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## SEQUENCE ENTROPY PAIRS AND COMPLEXITY PAIRS FOR A MEASURE

by Wen HUANG, Alejandro MAASS(\*) and Xiangdong YE (\*\*)

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### 1. Introduction.

Ergodic theory and topological dynamics exhibit a remarkable parallelism. Classical examples are the concepts of ergodicity, weak mixing and mixing in ergodic theory which can be considered as the analogues of transitivity, topological weak mixing and topological mixing in topological dynamics; or topological entropy and measure theoretical entropy which are related through the variational principle. This parallelism has allowed to tackle purely topological problems using measure-theoretical arguments, and in some cases it has become the only way to do it. In other cases it is the topological concept which has induced the measure-theoretical result.

The present work follows the research line developed since the introduction in topological dynamics of the concept of u.p.e. systems in [B1] which is an analogue of measure-theoretical K-systems in ergodic theory. In particular u.p.e. systems are disjoint from all minimal zero topological entropy systems [B2]. This last work is the starting point to the theory of topological entropy pairs which allowed to localize topological entropy and in [BL] served to construct the maximal zero topological entropy factor of a system, which corresponds to a parallel notion for the Pinsker factor.

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Later on, Glasner and Weiss [GW] have shown that if a topological dynamical system admits a K-measure with full support then it is u.p.e. In [B-R] the authors were able to define entropy pairs for a measure in topological dynamics and have shown that the set of entropy pairs for an invariant measure is contained in the set of topological entropy pairs, generalizing the result in [GW]. Also entropy pairs for a measure allowed to construct the maximal topological factor of zero measure theoretical entropy. It is a measure-theoretical factor of the Pinsker factor and it is a topological extension of the maximal zero topological entropy factor. In [BGH] it is shown the converse result of [B-R] stating a deep refinement of the classical variational principle. Characterizing the set of entropy pairs for an invariant measure as the support of some measure, Glasner [G1] has shown that the product of two u.p.e. systems is u.p.e. The same characterization has been fundamental to prove that Li-Yorke chaos is implied by positive topological entropy in [BGKM]. Recently in [HY] concepts of topological and measure theoretical entropy pairs were generalized to entropy tuples.

Following the idea of entropy pairs in order to consider systems with zero topological entropy one can also define complexity pairs [BHM] and sequence entropy pairs [H-Y]. It turns out that a system is topologically weakly mixing if and only if each pair (not in the diagonal) is a sequence entropy pair and for each system there is a maximal null factor (sequence entropy is zero for each sequence). In ergodic theory the topological concepts of maximal null factor and maximal equicontinuous factor are related with the Kronecker factor. In this paper we explore topological factors in between the Kronecker and the maximal equicontinuous factor and maximal null factor. In this purpose we introduce sequence entropy tuples for a measure and we show that the set of sequence entropy tuples for a measure is contained in the set of topological sequence entropy tuples. The reciprocal is not true. Moreover, we show that for each system there is a maximal  $M$ -null factor, that is, for each invariant measure the sequence entropy with respect to the measure is zero. We also define  $M$ -supe systems (which can be seen as “dual” to  $M$ -null systems) and show that the product of such systems is of the same type and each  $M$ -supe system is disjoint from any  $M$ -null system.

In the last section we introduce two notions of complexity pairs for a measure and we study their relation with sequence entropy pairs and topological complexity pairs. We prove that in general the strongest notion is strictly contained in between sequence entropy pairs and topological complexity pairs.

## 2. Preliminaries.

By a *topological dynamical system* (t.d.s.) we mean a pair  $(X, T)$ , where  $X$  is a compact metric space and  $T : X \rightarrow X$  is a homeomorphism from  $X$  to  $X$ . The set of  $T$ -invariant probability measures defined on Borel sets of  $X$ ,  $\mathcal{B}(X)$ , is denoted by  $\mathcal{M}(X, T)$ . In this context measurability will be always related to  $\mathcal{B}(X)$ . A probability measure  $\mu \in \mathcal{M}(X, T)$  induces a *measure theoretical dynamical system* (m.t.d.s.)  $(X, \mathcal{B}(X), \mu, T)$  (or just  $(X, \mu, T)$ ), that is,  $T : X \rightarrow X$  is measurable and  $T\mu = \mu$ . In this article we assume all sigma algebras are complete.

Let  $\mathcal{A} = \{0 \leq t_1 < t_2 < \dots\} \subseteq \mathbb{N}$  be an increasing sequence of natural numbers and  $\mathcal{U}$  be a finite cover of  $X$ . The *topological sequence entropy* of  $\mathcal{U}$  with respect to  $(X, T)$  along  $\mathcal{A}$  is defined by

$$h_{\text{top}}^{\mathcal{A}}(T, \mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N} \left( \bigvee_{i=1}^n T^{-t_i} \mathcal{U} \right),$$

where  $\mathcal{N}(\bigvee_{i=1}^n T^{-t_i} \mathcal{U})$  is the minimal cardinality among all cardinalities of sub-covers of  $\bigvee_{i=1}^n T^{-t_i} \mathcal{U}$ . The *topological sequence entropy* of  $(X, T)$  along  $\mathcal{A}$  is

$$h_{\text{top}}^{\mathcal{A}}(T) = \sup_{\mathcal{U}} h_{\text{top}}^{\mathcal{A}}(T, \mathcal{U}),$$

where supremum is taken over all finite open covers of  $X$  (that is, made of open sets). If  $\mathcal{A} = \mathbb{N}$  we recover standard topological entropy. In this case we omit the superscript  $\mathbb{N}$ .

Analogously, given  $\mu \in \mathcal{M}(X, T)$  and  $\alpha$  a finite measurable partition of  $X$  we define the *sequence entropy* of  $\alpha$  with respect to  $(X, \mu, T)$  along  $\mathcal{A}$  by

$$h_{\mu}^{\mathcal{A}}(T, \alpha) = \limsup_{n \rightarrow \infty} -\frac{1}{n} \sum_{A \in \bigvee_{i=1}^n T^{-t_i} \alpha} \mu(A) \log \mu(A).$$

The *sequence entropy* of  $(X, T, \mu)$  along  $\mathcal{A}$  is

$$h_{\mu}^{\mathcal{A}}(T) = \sup_{\alpha} h_{\mu}^{\mathcal{A}}(T, \alpha),$$

where supremum is taken over all finite measurable partitions. As in the topological case, when  $\mathcal{A} = \mathbb{N}$  we recover entropy of  $T$  with respect to  $\mu$ . In this case we omit the superscript  $\mathbb{N}$ . For the classical theory of measure-theoretical entropy see [P] and classical theory of topological entropy can be found in [DGS]. For sequence entropy see [Ku].

Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . On the complex Hilbert space  $H = L^2(X, \mathcal{B}(X), \mu)$  we define the unitary operator  $U_T : H \rightarrow H$  by  $U_T(f) = f \circ T$ . We recall the spectral mixing theorem of Koopman-Von Neumann (see [Be]).

PROPOSITION 2.1. — *The Hilbert space  $H$  can be decomposed as  $H = H_K \oplus H_{wm}$ , where*

$$H_K = \overline{\text{Span}\{f \in H : \exists \lambda \in \mathbb{C}, U_T(f) = \lambda f\}}, \text{ and}$$

$$H_{wm} = \{f \in H : \exists S \subseteq \mathbb{N}, d(S) = 0, \forall g \in H, \lim_{n \rightarrow \infty, n \in S^c} \langle U_T^n f, g \rangle = 0\},$$

where  $d(S)$  is the density of  $S$  and  $\langle \cdot, \cdot \rangle$  is the inner product of  $H$ .

It is known (see [Hu]) that there exists a  $T$ -invariant  $\sigma$ -algebra  $\mathcal{D}_\mu \subseteq \mathcal{B}(X)$  such that  $L^2(X, \mathcal{D}_\mu, \mu) = H_K$ . In fact,  $\mathcal{D}_\mu = \{A \subset X : 1_A \in H_K\}$  and  $(X, \mathcal{D}_\mu, \mu, T)$  is the Kronecker factor of  $(X, \mu, T)$  [Kr]. For any  $A \in \mathcal{D}_\mu$  we have  $h_\mu^A(T, \{A, A^c\}) = 0$  for any increasing sequence of natural numbers  $\mathcal{A}$  (see [Ku]). We also recall that  $(X, \mu, T)$  is weakly mixing if and only if  $H_K$  is one-dimensional if and only if  $\mathcal{D}_\mu$  is trivial. If  $H_K = H$ , we then say that  $(X, \mu, T)$  has discrete spectrum.

If  $(X, T)$  is a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ , then for any finite measurable partition  $\alpha$  of  $X$ , it holds  $\lim_{n \rightarrow \infty} h_\mu(T^n, \alpha) = H_\mu(\alpha | \mathbb{P}_\mu)$ , where  $\mathbb{P}_\mu$  is the Pinsker  $\sigma$ -algebra of  $(X, \mu, T)$ . Theorem 2.3 below states the same kind of property for  $\mathcal{D}_\mu$ .

LEMMA 2.2. — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . For any finite measurable partition  $\alpha$  of  $X$  and any increasing sequence of natural numbers  $\mathcal{A}$ ,  $h_\mu^{\mathcal{A}}(T, \alpha) \leq H_\mu(\alpha | \mathcal{D}_\mu)$ . Moreover, for any sequence  $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ ,*

$$\limsup_{j \rightarrow \infty} \frac{1}{n_j} H_\mu \left( \bigvee_{i=1}^{n_j} T^{-t_i} \alpha \right) \leq H_\mu(\alpha | \mathcal{D}_\mu).$$

*Proof.* — Since  $(X, \mathcal{B}(X))$  is separable there exists a countable set of finite  $\mathcal{D}_\mu$ -measurable partitions  $\{\beta_k : k \in \mathbb{N}\}$  such that  $\lim_{k \rightarrow \infty} H_\mu(\alpha | \beta_k) = H_\mu(\alpha | \mathcal{D}_\mu)$ . Thus for a fixed  $k \in \mathbb{N}$  and  $\mathcal{A} = \{0 \leq t_1 < t_2 < \dots\} \subseteq \mathbb{N}$

$$h_\mu^{\mathcal{A}}(T, \alpha) = \limsup_{n \rightarrow \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=1}^n T^{-t_i} \alpha \right)$$

$$\leq \limsup_{n \rightarrow \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=1}^n T^{-t_i} (\alpha \vee \beta_k) \right) - \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=1}^n T^{-t_i} \beta_k \right)$$

$$\begin{aligned}
 &= \limsup_{n \rightarrow \infty} \frac{1}{n} \left( H_\mu \left( \bigvee_{i=1}^n T^{-t_i} (\alpha \vee \beta_k) \right) - H_\mu \left( \bigvee_{i=1}^n T^{-t_i} \beta_k \right) \right) \\
 &= \limsup_{n \rightarrow \infty} \frac{1}{n} \left( H_\mu \left( \bigvee_{i=1}^n T^{-t_i} \alpha \mid \bigvee_{i=1}^n T^{-t_i} \beta_k \right) \right) \\
 &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{i=1}^n H_\mu(T^{-t_i} \alpha \mid T^{-t_i} \beta_k) \right) \\
 &= H_\mu(\alpha \mid \beta_k)
 \end{aligned}$$

where in the first inequality we use that  $h_\mu^A(T, \beta_k) = 0$  for  $\beta_k \subseteq \mathcal{D}_\mu$ . Since  $k$  is arbitrary we get  $h_\mu^A(T, \alpha) \leq H_\mu(\alpha \mid \mathcal{D}_\mu)$ . The second statement of the lemma follows by what we have proved.  $\square$

Now we prove,

**THEOREM 2.3.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . Given a finite measurable partition  $\alpha$  of  $X$  there exists an increasing sequence of natural numbers  $\mathcal{A} = \{0 \leq t_1 < t_2 < \dots\}$  such that  $h_\mu^A(T, \alpha) = H_\mu(\alpha \mid \mathcal{D}_\mu)$ .*

*Proof.* — Let us remark that for any  $A \in \mathcal{B}(X)$ ,  $1_A - \mathbb{E}(1_A \mid \mathcal{D}_\mu) \in H_{wm}$ . Therefore, there exists  $S \subset \mathbb{N}$  with  $d(S) = 0$  such that

$$\lim_{n \rightarrow \infty, n \in S^c} \langle U_T^n(1_A - \mathbb{E}(1_A \mid \mathcal{D}_\mu)), 1_B \rangle = 0$$

for all  $B \in \mathcal{B}(X)$ .

**Claim.** — For any finite measurable partition  $\beta$  of  $X$  and  $\epsilon > 0$ , there exist  $S \subset \mathbb{N}$  with  $d(S) = 0$  and  $M \in \mathbb{N}$  (depending on  $\beta$  and  $\epsilon$ ) such that when  $m \geq M, m \in S^c$ , one has  $H_\mu(T^{-m} \alpha \mid \beta) \geq H_\mu(\alpha \mid \mathcal{D}_\mu) - \epsilon$ .  $\square$

*Proof of the claim.* — Put  $\alpha = \{A_1, A_2, \dots, A_k\}$  and  $\beta = \{B_1, B_2, \dots, B_l\}$ . By previous remark, there exists  $S \subset \mathbb{N}$ ,  $d(S) = 0$ , such that for all  $1 \leq i \leq k, 1 \leq j \leq l$ ,

$$\lim_{n \rightarrow \infty, n \in S^c} \langle U_T^n(1_{A_i} - \mathbb{E}(1_{A_i} \mid \mathcal{D}_\mu)), 1_{B_j} \rangle = 0.$$

Hence

$$\begin{aligned}
 &\liminf_{n \rightarrow \infty, n \in S^c} H_\mu(T^{-n} \alpha \mid \beta) \\
 &= \liminf_{n \rightarrow \infty, n \in S^c} \sum_{i,j} -\mu(T^{-n} A_i \cap B_j) \log \left( \frac{\mu(T^{-n} A_i \cap B_j)}{\mu(B_j)} \right) \\
 &= \liminf_{n \rightarrow \infty, n \in S^c} \sum_{i,j} -\langle U_T^n \mathbb{E}(1_{A_i} \mid \mathcal{D}_\mu), 1_{B_j} \rangle \log \left( \frac{\langle U_T^n \mathbb{E}(1_{A_i} \mid \mathcal{D}_\mu), 1_{B_j} \rangle}{\mu(B_j)} \right).
 \end{aligned}$$

Let  $a_{ij} = -\langle U_T^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu), 1_{B_j} \rangle \log \left( \frac{\langle U_T^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu), 1_{B_j} \rangle}{\mu(B_j)} \right)$  and  $\mu_{B_j}(\cdot) = \mu(\cdot \cap B_j) / \mu(B_j)$ . From concavity of  $-x \log x$  we deduce that

$$a_{ij} \geq \int_{B_j} -U_T^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu) \log(U_T^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu)) d\mu.$$

We conclude that

$$\sum_{i,j} a_{ij} \geq \sum_i \int_X -U_T^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu) \log(U_T^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu)) d\mu = H_\mu(\alpha | \mathcal{D}_\mu)$$

and

$$\liminf_{n \rightarrow \infty, n \in S^c} H_\mu(T^{-n} \alpha | \beta) \geq H_\mu(\alpha | \mathcal{D}_\mu).$$

This finishes the proof of the claim.

Now we can define an increasing sequence of natural numbers  $\mathcal{A} = \{0 < t_1 < t_2 < \dots\}$  such that

$$H_\mu \left( T^{-t_n} \alpha \mid \bigvee_{i=1}^{n-1} T^{-t_i} \alpha \right) \geq H_\mu(\alpha | \mathcal{D}_\mu) - \frac{1}{2^n}.$$

As  $H_\mu(\bigvee_{i=1}^n T^{-t_i} \alpha) = H_\mu(\bigvee_{i=1}^{n-1} T^{-t_i} \alpha) + H_\mu(T^{-t_n} \alpha | \bigvee_{i=1}^{n-1} T^{-t_i} \alpha)$ , therefore

$$\begin{aligned} h_\mu^{\mathcal{A}}(T, \alpha) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H_\mu \left( T^{-t_k} \alpha \mid \bigvee_{i=1}^{k-1} T^{-t_i} \alpha \right) \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left( H_\mu(\alpha | \mathcal{D}_\mu) - \frac{1}{2^k} \right) = H_\mu(\alpha | \mathcal{D}_\mu). \quad \square \end{aligned}$$

In fact Lemma 2.2 and Theorem 2.3 can be stated for any m.t.d.s. in the general sense and thus Theorem 2.3 generalizes Theorem 3 of [Hu]. Moreover, the following corollary follows directly from Lemma 2.2 and Theorem 2.3.

**COROLLARY 2.4.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . Then,  $A \in \mathcal{D}_\mu$  if and only if  $h_\mu^{\mathcal{A}}(T, \{A, A^c\}) = 0$  for any increasing sequence of natural numbers  $\mathcal{A}$ . Particularly,  $(X, \mu, T)$  has discrete spectrum if and only if  $h_\mu^{\mathcal{A}}(T, \{A, A^c\}) = 0$  for any increasing sequence of natural numbers  $\mathcal{A}$  and  $A \in \mathcal{B}(X)$ .*

### 3. Sequence entropy $n$ -tuple.

Let us begin with some additional notations. Given a t.d.s.  $(X, T)$  and an integer  $n \geq 2$ , the  $n$ -th product system is the t.d.s.  $(X^{(n)}, T^{(n)})$  where  $X^{(n)}$  is the cartesian product of  $X$  with itself  $n$  times and  $T^{(n)}$  represents the simultaneous action of  $T$  in each coordinate of  $X^{(n)}$ . The product  $\sigma$ -algebra of  $X^{(n)}$  is denoted by  $\mathcal{B}^{(n)}$  and its diagonal by  $\Delta_n(X) = \{(x, \dots, x) \in X^{(n)} : x \in X\}$ .

Let  $\mu \in \mathcal{M}(X, T)$ . Define the measure  $\lambda_n(\mu)$  on  $\mathcal{B}^{(n)}$  by letting  $\lambda_n(\mu)(\prod_{i=1}^n A_i) = \int_X \prod_{i=1}^n \mathbb{E}(1_{A_i} | \mathcal{D}_\mu) d\mu$ , where  $\mathcal{D}_\mu$  is the  $T$ -invariant  $\sigma$ -algebra defined in Section 2.

Let  $(x_i)_{i=1}^n \in X^{(n)}$ . A finite cover of  $X$ ,  $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$ , is said to be an *admissible cover* with respect to  $(x_i)_{i=1}^n$  if for each  $1 \leq j \leq k$  there exists  $1 \leq i_j \leq n$  such that  $x_{i_j}$  is not contained in the closure of  $U_j$ . Analogously we define admissible partitions with respect to  $(x_i)_{i=1}^n$ .

DEFINITION 3.1. — Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . An  $n$ -tuple  $(x_i)_{i=1}^n \in X^{(n)}$ ,  $n \geq 2$ , is called

(1) a *sequence entropy  $n$ -tuple* if for some  $1 \leq i, j \leq n$ ,  $x_i \neq x_j$ , and for any admissible open cover  $\mathcal{U}$  with respect to  $(x_i)_{i=1}^n$  there exists an increasing sequence of natural numbers  $\mathcal{A}$  such that  $h_{\text{top}}^{\mathcal{A}}(T, \mathcal{U}) > 0$ ;

(2) a *sequence entropy  $n$ -tuple for  $\mu$*  if for some  $1 \leq i, j \leq n$ ,  $x_i \neq x_j$ , and for any admissible Borel partition  $\alpha$  with respect to  $(x_i)_{i=1}^n$  there exists an increasing sequence of natural numbers  $\mathcal{A}$  such that  $h_{\mu}^{\mathcal{A}}(T, \alpha) > 0$ .

We denote by  $SE_n(X, T)$  the set of sequence entropy  $n$ -tuples and by  $SE_n^{\mu}(X, T)$  the set of sequence entropy  $n$ -tuples for  $\mu$ . Sequence entropy 2-tuples are called sequence entropy pairs. The notion of sequence entropy pair was introduced in [H-Y] to study weak mixing property of t.d.s.

The following proposition states the basic properties of sequence entropy tuples. The proof is similar to the proof of the corresponding result in [B2].

PROPOSITION 3.2. — Let  $(X, T)$  be a t.d.s.

(a) If  $\mathcal{U} = \{U_1, \dots, U_n\}$  is an open cover of  $X$  with  $h_{\text{top}}^{\mathcal{A}}(T, \mathcal{U}) > 0$  for some increasing sequence of natural numbers  $\mathcal{A}$ , then for all  $1 \leq i \leq n$  there exists  $x_i \in U_i^c$  such that  $(x_i)_{i=1}^n$  is a sequence entropy  $n$ -tuple.

(b)  $SE_n(X, T) \cup \Delta_n(X)$  is a closed  $T^{(n)}$ -invariant subset of  $X^{(n)}$ .



(c) Let  $\pi : (Y, S) \longrightarrow (X, T)$  be a factor map of t.d.s.

(1) If  $(x_i)_{i=1}^n \in SE_n(X, T)$ , then for all  $1 \leq i \leq n$  there exists  $y_i \in Y$  such that  $\pi(y_i) = x_i$  and  $(y_i)_{i=1}^n \in SE_n(Y, S)$ .

(2) If  $(y_i)_{i=1}^n \in SE_n(Y, S)$  and  $(\pi(y_i))_{i=1}^n \notin \Delta_n(X)$ , then  $(\pi(y_i))_{i=1}^n \in SE_n(X, T)$ .

(d) Suppose  $W$  is a closed  $T$ -invariant subset of  $(X, T)$ . If  $(x_i)_{i=1}^n$  is a sequence entropy  $n$ -tuple of  $(W, T|_W)$ , then it is also a sequence entropy  $n$ -tuple of  $(X, T)$ .

Now we begin the study of the structure of  $SE_n^\mu(X, T)$ .

LEMMA 3.3. — Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . If  $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$  is a measurable cover of  $X$  with  $n \geq 2$ , then  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) > 0$  if and only if for any measurable finite partition  $\alpha$  finer than  $\mathcal{U}$  as a cover, there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^\mathcal{A}(T, \alpha) > 0$ .

*Proof.*

(i) Assume that for any measurable finite partition  $\alpha$  finer than  $\mathcal{U}$  as a cover, there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^\mathcal{A}(T, \alpha) > 0$  and  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) = 0$ .

Let  $C_i = \{x \in X : \mathbb{E}(1_{U_i^c} | \mathcal{D}_\mu) > 0\} \in \mathcal{D}_\mu$ , for  $1 \leq i \leq n$ . Therefore  $\mu(U_i^c \setminus C_i) = \int_{C_i^c} \mathbb{E}(1_{U_i^c} | \mathcal{D}_\mu) d\mu = 0$ . Put  $D_i = C_i \cup (U_i^c \setminus C_i)$ , then  $D_i \in \mathcal{D}_\mu$  and  $D_i^c \subset U_i$ .

For any  $s = (s(1), s(2), \dots, s(n)) \in \{0, 1\}^n$  let  $D_s = \bigcap_{i=1}^n D_i(s(i))$ , where  $D_i(0) = D_i$  and  $D_i(1) = D_i^c$ . Set  $D_0^j = (\bigcap_{i=1}^n D_i) \cap (U_j \setminus \bigcup_{k=1}^{j-1} U_k)$  for  $1 \leq j \leq n$ . Consider the measurable partition

$$\alpha = \{D_s : s \in \{0, 1\}^n \setminus \{(0, 0, \dots, 0)\}\} \cup \{D_0^1, D_0^2, \dots, D_0^n\}.$$

For any  $s \in \{0, 1\}^n$  with  $s \neq (0, 0, \dots, 0)$  one has  $s(i) = 1$  for some  $1 \leq i \leq n$ , then  $D_s \subset D_i^c \subset U_i$ . It is straightforward that for  $1 \leq j \leq n$ ,  $D_0^j \subset U_j$ . Thus  $\alpha$  is finer than  $\mathcal{U}$  and there exists by hypothesis a sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^\mathcal{A}(T, \alpha) > 0$ .

On the other hand, since  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) = 0$ , we get  $\mu(\bigcap_{i=1}^n D_i) = \mu(\bigcap_{i=1}^n C_i) = 0$ . Thus one has  $D_0^1, D_0^2, \dots, D_0^n \in \mathcal{D}_\mu$ . It is also clear that  $D_s \in \mathcal{D}_\mu$  for  $s \in \{0, 1\}^n \setminus \{(0, 0, \dots, 0)\}$  since  $D_1, D_2, \dots, D_n \in \mathcal{D}_\mu$ . Therefore each element of  $\alpha$  is  $\mathcal{D}_\mu$ -measurable and, by Lemma 2.2,  $h_\mu^\mathcal{A}(T, \alpha) \leq H_\mu(\alpha | \mathcal{D}_\mu) = 0$ , which is a contradiction.

(ii) Assume  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) > 0$ . Without loss of generality we may assume that any finite measurable partition  $\alpha$  which is finer than  $\mathcal{U}$  as a cover is of the type  $\alpha = \{A_1, A_2, \dots, A_n\}$  with  $A_i \subset U_i$ , for  $1 \leq i \leq n$ . Let  $\alpha$  be one of such partitions. We observe that

$$\int_X \prod_{i=1}^n \mathbb{E}(1_{A_i^c} | \mathcal{D}_\mu) d\mu \geq \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c} | \mathcal{D}_\mu) d\mu = \lambda_n(\mu) \left( \prod_{i=1}^n U_i^c \right) > 0.$$

Therefore,  $A_j \not\subset \mathcal{D}_\mu$  for some  $1 \leq j \leq n$ . We conclude by Theorem 2.3 that there exists a sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) = H_\mu(\alpha | \mathcal{D}_\mu) > 0$ . This finishes the proof.  $\square$

*Remark.* — For a measurable partition  $\alpha = \{A_1, A_2, \dots, A_n\}$  of  $X$  with  $n \geq 2$ . By Lemma 3.3, it is easy to see that  $\lambda_n(\mu)(\prod_{i=1}^n A_i^c) > 0$  if and only if there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ . In particular, for  $\alpha = \{A, A^c\}$ ,  $\lambda_2(\mu)(A \times A^c) > 0$  if and only if there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ .

**THEOREM 3.4.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . Then for any  $n \geq 2$ ,*

$$SE_n^\mu(X, T) = \text{supp}(\lambda_n(\mu)) \setminus \Delta_n(X).$$

*Proof.*

(i) Let  $(x_i)_{i=1}^n \in SE_n^\mu(X, T)$ . To show  $(x_i)_{i=1}^n \in \text{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$  we only need to prove that for any neighborhood  $\prod_{i=1}^n U_i$  of  $(x_i)_{i=1}^n$ ,  $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$ .

Set  $\mathcal{U} = \{U_1^c, U_2^c, \dots, U_n^c\}$ . Without loss of generality we may assume that  $\mathcal{U}$  is a measurable cover of  $X$ . It is clear that any measurable partition  $\alpha$  finer than  $\mathcal{U}$  as a cover is an admissible partition with respect to  $(x_i)_{i=1}^n$ . Therefore, there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ . By Lemma 3.3,  $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$ .

(ii) Let  $(x_i)_{i=1}^n \in \text{supp}(\lambda_n(\mu)) \setminus \Delta_n(X)$ . We will show that for any admissible partition  $\alpha = \{A_1, A_2, \dots, A_k\}$  with respect to  $(x_i)_{i=1}^n$  there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ .

Since  $\alpha$  is an admissible partition with respect to  $(x_i)_{i=1}^n$  there exist closed neighborhoods  $U_i$  of  $x_i$ ,  $1 \leq i \leq n$ , such that for each  $j \in \{1, 2, \dots, k\}$  we find  $i_j \in \{1, 2, \dots, n\}$  with  $A_j \subset U_{i_j}^c$ . That is,  $\alpha$  is finer than  $\mathcal{U} = \{U_1^c, U_2^c, \dots, U_n^c\}$  as a cover. Since  $\lambda_n(\mu)(\prod_{i=1}^n U_i) > 0$ , by Lemma 3.3, there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ .  $\square$

**THEOREM 3.5.** — Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . Let  $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$  be a measurable cover of  $X$  with  $n \geq 2$ . If  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) > 0$ , then there exists an increasing sequence of natural numbers  $\mathcal{A} = \{0 < t_1 < t_2 < \dots\}$  such that

$$(i) \inf_{\beta \succeq \mathcal{U}} h_\mu^A(T, \beta) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \inf_{\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}} H_\mu(\beta) > 0,$$

$$(ii) h_{\text{top}}^A(T, \mathcal{U}) > 0.$$

Thus, using Lemma 3.3 we have that  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) > 0$  if and only if there exists an increasing sequence of natural numbers  $\mathcal{A} = \{0 < t_1 < t_2 < \dots\}$  such that  $\inf_{\beta \succeq \mathcal{U}} h_\mu^A(T, \beta) > 0$  if and only if there exists an increasing sequence of natural numbers  $\mathcal{A} = \{0 \leq t_1 < t_2 < \dots\}$  such that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \inf_{\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}} H_\mu(\beta) > 0$ .

*Proof.* — Since  $\lambda_n(\mu)(\prod_{i=1}^n U_i^c) = \int_X \prod_{i=1}^n \mathbb{E}(1_{U_i^c} | \mathcal{D}_\mu) d\mu > 0$ , there is  $M \in \mathbb{N}$  such that  $\mu(D'_M) > 0$ , where

$$D'_M = \left\{ x \in X : \min_{1 \leq i \leq n} \mathbb{E}(1_{U_i^c} | \mathcal{D}_\mu)(x) \geq \frac{2}{M} \right\}.$$

For any  $s = (s(1), s(2), \dots, s(n)) \in \{0, 1\}^n$  set  $A_s = \bigcap_{i=1}^n U_i(s(i))$ , where  $U_i(0) = U_i$  and  $U_i(1) = U_i^c$ , and put  $\alpha = \{A_s : s \in \{0, 1\}^n\}$ .

Let  $(\gamma_j)_{j \in \mathbb{N}} \subseteq \mathcal{D}_\mu$  be an increasing sequence of finite  $\sigma$ -algebras with  $\bigvee_{j=1}^\infty \gamma_j = \mathcal{D}_\mu$ . By the martingale theorem,  $\lim_{j \rightarrow \infty} \mathbb{E}(1_{U_i^c} | \gamma_j) = \mathbb{E}(1_{U_i^c} | \mathcal{D}_\mu)$ ,  $j \in \{1, 2, \dots, n\}$ , in the sense of  $L^2(\mu, \mathcal{B}(X), \mu)$ . Hence there exists  $\gamma = \gamma_j$  for some  $j \in \mathbb{N}$  such that:

$$(1) \mu(D_M) > \frac{\mu(D'_M)}{2}, \text{ where } D_M = \{x \in X : \min_{1 \leq i \leq n} \mathbb{E}(1_{U_i^c} | \gamma)(x) \geq \frac{1}{M}\};$$

$$(2) H_\mu(\alpha | \gamma) < H_\mu(\alpha | \mathcal{D}_\mu) + \frac{\mu(D'_M)}{4M} \log\left(\frac{n}{n-1}\right).$$

The following property holds. □

**Claim.** —  $H_\mu(\alpha | \beta \vee \gamma) \leq H_\mu(\alpha | \gamma) - \frac{\mu(D_M)}{M} \log\left(\frac{n}{n-1}\right)$  for any finite measurable partition  $\beta$  which is finer than  $\mathcal{U}$  as a cover.

*Proof of the claim.* — Without loss of generality let  $\beta = \{B_1, B_2, \dots, B_n\}$  with  $B_i \subseteq U_i$ ,  $1 \leq i \leq n$ . Let  $\phi(x) = -x \log x$  for  $x > 0$  and  $\phi(0) = 0$ . Then

$$H_\mu(\alpha | \beta \vee \gamma) = \sum_{s \in \{0, 1\}^n} \int_X \sum_{i, s(i)=0} \mathbb{E}(1_{B_i} | \gamma) \phi\left(\frac{\mathbb{E}(1_{A_s \cap B_i} | \gamma)}{\mathbb{E}(1_{B_i} | \gamma)}\right) d\mu,$$

where we have used that for any  $s \in \{0, 1\}^n$  and  $1 \leq i \leq n$ , if  $s(i) = 1$  then  $A_s \cap B_i = \emptyset$ . Thus

$$\begin{aligned} &H_\mu(\alpha|\beta \vee \gamma) \\ &\leq \sum_{s \in \{0,1\}^n} \int_X \left( \sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma) \right) \phi \left( \sum_{i,s(i)=0} \frac{\mathbb{E}(1_{B_i}|\gamma)}{\sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma)} \frac{\mathbb{E}(1_{A_s \cap B_i}|\gamma)}{\mathbb{E}(1_{B_i}|\gamma)} \right) d\mu \\ &= \sum_{s \in \{0,1\}^n} \int_X \left( \sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma) \right) \phi \left( \frac{\mathbb{E}(1_{A_s}|\gamma)}{\sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma)} \right) d\mu \\ &= \sum_{s \in \{0,1\}^n} \left( \int_X \phi(\mathbb{E}(1_{A_s}|\gamma)) d\mu - \int_X \mathbb{E}(1_{A_s}|\gamma) \log \left( \frac{1}{\sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma)} \right) d\mu \right) \\ &= H_\mu(\alpha|\gamma) - \sum_{s \in \{0,1\}^n} \int_X \mathbb{E}(1_{A_s}|\gamma) \log \left( \frac{1}{\sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma)} \right) d\mu. \end{aligned}$$

We observe that if  $s(i) = 1$  for some  $1 \leq i \leq n$  then  $\sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma) \leq \mathbb{E}(1_{B_i^c}|\gamma)$ . Therefore,

$$\begin{aligned} &\sum_{s \in \{0,1\}^n} \int_X \mathbb{E}(1_{A_s}|\gamma) \log \left( \frac{1}{\sum_{k,s(k)=0} \mathbb{E}(1_{B_k}|\gamma)} \right) d\mu \\ &\geq \frac{1}{n} \sum_{i=1}^n \int_X \left( \sum_{s,s(i)=1} \mathbb{E}(1_{A_s}|\gamma) \right) \log \left( \frac{1}{\mathbb{E}(1_{B_i^c}|\gamma)} \right) d\mu \\ &= \frac{1}{n} \sum_{i=1}^n \int_X \mathbb{E}(1_{U_i^c}|\gamma) \log \left( \frac{1}{\mathbb{E}(1_{B_i^c}|\gamma)} \right) d\mu \\ &\geq \frac{1}{nM} \sum_{i=1}^n \int_{D_M} \log \left( \frac{1}{\mathbb{E}(1_{B_i^c}|\gamma)} \right) d\mu \\ &\geq \frac{1}{M} \int_{D_M} \log \left( \frac{n}{\sum_{i=1}^n \mathbb{E}(1_{B_i^c}|\gamma)} \right) d\mu \\ &= \frac{\mu(D_M)}{M} \log \left( \frac{n}{n-1} \right). \end{aligned}$$

Hence  $H_\mu(\alpha|\beta \vee \gamma) \leq H_\mu(\alpha|\gamma) - \frac{\mu(D_M)}{M} \log \left( \frac{n}{n-1} \right)$ . This ends the proof of the claim.

Put  $\epsilon = \frac{\mu(D_M)}{M} \log \left( \frac{n}{n-1} \right) > 0$ . By Theorem 2.3, there exists an increasing sequence of natural numbers  $\mathcal{A} = \{0 \leq t_1 < t_2 < \dots\}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) = H_\mu(\alpha|\mathcal{D}_\mu)$ .

Let  $n \in \mathbb{N}$  and  $\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}$ . Since  $T^{t_i} \beta$  is finer than  $\mathcal{U}$  for  $i \in \{1, \dots, n\}$ , one has

$$\begin{aligned}
H_\mu(\beta) &\geq H_\mu\left(\beta \middle| \bigvee_{i=1}^n T^{-t_i} \gamma\right) \\
&= H_\mu\left(\beta \vee \bigvee_{i=1}^n T^{-t_i} \alpha \middle| \bigvee_{i=1}^n T^{-t_i} \gamma\right) - H_\mu\left(\bigvee_{i=1}^n T^{-t_i} \alpha \middle| \beta \vee \bigvee_{i=1}^n T^{-t_i} \gamma\right) \\
&\geq H_\mu\left(\bigvee_{i=1}^n T^{-t_i} \alpha \middle| \bigvee_{i=1}^n T^{-t_i} \gamma\right) - \sum_{i=1}^n H_\mu(\alpha | T^{t_i} \beta \vee \gamma) \\
&\geq H_\mu\left(\bigvee_{i=1}^n T^{-t_i} \alpha\right) - H_\mu\left(\bigvee_{i=1}^n T^{-t_i} \gamma\right) - n\left(H_\mu(\alpha | \mathcal{D}_\mu) - \frac{\epsilon}{2}\right).
\end{aligned}$$

As  $h_\mu^A(T, \alpha) = H_\mu(\alpha | \mathcal{D}_\mu)$  and  $h_\mu^A(T, \gamma) \leq H_\mu(\gamma | \mathcal{D}_\mu) = 0$ , we have

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \frac{1}{n} \inf_{\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}} H_\mu(\beta) \\
&\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \left[ H_\mu\left(\bigvee_{i=1}^n T^{-t_i} \alpha\right) - H_\mu\left(\bigvee_{i=1}^n T^{-t_i} \gamma\right) - n\left(H_\mu(\alpha | \mathcal{D}_\mu) - \frac{\epsilon}{2}\right) \right] \\
&= -h_\mu^A(\gamma, T) + h_\mu^A(\alpha, T) - H_\mu(\alpha | \mathcal{D}_\mu) + \frac{\epsilon}{2} = \frac{\epsilon}{2} > 0.
\end{aligned}$$

Clearly,

$$\inf_{\beta \succeq \mathcal{U}} h_\mu^A(T, \beta) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \inf_{\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}} H_\mu(\beta) > 0.$$

Now we show (ii). For each  $n \in \mathbb{N}$ , there exists a finite measurable partition  $\beta_n \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}$  such that  $\log \mathcal{N}(\bigvee_{i=1}^n T^{-t_i} \mathcal{U}) \geq H_\mu(\beta_n)$ . Therefore

$$\begin{aligned}
h_{\text{top}}^A(T, \mathcal{U}) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^n T^{-t_i} \mathcal{U}\right) \\
&\geq \limsup_{n \rightarrow \infty} \frac{1}{n} H_\mu(\beta_n) > 0 \text{ (by (i)).} \quad \square
\end{aligned}$$

Now we can state a relation between sequence entropy tuples for a measure and sequence entropy tuples.

**COROLLARY 3.6.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . Then for each  $n \geq 2$ ,  $SE_n^\mu(X, T) \subset SE_n(X, T)$ .*

*Proof.* — Let  $(x_i)_{i=1}^n \in SE_n^\mu(X, T)$  and  $\mathcal{U}$  be any finite open cover of  $X$  admissible with respect to  $(x_i)_{i=1}^n$ . It is easy to see that any measurable

finite partition  $\alpha$  finer than  $\mathcal{U}$  as a cover, is an admissible partition with respect to  $(x_i)_{i=1}^n$ . By Definition 3.1, there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ . By Lemma 3.3 and Theorem 3.5, there exists an increasing sequence  $\mathcal{A} \subset \mathbb{N}$  such that  $h_{\text{top}}^{\mathcal{A}}(T, \mathcal{U}) > 0$ . Hence  $(x_i)_{i=1}^n \in SE_n(X, T)$ . This finishes the proof of the theorem.  $\square$

The following property states the way sequence entropy tuples for a measure pass through factors.

**THEOREM 3.7.** — *Let  $\pi : (X, T) \longrightarrow (Y, S)$  be a factor map of t.d.s.,  $\mu \in \mathcal{M}(X, T)$  and  $\nu = \pi(\mu)$ .*

(1) *For every  $(x_i)_{i=1}^n \in SE_n^\mu(X, T)$  let  $\pi(x_i) = y_i, i \in \{1, 2, \dots, n\}$ . If  $(y_i)_{i=1}^n \notin \Delta_n(Y)$ , then  $(y_i)_{i=1}^n \in SE_n^\nu(Y, S)$ .*

(2) *For every  $(y_i)_{i=1}^n \in SE_n^\nu(Y, S)$ , there exists  $(x_i)_{i=1}^n \in SE_n^\mu(X, T)$  with  $\pi(x_i) = y_i, i \in \{1, 2, \dots, n\}$ .*

*Proof.*

(1) It is direct from the definition, so we omit the proof.

(2) Let  $\mathcal{Z} = \pi^{-1}(\mathcal{D}_\nu)$ . We then have  $\mathcal{Z} = \pi^{-1}(\mathcal{B}(Y)) \cap \mathcal{D}_\mu$ .

Let  $(y_i)_{i=1}^n \in SE_n^\nu(Y, S)$ . Take any closed neighborhood  $V_i$  of  $y_i, i \in \{1, 2, \dots, n\}$ , with  $\bigcap_{i=1}^n V_i = \emptyset$ , then  $\lambda_n(\nu)(V_1 \times V_2 \times \dots \times V_n) > 0$  by Theorem 3.4. Let  $U_i = \pi^{-1}(V_i), i \in \{1, 2, \dots, n\}$ . We have the following property.

**Claim.** —  $\lambda_n(\mu)(U_1 \times U_2 \times \dots \times U_n) > 0$ .

*Proof of the claim.* — Assume  $\lambda_n(\mu)(U_1 \times U_2 \times \dots \times U_n) = 0$ . Let  $\mathcal{U} = \{U_1^c, U_2^c, \dots, U_n^c\}$ . Since  $\bigcap_{i=1}^n U_i = \emptyset, \mathcal{U}$  is a measurable cover of  $X$ . By Lemma 3.3, there exists a measurable partition  $\alpha = \{A_1, A_2, \dots, A_n\}$  of  $X$  with  $A_i \subset U_i^c$ , such that  $h_\mu^{\mathcal{A}}(T, \alpha) = 0$  for any increasing sequence of natural numbers  $\mathcal{A}$ . By Theorem 2.3,  $A_i \in \mathcal{D}_\mu$  for  $i \in \{1, 2, \dots, n\}$ .

It is well known that  $f \in L^2(X, \mathcal{D}_\mu, \mu)$  if and only if  $\text{cl}(\{T^k f : k \in \mathbb{Z}\})$  is a compact subset of  $L^2(X, \mathcal{B}(X), \mu)$ . Let  $f \in L^2(X, \mathcal{D}_\mu, \mu)$ , then  $\text{cl}(\{T^k f : k \in \mathbb{Z}\})$  is a compact subset of  $L^2(X, \mathcal{B}(X), \mu)$ . Since for any  $g \in L^2(X, \mathcal{B}(X), \mu)$  one has  $\|\mathbb{E}(g|\pi^{-1}(\mathcal{B}(Y)))\|_{L^2(X, \mathcal{B}(X), \mu)} \leq \|g\|_{L^2(X, \mathcal{B}(X), \mu)}$ ,  $\text{cl}(\{T^k \mathbb{E}(f|\pi^{-1}(\mathcal{B}(Y))) : k \in \mathbb{Z}\})$  is also a compact subset of  $L^2(X, \mathcal{B}(X), \mu)$ . Thus  $\mathbb{E}(f|\pi^{-1}(\mathcal{B}(Y))) \in L^2(X, \mathcal{D}_\mu, \mu)$ , i.e.  $\mathbb{E}(f|\pi^{-1}(\mathcal{B}(Y))) \in L^2(X, \mathcal{Z}, \mu)$ . In particular,  $B_i = \{x \in X : \mathbb{E}(1_{A_i}|\pi^{-1}(\mathcal{B}(Y))) > 0\} \in \mathcal{Z}$ , Moreover there

exists  $C_i \in \mathcal{D}_\nu$  such that  $B_i = \pi^{-1}(C_i)$ ,  $i \in \{1, 2, \dots, n\}$ .

Since  $\mathbb{E}(1_{A_i} | \pi^{-1}(\mathcal{B}(Y))) \leq \mathbb{E}(1_{U_i^c} | \pi^{-1}(\mathcal{B}(Y))) = 1_{U_i^c}$ , one has  $B_i \subset U_i^c$  and  $C_i \subset V_i^c$ . Observe that  $\sum_{i=1}^n \mathbb{E}(1_{A_i} | \pi^{-1}(\mathcal{B}(Y))) = 1$ , then  $\bigcup_{i=1}^n B_i = X$  and  $\bigcup_{i=1}^n C_i = Y$ . Take  $D_1 = C_1$ ,  $D_l = C_l \setminus \bigcup_{i=1}^{l-1} C_i$ ,  $l \in \{2, 3, \dots, n\}$ . We have that  $D_i \in \mathcal{D}_\nu$ ,  $i \in \{1, 2, \dots, n\}$ , and  $Q = \{D_1, D_2, \dots, D_n\}$  is a measurable partition of  $Y$  finer than  $\{V_1^c, V_2^c, \dots, V_n^c\}$ . By Theorem 2.3 and Lemma 3.3,  $\lambda_n(\nu)(V_1 \times V_2 \times \dots \times V_n) = 0$ , which contradicts the fact  $\lambda_n(\nu)(V_1 \times V_2 \times \dots \times V_n) > 0$ . This finishes the proof of the claim.

Now, from the claim,  $\text{supp}(\lambda_n(\mu)) \cap (U_1 \times U_2 \times \dots \times U_n) \neq \emptyset$ . As  $V_i$  is any closed neighborhood of  $y_i$ ,  $i \in \{1, 2, \dots, n\}$ , one deduces  $\text{supp}(\lambda_n(\mu)) \cap (\pi^{-1}(y_1) \times \pi^{-1}(y_2) \times \dots \times \pi^{-1}(y_n)) \neq \emptyset$ , that is, there exists  $(x_i)_{i=1}^n \in SE_n^\mu(X, T)$  with  $\pi(x_i) = y_i$ ,  $i \in \{1, 2, \dots, n\}$ .  $\square$

#### 4. $M$ -supe and maximal $M$ -null factor.

Applying the results obtained in the previous sections we now introduce the notions of  $M$ -supe and  $M$ -null systems. We remark that the smallest closed and invariant equivalence relation containing entropy pairs for a measure [B-R] define the maximal zero measure-theoretical entropy topological factor of a system. Here, using sequence entropy pairs for a measure we obtain the maximal  $M$ -null factor, one possible topological version of the Kronecker factor. But since there is no variational principle for the sequence entropy, the maximal  $M$ -null factor is not necessarily the maximal equicontinuous factor, even the maximal null factor.

Set

$$SE_n^M(X, T) = \overline{\bigcup_{\mu \in \mathcal{M}(X, T)} SE_n^\mu(X, T) \setminus \Delta_n(X)}.$$

By Corollary 3.6 we know that  $SE_n^M(X, T) \subset SE_n(X, T)$ . Any topologically weakly mixing t.d.s.  $(X, T)$  with a unique measure  $\mu \in \mathcal{M}(X, T)$  supported on a fixed point is an example for  $SE_n^M(X, T) = \emptyset$  and  $SE_n(X, T) = X^{(n)} \setminus \Delta_n(X)$  for any  $n \geq 2$  (see [H-Y]). Moreover, we have the following result.

**THEOREM 4.1.** — *Let  $(X, T)$  be a t.d.s. and  $n \geq 2$ . There is  $\nu \in \mathcal{M}(X, T)$  such that  $SE_n^\nu(X, T) = SE_n^M(X, T)$ . Hence,  $SE_n^M(X, T) =$*

$X^{(n)} \setminus \Delta_n(X)$  if and only if there is  $\nu \in \mathcal{M}(X, T)$  such that  $SE_n^\nu(X, T) = X^{(n)} \setminus \Delta_n(X)$ .

*Proof.* — Let  $((x_1^m, x_2^m, \dots, x_n^m))_{m \in \mathbb{N}}$  be a dense sequence of points in  $SE_n^M(X, T)$  with  $(x_1^m, x_2^m, \dots, x_n^m) \in SE_n^{\mu_m}(X, T)$  for some  $\mu_m \in \mathcal{M}(X, T)$ .

Let  $\nu = \sum_{m=1}^\infty \frac{1}{2^m} \mu_m$ . Since for any finite measurable partition  $\alpha$  of  $X$ , any increasing sequence of positive integers  $\mathcal{A}$  and  $m \in \mathbb{N}$ ,  $h_\nu^A(T, \alpha) \geq \frac{1}{2^m} h_{\mu_m}^A(T, \alpha)$ , then  $SE_n^{\mu_m}(X, T) \subset SE_n^\nu(X, T)$ . In particular,  $(x_1^m, x_2^m, \dots, x_n^m) \in SE_n^\nu(X, T)$ . We conclude,

$$SE_n^\nu(X, T) \supset \overline{\{(x_1^m, x_2^m, \dots, x_n^m) : m \geq 1\}} \setminus \Delta_n(X) = SE_n^M(X, T).$$

That is,  $SE_n^\nu(X, T) = SE_n^M(X, T)$ . □

*Remark 4.2.* — As mentioned before any topologically weakly mixing system  $(X, T)$  with a unique invariant measure  $\mu$  supported on a fixed point is an example for which  $SE_2(X, T) = X^{(2)} \setminus \Delta_2(X)$  and  $SE_2^\mu(X, T) = \emptyset$ . This shows that we cannot always find an invariant measure verifying  $SE_2^\mu(X, T) = SE_2(X, T)$ . An important reason is that there is no variational principle for sequence entropy [G]. According to [G] there is even a strictly ergodic system for which  $SE_2^\mu(X, T) \neq SE_2(X, T)$ .

We say  $(X, T)$  is *M-supe* if there is  $\mu \in \mathcal{M}(X, T)$  such that  $SE_2^\mu(X, T) = X^{(2)} \setminus \Delta_2(X)$ . Thus  $(X, T)$  is *M-supe* if and only if there is  $\mu \in \mathcal{M}(X, T)$  such that for any topologically non-trivial measurable partition  $\alpha$  made by two elements there is an increasing sequence of natural numbers  $\mathcal{A}$  such that  $h_\mu^A(T, \alpha) > 0$ . We observe that by a topologically non-trivial partition we mean a partition such that none of the atoms is dense in  $X$ .

It is clear that *M-supe* implies *supe*, that is,  $SE_2(X, T) = X^{(2)} \setminus \Delta_2(X)$ , and hence topologically weak mixing according to [H-Y] and Corollary 3.6. Moreover, for an *M-supe* system,  $\text{supp}(\mu) = X$  where  $\mu$  is a measure such that  $SE_2^\mu(X, T) = X^{(2)} \setminus \Delta_2(X)$ . Saleski showed that  $(X, \mu, T)$  is measure-theoretical weakly mixing if and only if  $\sup_{\mathcal{A}} h_\mu^A(T, \alpha) = H_\mu(\alpha)$  for each non-trivial finite measurable partition  $\alpha$  [S] (see also [Hu]). Call a system  $(X, \mu, T)$  *seq-K* if for each non-trivial finite measurable partition  $\alpha$  there is an increasing sequence of natural numbers  $\mathcal{A}$  such that  $h_\mu^A(T, \alpha) > 0$ . Remark that  $(X, \mu, T)$  is measure-theoretical weakly mixing if and only if  $\mathcal{D}_\mu$  is trivial. Thus, by Corollary 2.4, *seq-K* implies measure-theoretical weak mixing.



On the other hand, by Theorem 2.3 and [S], if  $(X, \mu, T)$  is measure-theoretical weakly mixing then  $(X, \mu, T)$  is *seq-K*. So  $(X, \mu, T)$  is measure-theoretical weakly mixing if and only if it is *seq-K* and a *seq-K* system is *M-supe* with  $SE_2^\mu(X, T) = X^{(2)} \setminus \Delta_2(X)$ . We mention that *M-supe* does not imply *seq-K* since there is a u.p.e. system (see [B2] for a definition) without any ergodic invariant measure with full support [HY].

**THEOREM 4.3.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . If there is an increasing sequence of natural numbers  $\mathcal{A}$  and a non-trivial finite measurable partition  $\alpha$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ , then  $SE_2^\mu(X, T) \neq \emptyset$ . Thus  $SE_2^\mu(X, T) = \emptyset$  if and only if  $(X, \mu, T)$  has discrete spectrum.*

*Proof.* — Assume there is an increasing sequence of natural numbers  $\mathcal{A}$  and a non-trivial finite measurable partition  $\alpha$  such that  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ . By Lemma 2.2,  $H_\mu(\alpha|\mathcal{D}_\mu) \geq h_\mu^{\mathcal{A}}(T, \alpha) > 0$ . Thus, there is  $A \in \alpha$  with  $A \notin \mathcal{D}_\mu$ . Hence  $H_\mu(\{A, A^c\}|\mathcal{D}_\mu) > 0$ . Then we may assume  $\alpha = \{A, A^c\}$  and  $h_\mu^{\mathcal{A}}(T, \alpha) > 0$ . By the remark after Lemma 3.3,  $\lambda_2(\mu)(A \times A^c) > 0$ . It follows that  $\lambda_2(\mu)(X^{(2)} \setminus \Delta_2(X)) > 0$ , and thus  $\text{supp}(\lambda_2(\mu)) \cap (X^{(2)} \setminus \Delta_2(X)) \neq \emptyset$ , i.e.  $SE_2^\mu(X, T) \neq \emptyset$ .

By Kushnirenko [Ku],  $(X, \mu, T)$  has discrete spectrum if and only if for each increasing sequence of natural numbers  $\mathcal{A}$  and each non-trivial finite measurable partition  $\alpha$ ,  $h_\mu^{\mathcal{A}}(T, \alpha) = 0$ . Thus the second statement of the theorem follows from this fact and the first property.  $\square$

In [P], Parry showed that the Pinsker  $\sigma$ -algebra of the product of two measure-theoretical dynamical systems is the product of the coordinate Pinsker  $\sigma$ -algebras. This property also holds for  $\mathcal{D}_\mu$ . The proof is a consequence of a previous result of Furstenberg (see Theorem 9.20 [G2]).

**LEMMA 4.4.** — *Let  $(X, T)$  and  $(Y, S)$  be t.d.s. and  $\mu \in \mathcal{M}(X, T)$ ,  $\nu \in \mathcal{M}(Y, S)$ . Then  $\mathcal{D}_{\mu \times \nu} = \mathcal{D}_\mu \times \mathcal{D}_\nu$ .*

From this lemma we obtain the following theorem. It is analogous to Theorem 3 in [G1].

**THEOREM 4.5.** — *Let  $(X, T)$  and  $(Y, S)$  be t.d.s. If  $(X, T)$  and  $(Y, S)$  are *M-supe*, so does  $(X \times Y, T \times S)$ .*

*Proof.* — Since  $(X, T)$  and  $(Y, S)$  are *M-supe*, there exist  $\mu \in \mathcal{M}(X, T)$  and  $\nu \in \mathcal{M}(Y, S)$  such that  $\text{supp}(\lambda_2(\mu)) = X^{(2)}$  and  $\text{supp}(\lambda_2(\nu)) = Y^{(2)}$ . By Lemma 4.4, we have  $\mathcal{D}_{\mu \times \nu} = \mathcal{D}_\mu \times \mathcal{D}_\nu$ . Therefore, for any  $U_i \times V_i \in$

$\mathcal{B}(X) \times \mathcal{B}(Y), i \in \{1, 2\},$

$$\lambda_2(\mu \times \nu) \left( \prod_{i=1}^2 U_i \times V_i \right) = \lambda_2(\mu) \left( \prod_{i=1}^2 U_i \right) \lambda_2(\nu) \left( \prod_{i=1}^2 V_i \right).$$

If  $U_i$  and  $V_i$  are nonempty open sets,  $i \in \{1, 2\}$ , then

$$\lambda_2(\mu) \left( \prod_{i=1}^2 U_i \right) > 0, \quad \lambda_2(\nu) \left( \prod_{i=1}^2 V_i \right) > 0 \quad \text{and} \quad \lambda_2(\mu \times \nu) \left( \prod_{i=1}^2 U_i \times V_i \right) > 0.$$

Therefore,  $\text{supp}(\lambda_2(\mu \times \nu)) = (X \times Y)^{(2)}$ . That is,  $(X \times Y, T \times S)$  is  $M$ -supe. □

A t.d.s. is  $M$ -null if for each  $\mu \in \mathcal{M}(X, T)$  and each increasing sequence of natural numbers  $\mathcal{A}$ ,  $h_\mu^{\mathcal{A}}(T) = 0$ . It is easy to see that  $(X, T)$  is  $M$ -null if and only if  $SE_2^M(X, T) = \emptyset$ . It turns out that each t.d.s. has a maximal  $M$ -null factor.

**THEOREM 4.6.** — *Each t.d.s. has a maximal  $M$ -null factor.*

*Proof.* — Let  $(X, T)$  be a t.d.s. and  $R$  be the smallest closed invariant equivalence relation containing  $SE_2^M(X, T)$ . Then  $R$  induces a factor  $(Y, S)$  of  $(X, T)$ . Let  $\pi : (X, T) \rightarrow (Y, S)$  be such factor map. We now show that  $(Y, S)$  is the maximal  $M$ -null factor of  $(X, T)$ .

First  $SE_2^M(Y, S) = \emptyset$ . In fact, if  $SE_2^M(Y, S) \neq \emptyset$ , then there is  $\nu \in \mathcal{M}(Y, S)$  such that  $SE_2^\nu(Y, S) \neq \emptyset$ . Let  $(y_1, y_2) \in SE_2^\nu(Y, S)$ . By Theorem 3.7, there are  $\mu \in \mathcal{M}(X, T)$  and  $(x_1, x_2) \in SE_2^\mu(X, T)$  with  $\pi\mu = \nu$  and  $\pi(x_1, x_2) = (y_1, y_2)$ . Since  $(x_1, x_2) \in SE_2^M(X, T)$  we get  $y_1 = \pi(x_1) = \pi(x_2) = y_2$ , which is a contradiction.

Let us prove  $(Y, S)$  is maximal. Assume that  $(Z, W)$  is an  $M$ -null factor of  $(X, T)$  which is induced by a closed invariant equivalence relation  $R'$  on  $X$ . It is clear, by Theorem 3.7, that  $SE_2^M(X, T) \subseteq R'$ . Thus  $R \subset R'$  and  $(Z, W)$  is a factor of  $(Y, S)$ . □

Recall that  $E(X, T)$ ,  $SE(X, T)$  and  $\text{Com}(X, T)$  are the sets of entropy pairs [B2], sequence entropy pairs [H-Y] and complexity pairs [BHM] respectively. It is easy to see that  $E(X, T) \subset SE_2^M(X, T) \subseteq SE(X, T) \subseteq \text{Com}(X, T)$  and that they induce the maximal zero entropy factor, the maximal  $M$ -null factor, the maximal null factor and the maximal equicontinuous factor.

In [G, Proposition 6.2], Goodman presents an example of a strictly ergodic t.d.s.  $(X, T)$  such that  $SE(X, T) = \emptyset$  and  $\text{Com}(X, T) \neq \emptyset$ . It is

an open question (see [H-Y]) if there is a t.d.s with  $SE(X, T) = \emptyset$  and  $\text{Com}(X, T) = X^{(2)} \setminus \Delta_2(X)$ , that is, 2-scattering (see [BHM]).

*Example 4.7.* — There exists a t.d.s.  $(X, T)$  such that  $E(X, T) = \emptyset$  and  $SE_2^M(X, T) = X^{(2)} \setminus \Delta_2(X)$ , that is, it is  $M$ -supe.

*Proof.* — The Chacon's system can be defined as a subshift  $X$  of the fullshift on two symbols  $\{0, 1\}^{\mathbb{Z}}$ . First define the sequence of finite words,

$$A_0 = 0, A_1 = 0010, \dots, A_{n+1} = A_n A_n 1 A_n,$$

for  $n \in \mathbb{N}$ . Then  $X$  contains all the two-sided sequences of 0's and 1's such that any finite sub-word of them is a sub-word of one  $A_n$  for some  $n \in \mathbb{N}$ . It is known that the Chacon's system is strictly ergodic and weakly mixing with respect to the unique measure and it has zero entropy. Thus  $E(X, T) = \emptyset$  and  $SE_2^M(X, T) = X^{(2)} \setminus \Delta_2(X)$ .  $\square$

Disjointness of two t.d.s. is defined in [F]. Following ideas in [B2] which essentially needs the properties of sequence entropy pairs for a measure stated in Theorems 3.4 and 3.7 it is easy to prove the following theorem.

**THEOREM 4.8.** — *Each  $M$ -supe system is disjoint from any minimal  $M$ -null system.*

## 5. Complexity pairs for a measure.

In this section we introduce two notions of complexity pair for a measure and study their dynamical properties. We remark that we only emphasize the fact that this is an analogue notion of complexity pair in the measure-theoretical context, not the complexity of the system (see [Fe] for a global approach).

**DEFINITION 5.1.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ .*

(1) *We say that  $(x_1, x_2) \in X^{(2)}$  is a complexity pair if  $x_1 \neq x_2$  and if whenever  $U_1, U_2$ , are closed mutually disjoint neighborhoods of the points  $x_1$  and  $x_2$ , one has*

$$\lim_{n \rightarrow \infty} \mathcal{N} \left( \bigvee_{i=0}^{n-1} T^{-i} \{U_1^c, U_2^c\} \right) = \infty$$

(see [BHM]).

(2) We say that  $(x_1, x_2) \in X^{(2)}$  is a weak  $\mu$ -complexity pair if  $x_1 \neq x_2$  and for every Borel partition  $\alpha = \{A_1, A_2\}$  of  $X$  with  $x_i \in \text{int}(A_i)$ ,  $i \in \{1, 2\}$ ,

$$\lim_{n \rightarrow \infty} H_\mu \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) = \infty.$$

(3) We say that  $(x_1, x_2) \in X^{(2)}$  is a strong  $\mu$ -complexity pair if  $x_1 \neq x_2$  and if whenever  $U_1, U_2$ , are closed mutually disjoint neighborhoods of the points  $x_1$  and  $x_2$ , one has

$$\lim_{n \rightarrow \infty} \inf_{\beta \succeq \bigvee_{i=0}^{n-1} T^{-i} \{U_1^c, U_2^c\}} H_\mu(\beta) = \infty.$$

Denote by  $\text{Com}(X, T)$  the set of all complexity pairs, by  $\text{Com}_\mu^+(X, T)$  the set of all weak  $\mu$ -complexity pairs and by  $\text{Com}_\mu^-(X, T)$  the set of all strong  $\mu$ -complexity pairs. It is clear that  $\text{Com}_\mu^-(X, T) \subseteq \text{Com}_\mu^+(X, T)$ .

**THEOREM 5.2.** — *Let  $(X, T)$  be a t.d.s. and  $\mu \in \mathcal{M}(X, T)$ . Then*

$$SE_2^\mu(X, T) \subseteq \text{Com}_\mu^-(X, T) \subseteq \text{Com}(X, T).$$

*Proof.*

(i) Let  $(x_1, x_2) \in \text{Com}_\mu^-(X, T)$  and let  $U_1, U_2$  be closed mutually disjoint neighborhoods of points  $x_1$  and  $x_2$  respectively. Consider  $\mathcal{U} = \{U_1^c, U_2^c\}$ .

Observe that

$$\log \mathcal{N} \left( \bigvee_{i=0}^{n-1} T^{-i} \mathcal{U} \right) \geq \inf_{\beta \succeq \bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}} H_\mu(\beta)$$

and

$$\lim_{n \rightarrow \infty} \inf_{\beta \succeq \bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}} H_\mu(\beta) = \infty.$$

We conclude  $\lim_{n \rightarrow \infty} \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}) = \infty$ . Hence  $(x_1, x_2) \in \text{Com}(X, T)$  and  $\text{Com}_\mu^-(X, T) \subseteq \text{Com}(X, T)$ .

(ii) Let  $(x_1, x_2) \in SE_2^\mu(X, T)$ . Let  $U_1, U_2$  be closed mutually disjoint neighborhoods of points  $x_1$  and  $x_2$  respectively. Consider the open cover  $\mathcal{U} = \{U_1^c, U_2^c\}$ . Since  $(x_1, x_2) \in \text{supp}(\lambda_2(\mu))$  then  $\lambda_2(\mu)(U_1 \times U_2) > 0$ .

By Theorem 3.5 there exists an increasing sequence of natural numbers  $\mathcal{A}$  such that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \inf_{\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}} H_\mu(\beta) > 0$ . This implies  $\lim_{n \rightarrow \infty} \inf_{\beta \succeq \bigvee_{i=1}^n T^{-t_i} \mathcal{U}} H_\mu(\beta) = \infty$ . Hence  $(x_1, x_2) \in \text{Com}_\mu^-(X, T)$ .  $\square$

In the following we will discuss the relation between  $\text{Com}_\mu^+(X, T)$  and  $\text{Com}_\mu^-(X, T)$ . We will show that there exists a t.d.s  $(X, T)$  such that  $\text{Com}_\mu^+(X, T) = X^{(2)} \setminus \Delta_2(X)$  and  $\text{Com}_\mu^-(X, T) = \text{Com}(X, T) = \emptyset$ .

Let  $(X, T)$  be a t.d.s and  $\mu \in \mathcal{M}(X, T)$ . Recall that  $(X, \mu, T)$  is *totally ergodic* if  $(X, \mu, T^k)$  is ergodic for each  $k \in \mathbb{N}$ .

**THEOREM 5.3.** — *Let  $(X, T)$  be a t.d.s and  $\mu \in \mathcal{M}(X, T)$ . Assume  $(X, \mu, T)$  is totally ergodic with  $\text{supp}(\mu) = X$ . Then  $\text{Com}_\mu^+(X, T) = X^{(2)} \setminus \Delta_2(X)$ .*

*Proof.* — Let  $(x_1, x_2) \in X^{(2)} \setminus \Delta_2(X)$ . Let  $\alpha = \{A_1, A_2\}$  be a partition of  $X$  with  $x_1 \in \text{int}(A_1), x_2 \in \text{int}(A_2)$ . We will show that  $\lim_{n \rightarrow \infty} H_\mu(\bigvee_{i=0}^{n-1} T^{-i} \alpha) = \infty$ .

**Claim.** — For each  $\epsilon > 0$  there exists  $M \in \mathbb{N}$  such that for any  $k \geq M$  and  $s \in \{1, 2\}^k, \mu(\bigcap_{i=0}^{k-1} T^{-i} A_{s(i)}) < \epsilon$ .

*Proof of the claim.* — Assume there exists  $\epsilon > 0$  such that for any  $M \in \mathbb{N}$ , there is  $k \geq M$  and  $s_M \in \{1, 2\}^k$  with  $\mu(\bigcap_{i=0}^{k-1} T^{-i} A_{s_M(i)}) \geq \epsilon$ . In this case, it is easy to see that there exists  $s \in \{1, 2\}^{\mathbb{Z}}$  such that  $\mu(\bigcap_{i \in \mathbb{Z}} T^{-i} A_{s(i)}) \geq \epsilon$ .

Since  $\mu$  is ergodic,

$$(5.3.1) \quad \mu\left(\bigcup_{j \in \mathbb{Z}} \bigcap_{i \in \mathbb{Z}} T^{-i} A_{(\sigma^j s)(i)}\right) = 1,$$

where  $\sigma : \{1, 2\}^{\mathbb{Z}} \rightarrow \{1, 2\}^{\mathbb{Z}}$  is the shift map. Observe that for  $t, r \in \{1, 2\}^{\mathbb{Z}}$  with  $t \neq r$

$$(5.3.2) \quad \bigcap_{i \in \mathbb{Z}} T^{-i} A_{t(i)} \bigcap \bigcap_{i \in \mathbb{Z}} T^{-i} A_{r(i)} = \emptyset.$$

By (5.3.1) and (5.3.2),  $s$  is a periodic point of the fullshift  $(\{1, 2\}^{\mathbb{Z}}, \sigma)$ . Let  $k$  be the period of  $s$ . Since  $\mu(A_1) > 0, \mu(A_2) > 0$ , one deduces  $0 < \mu(\bigcap_{i \in \mathbb{Z}} T^{-i} A_{(\sigma^j s)(i)}) < 1$  for every  $j \in \mathbb{Z}$ . Finally, the fact  $T^{kn}(\bigcap_{i \in \mathbb{Z}} T^{-i} A_{s(i)}) = \bigcap_{i \in \mathbb{Z}} T^{-i} A_{s(i)}$  for  $n \in \mathbb{Z}$  contradicts the ergodicity of  $(X, \mu, T^k)$ . This concludes the proof of claim.  $\square$

For any  $n \in \mathbb{N}$ , by using the claim, there exists  $M_n \in \mathbb{N}$  such that for any  $k \geq M_n$  and  $s \in \{1, 2\}^k$ ,  $\mu(\bigcap_{i=0}^{k-1} T^{-i} A_{s(i)}) < \frac{1}{n}$ . Let  $t_s = \mu(\bigcap_{i=0}^{k-1} T^{-i} A_{s(i)})$ . Thus

$$H_\mu\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha\right) = \sum_{s \in \{1,2\}^k} -t_s \log t_s \geq \log n \sum_{s \in \{1,2\}^k} t_s = \log n.$$

This implies  $\lim_{n \rightarrow \infty} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha) = \infty$  and hence  $(x_1, x_2) \in \text{Com}_\mu^+(X, T)$ . □

*Remark 5.4.* — Shannon-McMillan-Breiman Theorem states that if  $T$  is an ergodic measure-preserving transformation of the probability space  $(X, \mathcal{B}, \mu)$ ,  $\alpha$  is a finite measurable partition of  $X$  and  $A_n(x)$  is the atom of the partition  $\bigvee_{i=0}^{n-1} T^{-i}\alpha$  to which  $x$  belongs, then

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(A_n(x)) = h_\mu(T, \alpha)$$

$\mu$ -a.e. and in  $L^1(X, \mathcal{B}, \mu)$ . Hence, if  $h_\mu(T, \alpha) > 0$  one can say that  $\mu(A_n(x))$  goes to zero with exponential rate  $e^{-h_\mu(T, \alpha)}$ . The claim we just proved adds information on  $\mu(A_n(x))$  when  $h_\mu(T, \alpha) = 0$  under the assumption of total ergodicity.

*Example 5.5.* — Let  $K$  be the unit circle in the complex plane and  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ . Set  $X = K$ ,  $T = R_\alpha : K \rightarrow K$  with  $T(z) = z \cdot e^{2\pi i \alpha}$  and let  $\mu$  be the Haar measure on  $K$ . Then  $(X, \mu, T)$  is totally ergodic with  $\text{supp}(\mu) = X$ . By Theorem 5.3,  $\text{Com}_\mu^+(X, T) = X^{(2)} \setminus \Delta_2(X)$ . Since  $(X, T)$  is equicontinuous,  $\text{Com}(X, T) = \emptyset$  [BHM]. Moreover, by Theorem 5.2,  $\text{Com}_\mu^-(X, T) = \emptyset$ .

**THEOREM 5.6.** — *Let  $(X, T)$  be a t.d.s. and let  $\mu$  be an ergodic measure of  $(X, T)$  with full support. If  $x_1$  is a fixed point for  $T$ , then for  $x_2 \in X$ ,  $x_2 \neq x_1$ , one has  $(x_1, x_2) \in \text{Com}_\mu^-(X, T)$ .*

*Proof.* — Take closed neighborhoods  $U_1, U_2$ , of  $x_1$  and  $x_2$  respectively with  $U_1 \cap U_2 = \emptyset$ . Put  $V_i = U_i^c$ ,  $i \in \{1, 2\}$  and  $\mathcal{U} = \{V_1, V_2\}$ . First it holds,

**Claim.** — For any  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$\mu\left(\bigcap_{i=0}^{n-1} T^{-i} V_{s(i)}\right) < \epsilon$$

for any  $n \geq N$  and  $s \in \{1, 2\}^n$ .

*Proof of claim.* — If the claim is not true, there exist  $\epsilon > 0$  and  $s \in \{1, 2\}^{\mathbb{Z}}$  such that  $\mu(\bigcap_{i \in \mathbb{Z}} T^{-i} V_{s(i)}) \geq \epsilon$ .

Define  $\Omega = \{t \in \{1, 2\}^{\mathbb{Z}} : \mu(\bigcap_{i \in \mathbb{N}} T^{-i} V_{t(i)}) \geq \epsilon\}$ . It is easy to see that  $\Omega$  is a closed subset of  $\{1, 2\}^{\mathbb{Z}}$  and it is invariant under the shift map  $\sigma$ .

Take  $r \in \Omega$  with  $r$  a minimal point of the fullshift  $(\{1, 2\}^{\mathbb{Z}}, \sigma)$ . Since  $\mu(\bigcap_{i \in \mathbb{N}} T^{-i} V_{r(i)}) \geq \epsilon$ , there exists a generic point  $y \in \bigcap_{i \in \mathbb{Z}} T^{-i} V_{r(i)}$  of  $\mu$ . Clearly,  $r \neq (\dots, 2, 2, 2, 2, \dots)$  (or  $T^n y \in V_2$  for any  $n \in \mathbb{Z}$  which is impossible). Observe that  $r$  is a minimal point and  $\{i \in \mathbb{Z} : r(i) = 1\}$  is a syndetic set. Hence  $N(y, V_1) = \{i \in \mathbb{N} : T^i y \in V_1\}$  is a syndetic set. Since  $y$  is a transitive point and  $x_1$  is a fixed point,  $N(y, U_1)$  is a thick set, which contradicts the fact that  $N(y, V_1)$  is a syndetic set. This proves the claim.

Now, by using the claim, for any  $\epsilon > 0$ , there exist  $N \in \mathbb{N}$  such that  $\mu(\bigcap_{i=0}^{n-1} T^{-i} V_{s(i)}) < \epsilon$  for any  $n \geq N$  and  $s \in \{1, 2\}^n$ . Hence for any  $\beta \supseteq \bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}$  and  $C \in \beta$  one has  $\mu(C) < \epsilon$ . Therefore  $H_\mu(\beta) \geq \log(\frac{1}{\epsilon})$  and  $\lim_{n \rightarrow \infty} \inf_{\beta \supseteq \bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}} H_\mu(\beta) = \infty$ . We conclude  $(x_1, x_2) \in \text{Com}_\mu^-(X, T)$ . □

In the following, we will construct a t.d.s.  $(X, \mu, T)$  such that  $SE_2^\mu(X, T) = \emptyset$  and  $\text{Com}_\mu^-(X, T) \neq \emptyset$ .

DEFINITION 5.7. — A t.d.s.  $(X, T)$  is doubly minimal if for all  $x, y \in X, y \notin \{T^n x\}_{n \in \mathbb{Z}}, \{(T^j x, T^j y)\}_{j \in \mathbb{Z}}$  is dense in  $X \times X$ .

The following result is Theorem 5 in [W].

LEMMA 5.8. — Any ergodic system  $(Y, \mathcal{C}, \nu, S)$  with zero measure-theoretical entropy has a uniquely ergodic topological model  $(X, T)$  that is doubly minimal.

Example 5.9. — There are a t.d.s.  $(Z, R)$  and an ergodic measure  $\theta \in \mathcal{M}(Z, R)$ , such that  $SE_2^\theta(Z, R) = \emptyset$  and  $\text{Com}_\theta^-(Z, R) \neq \emptyset$ .

*Proof.* — Let  $(Y, \mathcal{C}, \nu, S)$  be an ergodic system with discrete spectrum and assume  $\nu$  is non-atomic. By Lemma 5.8, there is a uniquely ergodic doubly minimal system  $(X, T)$  which is a topological model of  $(Y, \mathcal{C}, \nu, S)$ . Set  $\mu$  the unique ergodic measure of  $(X, T)$ .

Put  $\Delta_k = \{(x, T^k x) : x \in X\}, k \in \mathbb{Z}$ . Since  $\mu$  is non-atomic,  $\mu \times \mu(\Delta_k) = 0$  for any  $k \in \mathbb{Z}$ . Thus  $\mu \times \mu(\bigcup_{k \in \mathbb{Z}} \Delta_k) = 0$ . Let  $\mu \times \mu = \int_\Omega \mu_\omega dm(\omega)$  be

the ergodic decomposition of  $\mu \times \mu$ . For any  $\omega \in \Omega$ , we have  $SE_2^{\mu\omega}(X \times X, T \times T) = \emptyset$ . Indeed, if  $((x_1, x_2), (y_1, y_2)) \in SE_2^{\mu\omega}(X \times X, T \times T)$  and  $\pi_i : X \times X \rightarrow X$  is the  $i$ -th projection map, since  $(X, T)$  is uniquely ergodic, then  $\pi_i(\mu_\omega) = \mu$ ,  $i = 1, 2$ . Now, by Theorem 3.7, one has  $(x_1, y_1) \in SE_2^\mu(X, T)$  or  $(x_2, y_2) \in SE_2^\mu(X, T)$  which contradicts the assumption  $SE_2^\mu(X, T) = \emptyset$ .

Take  $\omega \in \Omega$  such that  $SE_2^{\mu\omega}(X \times X, T \times T) = \emptyset$  and  $\mu_\omega(X \times X \setminus \bigcup_{k \in \mathbb{Z}} \Delta_k) = 1$ , and consider  $(x, y) \in X \times X \setminus \bigcup_{k \in \mathbb{Z}} \Delta_k$  a generic point of  $\mu_\omega$ . Since  $(X, T)$  is doubly minimal,  $\text{cl}(\bigcup_{n \in \mathbb{Z}} (T \times T)^n(x, y)) = X \times X$ . Hence,  $\text{supp}(\mu_\omega) = X \times X$ .

Now, by collapsing the diagonal  $\Delta_2(X)$  of  $X \times X$  into one point, we get a new space  $Z = X \times X / \Delta_2(X)$ . Let  $\pi : X \times X \rightarrow Z$  be the natural projection. Let  $\theta = \pi\mu_\omega$  and  $R$  be the map on  $Z$  induced by  $T$ . For the m.t.d.s.  $(Z, \theta, R)$ ,  $SE_2^\theta(Z, R) = \pi \times \pi(SE_2^{\mu\omega}(X \times X, T \times T)) = \emptyset$ . Clearly,  $(Z, R)$  has a fixed point  $z_1 = \pi(\Delta_2(X))$ ,  $\theta$  is ergodic and  $\text{supp}(\theta) = Z$ . By Theorem 5.6, we conclude  $\text{Com}_\theta^-(Z, R) \neq \emptyset$ . The m.t.d.s.  $(Z, \theta, R)$  is the system we are looking for.  $\square$

## BIBLIOGRAPHY

- [Be] V. BERGELSON, Ergodic Ramsey theory – an update, Ergodic theory of  $\mathbb{Z}^d$  actions (Warwick, 1993-1994), 1–61, London Math. Soc. Lecture Notes Ser., 228, Cambridge Univ. Press, Cambridge, 1996.
- [B1] F. BLANCHARD, Fully positive topological entropy and topological mixing, Symbolic dynamics and its applications, AMS Contemporary Mathematics, 135 (1992), 95–105.
- [B2] F. BLANCHARD, A disjointness theorem involving topological entropy, Bull. de la Soc. Math. de France, 121 (1993), 465–478.
- [BGH] F. BLANCHARD, E. GLASNER, B. HOST, Variations on the variational principle, Ergod. Th. and Dynam. Sys., 17 (1997), 29–53.
- [BGKM] F. BLANCHARD, E. GLASNER, S. KOLYADA, A. MAASS, On Li-Yorke pairs, Journal für die reine und angewandte Mathematik, 547 (2002), 51–68.
- [BHM] F. BLANCHARD, B. HOST, A. MAASS, Topological complexity, Ergod. Th. and Dynam. Sys., 20 (2000), 641–662.
- [BL] F. BLANCHARD and Y. LACROIX, Zero-entropy factors of topological flows, Proc. Amer. Math. Soc., 119 (1993), 85–992.
- [B-R] F. BLANCHARD, B. HOST, A. MAASS, S. MARTÍNEZ and D. RUDOLPH, Entropy pairs for a measure, Ergod. Th. and Dynam. Sys., 15 (1995), 621–632.
- [DGS] M. DENKER, C. GRILLENBERGER and C. SIGMUND, Ergodic theory on compact spaces, Lecture Notes in Math. 527, Springer-Verlag, New York.
- [Fe] S. FERENCZI, Measure-theoretic complexity of ergodic systems, Israel J. Math., 100 (1997), 189–207.



- [F] H. FURSTENBERG, Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation, *Math. System Th.*, 1 (1967), 1–55.
- [G1] E. GLASNER, A simple characterization of the set of  $\mu$ -entropy pairs and applications, *Israel J. Math.*, 102 (1997), 13–27.
- [G2] E. GLASNER, Ergodic theory via joinings, *Mathematical Surveys and Monographs*, 101 (2003).
- [G] T.N.T GOODMAN, Topological sequence entropy, *Proc. London Math. Soc.*, 29 (1974), 331–350.
- [GW] E. GLASNER, B. WEISS, Strictly ergodic, uniform positive entropy models., *Bull. Soc. Math. France*, 122-3 (1994), 399–412.
- [Hu] P. HULSE, Sequence entropy and subsequence generators, *J. London Math. Soc.*, 26 (1982), 441–450.
- [H-Y] W. HUANG, S. LI, S. SHAO and X. YE, Null systems and sequence entropy pairs, *Ergod. Th. and Dynam. Sys.*, 23-5 (2003), 1505–1523.
- [HY] W. HUANG and X. YE, Topological K-systems, a third approach, preprint 2001.
- [Kr] L. KRONECKER, Naherrungsweise ganzzahlige Auflosunglinear Gleichungen, *S.-B. Preuss. Akad. Wiss.*, 1179-93, 1271-99. *Werke III* (1), 47–109.
- [Ku] A. G. KUSHNIRENKO, On metric invariants of entropy type, *Russian Math. Surveys*, 22-5 (1967), 53–61.
- [P] W. PARRY, *Topics in Ergodic Theory*, Cambridge Tracks in Mathematics, Cambridge-New York, 1981.
- [S] A. SALESKI, Sequence entropy and mixing, *J. of Math. Anal. and Appl.*, 60 (1977), 58–66.
- [Wa] P. WALTERS, *An introduction to ergodic theory*, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.
- [W] B. WEISS, Multiple recurrence and doubly minimal systems, *AMS Contemporary Mathematics*, 215 (1998), 189–196.

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