ the invariant $\sigma(M,r)$ coincides with the one of Weinberger [122]. The relationship to symmetric signatures of manifolds-with-boundary, and to the necessity of Assumption 11.1, was pointed out by Weinberger (see [86], Section 4.1).

We will call $\sigma(M,r) \in K_0(C_r^*\Gamma)$ the $C_r^*\Gamma$-valued symmetric signature of (M,r). When ∂M is empty, this element $\sigma(M,r)$ agrees with the (Mischenko) symmetric signature we defined in 7.4. See also [109], p. 26 on this point.

11.2. Properties of the symmetric signature.

The main properties of this invariant will be that it occurs in a glueing formula, is a homotopy invariant and is related to higher signatures. More precisely:

THEOREM 11.3.

(a) *Glueing formula.*

Let M and N be two oriented compact $2m$-dimensional manifolds with boundary and let $\phi : \partial M \to \partial N$ be an orientation preserving diffeomorphism. Let $r : M \cup_\phi N^- \to B\Gamma$ be a reference map. Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption 11.1. Then

$$\sigma(M \cup_\phi N^-, r) = \sigma(M, r|_M) - \sigma(N, r|_N) \quad \text{in} \quad K_0(C_r^*\Gamma);$$

(b) *Cut-and-Paste invariance.*

Let M and N be two oriented compact $2m$-dimensional manifolds with boundary and let $\phi, \psi : \partial M \to \partial N$ be orientation preserving diffeomorphisms. Let

$(r : M \cup_\phi N^- \to B\Gamma)$ and $(s : M \cup_\psi N^- \to B\Gamma)$ be cut-and-paste equivalent.

Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption 11.1. Then

$$\sigma(M \cup_\phi N^-, r) = \sigma(M \cup_\psi N^-, s) \quad \text{in} \quad K_0(C^*_r\Gamma);$$

(c) *Homotopy invariance.*

Let M_0 and M_1 be two oriented compact $2m$-dimensional manifolds possibly with boundaries together with reference maps $r_i : M_i \to B\Gamma$ for $i = 0, 1$. Let $(f, \partial f) : (M_0, \partial M_0) \to (M_1, \partial M_1)$ be an orientation preserving homotopy equivalence of pairs with $r_1 \circ f \simeq r_0$. Suppose that $(\partial M_0, r_0|_{\partial M_0})$ satisfies Assumption 11.1. Then

$$\sigma(M_0, r_0) = \sigma(M_1, r_1).$$

The crux of the proof is Ranicki [109], Proposition 1.8.2 ii) and the underlying philosophical idea is the following: if $M, N,$ and D are compact oriented manifolds with boundary such that $\partial M = \partial N = \partial D$ then $M \cup D^- - N \cup D^-$ is cobordant to $M \cup N^-$.

11.3. On the cut-and-paste invariance of higher signatures on closed manifolds.

From Theorem 11.3 (b), we obtain the following corollary which extends [72], Corollary 0.4, i.e Theorem 10.44 above, to more general groups Γ.

COROLLARY 11.4. — *Recall that $n = 2m$. Let M and N be two oriented compact n-dimensional manifolds with boundary and let $\phi, \psi : \partial M \to \partial N$ be orientation preserving diffeomorphisms. Let $(r : M \cup_\phi N^- \to B\Gamma)$ and $(s : M \cup_\psi N^- \to B\Gamma)$ be cut-and-paste equivalent. Assume that the Γ-covering associated to $r|_{\partial M} : \partial M \to B\Gamma$ satisfies Assumption 11.1. Suppose furthermore that the assembly map $\mu : K_n(B\Gamma) \to K_n(C^*_r(\Gamma))$ is rationally injective. Then for all $c \in H^*(B\Gamma, \mathbb{Q})$*

$$\operatorname{sign}(M \cup_\phi N^-, r; [c]) = \operatorname{sign}(M \cup_\psi N^-, s; [c]). \tag{11.5}$$

In words, under the stated assumptions the higher signatures are cut-and-paste invariant.

Proof. — Since $\mu_{\mathbb{R}}$ is assumed to be injective we know that the equality of the symmetric signatures implies the equality of all the higher signatures, see Proposition 7.12. From Theorem 11.3 (b) we get immediately the result. □

Remark. — We have already remarked that for groups having the extension property the map $\mu_{\mathbb{R}}$ is injective. Thus Corollary 11.4 is indeed a generalization of Theorem 10.44.

12. Higher spectral flow and cut-and-paste invariance.

In the subsections 10.10, 11.3 we have extended to the higher context the index theoretic and topological proof of the cut-and-paste invariance of the lower signature. The goal of this Section is to (briefly) present the higher analogue of the third and last approach, the one employing the notion of spectral flow. Our strategy is to show, *analytically*, that under the same assumptions of Theorem 11.3 (b) above, the signature index classes of two cut-and-paste equivalent coverings $(r : M \cup_\phi N^- \to B\Gamma)$ and $(s : M \cup_\psi N^- \to B\Gamma)$ are equal in $K_*(C^*_r\Gamma)$. By Proposition 7.11 this will reprove Corollary 11.4.

We shall follow [79]. Notice that Michel Hilsum has also obtained these results by using the Kasparov intersection product and a somewhat different approach to boundary value problems in the noncommutative context. See [52].

12.1. Higher spectral flow.

First of all we need a definition for the *higher spectral flow*. This was defined in the family-case by Dai and Zhang, [34], and extended to the noncommutative context by F. Wu [124] and Leichtnam-Piazza [75] [79]. Let $(N, s : N \to B\Gamma)$ be an odd dimensional Galois covering and let $\mathcal{D}_{(N,s)}$ a generalized $C^*_r\Gamma$-linear Dirac operator. We assume that $\operatorname{Ind} \mathcal{D}^{\text{sign}}_{(N,s)} = 0$ in $K_1(C^*_r\Gamma)$. This is the case, for example, if $(N, s : N \to B\Gamma) = (\partial M, r_\partial : \partial M \to B\Gamma)$, with $(M, r : M \to B\Gamma)$ a Galois covering with boundary. According to Theorem 10.16 there exists spectral sections for $\mathcal{D}_{(N,r)}$. Recall that given two spectral section $\mathcal{Q}$ and $\mathcal{P}$, the difference class $[\mathcal{P} - \mathcal{Q}] \in K_0(C^*_r\Gamma)$ is well defined.

Assume now that we have a continuous one-parameter family of such operators, parametrized by a continuous family of inputs (metrics, connections, etc.); we denote by $(\mathcal{D}_u)_{u\in[0,1]}$ such a family. Recall that for any C^*-algebra Λ there exists an isomorphism $\mathcal{U} : K_1(C^0([0,1]; \mathbb{C}) \otimes \Lambda) \simeq$

$K_1(\Lambda)$ which is implemented by the evaluation map $f(\cdot) \otimes \lambda \to f(0)\lambda$. Using the above isomorphism $\mathcal{U}$ for $\Lambda = C_r^*\Gamma$, one gets that the index class associated to the $C^0[0,1] \otimes C_r^*\Gamma$-linear operator $(\mathcal{D}_u)_{u\in[0,1]}$ vanishes in $K_1(C^0([0,1]) \otimes \Lambda)$. Thus according to Theorem 10.16 the family $(\mathcal{D}_u)_{u\in[0,1]}$ admits a (total) spectral section $\mathcal{P} = (\mathcal{P}_u)_{u\in[0,1]}$.

DEFINITION 12.1. — *If $\mathcal{Q}_0$ (resp. $\mathcal{Q}_1$) is a spectral section associated with $\mathcal{D}_0$ (resp. $\mathcal{D}_1$) then the noncommutative (or higher) spectral flow* $\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_1)$ *from $(\mathcal{D}_0, \mathcal{Q}_0)$ to $(\mathcal{D}_1, \mathcal{Q}_1)$ through $(\mathcal{D}_u)_{u\in[0,1]}$ is the $K_0(C_r^*\Gamma)$-class:*

$$\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_1) = [\mathcal{Q}_1 - \mathcal{P}_1] - [\mathcal{Q}_0 - \mathcal{P}_0] \in K_0(C_r^*\Gamma).$$

This definition does not depend on the particular choice of the total spectral section $\mathcal{P} = (\mathcal{P})_{u\in[0,1]}$.

Theorem 1.4 in Dai-Zhang [34] proves that if Γ is trivial and $\mathcal{Q}_0 = \Pi_{\geqslant}(0)$, $\mathcal{Q}_0 = \Pi_{\geqslant}(1)$, then the above definition agrees with the usual one (net number of eigenvalues changing sign).

If the family is *periodic* (i.e. $\mathcal{D}_1 = \mathcal{D}_0$) and if we take $\mathcal{Q}_1 = \mathcal{Q}_0$ then the spectral flow $\mathrm{sf}((\mathcal{D}_u)_{u\in[0,1]}; \mathcal{Q}_0, \mathcal{Q}_0)$ does not depend on the choice of $\mathcal{Q}_1 = \mathcal{Q}_0$ and defines a K-theory class which is intrinsically associated to the given periodic family; we shall denote this class by $\mathrm{sf}((\mathcal{D}_u)_{u\in S^1})$.

More generally we can consider a periodic family of operators $(\mathcal{D}_u)$ as above but acting on the fibers of a fiber bundle $P \longrightarrow S^1$ with fibers diffeomorphic to our manifold M. Also in this case there is a well-defined noncommutative spectral flow $\mathrm{sf}((\mathcal{D}_u)_{u\in S^1}) \in K_0(C_r^*\Gamma)$. We shall encounter an example of this more general situation in the coming subsections.

12.2. The defect formula for cut-and-paste equivalent coverings.

The higher spectral flow fits into a variational formula for APS index classes; this formula is the analogue of formula (3.7) in subsection 3.3.1. Thus let $(\mathcal{D}_{(M,r)}(u))_{u\in[0,1]}$ be a 1-paramater family of $C_r^*\Gamma$-linear operator on a covering with boundary. Let $(\mathcal{D}_{(\partial M,r_\partial)}(u))_{u\in[0,1]}$ be the associated boundary family. Fix a spectral section $\mathcal{Q}_0$ for $\mathcal{D}_{(\partial M,r_\partial)}(0)$ and a spectral section for $\mathcal{D}_{(\partial M,r_\partial)}(1)$. Then the APS index classes $\mathrm{Ind}(\mathcal{D}_{(M,r)}(1), \mathcal{Q}_1)$ and

$\operatorname{Ind}(\mathcal{D}_{(M,r)}(0), \mathcal{Q}_0)$ are well defined in $K_0(C_r^*\Gamma)$ and the following formula holds:

(12.2)

$$\operatorname{Ind}(\mathcal{D}_{(M,r)}(1), \mathcal{Q}_1) - \operatorname{Ind}(\mathcal{D}_{(M,r)}(0), \mathcal{Q}_0) = \operatorname{sf}((\mathcal{D}_{(\partial M, r_\partial)}(u))_{u\in[0,1]}; \mathcal{Q}_1, \mathcal{Q}_0) \quad \text{in } K_0(C_r^*\Gamma)$$

Next, the gluing formula (3.9) given for the numeric indeces in subsubsection 3.3.3 can be extended to index classes. We state it directly for the signature operator: if

$$X = M \cup_F N^-, \quad \text{with} \quad F = \partial M = -\partial N^-$$

and $r : X \to B\Gamma$ is a classifying map, then

(12.3)

$$\operatorname{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(X,r)}\right) = \operatorname{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(M,r_{|M})}, \mathcal{P}\right) + \operatorname{Ind}\left(\mathcal{D}^{\mathrm{sign}}_{(N^-,r_{|N^-})}, \operatorname{Id} - \mathcal{P}\right), \text{ in } K_0(C_r^*\Gamma)$$

with $\mathcal{P}$ a spectral section for $\mathcal{D}^{\mathrm{sign}}_{(\partial M, r_{|\partial M})}$. This formula can be extended to $X_\phi = M \cup_\phi N^-$ with $\phi : \partial M \to \partial N$ an oriented diffeomorphism. Using these two formulae and proceeding as in the numeric case one can prove a *defect formula* for the difference $\operatorname{Ind}(\mathcal{D}^{\mathrm{sign}}_{(X_\phi,r)}) - \operatorname{Ind}(\mathcal{D}^{\mathrm{sign}}_{(X_\psi,s)})$, in $K_0(C_r^*\Gamma)$, associated to two cut-and-paste equivalent coverings $r : X_\phi := M \cup_\phi N^- \to B\Gamma$ and $s : X_\psi := M \cup_\psi N^- \to B\Gamma$:

THEOREM 12.4. — *There exists a periodic family of twisted signature operators on $F = \partial M$, $\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}$, such that*

$$\operatorname{Ind}\mathcal{D}^{\mathrm{sign}}_{(X_\phi,r)} - \operatorname{Ind}\mathcal{D}^{\mathrm{sign}}_{(X_\psi,s)} = \operatorname{sf}(\{\mathcal{D}_F(\theta)\}_{\theta\in S^1}) \text{ in } K_0(C_r^*\Gamma) \tag{12.5}$$

The family appearing on the right hand side of (12.5) is a S^1-family acting on the fibers of the mapping torus $M(F, \phi^{-1}\circ\psi) \to S^1$.

12.3. Vanishing higher spectral flow and the cut-and-paste invariance.

The equality of the index class with the Mishchenko symmetric signature, and the example given in section 9, show together that the right hand side of formula (12.5) is in general different from zero. This is in contrast with the numeric case. The following result is proved by making use of the *symmetric spectral sections* we alluded to in subsection 10.7.

THEOREM 12.6. — *Let M and N be two oriented compact $2m$-dimensional manifolds with boundary and let $\phi, \psi : \partial M \to \partial N$ be orientation preserving diffeomorphisms. We let $F = \partial M$. Let*

$(r : M \cup_\phi N^- \to B\Gamma)$ *and* $(s : M \cup_\psi N^- \to B\Gamma)$

be cut-and-paste equivalent coverings.

Suppose that $(\partial M, r|_{\partial M})$ satisfies Assumption 10.19. Then

$$\mathrm{sf}(\{\mathcal{D}_F(\theta)\}_{\theta \in S^1}) = 0 \quad in \quad K_0(C^*_r\Gamma) \tag{12.7}$$

Consequently, by 12.4, the signature index classes of $(r : M \cup_\phi N^- \to B\Gamma)$ and $(s : M \cup_\psi N^- \to B\Gamma)$ coincide. Thus, by Proposition 7.11, if the assembly map is rationally injective then for all $c \in H^*(B\Gamma, \mathbb{C})$

$$\mathrm{sign}(M \cup_\phi N^-, r; [c]) = \mathrm{sign}(M \cup_\psi N^-, s; [c]). \tag{12.8}$$

13. Open problems.

I. Let (M, r) be an even dimensional oriented manifold with boundary such that Assumption 10.19 (or 11.1) is satisfied. Then one observes that the $C^*_r\Gamma$-valued symmetric signature class $\sigma(M, r)$ constructed in [73] (see Subsection 11.1) and the signature index class of [72] $\mathrm{Ind}\, \mathcal{D}_C^{\mathrm{sign}, \lfloor, +}$ (see Subsection 10.9.1) have the same gluing and homotopy invariance properties. Moreover, when $\partial M = \emptyset$, these two classes coincide: see Theorem 7.5. Therefore it is natural to *conjecture* that

$$\sigma(M, r) = \mathrm{Ind}\, \mathcal{D}_C^{\mathrm{sign}, b, +} \quad \text{in} \quad K_0(C^*_r\Gamma). \tag{13.1}$$

II. Let $(M, \mathcal{F})$ and $(N, \mathcal{F}')$ be two foliated manifolds with boundary such that the leaves are even-dimensional oriented and transverse to the boundary. Then $\mathcal{F}$ has a product structure near ∂M. One should try to formulate for $(\partial M, \mathcal{F}_{|\partial M})$ an assumption analogous to 10.19 and then define for $(M, \mathcal{F})$ a signature index class which should be a leafwise homotopy invariant (see Baum-Connes [10] for the boundaryless case). Now let ϕ and ψ be two diffeomorphisms from ∂M to ∂N sending a leaf of $\mathcal{F}_{|\partial M}$ onto a leaf of $\mathcal{F}'_{|\partial N}$ and preserving the orientation. Then one gets two closed foliated manifolds $(M \cup_\phi N^-, \mathcal{F}_\phi)$ and $(M \cup_\psi N^-, \mathcal{F}_\psi)$. Let q denote the common codimension of $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ and consider the two corresponding Haefliger classifying maps (see [10], page 11):

$$h_\phi : M \cup_\phi N^- \to B\Gamma_q, \; h_\psi : M \cup_\psi N^- \to B\Gamma_q.$$

Then for each $\alpha \in H^*(B\Gamma_q, \mathbb{Q})$ one should try to compare

$$\int_{M\cup_\phi N^-} L(M \cup_\phi N^-) \cup h^*_\phi(\alpha) \text{ and } \int_{M\cup_\psi N^-} L(M \cup_\psi N^-) \cup h^*_\psi(\alpha).$$

Remark. — For the particular case of foliated bundles see the recent paper [81].

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