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A dimension formula for Ekedahl-Oort strata

A DIMENSION FORMULA FOR
EKEDAHL-OORT STRATA

by Ben MOONEN [*]

To an abelian variety X over a field k of characteristic p we can associate invariants such as the p-rank or the isogeny class of its p-divisible group. Such invariants can be used to define stratifications of the moduli space $\mathcal{A}_g$ in characteristic p. One example of such a stratification is the Newton stratification. Two points of $\mathcal{A}_g(k)$ are in the same Newton stratum iff the associated p-divisible groups are isogenous over $\overline{k}$. Mainly through the work of de Jong and Oort [3] and Oort [13], we now have a fairly complete picture of this stratification. For an overview we refer to Oort [15] or Rapoport's Bourbaki lecture [16].

Of more recent date is the EO-stratification, after Ekedahl and Oort. The starting point is that, given $g = \dim(X)$ and working over $k = \overline{k}$ with $\mathrm{char}(k) = p$, there is a finite list of isomorphism classes of group schemes $X[p]$. This was proven by Kraft [9] (unpublished), was seemingly forgotten for some time, and was then reobtained by Oort around 1995. The EO-stratification is defined by declaring that two points of $\mathcal{A}_g(k)$ are in the same stratum iff the associated group schemes $X[p]$ are isomorphic over $\overline{k}$. Several basic properties of this stratification were obtained by Ekedahl and Oort; see Oort [14]. Other questions, such as which strata occur in the boundary of a given one, remain open.

Naturally, people have tried to extend these theories to other Shimura varieties. A good class to start working on are the PEL moduli spaces at primes of good reduction. A generalized Newton stratification can be defined using the theory of isocrystals with additional structure, developed by Kottwitz [7] and Rapoport and Richartz [17]. The picture is, at present, still much less complete than in the Siegel modular case, though; see [16], §5.

Work on the generalized EO-stratification was taken up by Wedhorn [18] and the author [10]. (The Hilbert modular case was studied by Goren and Oort in [5].) The data that are fixed in the moduli problem give rise to a connected reductive group G and a conjugacy class $\mathbb{X}$ of parabolic subgroups. The "p-kernel" objects to consider are triples $\underline{Y} = (Y, \iota, \lambda)$ consisting of a group scheme Y killed by p, equipped with an action ι of a semi-simple $\mathbb{F}_p$-algebra and a polarization $\lambda : Y \xrightarrow{\sim} Y^D$. Writing W_G for the Weyl group of G and $W_{\mathbb{X}} \subset W_G$ for the subgroup associated to $\mathbb{X}$, the main result of [10] is that, fixing suitable discrete invariants, such triples $\underline{Y}$ are classified by the $W_{\mathbb{X}}$-cosets in W_G. (Here we assume that $p > 2$.) Wedhorn showed in [18] that this leads to a generalized EO-stratification on good reductions $\mathcal{A}_0$ of PEL moduli spaces. More precisely, k-valued points of $\mathcal{A}_0$ naturally give rise to triples $\underline{Y}$ as above, and with the appropriate assumptions we arrive at a stratification $\mathcal{A}_0 = \amalg_{w \in W_{\mathbb{X}} \backslash W_G} \mathcal{A}_0(w)$. Also in [18] we find a result about the dimension of the strata : If $\underline{Y}_w$ is the triple corresponding to the coset $w \in W_{\mathbb{X}} \backslash W_G$ and if $\mathcal{A}_0(w) \neq \emptyset$, then every irreducible component of $\mathcal{A}_0(w)$ has codimension equal to $\dim \left(\mathbf{Aut}(\underline{Y}_w) \right)$ in $\mathcal{A}_0$.

The main purpose of the present paper is to give an explicit dimension formula for the strata. To state the result, let us recall that once we fix a generating set of reflections $S \subset W_G$, every coset $w \in W_{\mathbb{X}} \backslash W_G$ has a distinguished representative $\dot{w}$. The actual element $\dot{w}$ depends on the choice of S, but its length $\ell(\dot{w})$ does not. Our main result is then the following.

THEOREM. — *If $\mathcal{A}_0(w) \neq \emptyset$ then its irreducible components all have dimension equal to $\ell(\dot{w})$.*

The strength of this formula lies in the fact that the lengths $\ell(\dot{w})$ are easily computable. In particular, an immediate consequence of our result is that there is at most one 0-dimensional stratum and that there is a unique stratum of maximal dimension. The latter takes the role of a generalized ordinary stratum and plays a central role in our paper [11], in which a

generalization of Serre-Tate theory is developed.

As suggested by the above, what we really compute in this paper are the dimensions of the automorphism group schemes $\mathbf{Aut}(\underline{Y})$. These computations are based on an explicit description of the Dieudonné modules of triples $\underline{Y}$ as in the above. The classification results of [10] are recalled in §1, the actual computation of the dimension of $\mathbf{Aut}(\underline{Y})$ is done in §2. The application to the study of EO-stratifications is discussed in §3.

Acknowledgements. I thank F. Oort, R. Pink and T. Wedhorn for stimulating discussions on the subject of this paper. The research for this paper was made possible by a Fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW). During my work on this paper I have been affiliated to the University of Utrecht (until June 2001) and the University of Amsterdam (from July 2001). I thank these institutions for their support.

Note added September 2003. Another proof of the main result of this paper can be obtained using the methods of [12]. See in particular the discussion of PEL moduli spaces in loc. cit., Section 6.

1. p-Kernel group schemes with additional structures.

1.1. Generalities on $\mathbf{BT}_1$.

We fix a prime number p. When dealing with polarized group schemes we assume $p \neq 2$.

1.1.1. Let S be a scheme. Write $S_0 \subset S$ for the closed subscheme defined by the ideal $(p) \subset O_S$. By a BT_1 *over* S (short for "truncated Barsotti-Tate group of level 1") we mean a commutative finite locally free S-group scheme Y such that, with the notation $Y_0 := Y \times_S S_0$, the sequence

$$Y_0 \xrightarrow{F} Y_0^{(p)} \xrightarrow{V} Y_0$$

is exact. Here F and V denote the relative Frobenius and Verschiebung of Y_0 over S_0. For further details see Illusie [6].

If Y is a BT_1 over S then we write Y^D for its Cartier dual. There is a canonical isomorphism $Y \xrightarrow{\sim} Y^{DD}$, which we take as an identification. If $\epsilon \in \{\pm 1\}$ then by an ϵ-*duality* of Y we mean an isomorphism $\lambda : Y \xrightarrow{\sim} Y^D$

such that $\lambda = \epsilon \cdot \lambda^D$. Such an ϵ-duality induces an involution $f \mapsto f^\dagger$ on the ring $\mathrm{End}_S(Y)$. We also refer to an ϵ-duality as a polarization.

1.1.2. Let B be an $\mathbb{F}_p$-algebra. By a BT_1 *with B-structure* over a basis S we mean a pair $\underline{Y} = (Y, \iota)$ where Y is a BT_1 over S and $\iota\colon B \to \mathrm{End}_S(Y)$ is a homomorphism of $\mathbb{F}_p$-algebras.

Suppose B is equipped with an $\mathbb{F}_p$-linear involution $b \mapsto b^*$. Let $\epsilon \in \{\pm 1\}$. By a BT_1 *with $(B, *, \epsilon)$-structure* over S we mean a triple $\underline{Y} = (Y, \iota, \lambda)$ where (Y, ι) is a BT_1 with B-structure and $\lambda\colon Y \to Y^D$ is an ϵ-duality, such that $\iota(b^*) = \iota(b)^\dagger$ for all $b \in B$.

1.1.3. We use contravariant Dieudonné theory as in Fontaine [4]. Let K be a perfect field, $\mathrm{char}(K) = p$. Then a BT_1 with B-structure over K corresponds to a 4-tuple (N, F, V, ι), where

- N is a finite dimensional K-vector space,

- $F\colon N \to N$ is a Frob_K-linear endomorphism,

- $V\colon N \to N$ is a Frob_K^{-1}-linear endomorphism, and

- $\iota\colon B \to \mathrm{End}(N, F, V)$ is an $\mathbb{F}_p$-linear homomorphism,

with $\mathrm{Ker}(F) = \mathrm{Im}(V)$ and $\mathrm{Im}(F) = \mathrm{Ker}(V)$. Using these last relations one can show that there exists a filtration

$$\mathcal{C}_\bullet : \quad (0) = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_r = N$$

that is the coarsest filtration with the properties that

(i) for every j there exists an index $f(j) \in \{0, 1, \ldots, r\}$ with $F(\mathcal{C}_j) = \mathcal{C}_{f(j)}$;

(ii) for every j there exists an index $v(j) \in \{0, 1, \ldots, r\}$ with $V^{-1}(\mathcal{C}_j) = \mathcal{C}_{v(j)}$.

We refer to this filtration as the *canonical filtration* of N. See also [10], 2.5.

Similarly, a BT_1 with $(B, *, \epsilon)$-structure corresponds to a 5-tuple $(N, F, V, \varphi, \iota)$, where (N, F, V, ι) is as above and where $\varphi\colon N \times N \to K$ is a perfect, ϵ-symmetric bilinear form, such that

(1.1.3.1) $\quad \varphi(Fn_1, n_2) = \varphi(n_1, Vn_2)^p \quad$ for all $n_1, n_2 \in N$;

(1.1.3.2) $\quad \varphi(bn_1, n_2) = \varphi(n_1, b^*n_2) \quad$ for all $b \in B$ and $n_1, n_2 \in N$.

1.2. BT_1 with given endomorphisms.

Let B be a finite dimensional semi-simple $\mathbb{F}_p$-algebra. Let k be an algebraically closed field of characteristic p. The first problem studied in [10] is the classification of BT_1 with B-structure over k. This generalizes the work of Kraft [9], who classified group schemes killed by p without additional structure. We shall briefly review our results.

1.2.1. Write κ for the center of B. Then κ is a product of finite fields, say $\kappa = \kappa_1 \times \cdots \times \kappa_\nu$. Let $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\nu$ be the set of homomorphisms $\kappa \to k$.

Consider pairs (N, L) consisting of a finitely generated $B \otimes_{\mathbb{F}_p} k$-module N and a submodule $L \subset N$. Note that the simple factors of $B \otimes_{\mathbb{F}_p} k$ are indexed by $\mathcal{I}$, so we get canonical decompositions $N = \oplus_{i \in \mathcal{I}} N_i$ and $L = \oplus_{i \in \mathcal{I}} L_i$. Define two functions d, $\mathfrak{f}: \mathcal{I} \to \mathbb{Z}_{\geqslant 0}$ by $d(i) = \mathrm{length}(N_i)$ and $\mathfrak{f}(i) = \mathrm{length}(L_i)$, taking lengths as $B \otimes_{\mathbb{F}_p} k$-modules. The pair $(d, \mathfrak{f})$ determines the pair (N, L) up to isomorphism.

To the pair (N, L) we associate an algebraic group G over k and a conjugacy class $\mathbb{X}$ of parabolic subgroups of G. First we define

$$G := \mathrm{GL}_{B \otimes_{\mathbb{F}_p} k}(N) \cong \prod_{i \in \mathcal{I}} \mathrm{GL}_{d(i), k} \, .$$

Then the stabilizer $P := \mathrm{Stab}(L)$ is a parabolic subgroup of G, and we define $\mathbb{X}$ as the conjugacy class of parabolic subgroups of G containing P.

1.2.2. Let $\underline{Y} = (Y, \iota)$ be a BT_1 with B-structure over k. Write N for the Dieudonné module of Y and let $L := \mathrm{Ker}(F) \subset N$. Let $(d, \mathfrak{f})$ be the corresponding pair of functions. It can be shown (see [10], 4.3) that the function d is constant on each of the subsets $\mathcal{I}_n \subset \mathcal{I}$. We refer to $(d, \mathfrak{f})$ as the *type* of (Y, ι).

1.2.3. Fix a pair $(d, \mathfrak{f})$ with d constant on each subset $\mathcal{I}_n \subset \mathcal{I}$. Fix a pair of $B \otimes_{\mathbb{F}_p} k$-modules $L_0 \subset N_0$ of type $(d, \mathfrak{f})$. Let $(G, \mathbb{X})$ be the associated algebraic group and conjugacy class of parabolic subgroups. Let W_G be the Weyl group of G, and let $W_{\mathbb{X}} \subset W_G$ be the subgroup corresponding to $\mathbb{X}$.

To a pair $\underline{Y} = (Y, \iota)$ of type $(d, \mathfrak{f})$ we associate an element $w(\underline{Y}) \in W_{\mathbb{X}} \backslash W_G$. This is done as follows. Write N for the Dieudonné module

of Y and let $L := \mathrm{Ker}(F) \subset N$. Choose an isomorphism $\xi : N \xrightarrow{\sim} N_0$ that restricts to $L \xrightarrow{\sim} L_0$. This allows us to view the canonical filtration $\mathcal{C}_\bullet$ of N as a filtration of N_0. Choose any refinement $\mathcal{F}_\bullet$ of $\mathcal{C}_\bullet$ to a complete flag. The relative position of L_0 and $\mathcal{F}_\bullet$ is given by an element $w(L_0, \mathcal{F}_\bullet) \in W_{\mathbb{X}} \backslash W_G$. It can be shown that this element is independent of the choice of ξ and the refinement $\mathcal{F}_\bullet$; see [10], especially 4.6 for details. Now define $w(\underline{Y}) := w(L_0, \mathcal{F}_\bullet)$.

With these notations, the first main result of [10] can be stated as follows.

1.2.4 THEOREM. — *Assume that $k = \overline{k}$. Fix a type $(d, \mathfrak{f})$. The map $\underline{Y} \mapsto w(\underline{Y})$ gives a bijection*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \underline{Y} \text{ of type } (d, \mathfrak{f}) \end{array} \right\} \xrightarrow{\sim} W_{\mathbb{X}} \backslash W_G.$$

1.3. BT$_1$ with given endomorphisms and a polarization.

Let B be a finite dimensional semi-simple $\mathbb{F}_p$-algebra equipped with an involution $b \mapsto b^*$. Let $\epsilon \in \{\pm 1\}$. Let k be an algebraically closed field of characteristic $p > 2$. The second problem studied in [10] is the classification of BT$_1$ with $(B, *, \epsilon)$-structure over k. We shall briefly review the result. As in section 1.2, this involves an algebraic group G. In the polarized case there are two versions of the result : one using a possibly non-connected group G, the other using the identity component G^0. We shall need both variants ; see 1.3.6–1.3.8 below ; see also [10], especially 3.8 and 5.7.

1.3.1. Let $(B, *, \epsilon)$ be as above. We can decompose $(B, *)$ as a product of simple factors, say

$$(B, *) = (B_1, *_1) \times \cdots \times (B_\nu, *_\nu).$$

If $*_n$ is an orthogonal involution, set $\epsilon_n := +1$; if $*_n$ is symplectic, set $\epsilon_n := -1$. The simple factors $(B_n, *_n)$ come in three kinds :

Type C : $B_n \cong M_r(\kappa_n)$ for some finite field κ_n, with $*_n$ of the first kind and
$\epsilon \cdot \epsilon_n = -1$;

Type D : $B_n \cong M_r(\kappa_n)$ for some finite field κ_n, with $*_n$ of the first kind and
$\epsilon \cdot \epsilon_n = +1$;

Type A : $B_n \cong M_r(\tilde{\kappa}_n)$, where $\tilde{\kappa}_n$ is an étale quadratic extension of a finite
field κ_n, and $*_n$ is of the second kind.

Note that the labelling depends on ϵ. For factors of type A, if $\kappa_n \cong \mathbb{F}_q$ then
either $\tilde{\kappa}_n \cong \mathbb{F}_{q^2}$ or $\tilde{\kappa}_n = \kappa_n \times \kappa_n$.

Let $\tilde{\kappa}$ be the centre of B, and define $\kappa := \{z \in \tilde{\kappa} \mid z^* = z\}$. We
have $\tilde{\kappa} = \tilde{\kappa}_1 \times \cdots \times \tilde{\kappa}_\nu$ and $\kappa = \kappa_1 \times \cdots \times \kappa_\nu$, where the κ_n are finite
fields, $\tilde{\kappa}_n = \kappa_n$ if $(B_n, *_n)$ is of type C or D, and $\tilde{\kappa}_n$ is an étale quadratic
extension of κ_n if $(B_n, *_n)$ is of type A. Let $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\nu$ be the set of
homomorphisms $\kappa \to k$. For $X \in \{A, C, D\}$ we say that $i \in \mathcal{I}$ is of type X if
$i \in \mathcal{I}_n \subset \mathcal{I}$ and $(B_n, *_n)$ is of type X. Let $\mathcal{I}^X \subset \mathcal{I}$ be the subset of elements
of type X. Let $\widetilde{\mathcal{I}}$ be the set of homomorphisms $\tilde{\kappa} \to k$. We have a restriction
map res: $\widetilde{\mathcal{I}} \to \mathcal{I}$. For $\tau \in \widetilde{\mathcal{I}}$ define $\bar{\tau} := \tau \circ *$. If $i \in \mathcal{I}$ is of type C or D then
there is a unique $\tau \in \widetilde{\mathcal{I}}$ with res$(\tau) = i$, and $\tau = \bar{\tau}$; if i is of type A then
there are precisely two elements $\tau, \bar{\tau} \in \widetilde{\mathcal{I}}$ that restrict to the embedding i
on κ.

1.3.2. Consider triples (N, L, φ) consisting of a finitely generated
$B \otimes_{\mathbb{F}_p} k$-module N, a perfect, ϵ-symmetric bilinear form $\varphi: N \times N \to k$
satisfying (1.1.3.2), and a maximal isotropic submodule $L \subset N$. With a
similar construction as in 1.2.1, such a triple is classified, up to isomorphism,
by a pair $(d, \mathfrak{f})$ consisting of functions $d: \mathcal{I} \to \mathbb{Z}_{\geq 0}$ and $\mathfrak{f}: \widetilde{\mathcal{I}} \to \mathbb{Z}_{\geq 0}$ such that
$\mathfrak{f}(\tau) + \mathfrak{f}(\bar{\tau}) = d(i)$ for all $\tau \in \widetilde{\mathcal{I}}$ and $i = \text{res}(\tau) \in \mathcal{I}$. Note that if i is of type
C or D then there is a unique $\tau = \bar{\tau}$ with res$(\tau) = i$ and we get the relation
$d(i) = 2\mathfrak{f}(\tau)$.

To a triple (N, L, φ) as above we associate a pair $(G, \mathbb{X})$. First,
define $G := \mathrm{U}_{B \otimes_{\mathbb{F}_p} k}(N, \varphi)$, the algebraic group (over k) of $B \otimes_{\mathbb{F}_p} k$-linear
automorphisms of N that preserve the form φ. We have $G = \prod_{i \in \mathcal{I}} G_i$, with
G_i isomorphic to $\mathrm{Sp}_{d(i),k}$ if i is of type C, to $\mathrm{O}_{d(i),k}$ if i is of type D and
to $\mathrm{GL}_{d(i),k}$ if i is of type A. Note that $d(i)$ is even if i is of type C or D.
In the presence of (non-connected) orthogonal factors, the set $\mathbb{X}$ that we
want to consider is not simply a conjugacy class of parabolic subgroups
of G. Instead we consider (partial) hermitian flags in N, i.e., filtrations by
$B \otimes_{\mathbb{F}_p} k$-submodules

$$(0) = C_0 \subset C_1 \subset \cdots \subset C_r = N$$

with the property that $C_j^\perp = C_{r-j}$ for all j. (Here $C_j^\perp := \{n \in N \mid \varphi(C_j, n) = 0\}$.) The set $\mathrm{Flag}(N, \varphi)$ of all such hermitian flags has the

structure of a (generally non-thick) building, on which G acts in a strongly transitive, type-preserving manner. Then we define $\mathbb{X}$ to be the G-orbit in $\mathrm{Flag}(N, \varphi)$ of the flag $(0) \subset L \subset N$. Equivalently, $\mathbb{X}$ is the set of all maximal isotropic subspaces in N. Note that if there are no factors of type D then $C_\bullet \mapsto \mathrm{Stab}(C_\bullet)$ gives a bijective correspondence between hermitian flags and parabolic subgroups of G; in this case we could also define $\mathbb{X}$ to be the conjugacy class of parabolic subgroups of G that contains $\mathrm{Stab}(L)$.

1.3.3. Let $\underline{Y}$ be a BT_1 with $(B, *, \epsilon)$-structure over k. Let N be the Dieudonné module of Y, let $L = N[F] := \mathrm{Ker}(F)$, and let φ be the ϵ-hermitian form on N corresponding to the given ϵ-duality. Then (N, L, φ) is a triple as in 1.3.2. Let $(d, \mathfrak{f})$ be the corresponding pair of functions; we refer to this pair as the type of $\underline{Y}$. It can be shown ([10], 4.3, 5.3 and 6.5) that the function d is constant on each of the subsets $\mathcal{I}_n \subset \mathcal{I}$.

1.3.4. Fix $(d, \mathfrak{f})$ as in 1.3.2, with d constant on each subset $\mathcal{I}_n \subset \mathcal{I}$. Choose a corresponding triple (N_0, L_0, φ_0). To this triple we associate a pair $(G, \mathbb{X})$ as explained above. The Coxeter group associated to the building $\mathrm{Flag}(N_0, \varphi_0)$ is just the Weyl group W_G of G. Note, however, that we work with a possibly non-connected group G; an orthogonal factor O_{2q} contributes a factor of type B_q (not D_q) to W_G. Write $W_{\mathbb{X}} \subset W_G$ for the subgroup corresponding to $\mathbb{X}$.

Let $\underline{Y}$ be a BT_1 with $(B, *, \epsilon)$-structure over k of type $(d, \mathfrak{f})$. Let (N, L, φ) be the associated triple, as in 1.3.3. To $\underline{Y}$ we associate an element $w(\underline{Y}) \in W_{\mathbb{X}} \backslash W_G$. This works essentially the same as in the non-polarized case. First we choose an isometry $\xi \colon (N, \varphi) \xrightarrow{\sim} (N_0, \varphi_0)$ that restricts to $L \xrightarrow{\sim} L_0$. Via ξ we can view the canonical filtration $C_\bullet$ as an element of $\mathrm{Flag}(N_0, \varphi_0)$. Choose a refinement of $C_\bullet$ to a full hermitian flag $\mathcal{F}_\bullet$ in N_0. (i.e., choose a chamber of which $C_\bullet$ is a face). The relative position of L_0 and $\mathcal{F}_\bullet$ is measured by an element $w(L_0, \mathcal{F}_\bullet) \in W_{\mathbb{X}} \backslash W_G$. It can be shown that this element is independent of the choice of ξ and the refinement $\mathcal{F}_\bullet$; see [10], especially 5.6 and 6.6 for details. Now define $w(\underline{Y}) := w(L_0, \mathcal{F}_\bullet)$.

With these notations, the second main result of [10] is the following.

1.3.5 THEOREM. — *Let k be an algebraically closed field, $\mathrm{char}(k) > 2$. Fix a type $(d, \mathfrak{f})$. Sending a BT_1 with $(B, *, \epsilon)$-structure $\underline{Y}$ to the element $w(\underline{Y})$ gives a bijection*

$$\left\{ \begin{matrix} \text{isomorphism classes of} \\ \underline{Y} \text{ of type } (d, \mathfrak{f}) \end{matrix} \right\} \xrightarrow{\sim} W_{\mathbb{X}} \backslash W_G.$$

(In [10] this result is only stated for BT_1 with $(B, *, -1)$-structure. However, by Morita equivalence one can always reduce to the case that $\epsilon = -1$.)

1.3.6. We describe a variant of the theorem that uses the Weyl group of the identity component G^0. To explain this we need to introduce another invariant, which is a function $\delta \colon \mathcal{I}^{\mathrm{D}} \to \mathbb{Z}/2\mathbb{Z}$. If there are no factors of type D then $\mathcal{I}^{\mathrm{D}} = \emptyset$ and the invariant δ is void. If $i \in \mathcal{I}^{\mathrm{D}}$, let $\tau \in \widetilde{\mathcal{I}}$ be the unique element with $\mathrm{res}(\tau) = i$. As we have seen, the Dieudonné module N decomposes as $N = \oplus_{\tau \in \widetilde{\mathcal{I}}} N_\tau$. Let $N_\tau[F] := \mathrm{Ker}(F_{|N_\tau})$ and $N_\tau[V] = \mathrm{Ker}(V_{|N_\tau})$. Now define $\delta(i)$ to be the length of the $B \otimes_{\mathbb{F}_p} k$-module $N_\tau[F]/\big(N_\tau[F] \cap N_\tau[V]\big)$ modulo 2.

As in the above, let $G := \mathrm{U}_{B \otimes_{\mathbb{F}_p} k}(N_0, \varphi_0)$. Define $\mathbb{X}^0$ to be the conjugacy class of parabolic subgroups of G^0 containing $\mathrm{Stab}(L_0)$. Write W_{G^0} for the Weyl group of G^0, and let $W_{\mathbb{X}^0} \subset W_{G^0}$ be the subgroup corresponding to $\mathbb{X}^0$.

In addition to the type $(d, \mathfrak{f})$, also fix $\delta \colon \mathcal{I}^{\mathrm{D}} \to \mathbb{Z}/2\mathbb{Z}$. Let $\underline{Y}$ be a BT_1 with $(B, *, \epsilon)$-structure of type $(d, \mathfrak{f}, \delta)$. With a similar procedure as above we associate to $\underline{Y}$ an element $w^0(\underline{Y}) \in W_{\mathbb{X}^0} \backslash W_{G^0}$. (If there are no factors of type D then $w^0(\underline{Y})$ is just the same as $w(\underline{Y})$).

With these notations we have the following variant of Theorem 1.3.5.

1.3.7 THEOREM. — *Let $k = \overline{k}$, $\mathrm{char}(k) > 2$. Fix a type $(d, \mathfrak{f}, \delta)$. Then the map $\underline{Y} \mapsto w^0(\underline{Y})$ gives a bijection*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \underline{Y} \text{ of type } (d, \mathfrak{f}, \delta) \end{array} \right\} \xrightarrow{\ \sim\ } W_{\mathbb{X}^0} \backslash W_{G^0}.$$

1.3.8. If $m = \#\mathcal{I}^{\mathrm{D}}$ then $\#(W_{\mathbb{X}} \backslash W_G) = 2^m \cdot \#(W_{\mathbb{X}^0} \backslash W_{G^0})$. This corresponds to the fact that, given $(d, \mathfrak{f})$, there are 2^m choices for δ.

In some applications it is more convenient to work with the element $w(\underline{Y})$; in other cases $w^0(\underline{Y})$ is easier to use. At any rate, once we fix δ we have a bijection

$$\left\{ \begin{array}{c} \text{elements of type } \delta \\ \text{in } W_{\mathbb{X}} \backslash W_G \end{array} \right\} \xrightarrow{\ \sim\ } W_{\mathbb{X}^0} \backslash W_{G^0}$$

that can be made completely explicit. For details we refer to [10], 3.8.

2. Automorphism group schemes.

2.1. Statement of the main result.

2.1.1. Let (W, S) be a Coxeter system, $\mathbb{X} \subset S$ a subset, $W_{\mathbb{X}} \subset W$ the subgroup generated by $\mathbb{X}$. Write ℓ for the length function on W. Every coset $w \in W_{\mathbb{X}} \backslash W$ has a unique representative $\dot{w}$ of minimal length, called the $(\mathbb{X}, \emptyset)$-reduced representative. (See Bourbaki [2], Chap. IV, Exercise 3.) If $W_{\mathbb{X}} \subset W$ and $w \in W_{\mathbb{X}} \backslash W$ are given then $\dot{w}$ in general depends on the choice of S, but $\ell(w) := \ell(\dot{w})$ is independent of S.

This section is devoted to the proof of the following result.

2.1.2 THEOREM. — (i) *Let B be a finite dimensional semi-simple $\mathbb{F}_p$-algebra. Let $\underline{Y}$ be a BT_1 with B-structure over a field $k = \bar{k}$ with $\mathrm{char}(k) = p$. Let $(d, \mathfrak{f})$ be the type of $\underline{Y}$, define G and $\mathbb{X}$ as in 1.2.1, and let $w := w(\underline{Y}) \in W_{\mathbb{X}} \backslash W_G$. Then the automorphism group scheme $\mathbf{Aut}(\underline{Y})$ has dimension equal to $\dim(\mathbb{X}) - \ell(\dot{w})$.*

(ii) *Assume that $p > 2$. Let $*$ be an involution on B and let $\epsilon \in \{\pm 1\}$. Let $\underline{Y}$ be a BT_1 with $(B, *, \epsilon)$-structure over k. Let $(d, \mathfrak{f}, \delta)$ be the type of $\underline{Y}$, define G^0 and $\mathbb{X}^0$ as in 1.3.6, and let $w^0 := w^0(\underline{Y}) \in W_{\mathbb{X}^0} \backslash W_{G^0}$. Then the automorphism group scheme $\mathbf{Aut}(\underline{Y})$ has dimension equal to $\dim(\mathbb{X}^0) - \ell(\dot{w}^0)$.*

2.1.3. Let S be a scheme of characteristic p. Following Wedhorn [18] we define a *Dieudonné space over S* to be a triple $(N, F^\sharp, V^\flat)$ where N is a locally free $\mathcal{O}_S$-module of finite type and where $F^\sharp : N^{(p)} \to N$ and $V^\flat : N \to N^{(p)}$ are $\mathcal{O}_S$-linear homomorphisms such that $F^\sharp \circ V^\flat = 0$ and $V^\flat \circ F^\sharp = 0$. (We use F and V for semi-linear endomorphisms, $F^\sharp$ and $V^\flat$ for their linearizations.)

In the following, $\mathcal{D}$ will either denote a finite dimensional semi-simple $\mathbb{F}_p$-algebra B or a triple $(B, *, \epsilon)$ as in section 1.3. We have the notion of a Dieudonné space with $\mathcal{D}$-structure over S; see [18], § 5 for details.

Let $\underline{Y}$ be a BT_1 with $\mathcal{D}$-structure over k. The Dieudonné module of $\underline{Y}$ defines, by linearization of F and V, a Dieudonné space with $\mathcal{D}$-structure $\underline{N}$ over k. Both $\underline{Y}$ and $\underline{N}$ give rise to an automorphism group scheme. Concretely, if $f : T \to \mathrm{Spec}(k)$ is a k-scheme then the T-valued points of $\mathbf{Aut}(\underline{Y})$ are the automorphisms of the pull-back $f^*(\underline{Y})$ as a BT_1 with $\mathcal{D}$-structure over T. Similarly, the T-valued points of $\mathbf{Aut}(\underline{N})$ are

the automorphisms of the pull-back $f^*(\underline{N})$ as a Dieudonné space with $\mathcal{D}$-structure over T.

We shall make use of the following result of Wedhorn.

2.1.4 PROPOSITION (Wedhorn, [18]). — *Notations as above. There is a natural homomorphism of k-group schemes*

$$\Delta \colon \mathbf{Aut}(\underline{Y}) \longrightarrow \mathbf{Aut}(\underline{N}).$$

If K is any perfect field then Δ induces an isomorphism on K-valued points. This implies that $\dim\big(\mathbf{Aut}(\underline{Y})\big) = \dim\big(\mathbf{Aut}(\underline{N})\big)$.

Strictly speaking, [18] only deals with polarized BT_1. In the non-polarized case the same arguments apply. Alternatively, one can reduce to the polarized case by passing from a BT_1 with B-structure $\underline{Y}$ to $Y \times Y^D$ with its natural $B \times B$-structure and the obvious ϵ-duality.

To prove Theorem 2.1.2 we shall compute the dimension of $\mathbf{Aut}(\underline{N})$. Let us note that $\mathbf{Aut}(\underline{Y})$ and $\mathbf{Aut}(\underline{N})$ are in general non-reduced (see also 3.1.7), and that Δ is in general not an isomorphism.

2.2. The non-polarized case.

2.2.1. We first prove (i) of Theorem 2.1.2. The notions involved in the proof are illustrated in an example in 2.2.12 below. There is an easy reduction to the case that $B = \kappa$ is a finite field. In fact, as the Brauer group of a finite field is trivial we have $B \cong M_{r_1}(\kappa_1) \times \cdots \times M_{r_\nu}(\kappa_\nu)$ where the κ_n are finite fields. Fixing such an isomorphism, every BT_1 with B-structure $\underline{Y} = (\underline{Y}_1)^{r_1} \times \cdots \times (\underline{Y}_\nu)^{r_\nu}$, where $\underline{Y}_n$ is a BT_1 with κ_n-structure. If the theorem holds for each $\underline{Y}_n$ then one readily checks that it also holds for $\underline{Y}$.

2.2.2. Suppose $B = \kappa$ is a field of p^m elements. Recall that we write $\mathcal{I}$ for the set of embeddings $\kappa \to k$. Then $\mathcal{I}$ is a set of m elements that comes equipped with a natural cyclic ordering : if $i \in \mathcal{I}$ then we write $i + 1$ for $\mathrm{Frob}_k \circ i$.

A type $(d, \mathfrak{f})$ consists of a positive integer d and a function $\mathfrak{f} \colon \mathcal{I} \to \{0, \ldots, d\}$. As in 1.2.1 we associate to the type $(d, \mathfrak{f})$ a pair $(G, \mathbb{X})$;

in this case we have $G = \prod_{i \in \mathcal{I}} \mathrm{GL}_{d,k}$. After a suitable choice of coordinates we can identify

$$W_G = \prod_{i \in \mathcal{I}} \mathfrak{S}_d \quad \text{and} \quad W_{\mathbb{X}} = \prod_{i \in \mathcal{I}} \mathfrak{S}\{1, \ldots, \mathfrak{f}(i)\} \times \mathfrak{S}\{\mathfrak{f}(i) + 1, \ldots, d\}.$$

In each factor $\mathfrak{S}_d$ we take the transpositions $(j \ j+1)$ as a set of generators.

As recalled in 2.1.1, each coset $w \in W_{\mathbb{X}} \backslash W_G$ has a distinguished representative $\dot{w} = (\dot{w}_i)_{i \in \mathcal{I}}$ in W_G. The permutations $\dot{w}_i$ that arise in this way are characterized by their property that
(2.2.2.1)
$$\text{if} \quad j' < j \text{ and } \dot{w}_i(j') > \dot{w}_i(j) \quad \text{then we have} \quad \dot{w}_i(j) \leqslant \mathfrak{f}(i) < \dot{w}_i(j').$$

The length of the element $\dot{w}$ is given by $\ell(\dot{w}) = \sum_{i \in \mathcal{I}} \ell(\dot{w}_i)$, with

$$(2.2.2.2) \qquad \ell(\dot{w}_i) = \sum_{n=1}^{\mathfrak{f}(i)} \left(\dot{w}_i^{-1}(n) - n \right) = \sum_{n=\mathfrak{f}(i)+1}^{d} \left(n - \dot{w}_i^{-1}(n) \right).$$

2.2.3. Let $w = w(\underline{Y}) \in W_{\mathbb{X}} \backslash W_G$. Let $\underline{N} = (N, F, V, \iota)$ be the Dieudonné module of $\underline{Y}$. We recall from [10], 4.9 an explicit description of $\underline{N}$ in terms of the distinguished representative $\dot{w}$. Namely, let N be the k-vector space with basis $e_{i,j}$ for $i \in \mathcal{I}$ and $j \in \{1, \ldots, d\}$. Write $N_i := \sum_{j=1}^{d} k \cdot e_{i,j}$, so that $N = \oplus_{i \in \mathcal{I}} N_i$. We let $a \in \kappa$ act on N_i as multiplication by $i(a)$. Next define Frob_k-linear maps $F_i \colon N_i \to N_{i+1}$ and Frob_k^{-1}-linear maps $V_i \colon N_i \leftarrow N_{i+1}$ by

$$F_i(e_{i,j}) \quad = \begin{cases} 0 & \text{if } \dot{w}_i(j) \leqslant \mathfrak{f}(i); \\ e_{i+1,n} & \text{if } \dot{w}_i(j) = \mathfrak{f}(i) + n; \end{cases}$$

and

$$V_i(e_{i+1,j}) \quad = \begin{cases} 0 & \text{if } j \leqslant d - \mathfrak{f}(i); \\ e_{i,n} & \text{if } j = d - \mathfrak{f}(i) + \dot{w}_i(n). \end{cases}$$

This defines a Dieudonné module with κ-action such that the corresponding BT_1 is isomorphic to $\underline{Y}$.

2.2.4. We find it useful to draw pictures of the Dieudonné modules $\underline{N}$ obtained in this way; see Figure 1. The boxes represent base vectors as indicated. An edge connecting boxes (i, j) and $(i + 1, j')$ either represents the relation $F(e_{i,j}) = e_{i+1,j'}$ (an F-edge) or the relation $V(e_{i+1,j'}) = e_{i,j}$

(a V-edge). To determine in which of the two cases we are we represent the base vectors $e_{i,j}$ with $F(e_{i,j}) = 0$ by shaded boxes; thus in the module N_i there are $\mathfrak{f}(i)$ shaded boxes. Note that in most cases we do not need the shading: (2.2.2.1) implies that two edges of the same kind never cross. Since the image of $F\colon N_i \to N_{i+1}$ is spanned by the *first* $d - \mathfrak{f}(i)$ base vectors, it follows that an F-edge in the picture always has slope $\leqslant 0$ and a V-edge always has slope $\geqslant 0$.
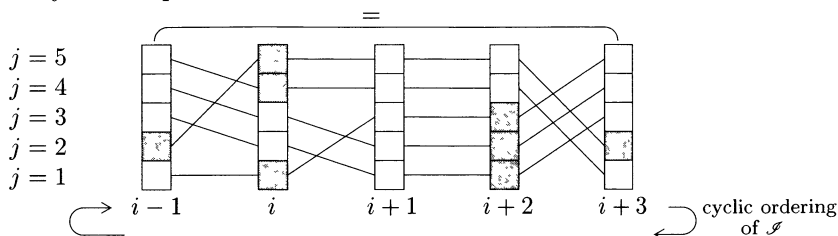
Figure 1

In the example drawn in Figure 1 we take $d = 5$ and $m = 4$, with $\mathfrak{f}(i-1) = 1$, $\mathfrak{f}(i) = 3$, $\mathfrak{f}(i+1) = 0$ and $\mathfrak{f}(i+2) = 3$. The distinguished representative $\dot{w}$ is given by

$$\dot{w}_{i-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix}, \quad \dot{w}_i = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{bmatrix},$$

$$\dot{w}_{i+1} = \dot{w}_{i+2} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

One should think of $\underline{N}$ as a roundabout (carousel); note that in Figure 1 (with $\#\mathcal{I} = 4$) the summands N_{i-1} and N_{i+3} have to be identified.

2.2.5. We recall some facts proven in [10], Lemma 4.5. The canonical filtration of $N = \oplus_{i \in \mathcal{I}} N_i$ induces filtrations

$$C_{i,\bullet}\colon \quad (0) = C_{i,0} \subset C_{i,1} \subset \cdots \subset C_{i,\ell} = N_i.$$

In the description of N given in 2.2.3, if $n = \dim(C_{i,r})$ then $C_{i,r} = k \cdot e_{i,1} + \cdots + k \cdot e_{i,n}$. The length ℓ of the filtrations $C_{i,\bullet}$ is independent of $i \in \mathcal{I}$. We refer to the vector spaces $B_{i,j} := C_{i,j}/C_{i,j-1}$ as the *canonical blocks* of N; they are indexed by the set $\mathcal{A} := \mathcal{I} \times \{1, \ldots, \ell\}$. For each pair $(i,j) \in \mathcal{A}$ there is a unique index $\rho_i(j) \in \{1, \ldots, \ell\}$ such that

either: $F_i\colon N_i \to N_{i+1}$ induces a Frob_k-linear bijection $B_{i,j} \xrightarrow{\sim} B_{i+1,\rho_i(j)}$,

or: $V_i\colon N_i \leftarrow N_{i+1}$ induces a Frob_k^{-1}-linear bijection $B_{i,j} \xleftarrow{\sim} B_{i+1,\rho_i(j)}$.

Then $\rho(i,j) = \big(i+1, \rho_i(j)\big)$ defines a permutation ρ of the set $\mathcal{A}$, and for each $a = (i,j) \in \mathcal{A}$ we have a Frob$_k$-linear bijection $t_a \colon B_a \overset{\sim}{\longrightarrow} B_{\rho(a)}$ which is either induced by F_i or is the inverse of the bijection induced by V_i.

If $f \in \mathbf{Aut}(\underline{N})(R)$ for some k-algebra R then f preserves the filtrations $C_{i,\bullet}$. This allows us to define a normal subgroup scheme $U \subset \mathbf{Aut}(\underline{N})$ by

$$U(R) := \Big\{ f \in \mathbf{Aut}(\underline{N})(R) \,\Big|\, \mathrm{gr}_a(f) = \mathrm{id}_{B_a} \quad \text{for all } a \in \mathcal{A} \Big\};$$

see also [10], 5.11.

2.2.6 LEMMA. — *The identity component of* $\mathbf{Aut}(\underline{N})$ *is contained in* U.

Proof. — By construction, U is the kernel of the homomorphism

$$h \colon \mathbf{Aut}(\underline{N}) \longrightarrow \prod_{a \in \mathcal{A}} \mathrm{GL}(B_a) \quad \text{given by } f \mapsto \big(\mathrm{gr}_a(f) \big).$$

It suffices to show that h factors through a finite étale subgroup scheme of $\prod \mathrm{GL}(B_a)$.

Take $a \in \mathcal{A}$, and let r be the smallest positive integer such that $\rho^r(a) = a$. The composition

$$\mathcal{T}_a := \Big(B_a \overset{t_a}{\longrightarrow} B_{\rho(a)} \overset{t_{\rho(a)}}{\longrightarrow} \cdots \overset{t_{\rho^{r-1}(a)}}{\longrightarrow} B_{\rho^r(a)} = B_a \Big)$$

is a Frob$_k^r$-linear automorphism of B_a. Then $\mathcal{B}_a := \{b \in B_a \mid \mathcal{T}_a(b) = b\}$ is an $\mathbb{F}_{p^r}$-subspace of B_a such that the natural map $\mathcal{B}_a \otimes_{\mathbb{F}_{p^r}} k \to B_a$ is an isomorphism. It follows that the subgroup scheme $\Gamma \subset \prod \mathrm{GL}(B_a)$ given by the automorphisms that commute with all $\mathcal{T}_a$ is finite étale. On the other hand, if $f \in \mathbf{Aut}(\underline{N})$ then for every $\alpha \in \mathcal{A}$ we have the relation $\mathrm{gr}_{\rho(\alpha)}(f) \circ t_\alpha = t_\alpha \circ \mathrm{gr}_\alpha(f)$. Iterating this we find that $\mathrm{gr}_a(f)$ commutes with $\mathcal{T}_a$. Hence h factors through Γ. $\qquad\square$

As we shall see later, U is connected; hence it is the identity component of $\mathbf{Aut}(\underline{N})$.

2.2.7. We describe the group scheme U in more detail; see also [10], 5.11. Let R be a k-algebra, and let $f \in \mathbf{Aut}(\underline{N})(R)$. Then f respects the decomposition $N \otimes R = \oplus_{i \in \mathcal{I}} N_i \otimes R$. Write $A_f(i, j', j)$

for the (j',j)-th matrix coefficient (with respect to the basis $e_{i,j}$) of the automorphism of $N_i \otimes R$ induced by f. Clearly f is fully determined by the map $A_f \colon \mathcal{I} \times \{1, \ldots, d\}^2 \to R$. Our task is to describe which maps $A \colon \mathcal{I} \times \{1, \ldots, d\}^2 \to R$ arise as A_f for some $f \in U$. The first obvious condition is that for all $f \in U$ the map $A = A_f$ satisfies

$$(2.2.7.1) \qquad A(i, j', j) = 0 \quad \text{if } j' > j\,, \text{ and} \quad A(i, j', j) = 1 \quad \text{if } j' = j\,.$$

Consider the set $\mathcal{I} \times \{1, \ldots, d\}$. Given (i, j) there is a unique index $t_i(j) \in \{1, \ldots, d\}$ such that either $F(e_{i,j}) = e_{i+1, t_i(j)}$ or $e_{i,j} = V(e_{i+1, t_i(j)})$. The t_i are permutations of $\{1, \ldots, d\}$. Given $\mathfrak{f}(i)$ we can easily compute t_i from $\dot{w}_i$ and vice versa; see the formulas for F and V in 2.2.3. When we draw a Dieudonné module as in Figure 1, it is the permutations t_i that we "see" in the picture. Write T for the permutation of $\mathcal{I} \times \{1, \ldots, d\}^2$ given by $T(i, j', j) = \big(i+1, t_i(j'), t_i(j)\big)$.

Let $(i_0, j'_0, j_0) \in \mathcal{I} \times \{1, \ldots, d\}^2$. For $n \in \mathbb{Z}$ write $(i_n, j'_n, j_n) := T^n(i_0, j'_0, j_0)$. Suppose $j'_0 < j_0$. We can think of the triple (i_0, j'_0, j_0) as a pair of base vectors e_{i_0, j'_0} and e_{i_0, j_0} with e_{i_0, j'_0} "below" e_{i_0, j_0}. Going from (i_0, j'_0, j_0) to (i_1, j'_1, j_1) there are four possible configurations, as shown in Figure 2. (For the interpretation of these illustrations see 2.2.4; cf. also the figures in [10].) In cases 1 and 2 we say that T is *parallel* at (i_0, j'_0, j_0); in case 3 we say T is *down-up*, in case 4 we say it is *up-down*.
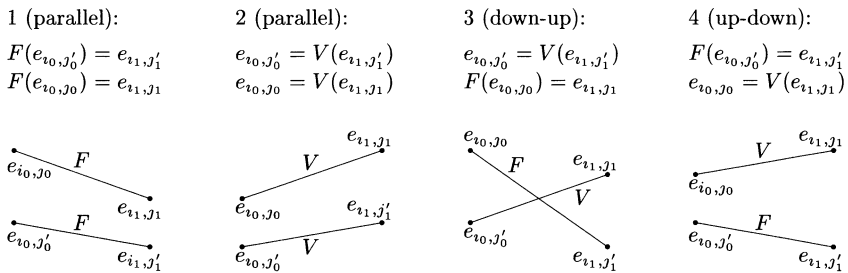
1 (parallel):	2 (parallel):	3 (down-up):	4 (up-down):
$F(e_{i_0,j'_0}) = e_{i_1,j'_1}$	$e_{i_0,j'_0} = V(e_{i_1,j'_1})$	$e_{i_0,j'_0} = V(e_{i_1,j'_1})$	$F(e_{i_0,j'_0}) = e_{i_1,j'_1}$
$F(e_{i_0,j_0}) = e_{i_1,j_1}$	$e_{i_0,j_0} = V(e_{i_1,j_1})$	$F(e_{i_0,j_0}) = e_{i_1,j_1}$	$e_{i_0,j_0} = V(e_{i_1,j_1})$

Figure 2

2.2.8 LEMMA. — *Suppose given a map $A \colon \mathcal{I} \times \{1, \ldots, d\}^2 \to R$ that satisfies (2.2.7.1). Then $A = A_f$ for some $f \in \mathbf{Aut}(\underline{N})(R)$ if and only if the following two conditions are satisfied for all (i_0, j'_0, j_0) with $j'_0 < j_0$:*

(2.2.8.1) *if T is parallel at (i_0, j'_0, j_0) then $A(i_1, j'_1, j_1) = A(i_0, j'_0, j_0)^p$;*

(2.2.8.2) *if T is up-down at (i_0, j'_0, j_0) then $A(i_0, j'_0, j_0)^p = 0$.*

Proof. — First suppose that $A = A_f$ for some $f \in \mathbf{Aut}(\underline{N})(R)$. That (2.2.8.1) and (2.2.8.2) hold is immediate from the fact that f is compatible with the linear maps $F^\sharp$ and $V^\flat$. (See 2.1.3 for notation.) For instance, suppose T is up-down at (i_0, j'_0, j_0). Writing out elements on the basis $e_{i_0,1}, \ldots, e_{i_0,d}$ we have

$$f(e_{i_0,j_0}) = e_{i_0,j_0} + \cdots + A(i_0, j'_0, j_0) \cdot e_{i_0,j'_0} + \cdots.$$

Applying $F^\sharp$ to both sides, and using that $F^\sharp \circ f^{(p)} = f \circ F^\sharp$, we find a relation $0 = 0 + \cdots + A(i_0, j'_0, j_0)^p \cdot e_{i_1,j'_1} + \cdots$. So indeed $A(i_0, j'_0, j_0)^p = 0$.

Conversely, suppose f is a $\kappa \otimes_{\mathbb{F}_p} R$-linear automorphism of N such that the corresponding matrices $A(i, j', j)$ satisfy (2.2.7.1), (2.2.8.1) and (2.2.8.2). We have to verify that f is compatible with $F^\sharp$ and $V^\flat$. To check the compatibility with $V^\flat$, let us consider a base vector e_{i_1,j_1}. Note that for each i the image of $F_{i-1}: N_{i-1} \to N_i$ is spanned by the *first* $d - \mathfrak{f}(i-1)$ base vectors $e_{i,j}$. Hence if $V^\flat(e_{i_1,j_1}) = 0$ then also $V^\flat(e_{i_1,j'_1}) = 0$ for all $j'_1 < j_1$, and it follows that $f^{(p)} V^\flat(e_{i_1,j_1}) = 0 = V^\flat f(e_{i_1,j_1})$. The other possibility is that $V^\flat(e_{i_1,j_1}) = e_{i_0,j_0}$. In this case we have

$$fV^\flat(e_{i_1,j_1}) = e_{i_0,j_0} + \sum_{j'_0 < j_0} A(i_0, j'_0, j_0)^p \cdot e_{i_0,j'_0}$$

$$V^\flat f(e_{i_1,j_1}) = e_{i_0,j_0} + \sum_{j'_1 < j_1} A(i_1, j'_1, j_1) \cdot V(e_{i_1,j'_1}).$$

If $j'_1 < j_1$, write $j'_1 = t_{i_0}(j'_0)$. Either $j'_0 > j_0$, in which case T is down-up at (i_0, j'_0, j_0) and $V(e_{i_1,j'_1}) = 0$, or $j'_0 < j_0$, in which case T is parallel at (i_0, j'_0, j_0) and (2.2.8.1) gives $A(i_1, j'_1, j_1) = A(i_0, j'_0, j_0)^p$. Combining this we find that $f^{(p)} V^\flat = V^\flat f$. We leave it to the reader to verify, in a similar manner, that $F^\sharp \circ f^{(p)} = f \circ F^\sharp$. $\square$

2.2.9. As before, suppose $j'_0 < j_0$. It can be shown (see [10], 4.16 and 4.19) that T is parallel at (i_n, j'_n, j_n) for all $n \in \mathbb{Z}$ if and only if (i_0, j'_0) and (i_0, j_0) belong to the same canonical block, by which we mean that there is an index r such that e_{i_0,j'_0} and e_{i_0,j_0} both lie in $C_{i_0,r} \backslash C_{i_0,r-1}$. Hence in order to have $A = A_f$ for some $f \in U$, the map A should satisfy the additional requirement
(2.2.9.1)
if $j'_0 < j_0$ and T is parallel at (i_n, j'_n, j_n) for all n then $A(i_0, j'_0, j_0) = 0$.

2.2.10. To summarize our conclusions, let us define a *track* in $\mathcal{I} \times \{1, \ldots, d\}^2$ to be a sequence

$$(i_0, j'_0, j_0), (i_1, j'_1, j_1), \ldots, (i_b, j'_b, j_b)$$

with $(i_n, j'_n, j_n) = T^n(i_0, j'_0, j_0)$ for all n, such that

- $j'_n < j_n$ for all $0 \leqslant n \leqslant b$;

- T is parallel at (i_n, j'_n, j_n) for all $0 \leqslant n \leqslant b - 1$;

- T is not parallel at (i_b, j'_b, j_b);

- T is not parallel at $(i_{-1}, j'_{-1}, j_{-1}) := T^{-1}(i_0, j'_0, j_0)$.

We call (i_0, j'_0, j_0) the start of the track, (i_b, j'_b, j_b) its end, and b its length. If T is up-down at (i_b, j'_b, j_b) then we call the track a *ud-track*; if T is down-up at (i_b, j'_b, j_b) then we call it a *du-track*.

Suppose there are μ different du-tracks in $\mathcal{I} \times \{1, \ldots, d\}^2$ and that there are ν different ud-tracks, of lengths $b_1, \ldots, b_\nu$. Then we find an isomorphism of k-schemes

$$U \xrightarrow{\sim} \mathrm{Spec}\left(k[x_1, \ldots, x_\mu, y_1, \ldots, y_\nu] / (y_1^{p^{b_1}}, \ldots, y_\nu^{p^{b_\nu}}) \right)$$

by associating to $f \in U$ the matrix coefficients $A(i_0, j'_0, j_0)$ for all starting points (i_0, j'_0, j_0) of a track. In particular, U is connected, and together with Lemma 2.2.6 it follows that $U = \mathbf{Aut}(\underline{N})^0$.

2.2.11. To compute $\mu = \dim(U)$ we count the du-tracks by their end points. These end points are simply all triples (i, j', j) at which T is down-up. Hence if for $i \in \mathcal{I}$ we define

$$S(i) := \{(j', j) \mid 1 \leqslant j' < j \leqslant d, F(e_{i,j'}) = 0 \text{ and } F(e_{i,j}) \neq 0\}$$
$$= \{(j', j) \mid 1 \leqslant j' < j \leqslant d \text{ and } \dot{w}_i(j') \leqslant \mathfrak{f}(i) < \dot{w}_i(j)\}$$

then the result is that $\dim(U) = \sum_{i \in \mathcal{I}} \#S(i)$. It follows from (2.2.2.1) that if $\dot{w}_i(j') = n \leqslant \mathfrak{f}(i)$ then there are $j' - n$ elements in the range $1, \ldots, j'$ for which $\dot{w}_i$ takes a value $> \mathfrak{f}(i)$. So there are $(d - \mathfrak{f}(i)) - (j' - n)$ elements j such that $(j', j) \in S(i)$. This gives, using (2.2.2.2):

$$\#S(i) = \sum_{n=1}^{\mathfrak{f}(i)} (d - \mathfrak{f}(i)) - (\dot{w}_i^{-1}(n) - n)$$

$$= \mathfrak{f}(i) \cdot (d - \mathfrak{f}(i)) - \sum_{n=1}^{\mathfrak{f}(i)} (\dot{w}_i^{-1}(n) - n) = \dim(\mathbb{X}_i) - \ell(\dot{w}_i),$$

which is the formula we want. $\square$

2.2.12 *Example.* — We consider the Dieudonné module illustrated in Figure 3; here $m = 4$ and $d = 4$, with $\mathfrak{f}(i) = 1$, $\mathfrak{f}(i + 1) = 0$, and $\mathfrak{f}(i + 2) = \mathfrak{f}(i + 3) = 2$.
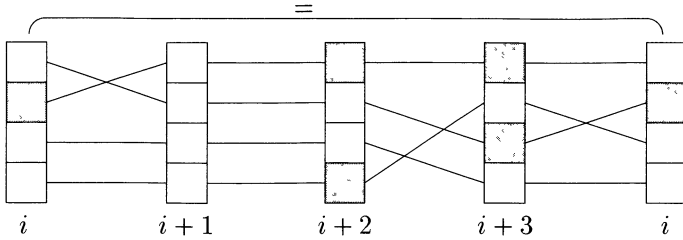

Figure 3

The distinguished representative of the element w is given by

$$\dot{w}_i = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}, \quad \dot{w}_{i+1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

$$\dot{w}_{i+2} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}, \quad \dot{w}_{i+3} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix},$$

which gives $\ell(\dot{w}) = 2+0+2+3 = 7$. As $\dim(\mathbb{X}) = 1\cdot3+0\cdot4+2\cdot2+2\cdot2 = 11$ we should find a 4-dimensional automorphism group. Note that there are indeed precisely four du-tracks; they are the sequences

$(i, 1, 4) \to (i + 1, 1, 3) \to (i + 2, 1, 3); \quad (i + 3, 2, 4) \to (i, 3, 4); \quad (i + 3, 2, 3)$

and

$(i + 3, 1, 3) \to (i, 1, 2) \to (i + 1, 1, 2) \to (i + 2, 1, 2).$

(When we draw a Dieudonné module as is done in Figure 3, we can quickly count the number of du-tracks as the number of intersections in the F-V-pattern.) A straightforward but tedious calculation shows that an automorphism of the Dieudonné module (over a field) is given by the four-tuple of matrices

$$\left[\begin{pmatrix} \zeta & w^p & t & \\ & \zeta^{p^4} & & \\ & & \eta & u^p \\ & & & \eta^{p^4} \end{pmatrix}, \begin{pmatrix} \zeta^p & w^{p^2} & t^p & \\ & \zeta^{p^5} & & \\ & & \eta^{p^5} & \\ & & & \eta^p \end{pmatrix}, \begin{pmatrix} \zeta^{p^2} & w^{p^3} & t^{p^2} & \\ & \zeta^{p^6} & & \\ & & \eta^{p^6} & \\ & & & \eta^{p^2} \end{pmatrix}, \begin{pmatrix} \zeta^{p^7} & & w & \\ & \eta^{p^7} & v & u \\ & & \zeta^{p^3} & \\ & & & \eta^{p^3} \end{pmatrix} \right],$$

where t, u, v and w are chosen arbitrarily, and ζ and η satisfy $\zeta^{p^8} = \zeta$ and $\eta^{p^8} = \eta$. So we indeed find a 4-dimensional automorphism group. The

identity component $U \subset \mathbf{Aut}(\underline{N})$ corresponds to setting $\zeta = \eta = 1$. Note that the variables t, u, v and w, and the places where they occur in the above four-tuple of matrices, correspond exactly to the four du-tracks. Note further that this only describes the points of $\mathbf{Aut}(\underline{N})$, which is non-reduced, with values in a field; as there are seven ud-tracks the Lie algebra of $\mathbf{Aut}(\underline{N})$ has dimension $4 + 7 = 11$.

2.2.13. Retaining the notations used above, we find that $\mathbf{Aut}(\underline{N})^0_{\mathrm{red}} \cong \mathbb{A}^{\mu}_k$ as schemes. The isomorphism is obtained by sending a point of $U = \mathbf{Aut}(\underline{N})^0_{\mathrm{red}}$ to the matrix coefficients $A(i_0, j'_0, j_0)$ for all triples (i_0, j'_0, j_0) that form the start of a du-track. If $(i_n, j'_n, j_n) = T^n(i_0, j'_0, j_0)$ then we have seen that $A(i_n, j'_n, j_n) = A(i_0, j'_0, j_0)^{p^n}$. As a result, when we identify the Lie algebra of $\mathrm{GL}_{\kappa \otimes k}(N) = \prod_{i \in \mathcal{I}} \mathrm{GL}_{d,k}$ with $\prod_{i \in \mathcal{I}} M_d(k)$, we have

$$\mathrm{Lie}\left(\mathbf{Aut}(\underline{N})_{\mathrm{red}}\right) = \left\{ (X_i) \in \prod_{i \in \mathcal{I}} M_d(k) \;\middle|\; \begin{array}{l} X_i(j', j) \neq 0 \text{ only if } (i, j', j) \\ \text{is the start of a du-track} \end{array} \right\}.$$

2.3. The polarized case.

Throughout this section we work over an algebraically closed field k with $\mathrm{char}(k) = p > 2$. Let $\mathcal{D} = (B, *, \epsilon)$ be a triple as in section 1.3. Let $\underline{Y}$ be a BT_1 with $\mathcal{D}$-structure over k, of type $(d, \mathfrak{f}, \delta)$. We write $\underline{N} = (N, F, V; \varphi, \iota)$ for the associated Dieudonné space with $\mathcal{D}$-structure and $\underline{N}' := (N, F, V, \iota)$ for the Dieudonné space with B-structure obtained by forgetting the polarization form φ. Define $(G, \mathbb{X})$ as in 1.3.2, and let $(G^0, \mathbb{X}^0)$ be as in 1.3.6. Let $w := w(\underline{Y}) \in W_{\mathbb{X}} \backslash W_G$ and $w^0 := w^0(\underline{Y}) \in W_{\mathbb{X}^0} \backslash W_{G^0}$. As explained in 1.3.6 and 1.3.8, there is only a difference between w and w^0 if there are factors of type D, and the two determine each other once we fix the invariant δ.

As a general notational convention, we use a prime $'$ for objects obtained by "forgetting the polarization form φ". E.g., we write $G' := \mathrm{GL}_{B \otimes_{\mathbb{F}_p} k}(N)$. In particular, our convention means that the objects studied in section 2.2 will now appear on stage equipped with a prime.

2.3.1 LEMMA. — *We have* $\mathbf{Aut}(\underline{N})_{\mathrm{red}} = \mathbf{Aut}(\underline{N}')_{\mathrm{red}} \cap G$.

Proof. — Recall that $G = \mathrm{U}_{B \otimes_{\mathbb{F}_p} k}(N, \varphi)$. Let $A := \mathbf{Aut}(\underline{N}')_{\mathrm{red}} \cap G$. It is clear that $\mathbf{Aut}(\underline{N})_{\mathrm{red}} \subset A \subset \mathbf{Aut}(\underline{N})$, so it suffices to show that

A is smooth at the origin. For $j = 1, 2$, let $(R_j, \mathfrak{m}_j)$ be an artinian local k-algebra with $R_j/\mathfrak{m}_j \cong k$. Let $\pi: R_1 \twoheadrightarrow R_2$ be a surjective homomorphism such that $I := \mathrm{Ker}(\pi)$ satisfies $\mathfrak{m}_1 \cdot I = 0$. Let $\alpha_2 \in A(R_2)$ be an element which reduces to the identity modulo $\mathfrak{m}_2$. We have to show that α_2 can be lifted to an element $\alpha_1 \in A(R_1)$.

As $\mathbf{Aut}(\underline{N}')_{\mathrm{red}}$ is a smooth k-group scheme we can lift α_2 to an R_1-valued point α_1 of $\mathbf{Aut}(\underline{N}')_{\mathrm{red}}$. If $\mathfrak{a}$ is the Lie algebra of $\mathbf{Aut}(\underline{N}')_{\mathrm{red}}$ then the set of all such liftings of α_2 is a principal homogeneous space under $\mathfrak{a} \otimes_k I$. Our task is to show that there exists an element $\beta \in \mathfrak{a} \otimes_k I$ such that $\widetilde{\alpha}_1 := \alpha_1 + \beta$ preserves the form φ.

As φ satisfies (1.1.3.2), it gives rise to an involution $g \mapsto g^*$ on $G' = \mathrm{GL}_{B \otimes k}(N)$. It follows from the relation (1.1.3.1) that this involution preserves the closed subgroup scheme $\mathbf{Aut}(\underline{N}')$; hence it also preserves $\mathbf{Aut}(\underline{N}')_{\mathrm{red}}$. It follows that α_1^* is also an R_1-valued point of $\mathbf{Aut}(\underline{N}')_{\mathrm{red}}$. By construction, $\alpha_1^* \alpha_1$ is congruent to the identity modulo I, and it follows that $\gamma := \mathrm{id} - \alpha_1^* \alpha_1$ is an element of $\mathfrak{a} \otimes_k I$.

Consider the bilinear form

$$\Psi: (N \otimes_k R_1) \times (N \otimes_k R_1) \to R_1 \text{ given by } \Psi(n, n') = \varphi(n, n') - \varphi(\alpha_1 n, \alpha_1 n').$$

Because α_1 lifts α_2 and α_2 preserves φ, the form Ψ takes values in I. Further, $\Psi(n, n')$ only depends on the classes of n and n' modulo $\mathfrak{m}_1$, as $\mathfrak{m}_1 \cdot I = 0$. In sum, Ψ gives rise to a bilinear form $\psi: N \times N \longrightarrow I$. The element $\gamma \in \mathfrak{a} \otimes_k I \subset \mathrm{End}_{B \otimes k}(N) \otimes_k I$ is the unique element such that $\psi(n, n') = \varphi(\gamma n, n')$ for all $n, n' \in N$. Note that $\gamma^* = \gamma$.

If $\widetilde{\alpha}_1 := \alpha_1 + \beta$ for some $\beta \in \mathrm{End}_{B \otimes k}(N) \otimes_k I$ then the corresponding forms ψ and $\widetilde{\psi}$ are related by

$$\widetilde{\psi}(n, n') = \psi(n, n') - \varphi\big((\beta^* \bar{\alpha}_1 + \bar{\alpha}_1^* \beta) n, n'\big),$$

where $\bar{\alpha}_1$ is the reduction of α_1 modulo $\mathfrak{m}_1$. By assumption we have $\bar{\alpha}_1 = \mathrm{id}_N$. As $\gamma^* = \gamma$ and $\mathrm{char}(k) \neq 2$ we can take $\beta = \gamma/2$. This gives a lifting of α_2 as desired. $\square$

2.3.2. To prove (ii) of Theorem 2.1.2 it suffices to consider the following four cases. In each case we further describe the possibilities for the type $(d, \mathfrak{f}, \delta)$, with notation as in section 1.3.

Case C : $B = \tilde{\kappa} = \kappa$ is a finite field with $* = \mathrm{id}$ and $\epsilon = -1$. The type $(d, \mathfrak{f})$ is fully determined by a number $q \in \mathbb{Z}_{>0}$ via $d = 2q$ and $\mathfrak{f}(i) = q$ for all $i \in \mathcal{I} = \mathrm{Hom}(\kappa, k)$.

Case D   : $B = \tilde{\kappa} = \kappa$ is a finite field with $* = $ id and $\epsilon = +1$. The type
$(d, \mathfrak{f})$ is as in Case C. Further we have an invariant δ, which is an
arbitrary function $\mathcal{I} \to \mathbb{Z}/2\mathbb{Z}$.

Case AU : $B = \tilde{\kappa}$ is a finite field, with $*$ an automorphism of order 2 and
$\epsilon = 1$. Let κ be the fixed field of $*$. The type $(d, \mathfrak{f})$ consists of a
positive integer d and a function $\mathfrak{f} \colon \widetilde{\mathcal{I}} \to \mathbb{Z}_{\geqslant 0}$ with $\mathfrak{f}(\tau) + \mathfrak{f}(\bar{\tau}) = d$
for all $\tau \in \widetilde{\mathcal{I}}$.

Case AL : $B = \tilde{\kappa} = \kappa \times \kappa$ with κ a finite field, with $(x_1, x_2)^* = (x_2, x_1)$,
and $\epsilon = 1$. The type $(d, \mathfrak{f})$ is as in Case AU. Note that in this
case $\widetilde{\mathcal{I}} = \mathcal{I} \amalg \mathcal{I}$, where the two copies of $\mathcal{I}$ are interchanged under
$\tau \mapsto \bar{\tau}$; hence $\mathfrak{f}$ is determined by its restriction to the first copy
of $\mathcal{I}$, which is a function $\mathfrak{f} \colon \mathcal{I} \to \mathbb{Z}_{\geqslant 0}$ with $\mathfrak{f}(i) \leqslant d$ for all $i \in \mathcal{I}$.

The reduction to these basic cases is an application of Morita equi-
valence. See also [10], 5.2 and 6.2. Note that the labels introduced in 1.3.1
correspond, via Morita equivalence, to the above labels, with type A
subdivided into two cases.

In Case AL, every BT_1 with $(B, *, \epsilon)$-structure is of the form
$\underline{Y} \times \underline{Y}^D$ where $\underline{Y}$ is a BT_1 with κ-structure, and where the ϵ-duality
$\lambda \colon \underline{Y} \times \underline{Y}^D \to \underline{Y}^D \times \underline{Y}$ is given by $(y_1, y_2) \mapsto (y_2, \epsilon y_1)$. This reduces Case AL
to the study of non-polarized BT_1, and one checks without difficulty that
(ii) of the theorem follows in this case from (i).

2.3.3. Suppose we are in Case C or Case D. We have $G = \prod_{i \in \mathcal{I}} G_i$
with $G_i \cong \mathrm{Sp}_{2q,k}$ (Case C) or $G_i \cong \mathrm{O}_{2q,k}$ (Case D). The Weyl group of G_i
can be described as

$$(2.3.3.1) \quad W_{G_i} = \mathbb{H}_q := \left\{ \sigma \in \mathfrak{S}_{2q} \ \middle| \ \sigma(j) + \sigma(2q+1-j) = 2q+1 \quad \text{for all } j \right\}.$$

In each factor W_{G_i} we take as generators the $q - 1$ elements of the form
$(j, j+1)(2g+1-j, 2g-j)$ together with the transposition $(q, q+1)$. The
subgroup $W_{\mathbb{X}} \subset W_G$ is the product of the groups

$$W_{\mathbb{X}_i} = \left\{ \sigma \in W_{G_i} \ \middle| \ \sigma\{1, \dots, q\} = \{1, \dots, q\} \right\} \cong \mathfrak{S}_q.$$

Consider the element $w \in W_{\mathbb{X}} \backslash W_G$. Write $\dot{w} = (\dot{w}_i)_{i \in \mathcal{I}}$ for its
distinguished representative. The length of $\dot{w}$ is given by $\ell(\dot{w}) = \sum_{i \in \mathcal{I}} \ell(\dot{w}_i)$,
with

$$(2.3.3.2) \qquad\qquad \ell(\dot{w}_i) = \sum_{n=1}^{l_i} \left(q + 1 - \dot{w}_i^{-1}(q + n) \right);$$

here $l_i \in \{1, \ldots, q\}$ is the largest index for which $\dot{w}_i^{-1}(q + l_i) \leqslant q$.

In Case D the structure of the Dieudonné module $\underline{N}$ is easiest to describe in terms of the classifying element $w \in W_{\mathbb{X}} \backslash W_G$. However, in the result that we want to prove it is not the length of $\dot{w}$ that matters but rather the length of the distinguished representative of $w^0 := w^0(\underline{Y}) \in W_{\mathbb{X}^0} \backslash W_{G^0}$. The length of $\dot{w}^0$ can be expressed directly in terms of $\dot{w}$; the result is that

$$(2.3.3.3) \qquad \ell(\dot{w}^0) = \sum_{i \in \mathcal{I}} \ell(\dot{w}_i^0) \quad \text{with} \quad \ell(\dot{w}_i^0) = \sum_{n=1}^{l_i} \big(q - \dot{w}_i^{-1}(q + n)\big);$$

here $l_i \in \{1, \ldots, q\}$ is again the largest index for which $\dot{w}_i^{-1}(q + l_i) \leqslant q$. Let us further notice that, still in Case D, the invariant $\delta \colon \mathcal{I} \to \mathbb{Z}/2\mathbb{Z}$ can be read from the element $\dot{w}$ by the formula

$$\delta(i) = \#\{j \leqslant q \mid \dot{w}_i(j) > q\} \bmod 2.$$

Similar to what was done in 2.2.3 we have an explicit description of $\underline{N}$ in terms of the element $\dot{w}$. Namely, we can choose a k-basis $e_{i,j}$ for N, for $i \in \mathcal{I}$ and $j \in \{1, \ldots, 2q\}$, such that $a \in \kappa$ acting on $e_{i,j}$ as multiplication by $i(a) \in k$, and such that F and V are given by

$$
\begin{aligned}
F(e_{i,j}) &= e_{i+1,n} \quad \text{and} \quad & V(e_{i+1,n}) &= 0 \quad & \text{if } \dot{w}_i(j) = q + n; \\
F(e_{i,j}) &= 0 \quad \text{and} \quad & V(e_{i+1,n}) &= e_{i,j} \quad & \text{if } \dot{w}_i(j) = -q + n.
\end{aligned}
$$

The ϵ-symmetric form φ on N is the orthogonal sum of forms $\varphi_i \colon N_i \times N_i \to k$. With respect to the basis $\{e_{i,j}\}_{j=1,\ldots,2q}$ the form φ_i is given by an (invertible) anti-diagonal matrix Φ_i. In Case D we can choose our basis such that $\Phi_i = \mathrm{antidiag}(1, \ldots, 1)$. In Case C there is not, in general, a natural choice of a normal form for the Φ_i. Note that (1.1.3.1) gives a relation between Φ_i and Φ_{i+1}. Any collection of anti-diagonal matrices $\{\Phi_i\}_{i \in \mathcal{I}}$ satisfying the relations imposed by (1.1.3.1) gives a Dieudonné module with $(\kappa, \mathrm{id}, -1)$-structure, and up to isomorphism this object is independent of the chosen collection $\{\Phi_i\}$. We refer to [10], 5.8 for details.

Via the chosen ordered bases $\{e_{ij}\}$ we may identify the Lie algebra of $G' = \prod_{i \in \mathcal{I}} \mathrm{GL}_{2q,k}$ with $\prod_{i \in \mathcal{I}} M_{2q}(k)$. As we have seen in 2.2.13, the Lie algebra of $\mathbf{Aut}(\underline{N}')_{\mathrm{red}}$ is given by

$$\mathrm{Lie}\left(\mathbf{Aut}(\underline{N}')_{\mathrm{red}}\right) = \left\{ (X_i) \in \prod_{i \in \mathcal{I}} M_{2q}(k) \ \middle|\ \begin{array}{l} X_i(j', j) \neq 0 \text{ only if } (i, j', j) \\ \text{is the start of a du-track} \end{array} \right\}.$$

Note that if we subdivide each matrix X_i into four blocks of size $q \times q$, say

$$X_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix},$$

then for $(X_i)_{i \in \mathcal{I}} \in \mathrm{Lie}\left(\mathbf{Aut}(\underline{N}')_{\mathrm{red}}\right)$, only the blocks β_i can be nonzero; this corresponds to the fact that if (i, j', j) is the start of a track then $j' \leqslant \mathfrak{f}(i) = q < j$.

Write $J = J_q$ for the anti-diagonal matrix of size $q \times q$ with all anti-diagonal coefficients equal to 1. Given a square matrix A of size $q \times q$, write $^sA = J \cdot {}^tA \cdot J$ for its reflection in the anti-diagonal. So $\left(^sA\right)_{t,u} = A_{q+1-u,q+1-t}$.

In the symplectic case (Case C) there are diagonal matrices C_i such that

$$\Phi_i = \begin{pmatrix} C_i & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \cdot \begin{pmatrix} C_i & 0 \\ 0 & 1 \end{pmatrix}.$$

(Here we write matrices in block form.) One computes that the Lie algebra of the symplectic group $\mathrm{Sp}(N_i, \varphi_i)$ is then given by

$$\mathfrak{sp}(N_i, \varphi_i) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2q}(k) \,\middle|\, \begin{matrix} C_i \cdot \alpha \cdot C_i^{-1} = -{}^s\delta, \ \beta = C_i^{-1} \cdot {}^s\beta \cdot {}^sC_i \\ \text{and } {}^s\gamma = {}^sC_i \cdot \gamma \cdot C_i^{-1} \end{matrix} \right\}.$$

In the orthogonal case (Case D), $\Phi_i = \mathrm{antidiag}(1, \ldots, 1)$. This gives

$$\mathfrak{o}(N_i, \varphi_i) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_{2q}(k) \,\middle|\, \begin{matrix} {}^s\beta = -\beta, \, {}^s\gamma = -\gamma \\ \text{and } \alpha + {}^s\delta = 0 \end{matrix} \right\}.$$

Given an element $(i, j', j) \in \mathcal{I} \times \{1, \ldots, d\}^2$, write $^s(i, j', j) := (i, 2q + 1 - j, 2q + 1 - j')$. Observe that $^s\left(^s(i, j', j)\right) = (i, j', j)$. If

$$\mathbf{t}: \quad (i_0, j'_0, j_0), (i_1, j'_1, j_1), \ldots, (i_b, j'_b, j_b)$$

is a track, write

$$^s\mathbf{t} := \quad {}^s(i_0, j'_0, j_0), {}^s(i_1, j'_1, j_1), \ldots, {}^s(i_b, j'_b, j_b).$$

We claim that $^s\mathbf{t}$ is again a track in $\mathcal{I} \times \{1, \ldots, d\}^2$. This is a direct consequence of the definitions once we know that the permutation t_i defined in 2.2.7 is an element of the group $\mathbb{H}_q$ defined in (2.3.3.1). Using that $\dot{w}_i \in \mathbb{H}_q$, this can be checked using the formulas for F and V given above.

We can now complete the proof of 2.1.2, (ii) in Cases C and D. First we do Case C. By Lemma 2.3.1 the dimension of $\mathbf{Aut}(\underline{N})$ equals the dimension of $\mathrm{Lie}\left(\mathbf{Aut}(\underline{N}')_{\mathrm{red}}\right) \cap \mathfrak{sp}_{\kappa \otimes k}(N, \varphi)$. By what was explained above this dimension equals the number of tracks modulo the equivalence relation $\mathfrak{t} \sim {}^s\mathfrak{t}$. (The equations for $\mathfrak{sp}(N_i, \varphi_i)$ give one relation for every pair $\{\mathfrak{t}, {}^s\mathfrak{t}\}$ with $\mathfrak{t} \neq {}^s\mathfrak{t}$.)

As in 2.2.11 we count the tracks by their end points. These are the triples (i, j', j) with $1 \leqslant j' < j \leqslant 2q$ and $\dot{w}_i(j') \leqslant q < \dot{w}_i(j)$. Precisely one of the tracks $\mathfrak{t}$ and ${}^s\mathfrak{t}$ has an end point (i, j', j) with $j \leqslant 2q + 1 - j'$, so we impose this as an extra condition. Writing

$$S_{\mathrm{C}}(i) := \left\{(j', j) \mid 1 \leqslant j' < j \leqslant 2q + 1 - j' \text{ and } \dot{w}_i(j') \leqslant q < \dot{w}_i(j)\right\}$$

this gives

$$\dim\left(\mathbf{Aut}(\underline{N})\right) = \sum_{i \in \mathcal{I}} \#S_{\mathrm{C}}(i).$$

Suppose $j' \leqslant q$ is an index with $\dot{w}_i(j') = n \leqslant q$. We subdivide the set $\{1, \ldots, 2q\}$, and in each range we count the numbers of times $\dot{w}_i$ takes a value $> q$:

range	$\#r$ with $\dot{w}_i(r) \leqslant q$	$\#r$ with $\dot{w}_i(r) > q$
$1, \ldots, j' - 1$	$n - 1$	$j' - n$
j'	1	0
$j' + 1, \ldots, 2q - j'$	$q - j'$	$q - j'$
$2q + 1 - j'$	0	1
$2q + 2 - j', \ldots, 2q$	$j' - n$	$n - 1$

Thus we find that there are $q - j'$ indices j for which $(j', j) \in S_{\mathrm{C}}(i)$. If, as above, we let $l_i \in \{1, \ldots, q\}$ be the largest number for which $\dot{w}_i^{-1}(q + l_i) \leqslant q$, we get

$$\#S_{\mathrm{C}}(i) = \sum_{n=1}^{q-l} q + 1 - \dot{w}_i^{-1}(n).$$

Combined with (2.3.3.2) this gives

$$\#S_{\mathrm{C}}(i) + \ell(\dot{w}_i) = \sum_{n=1}^{q-l}\left(q + 1 - \dot{w}_i^{-1}(n)\right) + \sum_{n=1}^{l}\left(q + 1 - \dot{w}_i^{-1}(q + n)\right)$$

$$= q(q+1) - \sum_{\mu=1}^{q} \mu = q(q+1)/2 = \dim(\mathbb{X}_i).$$

This proves our claim that $\dim\big(\mathbf{Aut}(\underline{N})\big) = \dim(\mathbb{X}) - \ell(\dot{w})$.

In the orthogonal case the computation is similar. This time we have to count only the pairs of tracks $\{\mathsf{t}, {}^s\mathsf{t}\}$ with $\mathsf{t} \neq {}^s\mathsf{t}$. Setting

$$S_{\mathrm{D}}(i) := \big\{(j', j) \mid d1 \leqslant j' < j < 2q + 1 - j' \text{ and } \dot{w}_i(j') \leqslant q < \dot{w}_i(j)\big\}$$

we get that $\dim\big(\mathbf{Aut}(\underline{N})\big) = \sum_{i \in \mathcal{I}} \#S_{\mathrm{D}}(i)$. Counting as above and using (2.3.3.3) gives

$$\#S_{\mathrm{D}}(i) + \ell(\dot{w}_i^0) = \sum_{n=1}^{l_i} q - \dot{w}_i^{-1}(q+n) + \sum_{n=1}^{q-l_i} q - \dot{w}_i^{-1}(n)$$

$$= q^2 - \sum_{\mu=1}^{q} \mu = q(q-1)/2 = \dim(\mathbb{X}_i),$$

which is what we want. $\square$

2.3.4. Finally we consider Case AU. Recall that we write $\widetilde{\mathcal{I}}$ for the set of embeddings of $\tilde{\kappa}$ into k. We have a natural $2:1$ map $\mathrm{res}\colon \widetilde{\mathcal{I}} \to \mathcal{I}$. For $\tau \in \widetilde{\mathcal{I}}$, let $\bar{\tau} := \tau \circ *$. The type $(d, \mathfrak{f})$ is given by an integer d and a function $\mathfrak{f}\colon \widetilde{\mathcal{I}} \to \mathbb{Z}_{\geqslant 0}$ with the property that $\mathfrak{f}(\tau) + \mathfrak{f}(\bar{\tau}) = d$ for all τ.

Given $(d, \mathfrak{f})$, the corresponding pair of Weyl groups $W_{\mathbb{X}} \subset W_G$ can be described as follows. Let $w_0 \in \mathfrak{S}_d$ be the permutation of order 2 given by $w_0(j) = d + 1 - j$. For $\pi \in \mathfrak{S}_d$, write $\tilde{\pi} := w_0 \circ \pi \circ w_0$. Then

$$W_G = \Big\{(\pi_\tau) \in \prod_{\tau \in \widetilde{\mathcal{I}}} \mathfrak{S}_d \ \Big|\ \tilde{\pi}_\tau = \pi_{\bar{\tau}} \quad \text{for all } \tau\Big\},$$

and

$$W_{\mathbb{X}} = \big\{(\pi_\tau) \in W_G \mid \pi_\tau\{1, \ldots, \mathfrak{f}(\tau)\} = \{1, \ldots, \mathfrak{f}(\tau)\} \quad \text{for all } \tau\big\}.$$

Let $\dot{w} = (\dot{w}_\tau)_{\tau \in \widetilde{\mathcal{I}}}$ for the distinguished representative of the coset $w = w(\underline{Y})$. Let $\mathcal{R} \subset \widetilde{\mathcal{I}}$ be a subset such that precisely one member of each pair $\{\tau, \bar{\tau}\}$ is in $\mathcal{R}$. For each pair $\{\tau, \bar{\tau}\}$ we have

$$(2.3.4.1) \quad \ell(\dot{w}_\tau) = \ell(\dot{w}_{\bar{\tau}}) = \sum_{n=1}^{\mathfrak{f}(\tau)} \big(\dot{w}_\tau^{-1}(n) - n\big) = \sum_{n=\mathfrak{f}(\tau)+1}^{d} \big(n - \dot{w}_\tau^{-1}(n)\big),$$

and $\ell(\dot{w}) = \sum_{\tau \in \mathcal{R}} \ell(\dot{w}_\tau)$. The dimension of $\mathbb{X}$ is given by

$$(2.3.4.2) \qquad \dim(\mathbb{X}) = \sum_{\tau \in \mathcal{R}} \mathfrak{f}(\tau) \cdot \big(d - \mathfrak{f}(\tau)\big).$$

(Observe that $\mathfrak{f}(\tau) \cdot \big(d - \mathfrak{f}(\tau)\big) = \mathfrak{f}(\bar{\tau}) \cdot \big(d - \mathfrak{f}(\bar{\tau})\big)$ because $\mathfrak{f}(\tau) + \mathfrak{f}(\bar{\tau}) = d$.)

The description of the Dieudonné module $\underline{N}$ is much the same as in the previous cases. Namely, we can choose a k-basis $e_{\tau,j}$ for $\tau \in \widetilde{\mathcal{I}}$ and $j \in \{1, \ldots, d\}$ such that $a \in \tilde{\kappa}$ acts on $e_{\tau,j}$ as multiplication by $\tau(a) \in k$, and such that Frobenius and Verschiebung are given by

$$\begin{aligned} F(e_{\tau,j}) &= e_{\tau+1,n} \text{ and } & V(e_{\tau+1,n}) &= 0 & \text{if } \dot{w}_\tau(j) = \mathfrak{f}(\tau) + n; \\ F(e_{\tau,j}) &= 0 \qquad \text{and } & V(e_{\tau+1,n}) &= e_{\tau,j} & \text{if } \dot{w}_\tau(j) = -d + \mathfrak{f}(\tau) + n. \end{aligned}$$

The form φ decomposes as an orthogonal sum of hermitian forms φ_i on $N_\tau \times N_{\bar{\tau}}$ (where $i = \mathrm{res}(\tau) = \mathrm{res}(\bar{\tau})$). We can choose the ordered basis $e_{\tau,1}, \ldots, e_{\tau,d}, e_{\bar{\tau},1}, \ldots, e_{\bar{\tau},d}$ such that φ_i is given by an invertible anti-diagonal matrix Φ_i. As in Case C there is, in general, not a natural choice of a normal form for these matrices.

For $(\tau, j', j) \in \widetilde{\mathcal{I}} \times \{1, \ldots, d\}^2$, write $^s(\tau, j', j) := (\bar{\tau}, d+1-j, d+1-j')$. Following the same procedure as in 2.3.3, if $\mathfrak{t}$ is a track we get a "mirrored" track $^s\mathfrak{t}$. The condition that the form ψ is preserved gives one relation for each pair of tracks $\{\mathfrak{t}, {}^s\mathfrak{t}\}$. We find, counting the tracks by their end points, that

$$\dim\big(\mathbf{Aut}(\underline{N})\big) = \sum_{\tau \in \mathcal{R}} \#S_{\mathrm{A}}(\tau),$$

with

$$S_{\mathrm{A}}(\tau) = \big\{(j', j) \mid 1 \leqslant j' < j \leqslant d \text{ and } \dot{w}_\tau(j') \leqslant \mathfrak{f}(\tau) < \dot{w}_\tau(j)\big\}.$$

If $\dot{w}_\tau(j') = n \leqslant \mathfrak{f}(\tau)$ then there are $d - \mathfrak{f}(\tau) + n - j'$ values of j with $(j', j) \in S_{\mathrm{A}}(\tau)$. Hence

$$\#S_{\mathrm{A}}(\tau) = \sum_{n=1}^{\mathfrak{f}(\tau)} \big(d - \mathfrak{f}(\tau) + n - \dot{w}_\tau^{-1}(n)\big) = \mathfrak{f}(\tau)\big(d - \mathfrak{f}(\tau)\big) + \sum_{n=1}^{\mathfrak{f}(\tau)} \big(n - \dot{w}_\tau^{-1}(n)\big).$$

Combining this with (2.3.4.1) and (2.3.4.2) we obtain the equality $\ell(\dot{w}) + \dim\big(\mathbf{Aut}(\underline{N})\big) = \dim(\mathbb{X})$ that we wanted to prove. $\qquad \square$

3. Application to Ekedahl-Oort stratifications.

3.1. The Ekedahl-Oort stratification
on moduli spaces of PEL type.

3.1.1. We consider a moduli problem of PEL type with good reduction at a prime $p > 2$. The data involved are the following.

- $(\mathcal{B}, *)$ is a finite dimensional semi-simple $\mathbb{Q}$-algebra with a positive involution;

- $\mathcal{V}$ is a finitely generated faithful left $\mathcal{B}$-module;

- $\varphi \colon \mathcal{V} \times \mathcal{V} \to \mathbb{Q}$ is a symplectic form ($\mathbb{Q}$-bilinear, alternating and perfect) with the property that $\varphi(bv_1, v_2) = \varphi(v_1, b^* v_2)$ for all $b \in \mathcal{B}$ and v_1, $v_2 \in \mathcal{V}$;

- p is a prime number > 2 such that $\mathcal{B} \otimes \mathbb{Q}_p$ is unramified, i.e., isomorphic to a product of matrix algebras over unramified field extensions of $\mathbb{Q}_p$;

- $O_{\mathcal{B}}$ is a $\mathbb{Z}_{(p)}$-order in $\mathcal{B}$, stable under $*$, such that $O_{\mathcal{B}} \otimes \mathbb{Z}_p$ is a maximal order in $\mathcal{B} \otimes \mathbb{Q}_p$;

- $\Lambda \subset \mathcal{V} \otimes \mathbb{Q}_p$ is a $\mathbb{Z}_p$-lattice which is also an $O_{\mathcal{B}}$-submodule, such that φ induces a perfect pairing $\Lambda \times \Lambda \to \mathbb{Z}_p$;

- $\mathcal{G} := \mathrm{CSp}(\Lambda, \varphi) \cap \mathrm{GL}_{O_{\mathcal{B}} \otimes \mathbb{Z}_p}(\Lambda)$ is the (not necessarily connected) reductive group over $\mathbb{Z}_p$ given by the symplectic similitudes of (Λ, φ) that commute with the action of $O_{\mathcal{B}}$;

- $\mathcal{X}$ is a $\mathcal{G}(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to \mathcal{G}_{\mathbb{R}}$ (with $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$) that define a Hodge structure of type $(-1, 0) + (0, -1)$ on $\mathcal{V}_{\mathbb{R}}$ for which either $2\pi i \cdot \varphi$ or $-2\pi i \cdot \varphi$ is a polarization form;

- $\mathfrak{c}$ is the $\mathcal{G}(\mathbb{C})$-conjugacy class of cocharacters of $\mathcal{G}_{\mathbb{C}}$ associated to $\mathcal{X}$; concretely, if $h \in \mathcal{X}$ then we have a cocharacter $\mu = \mu_h$ through which $z \in \mathbb{C}^\times$ acts on $\mathcal{V}^{-1, 0}$ (resp. $\mathcal{V}^{0, -1}$) as multiplication by z (resp. by 1);

- E is the reflex field, i.e., the field of definition of the conjugacy class $\mathfrak{c}$.

3.1.2. Fix data $\mathcal{D} = (\mathcal{B}, *, \mathcal{V}, \varphi, O_{\mathcal{B}}, \Lambda, \mathcal{X})$ as in 3.1.1. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. We fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let v be the corresponding place of E above (p). We write $O_{E,v}$ for the localization of O_E at v.

Let $C_p := \mathcal{G}(\mathbb{Z}_p)$. Let C^p be a compact open subgroup of $\mathcal{G}(\mathbb{A}_f^p)$, and put $C := C_p \times C^p$. We consider the moduli problem $\mathcal{A}_{\mathcal{D},C}$ over $\mathrm{Spec}(O_{E,v})$ defined by Kottwitz in [8], §5. If T is a locally noetherian $O_{E,v}$-scheme then the T-valued points of $\mathcal{A}_{\mathcal{D},C}$ are the isomorphism classes of four-tuples $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta})$ with

- A an abelian scheme up to prime-to-p isogeny over T;

- $\bar{\lambda} \in \big(\mathrm{NS}(A) \otimes \mathbb{Z}_p \big)/\mathbb{Z}_p^\times$ the class of a prime-to-p polarization;

- $\iota : O_\mathcal{B} \to \mathrm{End}_T(A) \otimes \mathbb{Z}_{(p)}$ a homomorphism of $\mathbb{Z}_{(p)}$-algebras with $\iota(b^*) = \iota(b)^\dagger$; here $\dagger$ is the Rosati involution associated to $\bar{\lambda}$;

- $\bar{\eta}$ a level structure of type C^p on A;

such that a certain determinant condition is satisfied. For precise details we refer to Kottwitz [8], §5. If C^p is sufficiently small, which we from now on assume, then $\mathcal{A}_{\mathcal{D},C}$ is representable by a smooth quasi-projective $O_{E,v}$-scheme.

3.1.3. Let $B := O_\mathcal{B}/pO_\mathcal{B}$. The involution $*$ induces an involution of B which we denote by the same symbol. Let $\widetilde{K}$ be the centre of $\mathcal{B}$, let $K := \{z \in \widetilde{K} \mid z^* = z\}$, and let $O_{\widetilde{K}}$ and O_K be the rings of integers. Then $O_{\widetilde{K}}/pO_{\widetilde{K}} = \tilde{\kappa}$ is the centre of B and $O_K/pO_K = \kappa := \{z \in \tilde{\kappa} \mid z^* = z\}$.

Let $\mathbb{Q}_p^{\mathrm{nr}} \subset \overline{\mathbb{Q}}_p$ be the maximal unramified extension of $\mathbb{Q}_p$. We write $\overline{\mathbb{F}}$ for its residue field. The assumptions on the data in 3.1.1 imply that v is an unramified prime, so $E_v \subset \mathbb{Q}_p^{\mathrm{nr}}$.

Consider an algebraically closed field k containing $\overline{\mathbb{F}}_p$. As usual we write $\widetilde{\mathcal{I}}$ (resp. $\mathcal{I}$) for the set of embeddings of $\tilde{\kappa}$ (resp. κ) into k. Using that v is unramified we get, via the chosen embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, natural identifications $\widetilde{\mathcal{I}} = \mathrm{Hom}(\widetilde{K}, \mathbb{C})$ and $\mathcal{I} = \mathrm{Hom}(K, \mathbb{C})$. Write $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\nu$ for the partition of $\mathcal{I}$ corresponding to the decomposition of κ as a product of finite fields.

The choice of $h \in \mathcal{X}$ gives $\mathcal{V}$ a $\mathbb{Q}$-Hodge structure of type $(-1,0) + (0,-1)$ equipped with an action of $\mathcal{B}$. In particular we have a pair a $\mathcal{B} \otimes_\mathbb{Q} \mathbb{C}$-modules $\mathcal{V}_\mathbb{C}^{-1,0} \subset \mathcal{V}_\mathbb{C}$. Via the above identifications this gives us a type $(d, \mathfrak{f})$ as in 1.3.2 with d constant on each of the subsets $\mathcal{I}_n \subset \mathcal{I}$ and such that (1.3.2.1) holds. This type is independent of the chosen h.

A k-valued point of $\mathcal{A}_{\mathcal{D},C}$ gives rise to a BT_1 with $(B, *, -1)$-structure $\underline{Y}$ by taking $Y := A[p]$ equipped with its B-action, and with

$\lambda: Y \xrightarrow{\sim} Y^D$ the -1-duality induced by the prime-to-p-polarization on A. The determinant condition mentioned in 3.1.2 implies that $\underline{Y}$ has type equal to $(d, \mathfrak{f})$.

3.1.4 LEMMA. — *Let S be a scheme of characteristic $p > 2$. Let $\underline{Y}$ be a BT_1 with $(B, *, \epsilon)$-structure over S. Let k be an algebraically closed field, $\mathrm{char}(k) = p$. For $s \in S(k)$ let $\delta_s: \mathcal{I}^D \to \mathbb{Z}/2\mathbb{Z}$ be the invariant associated to $\underline{Y}_s$ as in 1.3.6. Then the function $s \mapsto \delta_s$ is locally constant on $S(k)$.*

Proof. — It suffices to consider the case that S is the spectrum of a perfect valuation ring A. Write Q for the fraction field of A and $\mathfrak{m}_A$ for its maximal ideal. By Berthelot [1], we have a Dieudonné theory that generalizes the theory over a perfect field. (In fact, the Dieudonné functor is again given by taking homomorphisms into the Witt covectors.) So we have a 5-tuple $\underline{N} = (N, F, V, \iota, \varphi)$ as in 1.1.3 but now with N a free A-module of finite rank. After base change to Q (resp. to $A/\mathfrak{m}_A$) we retrieve the Dieudonné module of the generic (resp. special) fibre of $\underline{Y}$.

We know that $\mathrm{Ker}(F_Y) \subset Y$ and $\mathrm{Ker}(V_Y) \subset Y^{(p/S)}$ are free subgroup schemes; their Dieudonné modules can be identified with $\mathrm{Ker}(F_N) \subset N$ and $\mathrm{Ker}(V_N) \subset N$, respectively. The exactness of the sequences

$$1 \to \mathrm{Ker}(F_Y) \to Y \to \mathrm{Ker}(V_Y) \to 1 \text{ and } 1 \to \mathrm{Ker}(V_Y) \to Y \to \mathrm{Ker}(F_Y) \to 1$$

then implies, using [1], Prop. 2.4.1, that $\mathrm{Ker}(F_N)$ and $\mathrm{Ker}(V_N)$ are direct summands of N.

Using Morita equivalence one reduces the problem to the case where $B = \kappa$ is a finite field with $* = \mathrm{id}_\kappa$ and $\epsilon = +1$. Then the assertion reduces to the following fact : Let N be a free A-module of even rank $2q$ equipped with a perfect symmetric pairing $\varphi: N \times N \to A$. Let L_1 and L_2 be direct summands of N of rank q that are totally isotropic with respect to φ. Then $\dim\big((L_1 \otimes Q) \cap (L_2 \otimes Q)\big)$ has the same parity as $\dim\big((L_1 \otimes A/\mathfrak{m}_A) \cap (L_2 \otimes A/\mathfrak{m}_A)\big)$. $\qquad\square$

3.1.5. Write $\mathcal{A}_0 := \mathcal{A}_{\mathcal{D}, C} \otimes \overline{\mathbb{F}}$. The lemma implies that we can decompose $\mathcal{A}_0$ as a disjoint union of open and closed subscheme according to the value of δ, say $\mathcal{A}_0 = \amalg \mathcal{A}_{0, \delta}$.

Fix δ. To the type $(d, \mathfrak{f}, \delta)$ we associate a pair $(G^0, \mathbb{X}^0)$ as in 1.3.6. Note that, up to a central factor, G^0 can be identified with the special fibre of $\mathcal{G}$, and that $\mathbb{X}^0$ is closely related to $\mathcal{X}^0$. We shall not attempt to make

this more precise. Now our classification results of BT_1 with additional structure give rise to a generalized Ekedahl-Oort stratification

$$\mathcal{A}_{0,\delta} = \coprod_{w \in W_{\mathbb{X}^0} \backslash W_{G^0}} \mathcal{A}_{0,\delta}(w).$$

More precisely, we have a partition of $\mathcal{A}_{0,\delta}$ into a disjoint union of locally closed subspaces such that the Zariski closure of each stratum is a union of strata. These properties were proven by Wedhorn in [18], generalizing results of Ekedahl and Oort in the Siegel modular case, see [14].

Combining Theorem 2.1.2 with [18], Theorem (6.10) we arrive at the following dimension formula.

3.1.6 COROLLARY. — *If $\mathcal{A}_{0,\delta}(w) \neq \emptyset$ then all its irreducible components have dimension equal to $\ell(\dot{w})$.*

Note that w here takes the role of what in (ii) of Theorem 2.1.2 was called w^0. Further we are using that $\dim(\mathcal{A}_{0,\delta}) = \dim(\mathbb{X}^0)$; see [11], 4.1.8 for details.

3.1.7 *Remark.* — Let $\underline{Y}$ be a BT_1 with $(\mathcal{O}/p\mathcal{O}, *, \epsilon)$-structure associated to a k-valued point $s \in \mathcal{A}_{0,\delta}(w)$. The automorphism group scheme $\mathbf{Aut}(\underline{Y})$ is in general (highly) non-reduced. In fact, [18], (2.7) shows that the tangent space of $\mathbf{Aut}(\underline{Y})$ at the identity is isomorphic to the tangent space of the deformation functor $\mathrm{Def}(\underline{Y})$. But by Serre-Tate and [18], (2.17) the latter is isomorphic to the tangent space of $\mathcal{A}_{0,\delta}$ at the point s. Hence the Lie algebra of $\mathbf{Aut}(\underline{Y})$ has dimension equal to $\dim(\mathcal{A}_0)$, independent of w. Cf. Example (2.2.12).

3.2. Some examples.

An immediate consequence of 3.1.6 is the following.

3.2.1 COROLLARY. — *There is a unique Ekedahl-Oort stratum that is open in $\mathcal{A}_{0,\delta}$. There is a unique $w \in W_{\mathbb{X}^0} \backslash W_{G^0}$ such that the stratum $\mathcal{A}_{0,\delta}(w)$, if not empty, is 0-dimensional.*

The open Ekedahl-Oort stratum, corresponding to the class of the longest element of W_{G^0}, plays the role of the (generalized) ordinary locus in $\mathcal{A}_{0,\delta}$. It is studied in detail in our paper [11]. The unique 0-dimensional stratum (if not empty) takes the role of what in the Siegel modular case is called the superspecial stratum. (In the Siegel modular case it corresponds

to the abelian varieties that are isomorphic to E^g with E a supersingular elliptic curve).

Already in the simplest examples we find that for moduli points $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta})$ in the "generalized" ordinary locus, the underlying abelian variety A is not necessarily ordinary in the classical sense. A similar remark applies to the superspecial stratum. We shall illustrate this with an example.

3.2.2. We consider an example of type AL (see 2.3.2); in particular this means that there is no invariant δ to consider.

Fix PEL data as in 3.1.1 with $\mathcal{B} = \widetilde{K}$ a CM-field of degree $2m$ over $\mathbb{Q}$. Let $K \subset \widetilde{K}$ be the totally real subfield. Suppose p is a prime number that is totally inert (unramified) in the extension $\mathbb{Q} \subset K$ and that splits in the extension $K \subset \widetilde{K}$. Set $q := p^m$. Then $B = O_{\mathcal{B}}/pO_{\mathcal{B}}$ is isomorphic to $\mathbb{F}_q \times \mathbb{F}_q$, with involution (induced by complex conjugation on $\mathcal{B}$) given by $(x, y)^* = (y, x)$. Fixing such an isomorphism, every BT_1 with $(B, *, -1)$-structure is of the form $\underline{Y} \times \underline{Y}^D$, where $\underline{Y}$ is a BT_1 with $\mathbb{F}_q$-structure, and where the (-1)-duality $\lambda : Y \times Y^D \to Y^D \times Y$ is given by $(y, \eta) \mapsto (\eta, -y)$. In this way the study of BT_1 with $(B, *, -1)$-structure reduces to the study of BT_1 with $\mathbb{F}_q$-structure (without polarization).

Via the above mechanism, a k-valued point of $\mathcal{A}_0$ gives rise to a BT_1 with $\mathbb{F}_q$-structure. Its type $(d, \mathfrak{f})$ is determined by the chosen PEL data. More precisely, $d = \dim_{\mathcal{B}}(\mathcal{V})$ and the CM-type $\mathfrak{f}$ is determined by the choice of the $\mathcal{G}(\mathbb{R})$-conjugacy class $\mathcal{X} \subset \mathrm{Hom}(\mathbb{S}, \mathcal{G}_{\mathbb{R}})$.

For instance, suppose $m = 6$ and $d = 5$. The set $\mathcal{I}$ of embeddings of $\mathbb{F}_q$ into $k = \bar{k}$ is a set of 6 elements, equipped with a natural cyclic ordering. Suppose that the function $\mathfrak{f} : \mathcal{I} \to \{0, 1, \ldots, 5\}$ takes consecutive values 1, 3, 1, 1, 0 and 5. (This is a random choice.) The BT_1 corresponding to the stratum of maximal dimension then corresponds to the Dieudonné module given in Figure 4.
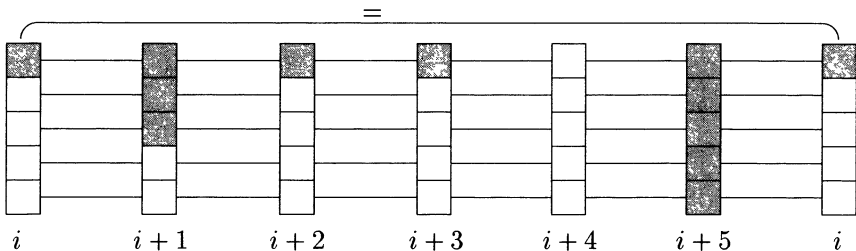

Figure 4

We see that in this example $\underline{Y}$ decomposes as a product, $\underline{Y} = \underline{Y}_1 \times \cdots \underline{Y}_5$, corresponding to the horizontal "layers" in the picture. This decomposition is not canonical but it can be shown that the coarser decomposition (referred to as the slope decomposition)

$$\underline{Y} = (\underline{Y}_1 \times \underline{Y}_2) \times (\underline{Y}_3 \times \underline{Y}_4) \times \underline{Y}_5$$

is canonical. It plays an important role in our generalization of Serre-Tate theory; we refer to [11] for further discussion. Note, however, that Y, the underlying BT_1 without additional structure, is not ordinary in the classical sense, i.e., it is not a product of factors $\mathbb{Z}/p\mathbb{Z}$ and μ_p. In fact, Y is a local-local group scheme.

At the other extreme, the BT_1 that corresponds to the 0-dimensional stratum is given by the Dieudonné module in Figure 5. The underlying group scheme Y is in this case indecomposable, quite in contrast to what happens on the superspecial stratum in the Siegel modular case.
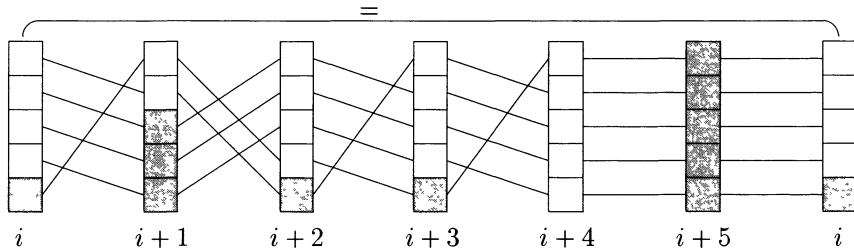

Figure 5

A variety of different examples is obtained by choosing different CM-types $\mathfrak{f}$. Let us add that this is typical for examples of type A (cases AU or AL in 2.3.2). In examples of type C there is no freedom of choice for the CM-type (due to the fact that the involution is of the first kind), and we find that the BT_1 (without additional structure) occurring on the stratum of maximal (resp. minimal) dimension is of the form $\left((\mathbb{Z}/p\mathbb{Z}) \times \mu_p\right)^g$ (resp. $E[p]^g$, for E a supersingular curve), as in the classical case.

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