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ORBIFOLDS, SPECIAL VARIETIES AND CLASSIFICATION THEORY

by Frédéric CAMPANA

A ma mère

Contents.

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Introduction.

The main observations and results of the present paper are the followings(*).

(1) The definition of the *orbifold base* of a fibration. This notion is standard for elliptic fibrations. Its extension to higher dimensions reveals the central role played by the category of orbifolds in classification theory, and is also used here as a leading thread for the appropriate definitions of the basic notions of the orbifold category (such as bimeromorphic maps, differential forms, ...). In fact, the definitions and constructions of the present paper easily extend to the category of orbifolds, using the definitions of terminal or canonical modification given in § 2 below. See also § 6.

(2) The notion of *special* manifold (or orbifold), and the associated construction of the *core* $c_X : X \dashrightarrow C(X)$, for an arbitrary compact Kähler

(*) added in proof: an introducting exposition of the present text can be found on math. AG/0402242.

manifold X. This fibration functorially "decomposes" X into its special (the fibres) and general type (the orbifold base) "components".

(3) The decomposition of the core as the canonical and functorial composition of orbifold Iitaka fibrations and rational quotients.

These two structure theorems show that the usual *trichotomy* of algebraic geometry (rational connectedness, $\kappa = 0$, general type) reduces to a *dichotomy* of a more fundamental nature (special vs general type), in which specialness is simply the orbifold combination of the first two terms (rational connectedness, $\kappa = 0$) of the initial trichotomy. The conjectural stability by deformation of all the fibrations constructed here may be seen as another indication of their fundamental nature.

(4) The fundamental nature of the dichotomy special vs general type, with the core as its concrete realisation, can be stressed by further conjectural aspects of the core and the class of special manifolds, at the levels of fundamental group, hyperbolicity, and arithmetics. These conjectures indeed claim that special and general type orbifolds have entirely opposite behaviours with respect to hyperbolicity and arithmetics, and that the core c_X also "decomposes" X with respect to hyperbolicity and arithmetics.

The core thus gives a very simple synthetic view of the global structure of any X, entirely unified from the points of view of geometry, positivity of cotangent sheaves, fundamental group, hyperbolicity and arithmetics.

(5) At a more technical level, the unavoidable consideration of orbifolds leads also to an orbifold extension of the $C_{n,m}$ additivity conjecture of S. Iitaka. Its solution here when the orbifold base is of general type is one of the main technical results of the present paper. Although this orbifold extension rests on the same techniques as the non-orbifold version, the range of applications is considerably extended, including now fibrations with general fibres having $\kappa = -\infty$.

This seems to be a general fact, that most constructions or statements of algebraic geometry can be extended at a low technical cost to the orbifold context, where they become more natural, and so cover many new situations. A second instance is given here with the orbifold version of the Kobayashi-Ochiai extension theorem, exposed in § 8.

We shall now give some short indications on the organisation of this paper, the table of contents lists the different sections. Each of them contains an introduction, describing and explaining the topics covered by each of its subsections.

Concerning the main themes treated in this paper:

In § 1, the basic definitions and properties of orbifold bases of fibrations are given, including the differential sheaf associated to a fibration. The computation of the orbifold base of the composition of two fibrations is not used before § 4, where it permits to define the orbifold base of a fibration in the orbifold context as well.

In § 2, special and general type fibrations are introduced, and their basic properties derived. General type fibrations are shown to correspond naturally to Bogomolov sheaves. We explain how to extend the usual definitions of Algebraic Geometry to the orbifold context, the proofs of the basic properties remaining exactly the same.

A first construction of the core, using chains of special subvarieties, is given in § 3. It gives a first, geometric, proof of the specialness of rationally connected manifolds. But the proof that the base orbifold of the core is of general type (or a point) is deferred to § 5, because it rests on the additivity theorem proved in § 4.

A second construction of the core, independent and shorter, is given in § 5. It also rests on the additivity result of § 4, but can be read independently of § 3.

The orbifold version of the Kobayashi-Ochiai extension theorem, shown in § 8, permits to solve a special case of conjecture III_H, which asserts that special manifolds are exactly the ones having a vanishing Kobayashi pseudometric.

The definition of a base orbifold of a fibration, and of a special manifold (or orbifold) given here was obtained after three other tentative definitions were rejected. The first definition for special had just meant: "without meromorphic map onto a (positive dimensional) manifold of general type". It was never considered, because unstable by finite étale covers. The second version precisely consisted in defining special manifolds as the ones having no finite étale cover with a meromorphic map onto a manifold of general type (these are called "weakly special" below). The third version just retained the multiple fibres in the classical sense (using gcd of the multiplicities of the various components of fibres). The definition given here is the fourth one, and replaces the above classical gcd by the infimum. It is interesting in the retrospect, to notice that the first two versions were considered earlier: the first one by D. Abramovich in [Ab97], and the second one by D. Abramovich and J.L. Colliot-Thélène (see [H-T00], and Section 9.4, for more details).

Acknowledgements. The present paper is an expanded version of [Ca01'], of October 2001. The only improvement (purely technical) to this first version is the proof of the orbifold additivity theorem, which permits to prove that the core is indeed a fibration of general type. These results were conjectured and proved only in special cases in [Ca01']. But the present approach was already described in details there, as a consequence of this additivity result.

I would like to thank F. Bogomolov, J.P. Demailly and C. Voisin for discussions which permitted me to improve § 9. P. Eyssidieux suggested me the study of the function field version of the conjectures III and IV of § 9. This version is discussed in [Ca01'], but this discussion is not included here.

After the first version of this work was put on the arXiv server, I got stimulating comments and references included here from D. Abramovich, J.L. Colliot-Thélène (see § 9.5) and B. Totaro, who independently suggested me to include Conjecture IV_A.

F. Bogomolov also noticed that general type fibrations might be linked with what is called here Bogomolov sheaves; this link is established in 2.26 (one direction was shown in the very first version in the transcendental context of Section 8).

I heartily thank J.P. Demailly, L. Manivel, C. Mourougane, M. Paun, E. Peyre, and especially L. Bonavero and S. Druel, for their carefull reading of the previous versions, which was exposed in Grenoble in July of 2002. And also for pointing many mistakes, gaps and inaccuracies. They suggested me to reorganise the exposition according to the technical order of proofs. The present version thus differs considerably from the preceding ones in this respect. My thanks also go to M. Paun, T. Peternell and J. Winkelmann, for their careful reading of the appendix (now a separate text), and their suggestions to improve its exposition.

Finally, special thanks are due to E. Viehweg, who gave me a decisive hint for the proof of the crucial Lemma 4.19.

1. Orbifold base of a fibration.

In this chapter, we introduce the notion of orbifold base of a holomorphic fibration $f : X \to Y$ (f is thus surjective, and its fibres are connected) between compact connected complex manifolds. This is simply (1.1.4) an effective $\mathbb{Q}$-divisor on Y, defined by the multiple fibres of f by means of a suitable ramification formula (which is the one for a virtual ramified cover of Y removing by base change the multiple fibres of f). This orbifold structure carries naturally a canonical bundle, a Kodaira dimension and a fundamental group (which would be the ones of this virtual ramified cover).

The Kodaira dimension $\kappa(Y, f)$ is the minimum of the Kodaira dimensions of the orbifold bases, when f' runs over all models of f, which are the fibrations bimeromorphically equivalent to f. This minimum is achieved by f itself when $\kappa(Y) \geqslant 0$ (by 1.14), but may be smaller in general when Y is at least two-dimensional. This is due to the fact that blowing up Y at the intersection of two components of the orbifold divisor may result in fibres of small multiplicity above the exceptional divisor.

We next show (1.8) that $\kappa(Y, f)$ behaves as in the "classical" case (where $f = id_X$) under composition with generically finite maps $u : X' \dashrightarrow X$ and Stein factorisation.

In the next section, we define canonically a rank one subsheaf F_f of Ω^p_X by saturating the inverse image of K_Y. This sheaf is an intrinsic birational invariant of the equivalence class of fibration. Its Kodaira dimension turns out to be $\kappa(Y, f)$, avoiding going to a suitable birational model of f. As an application, the orbifold base of f has Kodaira dimension $\kappa(Y, f)$ if f is what we call *neat* (see 1.2).

We next show (1.28) the countable upper semi-continuity of $\kappa(Y, f)$ in families, a technical result later used to construct the core.

Finally, we show how to compute, on suitable models, the orbifold structure on the base of a composition $g \circ f$ of two fibrations, from the orbifold bases of g and f. This computation plays an essential role in the proof of the shown case of the orbifold additivity conjecture in § 4. It is also the clue to the definition of the orbifold base of a fibration between orbifolds. See Section 1.6.

The basic technical results gathered in this chapter are of constant use in the next chapters.

1.1. Fibrations.

Before giving definitions, let us start with

1.1.1. A motivating example.

Example 1.1.— Let $X_0 := E \times \mathbb{P}_1(\mathbb{C})$, where E is an elliptic curve. Let $X := \tilde{X}/j$, where $\tilde{X} := E \times C$, where C is a hyperelliptic curve and $j = t \times h$ is the involution acting diagonally on $\tilde{X}$ by a translation t of order 2 on E, and by the hyperelliptic involution h on C. Then both X_0 and X have natural fibrations on $\mathbb{P}_1(\mathbb{C})$ with generic fibre E.

They cannot be distinguished by this information, although they differ fundamentally at the levels of Kodaira dimension, fundamental group, Kobayashi pseudo metric and arithmetics (for appropriate choice of E). If, however, we take into account the multiple fibres of the fibrations onto $\mathbb{P}_1(\mathbb{C})$, we see that in the second case (of X), the base is not really $\mathbb{P}_1(\mathbb{C})$, but rather the orbifold of general type C/h.

This we shall now generalise.

1.1.2. Fibrations.

A *fibration* $f : X \dashrightarrow Y$ is a surjective (*i.e.* dominant) meromorphic map with *irreducible* generic fibres between irreducible compact complex analytic spaces X and Y (see below for the precise definition of the fibres of a meromorphic map). Of course, this fibration is said to be holomorphic (or regular) if so is the map f. If X, Y are normal, a fibration thus has connected fibres.

Another fibration $f' : X' \dashrightarrow Y'$ is said to be *equivalent* to f if there exists bimeromorphic maps $u : X \dashrightarrow X'$ and $v : Y \dashrightarrow Y'$ such that $f' \circ u = v \circ f$. We denote by F or X_y the generic fibre of f. We also say that f' is a *model* or a *representative* of f (we shall not distinguish between f and its equivalence class).

A fibration $f : X \dashrightarrow Y$ canonically defines (see [Ca85]) a meromorphic map $\phi_f : Y \dashrightarrow \mathcal{C}(X)$, with $\mathcal{C}(X)$ the Chow-Scheme of X, by sending the generic $y \in Y$ to the point of $\mathcal{C}(X)$ parametrising the reduced fibre X_y of f over y. Let $\Phi_f \subset \mathcal{C}(X)$ be the image of Y by ϕ_f: it is a compact irreducible analytic subset of $\mathcal{C}(X)$ bimeromorphic to Y such that its incidence graph is bimeromorphic to X. The cycles of X parametrised by Φ_f are also called the *Chow-theoretic fibres* of f. The above correspondance induces a bijective map between equivalence classes of fibrations and compact irreducible analytic subsets of $\mathcal{C}(X)$ with incidence graph bimeromorphic to X.

1.1.3. Neat fibrations. Prepared fibrations.

For a fibration $f \ : X \to Y$ between compact complex spaces, the divisors of X mapped by f to a codimension at least two subset of Y will be the source of many troubles. We therefore introduce the following definition.

DEFINITION 1.2. — *Assume that $f : X \to Y$ is a holomorphic fibration. An irreducible Weil divisor D on X is said to be f-exceptional if $f(D)$ has codimension 2 or more in Y. We say that $f : X \to Y$ is neat if f is holomorphic, X, Y are smooth, and if there moreover exists a bimeromorphic holomorphic map $u : X \to X'$ with X' smooth such that each f-exceptional divisor of X is also u-exceptional.*

By allowing modifications, such fibrations always exist by the following lemma.

LEMMA 1.3. — *Let $f_0 : X_0 \dashrightarrow Y_0$ be a fibration and X' smooth bimeromorphic to X_0. Then, there exists a neat model $f \ : X \to Y$ of f_0 and a bimeromorphic map $u \ : X \to X'$ such that each f-exceptional divisor of X is also u-exceptional.*

Proof. — By Raynaud and Hironaka flattening theorems ([R74]), any fibration $f_0 : X_0 \dashrightarrow Y_0$ has a neat model, in which X' may be choosen arbitrarily (bimeromorphic to the domain X_0 of the initial fibration f_0). Indeed: first blow-up X' in such a way that $f_0 \circ b : X' \dashrightarrow Y_0$ is holomorphic, where $b : X' \dashrightarrow X_0$ is bimeromorphic; then flatten $f_0 \circ b$ by blowing-up Y_0 to get a smooth Y. Finally, take a smooth model X of $X' \times_{Y_0} Y$. □

Assume the fibration f is holomorphic. We say that the holomorphic fibration $f' : X' \to Y'$ with X', Y' smooth *dominates* f if there exists a commutative diagram in which u, v are bimeromorphic; obviously, f' is then equivalent to f:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

Observe that any f' dominating a neat fibration is itself neat.

More generally, we say that the fibration $f' : X' \dashrightarrow Y'$ dominates the fibration $f : X \dashrightarrow Y$ if there exists a diagram as above with u, v bimeromorphic. This notion is well-defined on equivalence classes.

Let $f : X \to Y$ be a holomorphic fibration with X, Y smooth. We say (as in [Vi82]) that f is *prepared* if the locus $Y^* \subset Y$ of points y with a smooth f-fibre X_y has a complement contained in a normal crossing divisor $D \subset Y$ such that, moreover, $f^{-1}(D)$ is also a divisor of normal crossings in X.

Any fibration is dominated by a prepared model, by an immediate application of Hironaka's results. The models of fibrations considered can thus always be assumed to be prepared.

1.1.4. Multiplicity divisor of a fibration.

Assume now X, Y to be smooth and $f : X \to Y$ to be a holomorphic fibration. For any irreducible divisor D of Y, write

$$f^*(D) := \sum_{j \in J(f,D)} m_*(f, D_j) D_j + R$$

where $J(f, D)$ is the set of all irreducible components of $f^*(D)$ which are mapped surjectively onto D by f, while R is f-exceptional. Then, define

$$m(f, D) := \inf\{m_*(f, D_j), j \in J(f, D)\}.$$

The integer $m(f, D)$ is called the *multiplicity of f along D*.

Let $|\Delta| \subset Y$ be the union of all codimension one irreducible components of the locus of y's $\in Y$ such that the scheme-theoretic fibre of f over y is not smooth. Remark that $m(f, D) = 1$ if D is not a component of $|\Delta|$. Finally, we introduce a $\mathbb{Q}$-divisor $\Delta(f)$, called the *multiplicity divisor of f* by the following:

$$\Delta(f) = \sum_{D \subset Y} \left(1 - \frac{1}{m(f, D)}\right) D,$$

where D ranges over the set of all irreducible divisors of Y (since $m(f, D) = 1$ if D is not a component of $|\Delta|$, the sum is finite and one can assume that D ranges over the set of all irreducible components of $|\Delta|$). The motivation for its introduction comes from the examples above and Example 1.4 below.

1.2. Orbifolds.

1.2.1. Notion of orbifold.

An *orbifold* (Y/Δ) is a pair consisting of a compact irreducible complex space Y together with a Weil $\mathbb{Q}$-divisor of the form: $\Delta =: \sum_{i\in I}(1 - 1/m_i)\Delta_i$ for distinct prime divisors Δ_i of Y, and positive integers m_i. We also say that Δ is an orbifold structure on Y. We write $|\Delta|$ for the support of Δ, in which the coefficient of each Δ_i is one. Or equivalently: each $m_i = +\infty$.

If Y is smooth, and if the support of Δ is a simple normal crossings divisor, such pairs (Y/Δ) were introduced and used by V. Shokurov in [Sh92] under the name of "standard pairs". We shall say that the orbifold (Y/Δ) is a *klt-orbifold* if the pair (Y, Δ) is klt (this is an abbreviation for "Kawamata-Log-terminal"). This seems to be the right category to consider, morphisms being the obvious ones.

The next example shows why such pairs are rather called orbifolds here. We also note that this term is commonly used in similar situations either in differential geometry, or when Y is a curve (in [F-M94] or [Lu01], for example).

Example 1.4. — Let $f : X\to Y$ be a finite (ramified) Galois cover between manifolds. Let its ramification divisor be: $\Delta =: \sum_{i\in I}(1-1/m_i)\Delta_i$, the order of ramification above the generic point of Δ_i being m_i.

This preceding example occurs in the construction of fibrations with multiple fibres, as in 1.1 for example.

1.2.2. Orbifold base of a fibration.

DEFINITION 1.5. — *Let f be a holomorphic fibration as in 1.1, and $\Delta =: \Delta(f)$ be the multiplicity divisor of f. We call (Y/Δ) the orbifold base of f.*

1.2.3. Orbifold invariants and fibrations.

DEFINITION 1.6. — *Let (Y/Δ) be an orbifold. We define its canonical bundle as the $\mathbb{Q}$-divisor $K_{Y/\Delta} =: K_Y + \Delta$ on Y and its Kodaira dimension as $\kappa(Y/\Delta) =: \kappa(Y, K_Y + \Delta)$.*

The Canonical Algebra of the orbifold (Y/Δ) *is the graded algebra:*

$K(Y/\Delta) := \oplus_{m\geqslant 0} H^0(Y, m.k.(K_Y + \Delta))$, *if* $k := l.c.m.(m_i's)$, *in the above notations.*

One can also associate to an orbifold its fundamental group, considered below.

The term "orbifold" arises in this context from the following reason: first from the special case when f is the quotient of a fibration $f' : X' \to Y'$ by a finite group G acting on some Galois cover X' of X, the action of G preserving the fibration f', and such that f' has no multiple fibre in codimension 1. In this case, $(Y/\Delta(f))$ is precisely the orbifold (Y'/G). We then say that the orbifold $Y/\Delta(f)$ has the *unfolding* (Y', G). When Y is a curve, such an unfolding exists, except when $Y = \mathbb{P}^1$ and Δ consists of one or two punctures with different multiplicities. In the general case, such unfoldings exist locally on Y (but not globally in general. M. Kapovitch explained me a very beautiful construction in dimension two for orbifolds of general type).

1.3. The Kodaira dimension of a fibration.

1.3.1. Kodaira dimension.

Define now in general, for $f : X \dashrightarrow Y$ a fibration between irreducible compact complex spaces X and Y:

$$\kappa(Y, f) := \inf\{\kappa(\bar{Y}/\Delta(\bar{f}))\},$$

where $\bar{f} : \bar{X} \to \bar{Y}$ ranges over all holomorphic fibrations $\bar{f}$ between manifolds $\bar{X}$ and $\bar{Y}$ which are equivalent to f. We call $\kappa(Y, f)$ *the Kodaira dimension of the fibration* f. Notice indeed that this notion depends only on the equivalence class of f.

DEFINITION 1.7. — *We shall say that* $f : X \to Y$ *is admissible if it is holomorphic, with* X, Y *smooth, and if* $\kappa(Y, f) = \kappa(Y, K_Y + \Delta(f))$.

In 1.14, we shall see that any $f : X \to Y$ is admissible if $\kappa(Y) \geqslant 0$. Notice also that it follows from 1.8 right below that if f' dominates f, and if f is admissible, so is f'.

1.3.2. Generically finite maps. Statement of main result.

We consider a commutative diagram of holomorphic surjective maps between compact irreducible normal complex spaces:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

We assume moreover that f, f' are holomorphic fibrations and u, v generically finite (and such that $f \circ u = v \circ f'$).

THEOREM 1.8.— *Let $f : X \to Y$ and $f' : X' \to Y'$ be two holomorphic fibrations and let holomorphic maps $u : X' \to X$ and $v : Y' \to Y$ such that $f \circ u = v \circ f'$ be given.*

1. *Assume also that u, v are bimeromorphic, then*
   a. $\kappa(Y/\Delta(f \circ u)) = \kappa(Y/\Delta(f))$ *and* $\kappa(Y'/\Delta(f')) \leqslant \kappa(Y/\Delta(f))$,
   b. *If, moreover,* $\kappa(Y) \geqslant 0$, *then equality holds, and* $\kappa(Y/\Delta(f)) = \kappa(Y, f)$.
2. *Assume that u and v are generically finite and surjective. Then:*
   a. $\kappa(Y', f') \geqslant \kappa(Y, f)$,
   b. $\kappa(Y', f') = \kappa(Y, f)$ *if u is étale, and X, X' are smooth.*

We shall prove Theorem 1.8 as a consequence of several lemmas, some of independent interest in the next subsection. A different, shorter, proof of 1.8 will be given in Section 1.4 below.

1.3.3. Generically finite maps. Proof of main result.

LEMMA 1.9.— *Assume u, v bimeromorphic. Then:*

a. $\Delta(f \circ u) = \Delta(f)$ *and so* $\kappa(Y/\Delta(f \circ u)) = \kappa(Y/\Delta(f))$,

b. $\Delta(f') = v^*(\Delta(f)) + E$, *for some $\mathbb{Q}$-divisor E supported on the exceptional locus of v. Therefore* $v_*(\Delta(f')) = \Delta(f)$ *and* $\kappa(Y'/\Delta(f')) \leqslant \kappa(Y/\Delta(f))$.

Proof. —

(a) For any $D \subset Y$, using notations of 1.1.4, we have:

$$(f\circ u)^*(D)=u^*(f^*(D))=u^*\Big(\sum_{j\in J(D,f)} m_jD_j+R\Big)$$
$$=\sum_{j\in J(D,f)} u^*(m_j(\overline{D_j}+R_j))+u^*(R),$$

where $\overline{D_j}$ is the strict transform by u of D_j, and R_j is the u-exceptional part of $u^*(D_j)$. Obviously, $u^*(R)$ is $(f\circ u)$-exceptional. For each component R_{jk} of R_j, its multiplicity in $(f\circ u)^*(D)$ is divisible by m_j, by the above equalities (and the factoriality of the smooth X). Thus $m(D,f\circ u)=m(D,f)$, and $\Delta(f\circ u)=\Delta(f)$, as claimed. (Observe that R_{jk} does not need to be $(f\circ u)$-exceptional, in general).

(b) By (a), we can and shall assume that $X=X'$, to ease notations. Let $D'\subset Y'$ be an irreducible divisor, and let $D:=v(D)$. We just need to show that $m(D,f)=m(D',f')$ if D' is not u-exceptional, that is: if D is a divisor of Y. Then: $v^*(D)=D'+E$, with E an effective v-exceptional divisor of Y'. Thus $\sum_{j\in J(D,f)} m_jD_j+R=f^*(D)=(f')^*(D')+(f')^*(E)$. Now observe that $(f')^*(E)$ is f-exceptional, and that each D_j is mapped surjectively onto D by f, and so must be surjectively mapped to D' by f'. Thus, $\sum_{j\in J(D,f)} m_jD_j=\sum_{j\in J(D',f')} m_jD'_j$, and we get the claim.

We thus get the first assertion of (b). The others easily follow from it: write $E=E^+-E^-$, with E^+ and E^- effective and v-exceptional. We thus get:

$$\kappa(Y'/\Delta(f'))\leqslant\kappa(Y',K_{Y'}+v^*(\Delta)+E^+)=\kappa(Y'/v^*(\Delta))=\kappa(Y/\Delta),$$

and so the conclusion.

Remark 1.10.— Modifications of X thus don't alter $\kappa(Y/\Delta(f))$, but modifications of the base may let it decrease. See the example below, which shows that strict inequality can actually occur in Theorem 1.8.

Example 1.11.— Probably the simplest example is when $Y=\mathbb{P}^2$ and Δ_{red} is a union of $2k\geqslant 6$ distinct lines meeting at one point $a\in\mathbb{P}^2$. Corresponding fibrations are easily constructed, as follows, let $p:Y^+\to Y$ be the double cover branched exactly along $\Delta_{\rm red}$, and h the involution of Y^+ exchanging the sheets of p. Let E be an elliptic curve and t a translation of order 2 of E. Let $X^+:=E\times Y^+$, $j:=t\times h$ the diagonal involution of

X^+, and $X_0 := X^+/G$, where G is the group of order 2 generated by j. Let $f_0 : X_0 \to Y$ and $f^+ : X^+ \to Y^+$ be the induced holomorphic fibrations. Let $d : X \to X_0$ be a desingularisation (induced by a desingularisation of Y^+, for example), and $f := d \circ f_0 : X \to Y$.

Then $\Delta(f) = (1/2)\Delta_{\mathrm{red}}$, and so $\kappa(Y/\Delta(f)) = -\infty$ (resp. 0; resp. 2) if $k \leqslant 2$ (resp. $k = 3$; resp. $k \geqslant 4$). On the other hand, let $v : Y' \to Y$ be the blow-up of $a \in \mathbb{P}^2$, let E be its exceptional divisor, and $f' : X' \to Y'$ be the lifting of the meromorphic map $v^{-1} \circ f : X \to Y'$ to a suitable modification X' of X. An easy local computation (for example) shows that $\Delta(f') = \overline{\Delta}$, the strict transform of $\Delta(f)$ by v. (This is just because the normalisation of $(Y^+ \times_Y Y')$ does not ramify over the generic point of E). We thus conclude that $\kappa(Y'/\Delta(f')) = -\infty$, whatever $k \geqslant 1$. (Because $(K_{Y'} + \Delta(f')) = (k-3)(v^*(H) - E) - 2E$, if H is the hyperplane line bundle on Y, and $v^*(H) - E$ defines the unique ruling of Y'). In particular, $\kappa(Y'/\Delta(f')) < \kappa(Y/\Delta(f))$ if $k \geqslant 3$. Finally, we observe that the same construction does not lead to this last (strict) inequality when $\Delta(f)$ is a normal crossings divisor (but such other examples should exist).

Remark 1.12. — If $f : X \to Y$ is admissible, the canonical algebra $K(Y/\Delta(f))$ is a bimeromorphic invariant of the fibration f. We call it the *Canonical Algebra of* f, and denote it by $K(f)$.

COROLLARY 1.13. — *Let $f : X \dashrightarrow Y$ a fibration between irreducible compact complex spaces X and Y.*

1. *There exists $\bar{f} : \bar{X} \to \bar{Y}$ an admissible holomorphic model of f between manifolds $\bar{X}$ and $\bar{Y}$, and bimeromorphic maps $u : \bar{X} \to X$ and $v : \bar{Y} \to Y$ such that $f \circ u = v \circ \bar{f}$.*
2. *For any X' bimeromorphic to X, there exists $f'' : X'' \to Y''$, a holomorphic admissible, neat and prepared model of f between manifolds X'' and Y''.*

Proof. — It is again an easy application of Hironaka smoothing and Hironaka-Raynaud Flattening theorems.

(1) First take an admissible model $\bar{f} : \bar{X} \to \bar{Y}$ of f. Modify $\bar{Y}$ to Y', smooth and dominating Y. Making base change by Y' over Y, and smoothing $X \times_{\bar{Y}} Y'$, we get the fibration f', with the desired properties.

(2) Start with the previous fibration f'. We can first modify X' so that the new X' dominates any given bimeromorphic model of X. Then flatten

the new f' by modifying Y', to get Y", smooth and the non-smooth locus of f' contained in a divisor of normal crossings. As before, make base change of Y" over Y', and smooth $X' \times_{Y'} Y$" to obtain a fibration $f" : X" \to Y"$ enjoying the claimed properties. □

PROPOSITION 1.14. — *Assume that $\kappa(Y) \geqslant 0$ in Lemma 1.9 above. Then,*

$$\kappa(Y/\Delta(f)) = \kappa(Y'/\Delta(f')) = \kappa(Y, f).$$

(In other words, any holomorphic model of f is then admissible).

Proof. — Because Y is smooth, using the notations of the proof of Lemma 1.9, we have $v^*(\Delta(f)) - bE \leqslant \overline{\Delta(f)} \leqslant \Delta(f')$, for some nonnegative rational number b, with $\overline{\Delta(f)}$ denoting the strict transform of $\Delta(f)$ by v. Also, here E denotes the reduced exceptional divisor of the map v, and the inequality: $D \leqslant D'$ between two $\mathbb{Q}$-divisors D, D' on Y' means that their difference $(D' - D)$ is an effective $\mathbb{Q}$-divisor.

Moreover, we have: $K_{Y'} \geqslant v^*(K_Y) + aE$, for some strictly positive rational number a (here we use the smoothness of Y, but Y having just terminal singularities would be sufficient). We can thus write, as $\mathbb{Q}$-divisors:

$$K_{Y'} + \Delta(f') \geqslant v^*(K_Y) + aE + v^*(\Delta(f)) - bE = v^*(K_Y + \Delta(f)) + (a-b)E.$$

We are thus finished if $(a - b) \geqslant 0$. So we now assume the contrary, $(a/b) < 1$.

From: $bE \geqslant v^*(\Delta(f)) - \overline{\Delta(f)}$, we get first, $aE = (a/b)bE \geqslant (a/b)(v^*(\Delta(f)) - \overline{\Delta(f)})$, and then:

$$\overline{\Delta(f)} + aE \geqslant (a/b)v^*(\Delta(f)) + (1 - (a/b))\overline{\Delta(f)}).$$

From which we deduce, $(K_{Y'} + \Delta(f')) \geqslant v^*(K_Y) + \overline{\Delta(f)} + aE \geqslant D$, with:

$$D := (a/b)v^*(K_Y+\Delta(f))+(1-a/b)(v^*(K_Y))+\overline{\Delta(f)} \geqslant (a/b)v^*(K_Y+\Delta(f)),$$

where the last inequality follows from our assumption that $\kappa(Y) \geqslant 0$. □

LEMMA 1.15. — *Let the situation be as in Theorem 1.8 above. Assume u and v are generically finite and surjective. Then, $\kappa(Y', f') \geqslant \kappa(Y, f)$.*

Proof. — We need only to show that for any admissible f' as above, we can find a holomorphic representative $f_0 : X_0 \to Y_0$ of f with: $\kappa(Y_0/\Delta(f_0)) \leqslant \kappa(Y'/\Delta(f'))$, because then:

$$\kappa(Y,f) \leqslant \kappa(Y_0/\Delta(f_0)) \leqslant \kappa(Y'/\Delta(f')) = \kappa(Y',f').$$

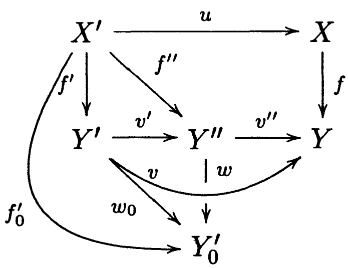
Modifying X', X, Y and Y', we can and shall thus assume f' to be admissible.

Let $v = v" \circ v'$ be the Stein factorisation of v, with $v' : Y' \to Y"$ connected (hence bimeromorphic), $v" : Y" \to Y$ finite, and $Y"$ normal.

We also denote here with $\Delta := \Delta(f)$, $\Delta' := \Delta(f')$, $\Delta" := \Delta(f")$, $\Delta_0 := \Delta(f_0)$,... the relevant multiplicity divisors for the corresponding fibrations f, f', $f" := v' \circ f'$, f_0,....

Let now $w : Y" \to Y'_0$ be a modification, and define $f'_0 := w \circ f" := w \circ v' \circ f'$, $w_0 := w \circ v' : Y' \to Y'_0$.

The relevant diagram is the following:



LEMMA 1.16. — *We can and shall further assume, modifying Y, X and X', Y' if needed, that such a w exists, with f'_0 admissible equivalent to f'.*

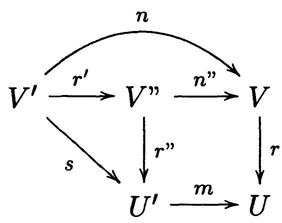
Proof. — Indeed, this existence follows from the next lemma, applied to our initial Y'_0, Y_0 in place of V, U, where f'_0 dominates f_0 (which means, there exists g, h such that $f_0 \circ h = g \circ f'_0$, and $f_0 : X_0 \to Y_0$ is equivalent to f, with $g : Y'_0 \to Y_0$ and $h : X'_0 \to X_0$ generically finite).

LEMMA 1.17. — *Let $r : V \to U$ be generically finite surjective, between irreducible normal compact complex spaces. There exist modifications $n : V' \to V$ and $m : U' \to U$ with smooth V' and U' such that:*

(1) $s := m^{-1} \circ r \circ n : V' \to U'$ is holomorphic,

(2) if $s = r" \circ r'$ is the Stein factorisation of s, then there exists a (holomorphic) factorisation $n" : V" \to V$ of $m \circ r" : V" \to U$ through r (ie: $r \circ n" = m \circ r"$). Here $r' : V' \to V"$ is connected and $r" : V" \to U'$ is finite.

The relevant diagram is the following:



Proof. — Just flatten the map r by suitably modifying U. □

To complete the proof of 1.15, we shall next need the following two properties **(a)**, **(b)**:

(a) Let us prove the following equality between Weil Divisors, with $R"$ effective:

$$(\text{*a})\ K_{Y"} + \Delta" = (v")^*(K_Y + \Delta) + R".$$

To see this, observe we may first assume that f' is deduced from f by base change by v and smoothing of a main component (this is simply because $\Delta(f \circ u) \geqslant \Delta(f)$ if $Y' = Y$, and u is generically finite. The argument is the same as in 1.9). Next, a simple computation in local analytic coordinates over the generic point $y"$ of $Y"$ shows that: if r is the ramification order of $v"$ at y' and m is the multiplicity of the fibre of f over $y := v"(y")$, then the multiplicity of the $f"$-fibre over $y"$ is $m' := (m/d)$, if $d := gcd(r, m)$ and $r' := r/d$ (y'' being generic means that $y"$ lies outside the codimension two analytic subset of $Y"$ consisting of points $y"$ which are either singular on $Y"$, or mapped by $v"$ to a point $y := v"(y")$ either singular on $\Delta(f) \cup S$, or with f-fibre of nongeneric multiplicity, where $S := v"(S")$, with $S"$ the Weil divisor of points of $Y"$ at which $v"$ ramifies). From this we see that near $y"$ on $Y"$, we have: $K_{Y"} + \Delta" = (v")^*(K_Y + (1 - 1/r)R) + (1 - 1/m')(1/r)D$, with $R := v"(R^+)$ ($S" = R^+$ the reduced ramification divisor of $v"$ near $y"$), and D the (unique) reduced component of Δ near y. If R^+ or D is empty, our equality (*a) is obvious. Otherwise, $R = D$, by our choice of $y"$, and (*a) follows from the inequality: $1 - 1/m \leqslant 1 - 1/mr' = 1 - 1/m'r = (1 - 1/r) + (1 - 1/m')1/r$.

(b) Moreover, we also have by an immediate check: $K_{Y"} + \Delta" = (v')_*(K_{Y'} + \Delta')$.

From the initial constructions made, we thus know that f'_0 is admissible, and dominates f_0, admissible (in the sense (**) just before the lemma).

We then get: $K_{Y"} + \Delta" = (v')_*(K_{Y'} + \Delta') = (v")^*(K_Y + \Delta) + R"$.

Hence: $K_{Y'_0} + \Delta'_0 = (w)_*(K_{Y"} + \Delta") = (w)_*((v")^*(K_Y + \Delta) + R")) = (w_0)_*(v^*(K_Y + \Delta) + R'))$.

And so: $\kappa(Y', f') = \kappa(Y'_0, K_{Y'_0} + \Delta'_0) \geqslant \kappa(Y'_0, (w_0)_*(v^*(K_Y + \Delta))) \geqslant \kappa(Y', v^*(K_Y + \Delta)) = \kappa(Y, K_Y + \Delta) \geqslant \kappa(Y, f)$, as claimed in 1.15, the proof of which is thus complete. □

The last property asserted in Theorem 1.8 is:

LEMMA 1.18. — *Let the situation be as in Theorem 1.8 above. Assume now that $u : X' \to X$ is étale (ie: unramified). Then, $\kappa(Y', f') = \kappa(Y, f)$.*

Proof. — By Lemma 1.15, we need only to show that $\kappa(Y', f') \leqslant \kappa(Y, f)$ for any f, f' as above.

We can assume f to be admissible, and still u to be étale. Indeed, just replace X' by $X'_1 := X' \times_X X_1$, if $m : X_1 \to X$ is a modification having an admissible fibration $f_1 : X_1 \to Y_1$ equivalent to f. Then X'_1 is smooth, and $u_1 : X'_1 \to X_1$ is étale. The Stein factorisation $f'_1 : X'_1 \to Y'_1$ of $f_1 \circ u_1 : X'_1 \to Y_1$ does not need to have Y_1 smooth. But a modification $m' : X'_2 \to X'_1$ of X'_1 allows to assume this to be true (accordingly modifying Y_1). Now the birational invariance of the fundamental group for complex manifolds shows that X'_2 may be assumed (after further modification) to be of the form $X' \times_X X_2$ for some modification X_2 of X. Let us check this.

We can indeed assume that $u : X'_1 := X' \to X$ is Galois, of group G, and that f' is the (connected part of the) Stein factorisation of $(f \circ u)$. The map $v : Y' \to Y$ is thus G-equivariant as well. But then, the modification $m' : X'_2 \to X'_1$ can be assumed to be also G-equivariant, which shows that X'_2 is obtained from u by base-change over a modification of X. This establishes our preliminary claim (that f can be chosen admissible).

Next, by Stein factorisation of $v = v" \circ v'$, we see as in the above proof of Lemma 1.8, that it is sufficient to show that:

$$(*)\ K_{Y"} + \Delta" = (v")^*(K_Y + \Delta)$$

(that is: $R"$ is empty).

Because then $K_{Y"} + \Delta" = (v')_*(K_{Y'} + \Delta')$, and so $K_{Y'} + \Delta' = (v)^*(K_Y + \Delta) + E'$, for some v-exceptional (not necessarily effective) $\mathbb{Q}$-divisor E' of Y', which implies the desired reverse inequality.

By the finiteness of the map $v"$, the equality (*) has only to be shown near any point $y \in Y$ lying outside a codimension two subset S of Y. We can thus assume that y is a generic point on some component of the support of $\Delta(f)$. Let us now cut Y by divisors in general position through y: we are reduced to the case when Y is a curve (the argument being local on Y in the analytic topology, we don't need any algebraicity assumption of Y). But then an easy local computation (see [Ca98], for example) shows that for any $y" \in Y"$, the order of ramification r of $v"$ at $y"$ divides the multiplicity m of the fibre X_y of f over $y := v"(y")$. The multiplicity $m"$ of the fibre $X'_{y"}$ of f' over $y"$ is thus $m' := m/r$, since u is étale. We now compute $K_{Y"} + \Delta"$ near $y"$: $K_{Y"} + \Delta" = (v")^*(K_Y + (1-1/r)[y]) + (1-1/m')[y"] = (v")^*(K_Y + ((1-1/r) + (1/r)(1-1/m))[y]) = (v")^*(K_Y + (1-1/m)[y])$, as claimed. (Here [y] is the reduced divisor on the curve Y supported by the point y). □

1.4. The sheaf of differential forms determined by a fibration.

In this section, we define canonically a rank one subsheaf F_f of Ω^p_X by saturating the inverse image of K_Y. This sheaf is an intrinsic invariant of the equivalence class of the fibration. Its Kodaira dimension is $\kappa(Y, f)$ if Y is smooth. As an application, the orbifold base of f has Kodaira dimension $\kappa(Y, f)$ if f is neat.

Definition 1.19. — *Let X be smooth, compact and connected and let $f : X \dashrightarrow Y$ be a meromorphic fibration, with Y reduced but not necessarily smooth. The rank one coherent subsheaf F_f of $\Omega^p_X, p := \dim(Y)$ is defined as the saturation in Ω^p_X of $f^*(K_{Y_0})$, if Y_0 is the smooth locus of Y.*

Let us remark that the subsheaf F_f so defined is a bimeromorphic invariant, it is preserved not only under modifications of X, but also under modifications of Y. In other words, F_f depends only on the equivalence class of f. We define by $\kappa(f)$ its Kodaira dimension.

Let us make more precise what is understood by $\kappa(f)$: for $m > 0$, define $H^0(X, F^m_f)$ to be the complex vector space of sections of the subsheaf of $(\Omega^p_X)^{\otimes m}$ which coincides with $F^{\otimes m}_f$ over the Zariski open subset of X (with codimension two or more complement) over which F_f is locally free. Then define κ in the usual way.

To see the usual property of κ (being an integer, or $-\infty$), notice that the data are bimeromorphically invariant on X. So that F_f can be considered as the injective image of some locally free rank one sheaf L on X (after some suitable modification). The claimed property is then obvious, since it holds for L. We can then always implicitely assume the existence of L in the sequel, and also the holomorphic character of any meromorphic map defined on X (such as, for example, the ones defined by linear systems $L^{\otimes m}$). We shall now describe F_f in more detail.

DEFINITION 1.20.— *In the preceding situation, define* $F(f) := f^*(K_X) \otimes \mathcal{O}_X(\lceil f^*(\Delta(f))\rceil)$, *where the symbol used is the usual round-up (defined as* $\lceil * \rceil := -[-(*)]$, *applied to the coefficients of the irreducible components of the effective* $\mathbb{Q}$*-divisor under consideration. Here [*] is the integral part, of course).*

DEFINITION 1.21.— *Let* $f : X \to Y$ *be a holomorphic fibration, and* S *an effective divisor on* X. *We say that* S *is partially supported on the fibres of* f *if* $f(S) \neq Y$ *and if for any irreducible component* T *of* $f(S)$ *of codimension one in* Y, *then* $f^{-1}(T)$ *contains an irreducible component mapped on* T *by* f, *but not contained in the support of* S.

Observe that if S is partially supported on the fibres of f, so are its positive multiples. The introduction of this notion is due to the following.

LEMMA 1.22.— *Let* $f : X \to Y$ *be a holomorphic fibration between manifolds, and* S *a divisor of* X *partially supported on the fibres of* f. *Let* L *be a line bundle on* Y. *The natural injection of sheaves* $L \subset f_*(f^*(L)+S)$ *is an isomorphism.*

Proof.— The assertion is of local nature on Y. So we can assume that L is trivial. We then just need to show that $f_*(\mathcal{O}_X(S)) \cong \mathcal{O}_Y$. We assume that $S \subset f^*(\mathcal{O}_Y(T))$, for some effective divisor T on Y. Local sections of the sheaf on the right are of the form $f^*(u/t)$, where u is holomorphic on Y, while t is a local equation of T. The sections of the sheaf on the left are meromorphic functions on X of the same form, but with poles contained in S. Because S is partially supported on the fibres of f, we get the claim (t divides u). □

We apply this to the following situation.

PROPOSITION 1.23.— *Let* $f : X \to Y$ *be a holomorphic fibration between manifolds. There exists a Zariski closed subset* $A \subset Y$ *of codimension at least* 2 *such that* $F(f) + S$ *and* F_f *are naturally isomorphic over*

$(X - B) := f^{-1}(Y - A)$, where $S \subset X$ is an effective divisor partially supported on the fibres of f.

Proof. — Let A be the union of the singular set of the support of $\Delta(f)$ and of images of all f-exceptional divisors on X. Let us remark that the above natural isomorphism is immediate outside of Δ, because if $T \subset Y$ is a one-codimensional component of the locus of non-smooth fibres of f, then $f^{-1}(T)$ contains a reduced component at the generic point of which f is smooth. So we consider the situation near a smooth point of some Δ_i not lying in A. In suitable local coordinates at the generic point of D_{ij}, in the notations of the lines preceding 1.1.4, we have: $(x) = (x_1, \ldots, x_n)$, and $(y) = (y_1, \ldots, y_p)$, with: $f(x) = (y_1 := x_1^{m_{ij}}, y_2 := x_2, \ldots, y_p := x_p)$.

And so: $\lceil f^*(K_Y + \Delta(f)) \rceil$ is generated by $x_1^{\lceil (m_{ij}/m_i)-1 \rceil} d(x')$, with $d(x') := dx_1 \wedge \cdots \wedge dx_p$. A simple check shows that this is exactly the claim. (One may even observe that the divisor S has the same description as the one given to define $F(f)$, by adding to Δ the one-codimensional components of the locus of non-smooth fibres of f). □

COROLLARY 1.24. — *Let $f : X \to Y$ be a fibration as in 1.23 above. Let $m > 0$ be a sufficiently divisible integer. Then:*

1. *The natural isomorphism between $F(f) + S$ and F_f over $(X - B)$ extends to a natural injection of $H^0(X, F_f^{\otimes m})$ into $H^0(X, m(F(f) + S)) \cong H^0(Y, m(K_Y + \Delta(f)))$.*
2. *If f is neat, this injection is bijective.*

Proof. — (1) We start by observing that, by 1.22, the bijection:

$$H^0(X, m(F(f) + S)) \cong H^0(Y, m(K_Y + \Delta(f)))$$

actually holds. The natural map at the level of sections of m-th powers induces an isomorphism over a codimension two subset of Y. Because $m(K_Y + \Delta(f))$ is locally free on Y, the said isomorphism thus extends as an injection, by Hartog's theorem.

(2) This is because B is mapped to a codimension two or more Zariski closed subset of X' if $u : X \to X'$ is a modification with X' smooth and sending the f-exceptional divisors of X in codimension 2 or more in X'. Then the sections of $m(F(f) + S)$ over $(X - B)$ extend to sections of $F_f^{\otimes m}$, as claimed. □

We now have the following easy but important consequence.

PROPOSITION 1.25. — *Let $f : X \dashrightarrow Y$ be a fibration, with X smooth, compact and connected. Then:*

(1) $\kappa(f) = \kappa(Y, f)$,

(2) $\kappa(f) = \kappa(Y/\Delta(f))$ *if Y is smooth and f is neat.*

Proof. — (2) is simply a restatement of 1.24. We deduce (1) by choosing a neat admissible model f' of f. Then, $\kappa(f) = \kappa(Y'/\Delta(f')) = \kappa(Y, f)$. □

Remark 1.26. — This allows us to give a short proof of the basic properties shown in Theorem 1.8. Indeed, using the notations there, we have a natural inclusion: $u^*(F_f) \subset F_{f'}$, which is an equality if u is étale. The conclusions follow from the standard properties of the Kodaira dimension. One can also use the sheaves F_f to simplify some of the geometric proofs given in section 2.2.

Remark 1.27. — As an immediate corollary of 1.25, we get that the canonical algebra $K(f)$ of f is nothing, but $\oplus_{m\geqslant 0}(H^0(X, (F_f)^{\otimes m})) \subset \oplus_{m\geqslant 0}(H^0(X, S^m\Omega^p_X))$, with S^mV the space of degree m symmetric tensors on the vector space V.

1.5. Semi-continuity of the Kodaira dimension.

PROPOSITION 1.28. — *Let $f : X \to Y$ and $g : Y \to Z$ be holomorphic fibrations, with X a connected compact complex manifold, and $h := g \circ f$. Let Z^* be the Zariski open subset of Z over which g and h are smooth. Let, for $z \in Z$, denote by $f_z : X_z \to Y_z$ the restriction of f to the z-fibre X_z of h, mapped by f to the z-fibre Y_z of g. Then,*

1. *There exist modifications $\mu : X' \to X$ and $\nu : Y' \to Y$ such that f'_z is admissible, for z general in Z, with $f' := \nu^{-1} \circ f \circ \mu$ holomorphic.*
2. *Let $d := \inf(\{\kappa(Y_z, f_z), z \in Z^*\})$. (So that $d \in \{-\infty, 0, 1, \dots, \dim(Y) - \dim(Z)\}$). Then, $d = \kappa(Y_z, f_z)$, for z general in Z.*
3. *Let A be the set of points z of Z^* such that $\kappa(Y_z, f_z) = \dim(Y) - \dim(Z)$. Then, either A contains the general point of Z, or A is contained in a countable union of closed proper analytic subsets of Z.*

Proof. — We don't mention modifications of X, since they don't change the Kodaira dimensions of fibrations with X as domain. For any modification $\nu : Y' \to Y$ and $d \in \{-\infty, 0, 1, \dots, \dim(Y) - \dim(Z)\}$, let $S_d^*(\nu) := \{z \in Z^*$ such that $\kappa(Y_z, \Delta(f_z')) \geqslant d\}$. From [Gr60], and as in [L-S75], we deduce that $S_d^*(\nu) = S_d(\nu) \cap Z^*$, where $S_d(\nu)$ is a countable union of Zariski closed subsets of Z.

Obviously, $S_{d+1}(\nu) \subset S_d(\nu)$ for any ν and d (this on Z^* at least, which is sufficient for our purposes).

If $\nu' : Y" \to Y$ dominates ν in the sense that there exists a $\nu" : Y" \to Y'$ with $\nu' = \nu" \circ \nu$, then obviously (by Theorem 1.8): $S_d(\nu') \subset S_d(\nu)$.

We can thus define, for any d, $S_d \subset Z$ as the intersection of all $S_d(\nu)$'s: it is again a countable intersection of Zariski closed subsets of Z.

Define now $d := \max\{d'$ such that $S_{d'} = Z\}$. There thus exists some ν such that $S_{d+1}(\nu) \neq Z$. Both claims then follow immediately from the constructions just made □

The third assertion is an immediate consequence of the second.

1.6. Composition of fibrations.

This section will not be used before Section 4.

Assume now X, Y, Z to be smooth and $f : X \to Y$ and $g : Y \to Z$ to be holomorphic fibrations. Our aim is to define, if H is an orbifold structure on Y (*i.e.* an effective $\mathbb{Q}$-divisor on Y with components having multiplicities of the form $(1 - 1/m)$, for m integer), an orbifold structure $\Delta(g, H)$ on Z in such a way that we have the equality $\Delta(g, H) = \Delta(g \circ f)$ when $H = \Delta(f)$, if $f : X \to Y$ is sufficiently "high", in a sense defined in 1.31 below.

1.6.1. Orbifold base of a fibration.

We shall now define the notion of *orbifold base of a fibration* $g : (Y/H) \to Z$, when the domain of the fibration is itself an orbifold (Y/H).

Writing $H := \sum_{i \in I}(1 - 1/m_i)H_i$, define first, for any irreducible reduced divisor $D' \subset Y$ its muliplicity $m(H; D')$ in H as being m_i if $D' = H_i$, and being 1 otherwise (i.e. if D' is not a component of the support of H).

For any irreducible divisor $D \subset Z$, write now as in 1.1.4

$$g^*(D) := \sum_{j \in J(g,D)} m_*(g, D_j) D_j + R$$

where $J(g, D)$ is the set of all irreducible components of $g^*(D)$ which are mapped surjectively onto D by g, while R is g-exceptional and define the *multiplicity* $m(g, H; D)$ *of* $g : (Y/H) \to Z$ *along* D by:

$$m(g, H; D) := \inf_{j \in J(g,D)} \{m_*(g, D_j) m(H; D_j)\}.$$

And, finally:

DEFINITION 1.29.— *Let $g : Y \to Z$ be a fibration, with Z smooth. Let H be an orbifold structure on Y. We define a $\mathbb{Q}$ divisor of Z called the orbifold base $\Delta(g, H)$ of the fibration $g : (Y/H) \to Z$ by:*

$$\Delta(g, H) := \sum_{D \subset Z} (1 - \frac{1}{m(g, H; D)}) D.$$

In general, it is not true that $\Delta(g, H) = \Delta(g \circ f)$ if $H = \Delta(f)$, but the following results at least are available.

PROPOSITION 1.30.— *Let $f : X \to Y$ and $g : Y \to Z$ be two holomorphic fibrations, with X, Y, Z smooth.*

(1) $\Delta(g \circ f) \leqslant \Delta(g, \Delta(f))$ *(recall that, for two $\mathbb{Q}$-divisors A, B on a variety Z, we write $A \leqslant B$ if $(B - A)$ is effective).*

(2) *For any $u : X' \to X$ and $v : Y' \to Y$ bimeromorphic with smooth X' and Y' and $f' : X' \to Y'$ such that $f \circ u = v \circ f'$, then $\Delta(g' \circ f') = \Delta(g \circ f)$ where $g' := g \circ v$ and $\Delta(g', \Delta(f')) \leqslant \Delta(g, \Delta(f))$.*

Proof.— For (1), we easily check that we have, for any prime divisor $D \subset Z$:

$$m(g, \Delta(f); D) = \inf_{j \in J(g,D)} \{m_*(g, D_j) m(\Delta(f); D_j)\}$$

and

$$m(g \circ f, D) = \inf_{k \in J(g \circ f, D)} \{m(g \circ f, D_k)\}.$$

Now, if $f(D_k) = D_j$ is a divisor, then: $m_*(g \circ f, D_k) = m_*(g, D_j) m(\Delta(f); D_j)$, by an easy check. Observe that the minimum of these values, taken over

$J(g,D)$, is precisely $m(g,\Delta(f);D)$. Thus, $m(g\circ f,D)=\inf\{m(g,\Delta(f);D), m(f,g)\}$, where $m(f,g) := \inf_{k\in J'(g\circ f,D)}\{m_*(g\circ f,D_k)\}$, and where $J'(g\circ f,D)$ is the set of irreducible components of $(g\circ f)^*(D)$ which are surjectively mapped to D by $(g\circ f)$, but are f-exceptional. From this the claim follows.

(2) Since $g'\circ f'=(g\circ f)\circ u$ the equality $\Delta(g'\circ f')=\Delta(g\circ f)$ follows from 1.9. We can and shall thus assume that $X'=X$ and $f=v\circ f$. Let $D\subset Z$ be a prime divisor. Then $J(g,D)\subset J(g',D)$, the difference consisting of the v-exceptional components of $(g')^*(D)$. Moreover, for each $j\in J(g,D)$, we have: $m(g,D_j)=m(g',\overline{D_j})$, with $\overline{D_j}$ the strict transform of D_j by v. Finally: $m(f,D_j)=m(f',\overline{D_j})$, by 1.8. This implies the claim. □

1.6.2. High and very high fibration.

Definition 1.31. — *We shall say that $g:(Y/H)\to Z$ is high (resp. very high) if there exists a modification $u_0:Y\to Y_0$ with Y_0 smooth such that (a) and (b) (resp. (a) and (b')) below are satisfied:*

(a) *every g-exceptional divisor of Y is u_0-exceptional,*

(b) *$\kappa(Y/H)=\kappa(Y_0/H_0)$, with $H_0:=(u_0)_*(H)$,*

(b') *$K_Y+H\geqslant (u_0)^*(K_{Y_0}+H_0)$.*

The following lemma is immediate.

Lemma 1.32. —

(1) *If g is very high, it is high.*

(2) *If g is high, then $\kappa(Y,K_Y+H+B)=\kappa(Y,K_Y+H)$, for any effective g-exceptional $\mathbb{Q}$-divisor B on Y.*

The main result of this section is the following.

Proposition 1.33. — *Assume X,Y,Z to be smooth and $f:X\to Y$ and $g:Y\to Z$ to be holomorphic fibrations. Then there exist $f':X'\to Y'$ a modification of f (with modifications $u:X'\to X$ and $v:Y'\to Y$ such that: $f\circ u=v\circ f'$) such that f', $g':=g\circ v$ and $g'\circ f'$ are prepared, admissible and high, and such that moreover :*

1. *$\Delta(g',\Delta(f'))\leqslant \Delta(g",\Delta(f"))$, for any modification $f"$ of f',*
2. *$\Delta(g'\circ f')=\Delta(g',\Delta(f'))$.*

Proof. — The existence of $f' : X' \to Y'$ a modification of f (with modifications $u : X' \to X$ and $v : Y' \to Y$ such that: $f \circ u = v \circ f'$) such that $\Delta(g', \Delta(f')) \leqslant \Delta(g", \Delta(f"))$, for any modification $f"$ of f', is simply due to the fact that there are only finitely many orbifold divisors on Z lying between $\Delta(g \circ f)$ and $\Delta(g, \Delta(f))$, and because of 1.30 above, which show that the first (resp. second) term is invariant (resp. decreases) under a modification.

Assume indeed that f (rather than f' to ease notations) is such a modification and by contradiction that we have $\Delta(g \circ f) < \Delta(g, \Delta(f))$. This means that there exists an f-exceptional prime divisor $D' \subset X$ such that $g(f(D')) := D \subset Z$ is a divisor in Z, and that the multiplicity m' of D' in $(g \circ f)^*(D)$ is equal to $m(g \circ f, D)$, and is so strictly less than $m(g, \Delta(f)); D)$. Take then a modification f' of f such that the strict transform of D' in X' is no longer f'-exceptional. The multiplicity of $D"$, the strict transform of D' in X' is then m' (by an easy check). Thus,

$$m(g', \Delta(f'); D) \leqslant m' < m(g, \Delta(f); D),$$

which gives the contradiction.

Finally, by modifying Z, we can assume that $(g \circ f)$ and g are admissible, high, and moreover that the non-smooth loci of these two fibrations are contained in a normal crossings divisor. By modifying next Y, we can assume that g is prepared, f admissible and high, and that the non-smooth locus of f is contained in a normal crossings divisor of Y. Finally modify X to get the remaining stated properties. □

2. Special fibrations and general type fibrations.

In §2, we first define general type fibrations $f : X \dashrightarrow Y$ as the ones having $\kappa(Y, f) = \dim(Y) > 0$, special manifolds X are defined as the ones having no f of general type, and special fibrations as the ones having special general fibres. We list without proof some examples of special manifolds. The two most important ones are the rationally connected manifolds, and manifolds with $\kappa = 0$.

Notice however that being special does not give any restriction on the Kodaira dimension, except for the top one. For example, any elliptic surface with base elliptic or rational is special if it has a section (or even no multiple fibre). Somewhat unexpectedly maybe, the moduli of fibrations do not play any role in our considerations. Notice that the consideration of special manifolds leads to a refinement, still stable by deformations, of the classical Enriques-Kodaira-Shafarevitch *et al.* classification of projective (or Kähler) surfaces.

We next show various geometric properties of special and general type fibrations. The most important one (2.7) is that any special fibration $f : X \dashrightarrow Y$ dominates any general type fibration $g : X \dashrightarrow Z$, in the sense that there exists $\phi : Y \dashrightarrow Z$ such that $g = \phi \circ f$.

From which one concludes that on any X there exists at most one fibration both special and either of general type, or constant. One of the main results of the present paper is the *existence* of such a fibration. We get it by the two possible approaches: either from "above", as the "lowest special" fibration on X, obtained by geometric means (see Section 3), or from "below", as the "highest general type fibration" on X (see 5.16). The second approach is much shorter, but less geometric.

The next subsection shows the bijective correspondance between Bogomolov sheaves and general type fibrations (2.26). From which we conclude that special manifolds are characterised by the absence of such sheaves. Applying a result from [Ca95], we obtain a first simple proof that X is special if either rationally connected or with $c_1(X) = 0$.

We next show (2.38), among other things, that the general fibre X_s of a fibration $h : X \to S$ is special if it is special for any s in a set E not contained in a countable union of Zariski closed subsets with empty interior of Z. This property, which we call Zariski regularity (for specialness) is, together with 2.7, one of the main ingredients in the proof that the core of a manifold is a special fibration (3.3).

This section ends with a brief sketch of the extension of the considerations of the present paper to the orbifold category. Most of them extend without any difficulty to the orbifold context, including the notions of orbifold modification and of differential form on an orbifold (Y/D) with smooth Y and the support of D of normal crossings. In particular the notion of Bogomolov sheaf and the sheaf F_f associated to a fibration between orbifolds make sense. The geometric aspect seems more delicate to handle (see Section 6.1 for some subtle problems on surface orbifolds which are classical in the non-orbifold context).

The notion of orbifold differential form we introduce interpolates between the usual one when D is empty, and the classical notion of $\Omega_Y(\log(D))$ when D is reduced, which corresponds to the limiting case of infinite multiplicities. It seems that this topic deserves by itself further developments.

2.1. Special or general type fibrations.

Recall that $\mathcal{C}$ is the class of compact complex spaces X which are bimeromorphic to (or, equivalently: dominated by) some compact Kähler manifold X' (depending on X). This class was introduced by A. Fujiki.

DEFINITION 2.1. — *Let $f : X \dashrightarrow Y$ be a fibration, with X, Y compact irreducible.*

1. *The fibration $f : X \dashrightarrow Y$ is said to be of general type if $\kappa(Y, f) = \dim(Y) > 0$.*
2. *The variety X is said to be special if X belongs to the Fujiki class $\mathcal{C}$ and if there is no meromorphic fibration $f : X \dashrightarrow Y$ of general type, for any Y.*
3. *The fibration $f : X \dashrightarrow Y$ is said to be special if $X \in \mathcal{C}$, and if its general fibre is special.*

DEFINITION 2.2. — *Recall (see [Ca81]) also that a point of a complex space Y is said to be general if it lies outside of a countable union of closed analytic subsets of Y, none of which containing any irreducible component of Y. Similarly, if $f : X \dashrightarrow Y$ is a fibration, one of its fibres X_y is general if it lies above a general point y of Y.*

Example 2.3. — We list (most proofs need tools developed below, and so are given later) some examples of special manifolds.

0. A variety of general type (and positive dimension) is *not* special. (Consider its identity map: it is a fibration of general type).
1. A *curve* is special iff its genus is 0 or 1, iff its Kodaira dimension is at most zero, iff its fundamental group is abelian, iff it is not hyperbolic. This is simply because a curve has only the two trivial fibrations (constant, and identity).

The two fundamental examples of special manifolds are direct generalisations:

2. A manifold which is *rationally connected* is special. See Theorem 3.22 for a geometric proof and definition of the notions involved. Another shorter (but more abstract) proof of the specialness of rationally connected manifolds is given in 2.28 below.

3. A manifold X with *vanishing Kodaira dimension* (*i.e.* $\kappa(X) = 0$) is special. See Theorem 5.1 for the proof. Another proof is given in 2.28 in the special case where $c_1(X) = 0$.

4. More generally, special manifolds are built up from manifolds either rationally connected (in a weak sense), or with Kodaira dimension zero by suitable compositions of fibrations with fibres of these two types. See Section 6.5 for a precise formulation.

5. For any $d > 0$ and $k \in \{-\infty, 0, ..., d-1\}$, there exists special projective manifolds of dimension d and Kodaira dimension k. See 2.19 for such examples.

6. A manifold $X \in \mathcal{C}$ is special if there exists a nondegenerate meromorphic map from $\mathbb{C}^n$ to X, where nondegenerate means: submersive at some point where it is holomorphic. For example, a complex torus, or a projective space are special (this follows also from [2.2 (2),(3)] above as well). See Theorem 8.2 for the proof of a more general version.

7. Kähler manifolds with nef anticanonical bundle are conjectured to be special. This conjecture implies most usual conjectures concerning these manifolds. See [D-P-S93], [Zh96], [Pa98]. This conjecture can be shown in the projective case, using the orbifold additivity theorem 4.2 below, even when the anticanonical bundle is pseudoeffective.

8. A manifold X of *algebraic dimension zero* (denoted $a(X) = 0$, to mean that all meromorphic functions on X are constant, so that meromorphic maps from X to projective varieties are constant) is also special (simply because any meromorphic map from X onto a projective manifold is constant).

9. More generally (see [Ue75], Chap. 12 for the notions used):

THEOREM 2.4.— *Let $a_X : X \dashrightarrow Alg(X)$ be the algebraic reduction of $X \in \mathcal{C}$. The generic fibre of a_X is special.*

Recall that the algebraic dimension of X, denoted $a(X)$, is the dimension of $Alg(X)$, and also the transcendance degree over $\mathbb{C}$ of the field of meromorphic functions on X. One says that X is *Moishezon* if

$a(X) = \dim(X)$. This also means that X has a modification which is projective.

The proof of the preceding result is given in 2.39 below.

QUESTION 2.5. — *Two important stability properties of the class of special manifolds are expected to hold, but are not proved in the present paper: are special varieties stable under deformation and specialisation?*

2.2. Special fibrations dominate general type fibrations. Statements.

The geometric study of special manifolds is based on the following theorem.

THEOREM 2.6. — *Let $h : V \dashrightarrow Z$ and $f : X \dashrightarrow Y$ be fibrations with f of general type and h having general fibres which are special. Let $g : V \dashrightarrow X$ be meromorphic surjective. Then, there exists $k : Z \dashrightarrow Y$ such that $f \circ g = k \circ h$.*

The situation is described in the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\ g\ } & X \\ {\scriptstyle h=special}\downarrow & & \downarrow{\scriptstyle f=gen.type} \\ Z & \xrightarrow{\ k\ } & Y \end{array}$$

The special case $V = X$ deserves special mention:

THEOREM 2.7. — *Let $h : X \dashrightarrow Z$ and $f : X \dashrightarrow Y$ be fibrations with f of general type and h special. Then, there exists $k : Z \dashrightarrow Y$ such that $f \circ g = k \circ h$ (We say that h dominates f).*

The corresponding diagram is

$$\begin{array}{ccc} X & & \\ {\scriptstyle h=special}\downarrow & \searrow^{\scriptstyle f=gen.type} & \\ Z & \xrightarrow{\ k\ } & Y \end{array}$$

Remark 2.8. — The special case where Y is of general type is obvious, by the easy addition theorem, because the covering family of Y by the subvarieties $h(X_z)$ has a generic member of general type. These subvarieties must be points.

2.3. Special fibrations dominate general type Fibrations. Proofs.

The proof of Theorem 2.6 rests on several preliminary resuts of independent interest that we now state and prove.

LEMMA 2.9. — *Let $g : X' \dashrightarrow X$ be meromorphic surjective (ie: dominant). Assume that X' is special. Then X is special, too.*

Proof. — Since being special is invariant by bimeromorphic maps, one can assume that g and all the maps occuring in the proof are holomorphic. Assume first that g is connected (i.e. a fibration). Let, if any, $f : X \to Y$ be a fibration of general type. We can assume that $f \circ g : X' \to Y$ is admissible. Then obviously, $\Delta(f \circ g) = \Delta(f) + E$, for some effective divisor $E \subset Y$ (because $m(f \circ g, D) \geqslant m(f, D)$, for any irreducible divisor $D \subset Y$). Thus $f \circ g$ is of general type, too. A contradiction. No such f does exist, which is what was claimed.

In the general case, Stein factorise g and use the first part to reduce to the case where g is generically finite. If f as in the first part exists, then we deduce from 1.8 that (the fibration part of) the Stein factorisation of $f \circ g$ also is of general type. Hence again a contradiction. □

PROPOSITION 2.10. — *Let $f : X \dashrightarrow Y$ be a fibration of general type. Let $j : Z \dashrightarrow X$ be meromorphic such that $f \circ j : Z \to Y$ is surjective. Let $f \circ j = g \circ h$ be the Stein factorisation of $f \circ j$, with $h : Z \dashrightarrow Y'$ connected and $g : Y' \dashrightarrow Y$ finite. Then: h is a fibration of general type.*

In particular, if $\dim(Z) = \dim(Y)$, we get:

COROLLARY 2.11. — *Let $f : X \dashrightarrow Y$ be a fibration of general type. Let $j : Z \dashrightarrow X$ be meromorphic and such that $f \circ j : Z \dashrightarrow Y$ is surjective. Then Z is a variety of general type.*

Proof (of 2.10 and 2.11). — Assume first that $f \circ j : Z \to Y$ is connected and admissible (as we can, then). For any component Δ_i of $\Delta := \Delta(f)$, we have (restricting to components surjectively mapped onto Δ_i by $(f \circ j)$): $(f \circ j)^*(\Delta_i) = j^*(f^*(\Delta_i)) = j^*(m_i D_i) = m_i j^*(D_i)$. Thus: $\Delta(f \circ j) = \Delta(f) + E$, for some effective $\mathbb{Q}$-divisor E of Y. And $(f \circ j)$ is thus of general type in this case.

We now consider the general case: let $f' : X' := (\widetilde{X \times_Y Y'}) \to Y'$ be deduced from the base change by g and smoothing of the fibre product. The map j lifts meromorphically to $j' : Z \dashrightarrow Y'$ by construction, because $f \circ j$ is surjective. But now $f' \circ j' : Z \to Y'$ is a fibration. Applying the first part, we get the claim.

For 2.11, notice that in this situation, h is bimeromorphic and of general type by 2.10. Thus Z is itself of general type, as claimed. □

Example 2.12. — We can now give two elementary examples of special varieties:

1. $\mathbb{P}_n(\mathbb{C})$ is special.
2. A product of special varieties is special.

Proof. — Indeed (for (1)): let $f : \mathbb{P}_n(\mathbb{C}) \dashrightarrow Y$ be any general type fibration, if any. Let $m := \dim(Y) > 0$. Choose $j : Z = \mathbb{P}_m(\mathbb{C}) \subset \mathbb{P}_n(\mathbb{C})$ such that $f \circ j$ is surjective to contradict 2.10.

The proof of (2) is similar. (We shall prove more general results in Section 3).

PROPOSITION 2.13. — *Let $f : X \dashrightarrow Y$ and $k : Y \dashrightarrow W$ be fibrations. Assume that f is of general type. Then $f_w : X_w \dashrightarrow Y_w$ is also of general type, for $w \in W$ general.*

Proof. — Let $w \in W$ be general, and recall that $f_w : X_w \to Y_w$ is nothing, but the restriction of f to X_w. But then, $\Delta(f_w) = \Delta(f)_{|Y_w} + E_w$, with E_w effective and empty for generic w in W. Moreover, $K_{Y_w} = K_{Y|Y_w}$, by adjunction. Thus $K_{Y_w} + \Delta(f_w) = (K_Y + \Delta(f))_{|Y_w}$ for general w. Now $(K_Y + \Delta(f))$ is big. Thus so is its restriction to Y_w. By modifying adequately X and Y, we can assume that f_w is admissible by the following lemma. We thus get the claimed property. □

LEMMA 2.14. — *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be fibrations. There exists representatives of f and g (also denoted f and g) such that f_z is admissible, for $z \in Z$ general.*

Proof. — This is a special case of 1.28. □

We now turn to the proof of Theorem 2.6.

Proof (of 2.6). — This is a direct consequence of 2.15 below. Indeed, if such a map k does not exist, then $f \circ g(V_z)$ is positive-dimensional, for generic $z \in Z$. But the Stein factorisation of $f \circ g_z$ is then of general type, by 2.15. This contradicts the assumption that g is special. □

PROPOSITION 2.15.— *Let $f : X \dashrightarrow Y$ and $h : V \dashrightarrow Z$ be fibrations. Assume f is of general type. Let $g : V \dashrightarrow X$ be a surjective meromorphic map. Let $g_z : V_z \dashrightarrow Y_z$, be the restriction of $f \circ g$ to V_z, with $Y_z := h_z(V_z)$. Assume Y_z is positive dimensional. Then, the Stein factorisation of g_z is of general type for z general in Z.*

Proof.— By 2.10, we can then replace X by V and f by $(f \circ g)$ without loosing the hypothesis that $(f \circ g)$ is of general type both when g is generically finite or is a fibration. We thus see that the (connected part of the) Stein factorisation f' of $f \circ g$ is a fibration of general type. Replace X by V and f by f', so that we are reduced to the case where $X = V$, and $f = (f \circ g)$, which we now treat.

Now we can replace Z by any subvariety $Z' \subset Z$ going through a general point of Z, and X by $X' \subset X$, defined by $X' := g^{-1}(Z')$, provided $f(X') = Y$. This is because of 2.10, which shows that the (Stein factorisation of the) restriction of f to X' is still of general type.

We shall then construct an appropriate $Z' \subset Z$. Let $c : Z \dashrightarrow \mathcal{C}(Y)$ be the meromorphic map sending a generic $z \in Z$ to the reduced cycle of Y supported on Y_z. Here $\mathcal{C}(Y)$ denotes the Chow Scheme of Y. Observe next that f being of general type, Y is Moishezon. By modifying suitably Y, we shall assume that Y is projective. Let $W \in \mathcal{C}(X)$ be the image of Z by c. Thus W is projective, too.

We next choose $W' \subset W$ to be an intersection of generic members of any very ample linear system on W, in such a way that the incidence graph $Y' \subset W' \times Y$ of the algebraic family of cycles of Y parametrised by W' is generically finite over Y. This means that if $p : Y' \to Y$ and $q : Y' \to W'$ are induced by the natural projections, then p is generically finite surjective. More concretely, the generic point of Y is contained in only finitely many of the Y_w's, for $w \in W'$. Define now $Z' := c^{-1}(W')$. (Remark that when Z is Moishezon, we don't need to consider c, and can just take intersections of ample divisors of a projective modification of Z directly).

We now replace X, Z, g, f respectively by X', Z', and their restrictions to X'.

LEMMA 2.16.— *$p : Y' \to Y$ is bimeromorphic.*

Proof. — Let c' be the restriction of the above map c to Z'. Then $Y' \subset (W' \times Y)$ is the image of $(c' \circ h) \times f : X' \dashrightarrow W' \times Y$. Thus $f : X' \dashrightarrow Y$ lifts to $f' : X' \dashrightarrow Y'$ such that $p \circ f' = f$. Because p is generically finite and f connected, we see that p is bimeromorphic. □

We have, by construction, $q \circ f = c' \circ h : X \dashrightarrow W'$. We now can conclude by applying successively 2.13 and 2.11 to X_w, for w general in W'. Indeed: from 2.13 we learn that the restriction $f_w : X_w \dashrightarrow Y_w$ of f to X_w is of general type. Further, for z generic in ${c'}^{-1}(w) := Z'_w$, we know that $f(X_z) = f(X_w) = Y_w$, and so by 2.10, the Stein factorisation of the restriction of f to X_z is of general type, as claimed. □

Corollary 2.17. — *Let $f : X \dashrightarrow Y$ be a special fibration, and let $j : Z \dashrightarrow X$ be such that $f \circ j : Z \dashrightarrow Y$ is onto. Assume that Z is special. Then X is special.*

Proof. — Let $h : X \dashrightarrow W$ be a fibration of general type (if any). By 2.6 with $V = X$, there is a factorisation $k : Y \dashrightarrow W$ such that $h = k \circ f$. Thus: $j \circ h : Z \dashrightarrow W$ is onto. We can thus apply 2.11, which says that the Stein factorisation $h' : Z \to W'$ of $h \circ j : Z \to W$ is of general type. But Z being special by assumption, this is a contradiction and X is special.

Remark 2.18. — It is not true true in general that X is special if it admits a special fibration $f : X \to Y$ with Y special (see 1.1, for example). But in some cases (if the fibres are for example, rationally connected), this is true (see 3.29).

Example 2.19. — For any $d > 0$ and $k \in \{-\infty, 0, ..., d-1\}$, there exists projective manifolds of dimension d and Kodaira dimension k which are special. In particular, it is not true that a special manifold has nonpositive Kodaira dimension.

To get examples with $k \geqslant 0$, just take indeed a general member of the linear system: $|\mathcal{O}_P(d-k+2, m)|$ on $P := \mathbb{P}^{d-k+1} \times \mathbb{P}^k$ for large m, and such that this member has a section over the base $\mathbb{P}^k$. If $k = -\infty$, just take $\mathbb{P}^d$.

2.4. A uniqueness result.

We now come to an important consequence of 2.7.

COROLLARY 2.20. — *Let $X \in \mathcal{C}$. There is at most one fibration defined on X which is both special and of general type. If it exists, such a fibration is both dominated by any other special fibration defined on X and dominates any other general type fibration defined on X.*

The proof is immediate, from 2.7. In other words, such a fibration is the "lowest special" and the "highest of general type" on X. The existence of such a fibration on any X is the main result of this paper.

These two descriptions provide us with two means of construction: by consideration of chains of special subvarieties, one geometrically constructs the "lowest special" fibration on any X. This is the way used in Section 3. Dually, by making fibre products of general type fibrations, one constructs the "highest general type" fibration on X. This is the approach followed in Section 5. In both cases, to show that the fibration so constructed has the missing property (special if general type, and conversely), we need the orbifold additivity result 4.2.

2.5. A result on almost holomorphic maps.

Recall the following definition:

DEFINITION 2.21. — *Let $f : X \dashrightarrow Y$ be a surjective meromorphic map between normal compact irreducible analytic complex spaces. We say that f is almost holomorphic if $f(J) \neq Y$, where J is the indeterminacy locus of f.*

More precisely: if $X' \subset X \times Y$ is the graph of f, and $f' : X' \to Y$ the (restriction of the) second projection, then $f(J) := f'(J')$, with J' being the set of all $x' \in X'$ such that $p^{-1}(x)$ does not reduce to x', or equivalently: is positive dimensional. (Here $p : X' \to X$ is the first projection (which is a proper modification), and $x := p(x')$).

THEOREM 2.22. — *Let $f : X \dashrightarrow Y$ be a meromorphic fibration of general type, with $X \in \mathcal{C}$ smooth. Then f is almost holomorphic. (In particular, if Y is a curve, then f is holomorphic).*

Remark 2.23. — The smoothness assumption is essential, as shown by the example of the cone X over a projective manifold of general type Y. The conclusion of the preceding theorem should however hold if the singularities of X are terminal, or even canonical.

Proof. — Resolve the indeterminacies of f by a sequence of smooth blow-ups $u : X' \to X$, with $f' := f \circ u : X' \to Y$ holomorphic. If f is not almost holomorphic, some irreducible component V of the exceptional divisor of u is mapped surjectively onto Y by f', in such a way that the fibres V_z of the restriction u' of u to V are mapped to positive-dimensional subvarieties $f(V_z)$ of Y. This contradicts 2.15, because V has two maps $u' : V \to Z := u(D) \subset X$, and $f" : V \to Y$, the restriction of f' to V. Now, by smoothness of X, u' is special since its generic fibre is a rational variety and 2.12 applies. Moreover, by 2.11, $f"$ is of general type (possibly after Stein factorisation).

From 2.15, we thus have a factorisation $\phi : Z \to Y$ with $f" = \phi \circ u'$. But this precisely contradicts $\dim(f(V_z)) > 0$, and we get the claim. □

2.6. General type fibrations and Bogomolov sheaves.

DEFINITION 2.24. — *Let $X \in \mathcal{C}$. A rank one coherent subsheaf F of Ω^p_X, $p > 0$ is said to be a (p-dimensional) Bogomolov sheaf on X if $\kappa(X, F) = p$.*

The properties of these Kodaira dimensions have been discussed in Section 1.4 to which we refer.

By the results of that section, any (equivalence class of a) general type fibration f defined on X canonically determines a Bogomolov sheaf F_f on X. We shall now see the converse direction.

By the results of [Bo79], any p-dimensional Bogomolov sheaf determines a meromorphic fibration $f_F : X \dashrightarrow Y_F$ with $\dim(Y_F) = p$, and such that $F = f_F^*(K_{Y_F})$ at the generic point of Y_F.

The proof given there applies only to X projective (because of the argument of cutting by transversal hyperplane sections), but (as is well-known) can be easily modified to apply to any X compact Kähler (or in $\mathcal{C}$), as follows:

THEOREM 2.25 [Bo79]. — *Let $F \subset \Omega^p_X$, $p > 0$, be a Bogomolov sheaf on $X \in \mathcal{C}$. Let $f_F : X \to Y_F$ be the fibration defined by the linear system $|L^{\otimes m}|$ for $m > 0$ sufficiently large and divisible. We can assume that f is holomorphic. Then $F = f_F^*(K_{Y_F})$ at the generic point of Y_F.*

Proof. — We can, using the covering trick argument of [Bo79], reduce to the case when $m = 1$, which we now treat.

We can thus select $(p+1)$ sections $s_i, i = 0, 1, \dots, p$ of F which are analytically independent (i.e. the linear system they define is f_F up to Stein factorisation, and so has p-dimensional image). Because F has rank one, there exists meromorphic functions y_i, $i = 1, \dots, p$ such that $s_i = y_i s_0$.

By Hodge theory (X being Kähler, or even just in $\mathcal{C}$), the holomorphic p-forms $s_i, i = 0, \dots, p$ on X are closed. From which we get $ds_0 = dy_i \wedge s_0 = 0$, $i = 1, \dots, p$.

The last equality shows by simple algebraic arguments the existence of a meromorphic function g on X such that $s_0 = g(dy_1 \wedge \cdots \wedge dy_p)$. The first equality shows that $g = f^*(h)$, for some meromorphic function h on Y, and so the claim, since the argument applies to $i > 0$ as well. □

We can thus sum up the preceding observations as follows.

THEOREM 2.26. — *Notations being as above, for any $X \in \mathcal{C}$, there are inverse bijective correspondances between Bogomolov sheaves F on X and (equivalence classes of) general type fibrations f defined on X. These correspondances are defined as follows.*

1. *If f is of general type, then F_f is a Bogomolov sheaf on X.*
2. *If F is a Bogomolov sheaf on X, then f_F is a fibration of general type.*

A direct application (and motivation) is:

THEOREM 2.27. — *The manifold $X \in \mathcal{C}$ is special if and only if there is no Bogomolov sheaf on X.*

Proof. — The Bogomolov subsheaves on X correspond bijectively to fibrations of general type with domain X. □

COROLLARY 2.28. — *The manifold X is special in the following two cases:*

1. *X is rationally connected (see Section 3.3 for this notion),*
2. *X is a compact Kähler manifold with $c_1(X) = 0$.*

Proof. — In both cases, it is shown in [Ca95] that $\kappa^+(X) \leqslant 0$, which means (in particular) that a coherent rank one subsheaf of Ω^p_X, $p > 0$ has Kodaira dimension negative or zero. Thus X has no Bogomolov subsheaf. It is thus special, by 2.27. The result in [Ca95] depends on Calabi-Yau's Theorem. But in the projective case, one can get algebro-geometric proofs using Miyaoka's generic semi-positivity Theorem. □

We shall see later that the weaker condition $\kappa(X) = 0$ is actually sufficient for X to be special.

Notice that the property shown in [Ca95] in the above two cases is considerably stronger than the absence of Bogomolov sheaves. This is not surprising, in view of the fact that these manifolds are the building blocks of the class of special manifolds, but do not exhaust this class, by far.

2.7. General type reduction.

2.7.1. Ordering of fibrations.

Recall from 1.1 that a meromorphic fibration $f : X \dashrightarrow Y$ canonically defines (see [Ca85]) a meromorphic map $\phi_f : Y \dashrightarrow \mathcal{C}(X)$. It is easy to show that Φ_f is an irreducible component of $\mathcal{C}(X)$ if f is almost holomorphic (see [Ca85]).

We now introduce an order on the set of (equivalence classes of) fibrations with domain X.

We say that f *dominates* the fibration $g : X \dashrightarrow Z$ if there exists a meromorphic fibration $\phi : Y \dashrightarrow Z$ such that $g = \phi \circ f$. Equivalently: each fibre of f is contained in some fibre of g. We write $f \geqslant g$. This binary relation defines an ordering on the set $\mathcal{F}(X)$ of all equivalence classes of fibrations (seen as a subset of $\mathcal{C}(\mathcal{C}(X))$).

There is now an easy,

LEMMA 2.29. — *If $\Lambda \subset \mathcal{F}(X)$ is any subset, it has in the ordered set $\mathcal{F}(X)$ a least upper bound, denoted Λ^+. Moreover, if any element of Λ is almost holomorphic, so is the least upper bound Λ^+ of the family.*

Proof. — Λ^+ is so constructed: let $\Lambda_0 := \{\lambda_1, \dots, \lambda_N\} \subset \Lambda$ be finite such that the product map $f := f_{\lambda_1} \times \cdots \times f_{\lambda_N}$ has an image of maximal dimension. Then take for Λ^+ the (fibration part of the) Stein factorisation of f. □

Example 2.30. — If Λ consists of fibrations onto varieties of general type, then Λ^+ is also a fibration onto a variety of general type.

This is easily reduced to the case when Λ has two elements, and then reduces to showing that if $Z \subset Y \times Y'$ is a subvariety of a product of two varieties of general type, then Z itself is of general type if it is mapped surjectively to Y, Y' by the first and second projections. This results easily from the additivity theorem for fibrations with base of general type (generalised orbifold versions will be proved in Section 4 below). □

DEFINITION 2.31. — *For any $X \in \mathcal{C}$, let $gt_X : X \dashrightarrow GT(X)$ be the least upper bound in $\mathcal{F}(X)$ of the family Λ_X of all (equivalence classes of) fibrations of general type $f_j : X \dashrightarrow Y_j$. (If X is special, we just take for gt_X the constant fibration.) We call gt_X the general type reduction of X.*

From 2.7.1 above, we deduce:

PROPOSITION 2.32. — *Suppose that $X \in \mathcal{C}$ is smooth. Then the map $gt_X : X \dashrightarrow GT(X)$ is almost holomorphic.*

We shall see in 5.16 the following two properties of gt_X, the proofs rest on the very different techniques of section 4.

PROPOSITION 2.33. — *Let $u : X \dashrightarrow U$ and $v : X \dashrightarrow V$ be fibrations of general type. Then the connected part of the Stein factorisation of the product map $(u \times v) : X \dashrightarrow W'$, with $W' := (u \times v)(X) \subset U \times V$, is a fibration of general type.*

COROLLARY 2.34. — *Let $X \in \mathcal{C}$. Then gt_X is either constant or a fibration of general type.*

2.7.2. Relative gt-reduction.

This subsection is devoted to the construction of relative gt-reduction.

THEOREM 2.35. — *Let $X \in \mathcal{C}$ and $f : X \dashrightarrow Y$ be any fibration. Then f admits a relative gt-reduction. This means that there exists a unique factorisation $f = h \circ g$ of f by fibrations $h : Z \dashrightarrow Y$ and $g : X \dashrightarrow Z$ such that for y general in Y, the restriction $g_y : X_y \dashrightarrow Z_y$ of g to X_y is the gt-reduction of X_y.*

Proof. — This construction is actually in essence already in [Ca80] to which we refer for more details. We can and shall assume that X is smooth and f holomorphic, due to the bimeromorphic invariance of the notions involved.

We shall actually show a more precise version:

LEMMA 2.36. — *Let $f : X \dashrightarrow Y$ be a fibration, with $X \in \mathcal{C}$. After a generically finite base change $v : Y' \to Y$ and proper modifications that we notationally ignore, there exists finitely many factorisations $f = h_i \circ g_i$, $i = 1, 2, \ldots, N$, with $g_i : X \dashrightarrow Z_i$, $h_i : Z_i \dashrightarrow Y$, such that:*

1. *the restriction $g_{i,y} : X_y \dashrightarrow (Z_i)_y$ of each g_i to the general fibre X_y of f is of general type,*
2. *if $g : X \dashrightarrow Z$ is the (fibration part of) the Stein factorisation of the product map $g_1 \times \cdots \times g_N : X \dashrightarrow Z_1 \times_Y Z_2 \times \cdots \times Z_{N-1} \times_Y Z_N$, then the restriction $g_{|X_y} : X_y \dashrightarrow Z_y$ of g to the general fibre X_y of f coincides with the gt-reduction $gt_{X_y} : X_y \dashrightarrow GT(X_y)$ of that fibre.*

Proof. — Let, for $y \in Y^*$, $gt_y : X_y \to Z_y$ be the gt-reduction of X_y, where Y^* is the Zariski open subset of Z over which f is smooth. For such a y, let Z'_y be the family of fibres of gt_y, defined as the image of the meromorphic map from Z_y to $\mathcal{C}(X_y)$ sending a generic point of Z_y to the point in $\mathcal{C}(X_y)$ parametrising its reduced gt_y-fibre in X_y.

Because gt_y is an almost holomorphic map by 2.22, Z'_y is an irreducible component of $\mathcal{C}(X_y)$.

Consider now the Zariski closed subset $\mathcal{C}(X/Y)$ of $\mathcal{C}(X)$ consisting of cycles contained in some fibre of f. It is naturally equipped with the holomorphic map $f_y : \mathcal{C}(X/Y) \to Y$ sending such a cycle to the fibre containing it. (Strictly speaking, one may need to weakly normalise first, to make f_* holomorphic, but this does not change the argument). Assume the fibre of f is not special, for y in a subset of Y which is of second category, in Baire's terminology. (Being of second category means: not contained in a countable union of closed subsets with empty interior. As we shall see later, the right topology here in our context is the Zariski topology, not the metric topology).

Because $X \in \mathcal{C}$, the irreducible components of $\mathcal{C}(X/Y)$ are compact. By the countability at infinity of $\mathcal{C}(X/Y)$, there is an irreducible component Γ' of $\mathcal{C}(X/Y)$ mapped surjectively onto Y by f_*, and such that the f_*-fibre Γ'_y of Γ' over y has a component equal to $Z'_y \subset \mathcal{C}(X_y)$, the family of fibres of a fibration of general type $g_{i,y} : X_y \dashrightarrow Z'_y$. This map is almost holomorphic, by 2.22. The Stein factorisation of f_* restricted to Γ' gives a finite base change for Y. This base change we shall notationally ignore, here, because they are irrelevant to our problem. So we deal as if the generic fibres of $(f_*)_{|\Gamma'}$ were irreducible. Thus, for some $y \in Y^*$, the fibre Γ'_y of $(f_*)_{|\Gamma'}$ is the family of fibres of some almost holomorphic fibration $g_{i,y} = \gamma_y : X_y \dashrightarrow Z_{y,\gamma}$

on X_y. By the (obvious) openness of almost holomorphicity, one deduces the existence of such a γ_y for the generic $y \in Y$. And so, using the graph of the family, we get a factorisation $f = \delta \circ \gamma$, with fibrations $\gamma : X \dashrightarrow Z_\gamma$, and $\delta : Z_\gamma \dashrightarrow Y$.

By our assumption, γ_y is a fibration of general type for y in $S \subset Y$ of second Baire category in Y. From 1.28, we conclude that γ_y is still of general type for y general in Y.

The construction of the g_i's is now obvious, by observing that if the map g resulting from a finite family of g_i's, $i = 1, 2, \ldots, N$, does not induce gt_{X_y} on the general X_y, there exists, by the same argument as above, a component Γ', inducing a general type fibration on the general X_y, and such that its (Stein factorised) fibre product over Y with the preceding ones will increase the dimension of the resulting Z. Contradiction. This shows the lemma, and so 2.35. □

Definition 2.37. — *A subset $A \subset V$ of a complex analytic space is said to be of second Zariski category in V if it is not contained in a countable union of Zariski closed subsets with empty interior of V. (Notice that the definition makes sense in the algebraico-geometric context as well).*

From the proof 2.35, we immediately get:

Corollary 2.38. — *Let $f : X \dashrightarrow Y$ be a fibration, with $X \in \mathcal{C}$. Assume that $\dim(GT(X_y)) = d$, for $y \in A$, where A is of second Zariski category in Y. Then, this equality holds for the general point y of Y. In particular, if X_y is special for y in a subset of second Zariski category in Y, the general fibre of f is special.*

Proof. — Let $f = h \circ g$ be the gt-reduction of f. By assumption, $\dim(GT(X_y)) = d$ for $y \in A$. But also $\dim(g(X_y)) = \dim(GT(X_y))$ for y general in Y, and $\dim(g(X_y)) = d$ for y generic in Y. Thus $\dim(GT(X_y)) = d$ for $y \in Y$ general. □

2.8. The algebraic reduction.

As an application of the preceding arguments, we show Theorem 2.39:

Theorem 2.39. — *Let $a_X : X \dashrightarrow Alg(X)$ be the algebraic reduction of $X \in \mathcal{C}$. Then the generic fibre of a_X is special.*

Proof. — Assume not. By Lemma 2.36 above, after a suitable finite base change over $Alg(X)$ (which we notationally ignore because it preserves the algebraic reduction and dimension), there exists a non-trivial factorisation $a_X = h \circ g$ with g a fibration inducing a fibration of general type over the general fibre of a_X. Write $a_X = h \circ g$, with $g : X \dashrightarrow Z$ and $h : Z \dashrightarrow Alg(X)$. Then, $\dim(Z) > \dim(Alg(X))$.

By construction, the line bundle $K_Y + \Delta(g)$ over Z is thus h-big. Thus Z is Moishezon, as one sees considering the line bundle $L := h^*(kH) + (K_Z + \Delta(g))$ on Z, which is big for k a large and positive integer, and H an ample line bundle on $Alg(X)$, which we obviously can choose to be projective. (See for example, the Proof of [Ue75], Theorem (12.1)). But this contradicts the definition of a_X, and proves the claim. □

2.9. The category of orbifolds.

We very briefly discuss without proofs the extension of part of our considerations to orbifolds, restricting here to prepared orbifolds (Y/Δ) with Y smooth and the support of Δ an s.n.c divisor of Y (but ultimately, one needs to consider *klt* orbifolds).

One of the main point is to define bimeromorphic equivalence. The right notion seems to be derived from *terminal modifications.*

Definition 2.40. — *The bimeromorphic holomorphic map $v : Y' \to Y$ is said to induce a bimeromorphic map: $v : (Y'/\Delta') \to (Y/\Delta)$ if it is terminal with respect to the orbifold structures, that is if: $K_{Y'} + \Delta' = v^*(K_Y + \Delta) + \sum_{j \in J} a_j E_j$, where (as usual) the a_j are all positive, and J is the collection of v-exceptional divisors on Y'.(One might of course also define similarly the notion of canonical modification).*

Notice that the orbifold Kodaira dimension is invariant under bimeromorphic equivalence of orbifold, which is the one generated by terminal modifications.

One can define for any fibration $g : (Y/\Delta) \to Z$ its orbifold base, as in 1.6. One can extend this notion to the case of meromorphic g, by first resolving the indeterminacies of g by a terminal modification.

The Kodaira dimension of this fibration is then the minimum of the Kodaira dimensions of the orbifolds bases of fibrations equivalent to g, these being defined on orbifolds (Y'/Δ') bimeromorphically equivalent to (Y/Δ).

Definition 2.41. — *The fibration $g : (Y/\Delta)\to Z$ is of general type if its Kodaira dimension is* $\dim(Z) > 0$. *The orbifold (Y/Δ) is special if it has no fibration of general type.*

Fundamental tools for the study of orbifolds are the locally free sheaves $\Omega^p_Y(\log(\Delta))$ of logarithmic forms along Δ (classically known when the multiplicities are infinite, or said differently, when Δ is reduced). We shall not give the definition here, but simply say that sections of this sheaf can be symbolically written locally in the standard normal crossing coordinates for fixed q as linear combinations of expressions of the form:

$$h(dy_{j_1}/y_{j_1}^{(1-1/m_{j_1})}) \wedge \cdots \wedge (dy_{j_s}/y_{j_s}^{(1-1/m_{j_s})}) \wedge dy_{j_{s+1}} \wedge \cdots \wedge dy_{j_q},$$

with h holomorphic, and $1 \leqslant j_1 < \cdots < j_s \leqslant r < j_{s+1} < \cdots < j_q$ if $y_1 \ldots y_r = 0$ is a local equation of $|\Delta|$, the multiplicities being given by the m_i's.

A section of $\Omega^q_Y(\log(\Delta))$ is thus an m-th root of a well-defined holomorphic tensor, when m is an integer divisible by each of the m_i's.

More precisely, a section s of this sheaf is defined as a pair (F, s), where F is a rank one coherent subsheaf of $\Omega^q_Y(\log|\Delta|)$, and s is a holomorphic section of $F^{\otimes m}$, for some m divisible by all the m_i's, and such that $s \in (\Omega^q_Y(\log|\Delta|))^{\otimes m}(-m\Delta^*)$, where Δ^* is the $\mathbb{Q}$-divisor on Y defined by: $\Delta^* := |\Delta| - \Delta = \sum_{j\in I}(1/m_j)\Delta_j$.

By lifting to a Δ-nice covering (see Section 4), these sections become standard p-forms. From which one deduces the important property of d-closedness of such $\log((\Delta))$-forms.

As we did above, one can then also define directly the Kodaira dimension of a fibration by introducing the saturation of the differential sheaf defined by g in $\Omega_Y(\log(\Delta))$. Because we may only consider high and divisible multiples to define the Kodaira dimensions, one does not need to define precisely $\Omega_Y(\log(\Delta))$ to define this orbifold Kodaira dimension, and directly look at rank one subsheaves F of $\Omega_Y(\log|\Delta|)$, and define as usual the Kodaira dimension of $(F^{\otimes m}(-m\Delta^*))$.

Their relevance to our topic is that special orbifolds (Y/Δ) are characterised by the absence of Bogomolov sheaves on (Y/Δ), defined as in 2.24 above when Δ is empty, just replacing there Ω^p_Y by $\Omega^p_Y(\log\Delta)$.

The correspondance between Bogomolov sheaves on (Y/Δ) and general type fibrations on this orbifold extends to this orbifold context. The orbifold additivity theorem then applies in this context.

The construction of the core for an orbifold can then be made by the second approach we followed (as the highest special fibration). The geometric approach seems more delicate than for varieties, and certainly needs some extra arguments, because one needs to take into account the order of contact of the subvarieties with the orbifold divisor.

3. The core.

In this third chapter, we construct and start the geometric study of the core $c_X : X \dashrightarrow C(X)$ of a manifold, together with its functoriality properties.

We first show (3.3) that its general fibres are special. This fails for general singular varieties, and does not follow from the original definition, obtained by applying the construction of meromorphic quotients recalled in [Ca04]. From this result, we immediately get (3.22) that rationally connected manifolds are special, simply because $\mathbb{P}^1$ is special. The notions around rational connectedness are recalled. We next deduce in 3.26 from [G-H-S01] that the rational quotient $R(X)$ of X (see 3.23 for this notion), coincides with the rational quotient of its core $C(X)$.

Up to this point, we do not know that the base orbifold of the core is either a point, or of general type. This property is only obtained as a consequence of the orbifold additivity theorems of the next chapter. Notice that a second construction of the core, shorter and independent from the results of the present chapter, is given in 5.7. The present chapter presents what can be reached without the techniques of the next chapter.

We next describe (3.31 and 3.38) the core and list the special manifolds in dimensions 2 and 3, after having introduced (see §3.6) the "higher Kodaira dimensions" of a compact complex manifold. From the description so obtained, we deduce that in these cases the core is a fibration of general type, when X is not special. The "decomposition theorem" (5.8) asserts that this is true in any dimension.

3.1. Construction of the core as the lowest special fibration.

We use the notations of the separate appendix [Ca04].

Definition 3.1. — *Let $X \in \mathcal{C}$ be normal. Let $A := A(X) \subset \mathcal{C}(X)$ be the family of special subvarieties of X. It is Z-regular. Let then $T(A)$ be the family of its components (see Proposition 2.4 in [Ca04]), and let $c_X : X \dashrightarrow C(X)$ be the $T(A)$ quotient of X. This almost holomorphic fibration will be called the core of X.*

In general, not much can be said about the fibres of c_X.

Example 3.2. — Let X be the cone over a variety of general type V. Then c_X is the constant map. But X is by no means special, since it has a $\mathbb{P}^1$-fibration over V. Note that this fibration is not almost holomorphic.

This example shows the role of singularities. In the smooth case, we have the following.

Theorem 3.3. — *Let $X \in \mathcal{C}$ be smooth. Let $c_X : X \dashrightarrow C(X)$ be the core of X. Then we have the following.*

1. *The general fibre of c_X is special.*
2. *If F is a general fibre of c_X, and if $Z \subset X$ is a special subvariety of X meeting F, then Z is contained in F. Such a fibre F will be said c_X-general.*
3. *The map c_X is almost holomorphic.*

Remark 3.4. — The above result should hold true when X is singular, provided it has at most canonical singularities.

Definition 3.5. — *The canonical algebra $K(c_X)$ of the core (see 1.12), for $X \in \mathcal{C}$, smooth, will be called the essential algebra of X, and will be denoted by $K(c_X) := Ess(X)$.*

Proof of Theorem 3.3. — For this, we shall simply apply Theorem 3.3 in [Ca04] to the family $A(X)$ of special subvarieties of X.

We know that $A(X)$ is Z-regular (see Section 2 in [Ca04] for this notion). It is thus sufficient to show that $A(X)$ is also stable (see Section 3 in [Ca04] for this notion). The property [stab2] is obtained by applying Corollary 2.9.

The property [stab1] is the content of the next Theorem, which thus establishes at the same time Theorem 3.3.

Theorem 3.6. — *Let $T \subset T(A(X)) \subset \mathcal{C}(X)$ be a special family as above, with X smooth in $\mathcal{C}$. Assume that each irreducible component of T*

is X-covering and let $q_T : X \dashrightarrow X_T$ be the T quotient of X. Then we have the following.

1. *If X is T-connected, that is, if q_T is the constant map, then X is special.*
2. *The general fibre of q_T is special.*

Proof. — Let $V \subset X \times T$ be the incidence graph of the family $(V_t)_{t \in T}$ and let ψ and g be the projections from V to X and T respectively.

The second assertion is a consequence of the first, because if X_y is a general fibre of q_T, then X_y is smooth since q_T is almost holomorphic, and the family T_y consisting of $t \in T$ such that $V_t \subset X_y$ is a finite union of covering families of X_y with general member special, and such that X_y is T_y-connected.

We thus only need to show the first statement. Assume that there exists a meromorphic fibration $f : X \dashrightarrow Y$ of general type. By 2.22, f is almost holomorphic, since X is supposed to be smooth. By Theorem 2.6 applied to each irreducible component V_i of V, we have a meromorphic factorisation $\phi : T \dashrightarrow Y$ such that $\phi \circ g = f \circ \psi : V \dashrightarrow Y$.

Assume first that f is holomorphic. Then f is constant on every V_t, and so on every T-chain. Because X is T-connected, f takes the same value on two arbitrary points of X. Thus f is constant and Y is a single point, in contradiction with the fact that it is of general type. So X is special, as claimed.

If f is only almost holomorphic, the same argument applies, provided we choose an f-regular point $y \in Y$. For every $t \in T$, if V_t meets X_y, then V_t is contained in X_y because of the factorisation property $\phi \circ g = f \circ \psi$, and the usual rigidity lemma. More precisely, approximate V_t in $\mathcal{C}(X)$ by a sequence V_{t_n}, such that $V_{t_n} \subset X_{y_n} \subset f^{-1}(U)$, for some Stein-or affine in the algebraic category-neighborhood U of y in Y. This is possible because the generic $V_{t'}$ is contained in a fibre of f. Then $V_t \subset X_y$, by an easy argument, based on the fact that f is holomorphic on $f^{-1}(U)$.

So we get that the generic member, hence every member, of the family T is contained in some fibre of f (here the notion of fibre of f is the usual Chow-scheme theoretic one, defined in Section 1.1). Thus every T-chain meeting X_y is contained in X_y. Because X is T-connected, X is contained in X_y and f is constant, and so not of general type, as assumed. Contradiction. □

We can now establish the following often useful characterisations of the core.

THEOREM 3.7.— *Let $f : X \dashrightarrow Y$ be a special fibration, with $X \in \mathcal{C}$ normal, and such that for any special fibration $g : X \dashrightarrow Z$, there exists a factorisation $\phi : Z \dashrightarrow Y$ such that $f = \phi \circ g$. Then f is the core of X. In particular, if $f : X \dashrightarrow Y$ is a special fibration of general type, then it is the core of X.*

Proof.— First, because c_X is special, there exists a factorisation $\psi : C(X) \dashrightarrow Y$ such that $f = \psi \circ c_X$. Let F be a c_X-general fibre of c_X. By the existence of ψ, it is contained in some fibre G of f. But G is special, because f is. By the defining property of c_X, we have the reverse inclusion $G \subset F$.

The last assertion follows now from Theorem 2.6, because f being of general type, for any special fibration g, Theorem 2.6 shows that the factorisation ϕ exists. □

The core can be constructed in a relative setting, as well, by a simple application of Theorem 2.7 in [Ca04].

THEOREM 3.8.— *Let $f : X \dashrightarrow Y$, with $X \in \mathcal{C}$ normal. There exists a unique factorisation $f = g_f \circ c_f$ by two fibrations $c_f : X \dashrightarrow C(f)$ and $g_f : C(f) \dashrightarrow Y$ such that, for $y \in Y$ general, the restriction $c_f : X_y \dashrightarrow C(f)_y$ is the core of X_y. We call the factorisation $f = g_f \circ c_f$ the core of f.*

3.2. Functoriality properties.

Notice that c_X is not, in general, a bimeromorphic invariant. But it is easily seen from 3.3 to be so if X is smooth. Indeed, if $m : Y \dashrightarrow X$ is bimeromorphic then $c_Y = c_X \circ m$. When X is smooth, we denote by $\mathrm{ess}(X)$ the dimension of $C(X)$, and call it the *essential dimension* of X. Thus $\mathrm{ess}(X) = 0$ iff X is special, and $\mathrm{ess}(X) = d = \dim(X)$ iff X is of general type, by Theorem 5.5 below. In the first case, the core is the constant map and in the second one, it is the identity map.

THEOREM 3.9.— *Let $X \in \mathcal{C}$ be normal. Let $c_X : X \dashrightarrow C(X)$ be its core, and let $a_X : X \dashrightarrow A(X)$ be its algebraic reduction. Then there exists a factorisation $b_X : A(X) \dashrightarrow C(X)$ of c_X such that $c_X = b_X \circ a_X$. In particular, $C(X)$ is always Moishezon.*

Proof. — By Theorem 2.4, the fibres of a_X are special, hence contained in the fibres of c_X. □

PROPOSITION 3.10. — *Let $X \in \mathcal{C}$ be normal. Let $h : Z \dashrightarrow X$ be any meromorphic map with $Z \in \mathcal{C}$ smooth. Assume $h(Z)$ meets some c_X-general fibre of c_X. Then there exists then a natural meromorphic map $c_h : C(Z) \dashrightarrow C(X)$ such that $c_h \circ c_Z = c_X \circ h$.*

Proof. — By the assumption, if $z \in Z$ is general, its image $h(z)$ in X belongs to a c_X-general fibre F_z of c_X. The fibre of c_Z through z is special and thus so is its image V_z by h. Since V_z meets F_z, it is contained in F_z, by property [2.] in Theorem 3.3. Hence the existence of c_h. □

COROLLARY 3.11. — *Let $h : Z \dashrightarrow X$ be as in proposition 3.10 above. Then c_h as above exists in the following cases.*

1. *The map $c_X \circ h : Z \dashrightarrow C(X)$ is surjective.*
2. *X is smooth and $Z \subset X$ is the general member of a family $(Z_t)_{t \in T}$ of submanifolds of X such that the varieties $c_X(Z_t)$ cover $C(X)$.*
3. *X is smooth and $Z \subset X$ is a general fibre of $\psi \circ c_X$, where $\psi : C(X) \dashrightarrow Y$ is any fibration. In this case, c_Z is simply the restriction of c_X to Z.*

Let us give some easy examples in which c_X can be described.

PROPOSITION 3.12. — *Let $X \in \mathcal{C}$ be smooth and assume that $f : X \dashrightarrow Y$ is a special fibration of general type. Then $f = c_X$. In particular, there is at most one fibration both special and of general type on X.*

Proof. — Because f is special, there is a factorisation $g : Y \dashrightarrow C(X)$ such that $g \circ f = c_X$. Indeed, a general fibre F of f is special and meets some general fibre C of c_X. Thus $F \subset C$. But f is of general type, and so by Theorem 2.6, there exists a factorisation $h : C(X) \dashrightarrow Y$ such that $f = h \circ c_X$. Thus $f = c_X$, as claimed. □

Remark 3.13. — We shall later prove (see Section 5.8), that c_X is the fibration of general type. So that c_X is the unique fibration of domain X both special and of general type.

COROLLARY 3.14. — *Let $X \in \mathcal{C}$ be a manifold of general type. Then c_X is the identity map of X and so* $\mathrm{ess}(X) = \dim(X)$.

Proof. — Indeed, id_X is then special and of general type. Apply then 3.12. □

Remark 3.15. — We shall see later in Theorem 5.5 that the converse also holds true, that is if $\mathrm{ess}(X) = \dim(X) > 0$, then X is of general type.

COROLLARY 3.16. — *Let $X \in \mathcal{C}$ be a manifold and let $n := \dim(X)$. Then $\mathrm{ess}(X) = n - 1$ in the following two cases.*

a. *If $\kappa(X) = n - 1$ and the Iitaka-Moishezon fibration J_X of X is a fibration of general type. Then $c_X = J_X$.*

b. *The rational quotient $R(X)$ of X is of dimension $n-1$ and of general type. (See 3.23 below for this notion)*

Proof. — Indeed, in case (a.) (resp. (b.)), the generic fibre of J_X (resp. r_X) is an elliptic (resp. a rational) curve. The fibration J_X (resp. r_X) is thus special. The other conditions imply that it is also of general type. It is thus the core of X. In particular, $\mathrm{ess}(X) = n - 1$. □

Remark 3.17. — We shall see later in Theorem 5.7, as a consequence of orbifold additivity theorems, that the converse also holds true, that is, if $\mathrm{ess}(X) = \dim(X) - 1 > 0$, then X is of the type (a) or (b). For the case $\mathrm{ess}(X) = n - 2$ see below 3.36.

COROLLARY 3.18. — *Let $f : X \dashrightarrow C$ be a special fibration, with $X \in \mathcal{C}$ smooth and C a curve. Then, either f is of general type and $f = c_X$, or f is not of general type and X is special.*

Proof. — In the first case, the claim follows from Proposition 3.12. In the second, it follows from the fact that if $g : X \dashrightarrow Z$ is a fibration of general type, then there exists by 2.6 a factorisation $h : C \dashrightarrow Z$ such that $g = h \circ f$. But C is curve, and f is not of general type. Thus Z is a point, and g is not of general type. Contradiction. □

3.3. Rationally connected manifolds.

We now come to our first basic example of special manifolds, the rationally connected ones. We recall first their definition and some properties. Recall ([Ca92],[K-M-M92]) that an irreducible compact complex space X

is said to be *rationally connected* if any two generic points of X are contained in a rational chain of X that is a connected projective curve of X, all irreducible components of which are rational, possibly singular, curves.

Examples of rationally connected manifolds include unirational, Fano manifolds and twistor spaces. This property is bimeromorphically stable among manifolds, but not among varieties (the cone over a projective manifold which is not rationally connected will again provide such an example). Of course, the above definition can be given for algebraic varieties defined over arbitrary fields. We refer to [K-M-M92] for some of the fundamental properties of this class of manifolds.

Rational connectedness has a slightly different characterisation, by the following fundamental result ([G-H-S01]).

THEOREM 3.19 ([G-H-S01]). — *Any fibration $f : X \to C$ over a projective curve C with X smooth and projective, and generic fibre rationally connected has a holomorphic section.*

DEFINITION ([CA95]) 3.20. — *Let $X \in \mathcal{C}$ be irreducible. We say that X is rationally generated if for any surjective meromorphic map $f : X \dashrightarrow Y$, Y is uniruled.*

Any rationally connected $X \in \mathcal{C}$ is thus rationally generated. But, conversely, we have the following.

THEOREM 3.21. — *Let $X \in \mathcal{C}$ be rationally generated. Then X is rationally connected.*

Proof. — By induction on the dimension. The complex space X is obviously uniruled. Let $r_X : X \dashrightarrow R(X)$ be the rational quotient of X (see Theorem 3.23). Then $R(X)$ is also rationally generated. By induction, it is rationally connected. Recall from [Ca81] that if any two points of $X \in \mathcal{C}$ can be joined by a chain of curves then X is Moishezon. So $R(X)$ is in particular Moishezon. Thus X too is Moishezon by [Ca85], which says among others that $X \in \mathcal{C}$ is Moishezon if there is a fibration $u : X \dashrightarrow Y$ with Y and the generic fibre F of u Moishezon and such that F has $q(F) = 0$. We can assume X to be projective, by the bimeromorphic invariance of the rational generatedness. But then the conclusion follows easily from [G-H-S01], which allows to lift rational curves from $R(X)$ to X. □

THEOREM 3.22. — *Let $X \in \mathcal{C}$ be rationally connected and smooth. Then X is special.*

Proof. — Let $c_X : X \dashrightarrow C(X)$ be the core of X. Assume it is not the constant map. Let F be a c_X-general fibre of c_X. Because X is rationally connected, some rational curve in X meets F, but is not contained in F. Contradiction. □

Notice that the smoothness of X is essential, as shown again by the cone over a projective manifold of general type.

3.4. The rational quotient and the core.

We now turn to the study of the *rational quotient* of X from the point of view of special varieties. The rational quotient of $X \in \mathcal{C}$ was introduced in [Ca92] as an application of T-quotients. It was also independently constructed in [K-M-M92], under the name of "maximal rationally connected fibration" (M.R.C for short), by a different method based on their "glueing lemma" for rational curves, in the algebraic context.

THEOREM 3.23. — *Let $X \in \mathcal{C}$ be normal. There exists a unique meromorphic fibration $r_X : X \dashrightarrow R(X)$ called the rational quotient of X such that the following holds.*

1. *The general fibre of r_X is rationally connected.*
2. *The general fibre of r_X contains any rational curve of X that it meets.*

As usual, r_X is almost holomorphic.

The proof is given in the separate appendix [Ca04]. Notice that, by Proposition 2.8 in [Ca04], the rational quotient also exists in relative version.

COROLLARY 3.24. — *Let $X \in \mathcal{C}$ be smooth. The rational quotient r_X of X is then a special fibration. There exists a factorisation $(cr)_X : R(X) \dashrightarrow C(X)$ such that $c_X = (cr)_X \circ r_X$.*

Proof. — This is simply because X, and so the generic fibre of r_X is special, by Theorem 3.22. This shows the first assertion. Notice the second is obvious, and does not require X to be smooth. □

We have the following easy property.

PROPOSITION 3.25. — *Let $X \in \mathcal{C}$ be smooth and let $f : X \dashrightarrow Y$ be a surjective meromorphic map, with $Y \in \mathcal{C}$ normal. Then f induces functorial maps $f_* : R(X) \dashrightarrow R(Y)$ and $f_* : C(X) \dashrightarrow C(Y)$.*

In the above proposition, taking $f := r_X$, we get a natural map $(r_X)_* : C(X) \dashrightarrow C(R(X))$. For the rational quotient, we have a particular property not valid for arbitrary special fibrations.

THEOREM 3.26. — *Let X be smooth and Moishezon. Let $r_X : X \dashrightarrow R(X)$ be the rational quotient of X and let $c_{R(X)} : R(X) \dashrightarrow C(R(X))$ be the core of $R(X)$. Then $(r_X)_* : C(X) \dashrightarrow C(R(X))$ is bimeromorphic or, equivalently, $c_{R(X)} \circ r_X : X \dashrightarrow C(R(X))$ is the core of X. In particular, $C(R(X)) = C(X)$.*

Remark 3.27. — The hypothesis that X is Moishezon can certainly be weakened to $X \in \mathcal{C}$. For this, it is sufficient to make the same weakening in the hypothesis for G in the Lemma 3.29 below.

Proof. — We have natural fibrations $\phi : R(X) \dashrightarrow C(X)$ and $\psi = (r_X)_* : C(X) \dashrightarrow C(R(X))$ defined above. We need to show that ψ is bimeromorphic, or that the general fibre F of $c_{R(X)} \circ r_X : X \dashrightarrow C(R(X))$ is special.

Observe that we have, by restricting r_X to F, a map $r_X : F \dashrightarrow G$, where $G = r_X(F)$ is the corresponding fibre of $c_{R(X)}$. Thus F is fibred over G, which is special, with fibres which are generically rationally connected. The claim thus follows from the next proposition. □

PROPOSITION 3.28. — *Let $f : F \to G$ be a fibration with $F \in \mathcal{C}$ smooth such that G is Moishezon and special, and the generic fibre of f is rationally connected. Then F is special.*

Proof. — Let, if any, $g : F \to H$ be an admissible holomorphic fibration of general type. Since f has special fibres, there exists by Theorem 2.6 a factorisation $\phi : G \to H$ of $g = \phi \circ f$. But now by Lemmas 3.29 and 3.30 below, we see that $\Delta(g) = \Delta(\phi)$. Thus ϕ is of general type, too. But this contradicts G being special. Such a g thus does not exist, and X is special. □

LEMMA 3.29. — *Let $f : F \to G$ be a fibration with generic fibres rationally connected, F smooth and G Moishezon. Then f is multiplicity free, that is $\Delta(f)$ is empty.*

Proof. — We may assume that G is projective. The claim then follows immediately from [G-H-S01], and is actually the most difficult part of the proof, by considering the restriction of f over a very ample curve of G meeting transversally any irreducible component of $\Delta(f)$. □

The proof of the following is immediate from the definition of multiplicities and the computation of the base orbifold divisor of a composed fibration, as in section 1.6.

LEMMA 3.30.— *Let $f : F \to G$ and $\phi : G \to H$ be fibrations. Assume that f is multiplicity free. Then $\Delta(\phi \circ f) = \Delta(\phi)$.*

3.5. Surfaces.

We can describe the core of a surface as follows, in terms of its rational quotient or Iitaka-Moishezon fibration. This will be extended to threefolds in the next section. A description in arbitrary dimension will be given in Section 6.5.

Recall that for any compact connected complex manifold X with $\kappa(X) \geqslant 0$, we denote by $J_X : X \dashrightarrow J(X)$ its Iitaka-Moishezon fibration. Let also $\kappa'(X) := \kappa(J(X), J_X)$. Obviously $\kappa(X) \geqslant \kappa'(X) \geqslant -\infty$.

THEOREM 3.31.— *Let X be a compact Kähler smooth surface. Then its core c_X is described as follows.*

1. *If $\kappa(X) = 2$, then $c_X = id_X$, and* $\mathrm{ess}(X) = 2$.
2. *If $\kappa(X) = \kappa'(X) = 1$, then $c_X = J_X$ and* $\mathrm{ess}(X) = 1$.
3. *If $\kappa(X) = 1 > \kappa'(X)$, then X is special.*
4. *If $\kappa(X) = 0$, then X is special.*
5. *If $\kappa(X) = -\infty$ and $q(X) \geqslant 2$, then $c_X = r_X$ and* $\mathrm{ess}(X) = 1$.
6. *If $\kappa(X) = -\infty$ and $q(X) \leqslant 1$, then X is special.*

Proof.— If $\kappa(X) = 2$, the claim is given by 3.14. If $\kappa(X) = 1$, the fibration $J_X : X \dashrightarrow C = J(X)$ is special, and the claim follows from 3.18. If $\kappa(X) = 0$, X is special from the facts just recalled above. If $\kappa(X) = -\infty$, from the classification of surfaces, X is bimeromorphic to $\mathbb{P}^1 \times C$, C a curve with $g(C) = q(X)$, and r_X is the projection to C if $q(X) > 0$, and the constant map if $q(X) = 0$. The claims are then obvious. □

Recall that a group G is said to be almost or virtually abelian if it has a finite index subgroup which is abelian.

COROLLARY 3.32.— *Let X be a compact Kähler surface. Either X is special and c_X is the constant map, or $\kappa(X) \geqslant 1$ and $c_X = J_X$, the Itaka* *fibration, or* $\kappa(X) = -\infty$*, and* $c_X = r_X$*, the rational quotient of* X*. One can compute* $\mathrm{ess}(X)$ *as follows.*

1. $\mathrm{ess}(X) = 2$ *iff* $\kappa(X) = 2$.
2. $\mathrm{ess}(X) = 1$ *iff* $\kappa(X) \in \{1, -\infty\}$ *and* $\pi_1(X)$ *is not virtually abelian.*
3. $\mathrm{ess}(X) = 0$ *iff* $\kappa(X) \leqslant 1$ *and* $\pi_1(X)$ *is virtually abelian.*

Proof. — All claims are deduced immediately from 3.31, except for the ones concerning the fundamental group, when $\kappa(X) \leqslant 1$. If $\kappa(X) = -\infty$, then $\pi_1(X) \cong \pi_1(C)$, with the notations of the proof of 3.31. The assertion is obvious. If $\kappa(X) = 0$, we know that $\pi_1(X)$ is almost abelian from classification theory. If $\kappa(X) = 1$, the assertion follows from Lemma 3.34, applied to J_X. □

COROLLARY 3.33. — *A compact Kähler surface* X *is special if and only if it has a finite étale cover which is bimeromorphic to one of the following surfaces.*

1. $\mathbb{P}_2(\mathbb{C})$.
2. $\mathbb{P}_1(\mathbb{C}) \times E$*, with* E *elliptic.*
3. *K3, or Abelian.*
4. *Elliptic over a curve* C *with* m *multiple fibres,* C *either rational and then* $m \leqslant 2$*, or elliptic and then* $m = 0$.

Proof. — The surfaces listed above are special, by 3.31 above. Thus so are their undercovers. Conversely, if X is special, it has a finite étale cover in the preceding list. This is clear if $\kappa(X) \leqslant 0$, by classification, and from the next Lemma 3.34 if $\kappa(X) = 1$. □

LEMMA 3.34. — *Let* $f : X \to C$ *be a relatively minimal elliptic fibration on the compact Kähler surface* X*.*

1. *Let* $f^*(c) := \sum_{j \in J} m_j D_j$ *be any scheme-theoretic fibre* X_c *of* f*. Then, its multiplicity* $m(c, f) := inf\{m_j\}$ *is also equal to* $m^+(c, f) := gcd\{m_j\}$*.*
2. *There exists a finite étale cover* $u : X' \to X$ *such that if* $v \circ f' = f \circ u$ *is the Stein factorisation of* $f \circ u$*, with* $f' : X' \to C'$ *connected and* $v : C' \to C$ *finite, then* f' *has no multiple fibre if* $g(C') \geqslant 1$*, and at most 2 multiple fibres of coprime multiplicities if* C' *is rational.*
3. *Moreover,* $g(C') = \kappa(C', f') = \kappa(C, f)$ *in the preceding situation.*

4. *X is special if and only if $\pi_1(X)$ is almost abelian.*

Proof. — (1) follows from Kodaira's classification of singular fibres of elliptic fibrations (see [B-P-V84] Chap. V.7). This equality actually also follows from an elementary argument in the more general case of fibrations with generic fibre a complex torus.

Assertion (2) follows from [Ca98] and [Na87]. Indeed, [Na87] shows that if a curve C with points $a_1, ..., a_m$, affected with multiplicities $n_1, ..., n_m$ is given, there exists a cover C' of C ramified above the a_i's only, each point above any a_i having ramification exactly n_i. The only exception is when $C = \mathbb{P}^1, m = 1, 2$, and when $n_1 \neq n_2$ if $m = 2$. In [Ca98], it is shown (it is a simple computation) that the base change over C' leads to the sought after étale cover $u : X' \to X$.

Then property (3) follows from Theorem 1.8 for the second equality, and from the fact that $m = 0$ if $g(C') \geqslant 1$.

We show (4). If X is special then $\kappa(C, f) \leqslant 0$, so that C' is rational or elliptic. We apply [Ca98], which shows that the natural sequence of maps

$$\pi_1(F') \to \pi_1(X') \to \pi_1(C') \to 1$$

is exact, with F' a generic fibre of f', so that F' is an elliptic curve, and $\pi_1(F') \cong \mathbb{Z}^{\oplus 2}$. Thus $\pi_1(X')$ is almost abelian if X' is special, which is true if so is X, because f' is special and $\kappa(C', f') = \kappa(C, f)$.

Conversely, assume that $\pi_1(X)$ is almost abelian. Then so is $\pi_1(C')$, and C' is either rational or elliptic. Thus $\kappa(C, f) \leqslant 0$, and X is special by 3.31. □

3.6. Higher Kodaira dimensions.

We shall define higher Kodaira dimensions (to be generalized in 6.4 below) of any connected manifold $X \in \mathcal{C}$ as follows. This works for compact connected manifolds as well.

The first Kodaira dimension of X is the usual one, $\kappa(X)$. If $\kappa(X) = -\infty$, the second Kodaira dimension of X is not defined. Otherwise, $\kappa(X) \geqslant 0$, and $J_X : X \dashrightarrow J(X)$, the Iitaka-Moishezon fibration of X, is defined. Let then

$$\kappa'(X) := \kappa(J(X), J_X).$$

We have

$$\kappa(X) := \dim(J(X)) \geqslant \kappa'(X) \geqslant \kappa(J(X)) \geqslant -\infty.$$

If $\kappa'(X) = -\infty$, the next Kodaira dimension $\kappa''(X)$ is not defined. Otherwise $\kappa'(X) \geqslant 0$. Let then

$$J_X^{(0,1)} : J(X) \dashrightarrow J'(X)$$

be the Iitaka fibration defined on $J(X)$ by the $\mathbb{Q}$-divisor $K_{J(X)} + \Delta(J_X)$, for any admissible model of the fibration J_X. Define

$$J'_X := J_X^{(0,1)} \circ J_X : X \dashrightarrow J'(X).$$

Define next

$$\kappa''(X) := \kappa(J'(X), J'_X) \geqslant \kappa(J'(X)) \geqslant -\infty.$$

Of course, we have

$$\kappa(X) \geqslant \kappa'(X) \geqslant \kappa''(X) \geqslant -\infty,$$

for the invariants so defined. Continuing inductively, we can define a decreasing sequence of invariants

$$\kappa(X) \geqslant \kappa'(X) \geqslant \ldots \geqslant \kappa^{(r)}(X) \geqslant -\infty,$$

and iterated orbifold Iitaka fibrations

$$J_X^{(r)} : X \dashrightarrow J^{(r)}(X).$$

If the sequence is defined til $J_X^{(r)}$, define

$$\kappa^{(r+1)}(X) := \kappa(J^{(r)}(X), J_X^{(r)}),$$

and if this is nonnegative, define $J_X^{(r,r+1)}$ as being the Iitaka fibration defined by the $\mathbb{Q}$-divisor $K_{J^{(r)}((X))} + \Delta(J_X^{(r)})$, for any admissible model of the fibration $J_X^{(r)}$. We thus have fibrations

$$J_X^{(r-1,r)} : J^{(r-1)}(X) \dashrightarrow J^{(r)}(X)$$

such that $J_X^{(r)} = J_X^{(r-1,r)} \circ J_X^{(r-1)}$.

Observe also that $\kappa(F_r, J_X^{(r)}) = 0$, if F_r is a general fibre of $J_X^{(r-1,r)}$, by the standard property of Iitaka fibrations.

This sequence stops at the first term, if any, equal to $-\infty$, and is stationary if any two terms $\kappa^{(r)}(X) = \kappa^{(r+1)}(X)$ are equal, and nonnegative, necessarily. This happens clearly if and only if the corresponding map $J_X^{(r)}$ is of general type.

The following is easily shown by induction on r.

Proposition 3.35. — *The sequence of higher Kodaira dimensions is invariant under bimeromorphic maps and finite étale covers.*

We shall later (in 6.4) extend these notions, and even conjecture that these higher Kodaira dimensions are invariant under deformation, for X Kähler.

As an illustration for the introduction of these invariants, we show the following.

Proposition 3.36. — *Let $X \in \mathcal{C}$ be smooth of dimension $n \geqslant 2$. Then* $\mathrm{ess}(X) = n-2$ *in each of the following cases (a-e). Moreover, the core c_X and its generic fibre F, a special surface, are described as follows.*

a. *$\kappa(X) = n-1$ and $\kappa'(X) = \kappa''(X) = n-2$. Then $c_X = J'_X$, $\kappa(F) = 1$ and $\kappa'(F) = 0$.*

b. *$\kappa(X) = n-1$, $\kappa'(X) = -\infty$ and there exists $r : J(X) \dashrightarrow Z$ with* $\dim(Z) = n-1$ *such that $f \circ J_X$ is of general type. Then $\kappa(F) = 1$, and $\kappa'(F) = -\infty$.*

c. *$\kappa(X) = \kappa'(X) = n-2$. Then $c_X = J_X$, and $\kappa(F) = 0$.*

d. *$R(X)$, the rational quotient of X, is of general type, and of dimension $n-2$. Then F is a rational surface.*

e. *$R(X)$ has dimension $n-1$, and $\kappa(R(X)) = \kappa'(R(X)) = n-2$. Then F is birationally elliptic ruled.*

Proof. — Case (a). Indeed, the fibre of J'_X is special, because F has an elliptic fibration $J : F \to C$ with $\kappa(C, J) = 0$, so the assertion follows from 3.31. By assumption, J'_X is of general type, because $\kappa'(X) = \kappa''(X)$. To show that $\kappa(F) = 1$, use the easy addition theorem, applied to X and J'_X: it says that $n-1 = \kappa(X) \leqslant \kappa(F) + \dim(J'(X)) = \kappa(F) + n - 2 \leqslant \dim(F) + n - 2 = n-1$.

We shall skip the proofs of the other cases, which are easier or similar. □

Remark 3.37.— We shall also nearly show the converse, as a consequence of additivity theorems, in Section 6.5. Actually, the converse holds under the general additivity conjecture.

The formulation of case (b) is unnatural. A natural formulation rests on the notion of rational quotient for orbifolds. See 6.5.

3.7. Threefolds.

We shall describe the core of a compact Kähler threefold. For this we shall need Theorem 5.1 shown later in Section 5, which says that X is special if $\kappa(X) = 0$, in all dimensions. (See Section 6.5 for the n-dimensional versions of the next two theorems).

THEOREM 3.38.— *Let $X \in \mathcal{C}$ be a nonspecial threefold. The core c_X of X is a fibration of general type. Moreover, one can describe c_X, its generic fibre F, and* ess(X) *as follows.*

1. ess$(X) = 3$ *iff* $\kappa(X) = 3$. *Then* $c_X = id_X$.
2. ess$(X) = 2$ *in the following two cases.*
   a. $\kappa(X) = \kappa'(X) = 2$. *Then* $c_X = J_X$ *is an elliptic fibration of general type.*
   b. $\kappa(X) = -\infty$, *$R(X)$ is a surface of general type, and $c_X = r_X$ is a $\mathbb{P}^1$-fibration over $R(X)$.*
3. ess$(X) = 1$ *iff one of the following cases occurs.*
   a. $\kappa(X) = 2, \kappa'(X) = \kappa''(X) = 1$. *Then $c_X = J'_X$ is a fibration of general type onto a curve, with F a special surface with $\kappa(F) = 1$ and $\kappa'(F) = 0$.*
   b. $\kappa(X) = 2$, $\kappa'(X) = -\infty$ *and X nonspecial. Then c_X is a fibration of general type onto a curve with $\kappa(F) = 1$ and $\kappa'(X) = -\infty$.*
   c. $\kappa(X) = \kappa'(X) = 1$. *Then $c_X = J_X$ is a fibration of general type onto a curve with $\kappa(F) = 0$.*
   d. *$c_X = r_X$ is a fibration onto a curve of general type, with F a rational surface.*

e. $c_X = J_{R(X)} \circ r_X$ *is a fibration of general type over a curve with* F *a birationally ruled elliptic surface.*

We now give a very rough list of the special threefolds.

THEOREM 3.39. — *Any special threefold* $X \in \mathcal{C}$ *is one of the following.*

1. $\kappa(X) = 2$.

   a. $\kappa'(X) = 1 > \kappa''(X)$ *and* J'_X *is a non-general type fibration over a curve with fibre* F *a special surface with* $\kappa(F) = 1 > \kappa'(F) \in \{0, -\infty\}$.

   b. $\kappa'(X) = 0$ *and* J_X *is an elliptic fibration with a klt orbifold base a normal surface with torsion canonical bundle* $K_S + \Delta$*, the log-Enriques case.*

   c. $\kappa'(X) = -\infty$ *and* X *has either a non-general type fibration over a curve with generic fibre a special surface* F *with* $\kappa(F) = 1, \kappa'(F) = -\infty$*, or an elliptic fibration with base orbifold a klt normal surface with Picard number one, and log-Del Pezzo, that is* $-(K_S + \Delta)$ *is ample.*

2. $\kappa(X) = 1$ *and* J_X *is a non-general type fibration over a curve with generic fibre* F *a surface with* $\kappa(F) = 0$.

3. $\kappa(X) = 0$.

4. $\kappa(X) = -\infty$ *and* X *is either rationally connected, or a fibration over an elliptic curve with generic fibre a rational surface, or a* $\mathbb{P}^1$*-fibration over a special surface* S *with* $\kappa(S) \geqslant 0$*. The case where* X *is simple non-Kummer conjecturally does not exist, but strictly speaking additionally belongs to the last part (4) of the above list, because* $\kappa(X) \leqslant 0$*, then.*

Proof. — We shall prove both results 3.38 and 3.39 at the same time.

The case where $\kappa(X) = -\infty$ is clear from the Section 3.4 above, because X is then uniruled if X is projective by [Mi88], and by [C-P00] otherwise if X is not simple.

So we proceed case-by-case, assuming $\kappa(X) \geqslant 0$.

If $\kappa(X) = 0$, we are done by Theorem 5.1. If X is non-projective, we could also have applied [C-P00], Theorem 8.1, which says that if X has a nonzero holomorphic 2-form, it is covered by either a torus, or by the product of an elliptic curve and a $K3$-surface. Thus X is special in this case, too. The existence of a nonzero 2-form when X is non-projective is a famous result of Kodaira.

We now classify the cases occuring according to the pairs $(\kappa(X), \kappa'(X))$ with $\kappa(X) \geqslant 1$. When the two terms are equal and positive, we conclude from 3.12 that J_X is the core, and of general type. If $\kappa(X) = 1$, and $\kappa'(X) \leqslant 0$, we conclude from 3.18 that X is special. We are thus left with the cases where $\kappa(X) = 2$ and $\kappa'(X) \leqslant 1$. We thus now assume that $\kappa(X) = 2$.

Assume first that $\kappa'(X) = 1$. Consider the map $J'_X : X \dashrightarrow J'(X)$, using the notations of 3.6. It has general special fibres, because if F is its general fibre, and $G{:=}J_X(F)$, then the restriction to F of J_X defines $J' : F \dashrightarrow G$, which has generic fibres elliptic curves, while G is a curve and, by the definition of $J_X^{0,1}$, we have $\kappa(G, J') = 0$. We conclude from 3.18 that F is a special surface. The easy addition theorem shows that $\kappa(F) = 1$.

If $\kappa''(X) = 1$, J'_X is a fibration both special and of general type. So we conclude from 3.12 that it is the core of X.

Otherwise, if $\kappa''(X) \leqslant 0$, X is special. We are thus done with this case ($\mathrm{ess}(X) = 0$, or 1).

We are now left with the more difficult case when $\kappa'(X) \leqslant 0$. Assume that X is not special. Let $f : X \dashrightarrow Y$ be a fibration of general type. The fibration J_X being special, we get from 2.6 a factorisation $\phi : J(X) \dashrightarrow Y$ of $f = \phi \circ J_X$. Thus Y has to be a curve.

To conclude the proof we thus just need to show that this does not happen if $\kappa'(X) = 0$, because of the Minimal Model Program applied to a klt surface orbifold, as described below. The assertion we need follows from Proposition 3.41 below. But we need first some definitions for its statement.

3.8. Orbifold surfaces.

A *surface orbifold* will be a klt pair (S, Δ), with S a normal projective surface, and Δ an orbifold divisor. We refer to [K-M98], §3.7 and §4.1 for the notions of canonical bundle and intersection numbers in this context, peculiar to surfaces.

If $g : S \to C$ is a holomorphic fibration, we define the *orbifold base* of $g : (S/\Delta) \to C$ as the pair $(C, \Delta_g := \Delta(g, \Delta))$, where $\Delta(g, \Delta)$ is the Weil $\mathbb{Q}$-divisor on C defined as in 1.29.

Remark 3.40. — As in 1.6, we see that if $h : X \to S$ is a fibration with $\Delta = \Delta(h)$, then $\Delta_g = \Delta(g \circ h)$ on a suitable model of h.

We define then as usual the canonical bundle and Kodaira dimension of (C/Δ_g) and (S/Δ) (by definition, $K_S + \Delta$ is supposed to be $\mathbb{Q}$-Cartier).

We then say that g *is of general type* if $\kappa(C/\Delta_g) = 1$, and that (S/Δ) is *special* if $\kappa(S/\Delta) < 2$, and if there is no holomorphic fibration of general type $g : (S/\Delta) \to C$ onto a curve.

PROPOSITION 3.41. — *Let (S/Δ) be a surface orbifold. If $\kappa(S/\Delta) = 0$, then (S/Δ) is special.*

Proof. — Assume there exists a general type fibration g on (S/Δ). We shall show that $\kappa(S/\Delta) \neq 0$.

We thus apply the MMP to our initial pair (S/Δ). This produces a sequence of elementary contractions of the form $k : (S/\Delta) \to (S'/\Delta')$, with (S'/Δ') still a klt pair, such that after at most $\rho(S) - 1$ steps, one gets for the final pair, denoted also (S'/Δ')), one of the three basic cases:

(1) $K_{(S'/\Delta')}$ is nef,

(2) There is a fibration $g' : S' \to C'$ onto a curve such that $-K_{(S'/\Delta')}$ is g'-ample, and $\rho(S') = 2$,

(3) -$K_{(S'/\Delta')}$ is ample, and $\rho(S') = 1$ ("log-Del Pezzo" case).

Notice that at each step, the Kodaira dimension of the pair (S/Δ) is preserved, and that the curve being contracted is rational smooth (because S itself is klt). We refer to [K-M98] and [F-M94] for the existence and usual properties of these reduction steps.

In cases (2) and (3), we have $\kappa(S'/\Delta') = \kappa(S/\Delta) = -\infty$. So these cases do not occur, because we assumed that $\kappa(S/\Delta) = 0$. Thus $K_{(S'/\Delta')}$ is nef.

We claim that $D' := K_{S'} + \Delta' \equiv 0$. Indeed, we have $(D')^2 = 0$, otherwise $\kappa(S/\Delta) = 2$. From [F-M-K92], (11.3), we get the claim (their theorem asserts that $\kappa(S'/\Delta')$ is equal to the numerical Kodaira dimension of (S'/Δ')).

The rest of the proof rests on the following two Lemmas 3.42 and 3.43. The first one will be proved at the end of this section.

LEMMA 3.42. — *Let $g : (S/\Delta) \to C$ be a fibration of general type, with (S/Δ) klt. Let E be a rational curve mapped surjectively onto C by g. Then $\Delta \cdot E > 2$.*

COROLLARY 3.43. — *Let $k : (S/\Delta) \to (S'/\Delta')$ be the contraction of an irreducible smooth rational curve E, with S and S' klt surfaces, $(K_S + \Delta) \cdot E \leqslant 0$, and $\Delta' := k_*(\Delta)$. If $g : (S/\Delta) \to C$ is a fibration of general type, then there exists $g' : (S'/\Delta') \to C$ such that $g = g' \circ k$. Moreover, g' is still of general type, and $K_S + \Delta = k^*(K_{S'} + \Delta')$ if $(K_S + \Delta) \cdot E = 0$.*

Proof. — The second assertion is clear, if the first one is. This is because $m(c, g', \Delta) \leqslant m(c, g', \Delta')$, since in the definition of the left hand side of the inequality, the infimum is taken over a smaller subset.

The first assertion is clear also if $g(C) \geqslant 1$, because the rational curve E contracted by k cannot be mapped surjectively to C by g.

We shall show that this also cannot happen when C is rational, because g is of general type. Actually, as the proof shows, the condition $\kappa(C, \Delta_g) \geqslant 0$ is sufficient, even. We use the following numerical conditions:

(a) $E^2 \leqslant 0$ (E exceptional),

(b) $(K_S + E) \cdot E = -2$ (E rational, smooth),

(c) $(K_S + \Delta) \cdot E \leqslant 0$.

So we assume by contradiction that E is mapped onto C by g. We then get

$$\Delta \cdot E \leqslant -K_S \cdot E = 2 + E^2 \leqslant 2,$$

which contradicts 3.42. □

We now complete the proof of 3.41.

We then apply the Minimal Model Program to S', but relative to $K_{S'}$ (ie. we take $\Delta' = 0$. Notice that the pair $(S', 0)$ remains klt, because Δ' is effective) At each contraction step again only smooth rational $K_{S'}$-negative curves are contracted, which have zero intersection number with $(K_{S'} + \Delta')$, so that this $\mathbb{Q}$-Weil divisor also remains numerically trivial at each of these contraction steps. Let (S'', Δ'') be the resulting pair. It has, by 3.43, all properties of (S', Δ'). In particular, $D'' := (K_{S''} + \Delta'') \equiv 0$, and $g'' : (X''/\Delta'') \to C$ is of general type. Put $K := K_{S''}$. Assume first that $\kappa(S'') \geqslant 0$. In addition to the above properties, K is nef. By our assumptions, the generic fibre of g'' is elliptic, and Δ'' is "vertical", that is contained in fibres of g''.

Thus $K^2 = 0$, $D'' \cdot \Delta'' = 0 = (D'')^2$. Thus $(\Delta'')^2 = 0$. And so Δ'' is a union of complete fibres of g'', by Zariski's Lemma. There exists thus an orbifold divisor δ'' on C such that $\Delta'' = (g'')^*(\delta'')$. Because $\kappa(S'') \geqslant 0$, and $\kappa(C/\delta'') = 1$, we easily get that $\kappa(S''/\Delta'') \geqslant 1$ (see Lemma 4.9 below, for example). A contradiction.

We now treat the remaining case in which $\kappa(S) = -\infty$. Now $-\Delta''$ is g''-ample, and every fibre of g'' has an irreducible reduction. As above, $\Delta'' = (g'')^*(\delta'')$, for an orbifold structure δ'' of general type on C. We thus have $\rho(S'') = 2$, and the arguments of [F-M-K92], Theorem 11.2.3, show that $K_{S''} + (\Delta'')^{\mathrm{hor}} \equiv \lambda F''$, $(\Delta'')^{\mathrm{hor}}$ is the horizontal part of Δ'', defined as usual, and F'' is any fibre of g''. Here $\lambda \in \mathbb{Q}$ is such that $\lambda \geqslant 2q - 2$, with q the genus of C. Thus $D'' \equiv (\lambda + \deg(\delta''))F''$ has Kodaira dimension at least 1, because $\deg(\delta'') > 2 - 2q$. Contradiction. □

We still have to show Lemma 3.42.

Proof of 3.42. — From Hurwitz's formula, we get $-2 = -2d + \sum_{e\in E}(r_e - 1)$, where d is the degree of the restriction $h : E \to C$ of g to E, and for each $e \in E$, r_e is the ramification order of h at e. Fix $c \in C$. Write $g^*(c) = \sum_{j\in J} m_j D_j$, $m := inf\{m_j n_j, j \in J\}$, $n_j := (1 - d_j)^{-1}$ with d_j:=multiplicity of D_j in Δ.

Let $E_c := E \cap S_c$, where $S_c := g^{-1}(c)$ is the fibre of g over c.

Claim. $\sum_{e\in E_c}(r_e - 1) \geqslant (1 - 1/m)d - \Delta_c \cdot E$, where Δ_c is the union of components of Δ contained in S_c, with their corresponding multiplicities.

Then Lemma 3.42 is an easy consequence of the claim. Indeed,

$$\begin{aligned}
-2 &= -2d + \sum_{e\in E}(r_e - 1)\\
&\geqslant -2d + \sum_{c\in C}\sum_{e\in E_c}(r_e - 1)\\
&\geqslant -2d + g^*(\delta)\cdot E - \Delta\cdot E, \text{if} \delta := \Delta(g, \Delta)\\
&= (\deg(\delta) - 2)d - \Delta\cdot E.
\end{aligned}$$

But $\deg(\delta) - 2 > 0$, because g is of general type. Hence the conclusion of 3.42.

To complete the proof, we establish the preceding *claim*.

Proof of the claim. —

$$\begin{aligned}(1-1/m)g^*(c)\cdot E &= (1-1/m)\sum_{j\in J} m_j D_j\cdot E\\ &= \sum_{j\in J}(m_j - m_j/m)D_j\cdot E\\ &\leqslant \sum_{j\in J}((m_j-1)+(1-1/n_j))D_j\cdot E\\ &= (\sum_{j\in J}(m_j-1)D_j\cdot E)+\Delta_c\cdot E.\end{aligned}$$

We are thus reduced to show that

$$\sum_{j\in J}(m_j-1)D_j\cdot E\leqslant \sum_{e\in E_c}(r_e-1).$$

Because $\sum_{j\in J} m_j D_j\cdot E = \sum_{e\in E_c} r_e = d$, we just need to establish that $\sum_{j\in j} D_j\cdot E \geqslant \sum_{e\in E_c} 1$, which itself follows from the inequality $\sum_{j\in J}(D_j\cdot E)_e \geqslant 1, \forall e\in E_c$, where the intersection number $(D_j\cdot E)_e$ is the local intersection number near e. Recall that S is only assumed to be normal.

We now show this last inequality. We have $\sum_{j\in J}(m_j D_j\cdot E)_e = r_e$, and the conclusion follows from the inequality $m_j\leqslant r_e, \forall j\in J$. To show this inequality, we make a base change over $h : E{\to}C$. Let S' be the normalisation of the fibre product $S\times_C E$, and let $k : S'{\to}S$, $g' : S'{\to}E$ such that $k\circ g = h\circ g$ be the natural maps induced from this base change. Let $E'\subset S'$ be the lift of E to S': it is a section of g'. Let e' be the point of E' lying above e. The components of $(g')^*(c')$ which contain e' are thus reduced. This easily implies the conclusion, by looking at a generic point of D_j, near which the projection g is locally given by the equation $g(t,z)=z^{m_j}$. The fibre product is thus locally given by an equation of the form $\zeta^{r_e}=z^{m_j}$.

Dividing by $d := gcd(r_e, m_j)$, we can assume that $d=1$, since we normalised the fibre product. Thus we have on S' local coordinates (t,s) with $\zeta = s^{m_j}$, and $z = s^{r_e}$. Locally, the projection g' is given in these coordinates by $g'(t,s)=\zeta$. From which we deduce that $m_j=1$. Thus m_j divides r_e. This in particular proves the claimed inequality. □

Remark 3.44. — What we actually proved in this section is the orbifold additivity for Kodaira dimensions in dimension 2. The general case is stated in the next section at 4.2.

4. Orbifold additivity.

This fourth chapter states (see 4.1) the orbifold version of Iitaka conjecture $C_{n,m}$, and shows (4.2) the special case when its orbifold base is of general type. This result is one of the main technical tools of the present paper, and can be used in many cases for fibrations with general fibres having negative Kodaira dimensions, where the classical statements do not give any conclusion. It should find many further applications in the future.

Despite of this, the proof rests on the same techniques as the classical case. It consists in extending the weak positivity results of Fujita, Kawamata and Viehweg for direct image sheaves of pluricanonical forms to the orbifold situation, by suitably introducing the orbifold divisor into the proofs, distinguishing its vertical and horizontal components. The vertical part increases the second term $\kappa(B)$, B the orbifold base, while the horizontal part contributes increasing the term $\kappa(F)$, F the general orbifold fibre. Also, because the base orbifold does not depend on the horizontal part of the orbifold divisor, this horizontal part may be allowed to have arbitrary rational coefficients between 0 and 1.

4.1. Orbifold conjecture $C_{n,m}$.

We use the notations and notions introduced in Section 1.6. So, if $g : (Y/H)\to Z$ is a holomorphic fibration from the manifold Y equipped with the orbifold divisor H, we defined the base orbifold $\Delta(g, H)$ on Z. The fundamental property of this definition is that, when $H = \Delta(f)$, for some fibration $f : X\to Y$, then $\Delta(g, H) = \Delta(g \circ f)$ for suitable models of f, g which can be choosen so that f, g and $g\circ f$ are prepared and admissible and g is high (see 1.33). We can now state the orbifold additivity conjecture $C^{orb}_{n,m}$.

CONJECTURE 4.1 ($C^{orb}_{n,m}$). — *Let $g : (Y/H)\to Z$ be a holomorphic fibration between manifolds, with $Y \in \mathcal{C}$. Assume g is prepared and high. Then*

$$\kappa(Y/H) \geqslant \kappa((Y/H)_z) + \kappa(Z/\Delta(g, H))$$

where $z \in Z$ is general and $(Y/H)_z := (Y_z/H_z)$.

Of fundamental importance for the considerations of the present paper is the following special case, shown by suitably adapting the classical methods of proof (T. Fujita, Y. Kawamata, E. Viehweg).

THEOREM 4.2 (C^{orb}_{gt}). — *Let $g : (Y/H) \to Z$ be a holomorphic fibration between manifolds, with $Y \in \mathcal{C}$ and Z projective. Assume g is prepared, high and of general type, that is $\kappa(Z/\Delta(g,H)) = \dim(Z)$. Then*

$$\kappa(Y/H) = \kappa((Y/H)_z) + \dim(Z),$$

where $z \in Z$ is general and $(Y/H)_z := (Y_z/H_z)$.

Of course, the above $C^{orb}_{n,m}$ is a simple generalisation and refinement of the classical conjecture of S. Iitaka, dictated by the constructions made in the previous chapters.

Remark 4.3. — In 4.2 and 4.1 above, it is sufficient that the horizontal part H^{hor} of H has all of its irreducible components having rational multiplicities lying in [0,1]. The proof given below applies in this broader situation.

Let us list some of its corollaries or special cases of the above conjecture.

PROPOSITION 4.4. — *Assume $C^{orb}_{n,m}$ holds. Let $f : X \to Y$ and $g : Y \to Z$ be fibrations, with $X \in \mathcal{C}$. Then $\kappa(Y,f) \geqslant \kappa(Y_z, f_z) + \kappa(Z, g \circ f)$, where $z \in Z$ is general and $f_z : X_z \to Y_z$ is the restriction of f.*

Proof. — By 1.6 and 1.28, we can choose models of g and f in such a way that f, g and $g \circ f$ are admissible, prepared, with g high, $\Delta(g \circ f) = \Delta(g, \Delta(f))$, and moreover such that f_z is admissible. We conclude then from $C^{orb}_{n,m}$ and the equalities $\kappa(Y/\Delta(f)) = \kappa(Y,f)$, $\kappa(Y_z/\Delta(f)_z) = \kappa(Y_z, f_z)$, and $\kappa(Z, g \circ f) = \kappa(Z/\Delta(g \circ f)) = \kappa(Z/\Delta(g, \Delta(f)))$. □

In the special case where $X = Y$, we get the following.

PROPOSITION 4.5. — *Assume $C^{orb}_{n,m}$ holds. Let $g : Y \to Z$ be a fibration, with $Y \in \mathcal{C}$. Then $\kappa(Y) \geqslant \kappa(Y_z) + \kappa(Z,g) \geqslant \kappa(Y_z) + \kappa(Z)$.*

The extreme inequality is of course the classical Iitaka conjecture. Let us first list some immediate consequences of Theorem 4.2.

COROLLARY 4.6. — *Let $f : X \to Y$ and $g : Y \to Z$ be fibrations, with $X \in \mathcal{C}$. If $g \circ f$ is of general type, then $\kappa(Y,f) = \kappa(Y_z, f_z) + \dim(Z)$.*

COROLLARY 4.7. — *Let $g : Y \to Z$ be a fibration with $Y \in \mathcal{C}$. If g is of general type, then $\kappa(Y) = \kappa(Y_z) + \dim(Z)$.*

We shall now give the proof of Theorem 4.2. It is classically done in two steps. First an easy reduction to weak-positivity statements for direct images by g of twisted pluricanonical forms and second the proof of semipositivity. The first step is entirely similar to the known cases, so we shall be brief on it. The second step is simply obtained by introducing the orbifold divisors at appropriate places in the classical proofs of Y. Kawamata and E. Viehweg. See also the initial work [F78].

4.2. Reduction to weak positivity.

We start by briefly recalling the notion of weak-positivity introduced in [Vi83] (see also the survey [Es80]).

A torsionfree coherent sheaf $\mathcal{F}$ on Z, projective, is said to be *weakly-positive* (written w.p for short) if for any ample line bundle A on Z, and every integer $a > 0$, there exists an integer $b > 0$ such that $S^{ab}(\mathcal{F}) \otimes A^b$ is generated over some nonempty open subset U of Z by its global sections (defined over Z). Here $S^{ab}(\mathcal{F})$ denotes the extension to Z of the sheaf denoted by the same symbol, naturally defined over the open subset where $\mathcal{F}$ is locally free.

LEMMA 4.8. — *The following properties are shown in [Vi82].*

(1) *If $\mathcal{F}$ is locally free and nef, it is w.p.*

(2) *Let $v : Z' \to Z$ be bimeromorphic, and $\mathcal{F} \subset \mathcal{G}$ a inclusion of torsionfree coherent sheaves of the same rank on Z'. If $v_*(\mathcal{F})$ is w.p, then so is $v_*(\mathcal{G})$.*

(3) *If $v : Z' \to Z$ is a ramified flat covering with Z and Z' smooth, and if $\mathcal{F}$ is torsionfree coherent on Z, then $\mathcal{F}$ is w.p if so is $v^*(\mathcal{F})$.*

We now state without proofs two lemmas, shown but not separately stated in [Es80], and in various more or less implicit forms in [Vi82] and [Ka81]. Together with the weak-positivity result shown in the next section, they imply immediately Theorem 4.2.

LEMMA 4.9. — *Let $g : Y \to Z$ be a fibration with Z projective. Let E and L be $\mathbb{Q}$-divisors on Y and Z respectively, such that L is big and $\kappa(Y, E) \geqslant 0$. Then $\kappa(Y, E + g^*(L)) = \dim(Z) + \kappa(Y_z, E_{|Y_z})$, for z general in Z.*

The crucial place where weak-positivity enters is to check that $\kappa(Y, E) \geqslant 0$.

LEMMA 4.10. — *Let $g : Y \to Z$ be a fibration, E a line bundle on Y and L a $\mathbb{Q}$-divisor on Z such that L is big and $g_*(E)$ is weakly positive and nonzero. Then $\kappa(Y, E + g^*(L)) \geqslant 0$.*

In the next section, we shall show the following theorem.

THEOREM 4.11. — *Let $g : Y \to Z$ a prepared holomorphic fibration $g : Y \to Z$, with Y and Z smooth and Z projective. Let H be an orbifold structure on Y. Let $m > 0$ be an integer such that all $\mathbb{Q}$-divisors involved are integral. There exists an effective g-exceptional divisor B on Y such that the sheaf $g_*(m(K_{(Y/(Z/\Delta(g,H)))} + H) + B)$ is weakly positive on Z.*

Remark 4.12. — The preceding Theorem 4.11 holds with the same proof when H satisfies the weaker condition stated in Remark 4.3 above. That is, if the horizontal part of H has components with coefficients rationals lying between 0 and 1.

Let us now explain how to deduce Theorem 4.2 from the preceding lemmas and Theorem 4.11. First apply Lemma 4.10 to $E := m(K_{(Y/(Z/\Delta(g,H)))} + H) + B$, with $m > 0$ an integer sufficiently divisible, so chosen that $g_*(E)$ is nonzero and w.p, and $L := (m/2)K_{(Z/\Delta(g,H))}$, which is big by hypothesis. We conclude that $\kappa(Y, E + g^*(L)) \geqslant 0$. Thus $E' = E + g^*(L)$ is such that $\kappa(Y, E') \geqslant 0$. Next apply Lemma 4.9 to E' and L to conclude that $\kappa(Y, K_Y + H + B/m)$ satisfies the inequality stated in 4.2. Use finally the fact that g is high to conclude the proof of 4.2, because $\kappa(Y, K_Y + H + B/m) = \kappa(Y, K_Y + H)$ (see 1.32). □

4.3. Orbifold weak-positivity.

Our objective in the next two sections is to establish Theorem 4.11 stated and used above, an orbifold generalisation of famous results of Y. Kawamata and E. Viehweg, initiated by T. Fujita in the case where Z is a curve. We shall actually essentially just reduce our case to the cases they treated.

We consider thus a prepared holomorphic fibration $g : Y \to Z$, with Y and Z smooth and Z projective. That g is prepared means that its non-smooth locus is contained in a simple normal crossing divisor of Z, and that the inverse image by g of this non-smooth locus is also a divisor of simple normal crossings on Y. We let $\Delta := \Delta(g)$ and $\Delta' := \Delta(g, H)$ be the orbifold divisors. Thus Δ is also of simple normal crossings. Notice that $\Delta' \geqslant \Delta$.

Notice that $\Delta(g, H) = \Delta(g, H^{\mathrm{vert}})$, where H^{vert} is the *g-vertical part of H* is defined as follows. If $H = \sum_{k\in K}(1 - 1/n_k)H_k$ then H^{vert} is equal to $\sum_{k\in K'}(1 - 1/n_k)H_k$, where $K' \subset K$ consists of the components of $|H|$ which are not mapped onto Z by g.

We shall also denote by H^{hor} the *g-horizontal part of H*, defined such that $H = H^{\mathrm{vert}} + H^{\mathrm{hor}}$.

We shall obtain Theorem 4.11 as the consequence of three intermediate steps. The first step is a generalisation of the standard weak positivity results for direct image sheaves of pluricanonical forms in Kähler geometry.

THEOREM 4.13. — *Let $g : Y \to Z$ be a prepared holomorphic fibration, with Y and Z smooth, Y Kähler and Z projective. Let $D = \sum_{j\in J} d_j D_j$ be a divisor with positive integer coefficients on Y, the D_j being pairwise distinct. Write $D = D^{\mathrm{vert}} + D^{\mathrm{hor}}$. Assume also that the support of D^{hor} is a divisor of simple normal crossings with positive integer coefficents. Let m be a positive integer such that $m \geqslant d_j^{\mathrm{hor}}$ for every $j \in J^{\mathrm{hor}}$. Then $g_*(mK_{Y/Z} + D)$ is weakly positive.*

This result will be proved in the next section. The second and third steps are given by the following lemma 4.14 and proposition 4.15.

LEMMA 4.14. — *Let $g : Y \to Z$ satisfy the same assumptions as the preceding 4.13. Let D be any divisor with integral coefficients on Y and let m be a nonegative integer. If $v : Z' \to Z$ is a flat finite map with Z' smooth, and if $g' : Y' \to Z'$ is deduced from g by smoothing the base change $\widehat{Y} := Y \times_Z Z'$ of Y by v, then there is a natural injection of sheaves*

$$g'_*(mK_{Y'/Z'} + D') \to v^* g_*(mK_{Y/Z} + D),$$

where $D' := u^*(D)$ and $u : Y' \to Y$ is the natural map obtained by composing the desingularisation $d' : Y' \to \widehat{Y}$ with the base change map $\widehat{u} : \widehat{Y} \to Y$.

PROPOSITION 4.15. — *Let g, D, m be as in 4.14 above. Let, in addition, $H = H^{\mathrm{vert}}$ be an orbifold vertical divisor on Y, which means that no component of H is mapped onto Z by g. There exists a finite flat map*

$v : Z' \to Z$, with Z' smooth, such that if $g' : Y' \to Z'$ is constructed as in 4.14 above from v, the above injection of sheaves extends to an injection

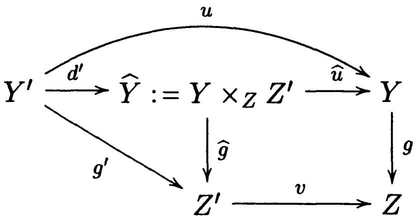
$$g'_*(mK_{Y'/Z'} + D') \to v^* g_*(m(K_{Y/(Z/\Delta(g,H))} + H) + D + B),$$

for some effective g-exceptional divisor B on Y.

We shall give below the proofs of the preceding lemma and proposition. Let us first show that they imply, together with 4.13, the Theorem 4.11. Write $mH = D + mH^{\text{vert}}$, with $D := mH^{\text{hor}}$. This is an integral divisor, if m is sufficiently divisible. Moreover, $0 < d_j = (1 - 1/n_j)m < m$, for each j. So that 4.13 applies to g' and $D' := u^*(D)$, if g' is deduced from g by any base change $v : Z' \to Z$ as in Proposition 4.15 above.

If we now apply the Proposition 4.15, we see that the conclusion of 4.11 holds. We shall now prove 4.14 and 4.15.

Proof of 4.14. — The relevant diagram is the following.



It is proved in [Vi83], Lemma 3.3, pp. 335-336 to which we refer, that in our situation, $d'_*(mK_{Y'/Z'})$ naturally injects into $\widehat{u}^*(mK_{Y/Z})$. This statement implies that $d'_*(mK_{Y'/Z'} + D')$ injects into $\widehat{u}^*(mK_{Y/Z} + D)$, since $D' = u^*(D)$. We then just need to apply $\widehat{g}_*$ to both sides, noticing that $\widehat{g}_*(\widehat{u}^*) = v^*(g_*)$, by flatness of v. □

Proof of 4.15. — We start with the construction of v. Write $\Delta = \sum_{i \in I}(1 - 1/m_i)\Delta_i$.

Definition 4.16. — *Let $g : Y \to Z$ be a prepared holomorphic fibration, with Y and Z smooth and Z projective. A finite covering $v : Z' \to Z$ is said to be $\widetilde{\Delta}$-nice in similarity to [Ka81] if the following hold.*

(1) *It is flat, and Z' is smooth.*

(2) *$v^*(\Delta_i) = \widetilde{m_i}\Delta_i$", for some reduced divisor Δ_i" $\subset Z'$, this for any $i \in I$. Here $\widetilde{m_i}$ is any integer divisible by lcm(m_{ij}'s), these being the same as above, used to define the multiplicity of g along Δ_i. Observe in particular that $v^*(\Delta)$ is Cartier on Z'.*

(3) $v^{-1}(|\Delta|)$ *is a divisor of normal crossings on* Z'.

By [Ka81] and [Vi82], such coverings exist. Let $v : Z' \to Z$ be a $\widetilde{\Delta}$-nice covering and let things be as in Lemma 4.14. Then 4.15 is an immediate consequence of 4.8 and the following.

PROPOSITION 4.17. — *In the preceding situation, there exists on* Z' *a natural injection of sheaves*

$$(g')_*(m(K_{Y'/Z'}+D'+(g')^*v^*(m\Delta(g,H))) \subset v^*g_*(m(K_{Y/Z}+H^{\mathrm{vert}})+D+B),$$

this for any integer $m > 0$ *sufficiently divisible, and some effective* g-*exceptional divisor* B *on* Y.

Proof. — The main point is that we just need to check this injection on the complement of a codimension two subvariety A of Z', since by 4.14, the result holds with $(g')^*v^*(m\Delta(g,H))$ deleted from the left-hand side, and H^{vert} deleted from the right-hand side, on all of Z'. This provided $\mathcal{O}_Y(B)$ is defined by the poles of maximal order acquired by an arbitrary extension across A of a section of $(g')_*(m(K_{Y'/Z'} + (g')^*v^*(m\Delta(g,H)))$ defined outside of A. Observe that we are working here on the fibre product $Y \times_Z Z'$ which is Cohen-Mac Caulay, so that the poles of the sections considered actually occur in codimension one.

We shall thus check the above injection only above the generic point z of some Δ_i. Write $g^*\Delta_i = \sum_{j\in J(g,\Delta_i)} m_{i,j}\Delta_{i,j} + R$ where $J(g,\Delta_i)$ is the set of all irreducible components of $g^*\Delta_i$ which are mapped surjectively onto Δ_i by g, while R is g-exceptional. Let U_i be a sufficiently small analytic open neighborhood of z in Z. Let $Y_U := g^{-1}(U)$, and let $W \subset Y_U$ be a small analytic neighborhood of y, a generic point of Δ_{ij}.

Thus m_{ij} divides $\widetilde{m_{ij}} := m_{ij}q_{i,j}$, for some integer, by the definition of a $\widetilde{\Delta}$-nice covering. Factorise $v := v' \circ v^*$ over U, with $v' : Z' \to Z^*$, and $v^* : Z^* \to Z$, in such a way that v^* (resp. v') ramifies at order exactly m_{ij} (resp. $q_{i,j}$) above Δ_i (resp. $\Delta_i^* := (v^*)^{-1}(\Delta_i)$). Construct Y^* from Y over Z by taking base change by v^*, followed by normalisation and then smoothing. We have also a natural fibration $g^* : Y^* \to Z^*$. Possibly modifying Y', we thus get a factorisation $u = u^* \circ u'$. Moreover, from the usual commutation properties, we have $v' \circ g' = g^* \circ u'$, and $v^* \circ g^* = g \circ u^*$. The relevant diagram is the following.

$$\begin{array}{ccccc} Y' & \xrightarrow{u'} & Y^* & \xrightarrow{u^*} & Y \supset W \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{v'} & Z^* & \xrightarrow{v^*} & Z \supset U \end{array}$$

The crucial property in this construction is that u^* is étale over W, if sufficiently small. This is an easy standard local computation which we already used several times before. Thus u^* is étale over the generic point of Δ_{ij}.

From the following Lemma 4.18 below, we deduce that, over the generic point of Δ_{ij}, we have a natural injection of sheaves

$$m(K_{(Y^*/Z^*)}) \subset (u^*)^*(m(K_{Y/(Z/\Delta(g,H))} + H^{\text{vert}})),$$

for any integer $m > 0$ sufficiently divisible.

From this injection, we can deduce the following, by tensorising with $\mathcal{O}_Y(D)$ and its lift $\mathcal{O}_{Y^*}(D^*)$ to Y^* by u^*

$$mK_{(Y^*/Z^*)} + D^* \subset (u^*)^*(m(K_{Y/(Z/\Delta(g,H))} + H^{\text{vert}}) + D).$$

The very same argument as in the proof of 4.14 above shows the existence, for any j, on which the preceding factorisations depend, of a natural injection of sheaves

$$g'_*(m(K_{Y'/Z'}) + D')) \subset (v')^*(g^*)_*(m(K_{Y^*/Z^*}) + D^*).$$

By composing the above injections, and restricting over W, we see that the sections of $g'_*(m(K_{Y'/Z'}) + D')$ actually belong to $v^*g_*(m(K_{Y/(Z/\Delta(g,H))} + H^{\text{vert}}) + D)$. We identify local sections of $g'_*(m(K_{Y'/Z'}) + D')$ over Z' and sections of $m(K_{Y'/Z'}) + D'$ over corresponding open subsets of Y'.

The conclusion now follows from Hartog's extension theorem, applied over Y_U, to extend the sections thus obtained across the intersection of two or more such $\Delta_{i,j}$'s. □

We used the following.

Lemma 4.18. — *With the above notations, over the generic point of* Δ_{ij}, *we have a natural injection of sheaves* $m(K_{(Y^*/Z^*)}) \subset (u^*)^*(m(K_{Y/(Z/\Delta(g,H))} + H^{\text{vert}}))$, *for any integer* $m > 0$ *sufficiently divisible.*

Proof. — We shall argue using, instead of sheaf injections, rather inequalities between $\mathbb{Q}$-divisors; the inequality $A \geqslant B$ meaning as usual that $A - B$ is effective.

From the equalities,

$$K_{Y^*} = (u^*)^*(K_Y) \quad (u^* \text{ is étale}),$$

and

$$K_{Z^*} = (v^*)^*(K_Z + (1 - 1/m_{i,j})\Delta_i) \quad \text{(ramification formula)},$$

we deduce that

$$K_{Y^*/Z^*} = (u^*)^*(K_{Y/Z} - (m_{i,j} - 1)\Delta_{i,j}),$$

because $g^*(\Delta_i) = m_{i,j}\Delta_{i,j}$ and $g \circ u^* = v^* \circ g^*$. On the other hand,

$$\begin{aligned}
&K_{Y/(Z/\Delta(g,H))} + H^{\mathrm{vert}}\\
&= K_{Y/Z} - g^*((1 - 1/m'_i)\Delta_i) + (1 - 1/n_{i,j})\Delta_{i,j}\\
&= K_{Y/Z} + (-m_{i,j}(1 - 1/m'_i) + (1 - 1/n_{i,j}))\Delta_{i,j}\\
&= K_{Y/Z} + (-m_{i,j} + 1 + (m_{i,j}/m'_i) - (1/n_{i,j}))\Delta_{i,j}\\
&\geqslant K_{Y/Z} - (m_{i,j} - 1)\Delta_{i,j},
\end{aligned}$$

since $m_{i,j}n_{i,j} \geqslant m'_i$, by the very definition of m'_i. This concludes the proof, by applying $(u^*)^*$. □

4.4. Twisted weak positivity.

The aim of this section is to prove Theorem 4.13. Write $D = D^{\mathrm{vert}} + D^{\mathrm{hor}}$. We have a natural injection $g_*(mK_{Y/Z} + D^{\mathrm{hor}}) \subset g_*(mK_{Y/Z} + D)$ and these sheaves have the same rank. Thus, we may assume $D = D^{\mathrm{hor}}$.

For the proof, we shall refer to the one given in [Vi83] of the classical case where D is empty, and simply indicate the changes needed. I would like to heartily thank E. Viehweg, who gave me a decisive hint for the proof of the following lemma.

The single change needed lies in the Corollary 5.2 of [Vi83]. We restate this corollary in the form we need.

LEMMA 4.19. — *Let $g : Y \to Z$ be as in 4.13. Let A^* be ample on Z, let $A = g^*(A^*)$. Assume that $S^N(g_*(mK_{Y/Z} + D + mA))$ is generated by its global sections on some nonempty open subset of Z. Then $g_*(mK_{Y/Z} + D + (m-1)A)$ is weakly positive on Z.*

Proof. — We shall closely follow the proof of the same Corollary 5.2, where D is empty, in [Vi83]. The proof in fact reduces to add D everywhere in an appropriate way.

Observe also that the proof given there uses only the fact that $g_*(K_{Y/Z})$ is weakly positive when Y is projective. The case when Y is Kähler can be obtained from the different proof of this fact sketched in [Vi86], based on [Ko86] (see the references in [Vi86] for more details), because it uses only the Hodge-theoretic result due to P. Deligne that holomorphic forms with logarithmic poles on a compact Kähler manifold are d-closed.

Take now $L := K_{Y/Z} + D + A$. Define next,

$$M := \text{ Image } (g^*(g_*(mK_{Y/Z} + D + mA)) \to m(K_{Y/Z} + D + A)).$$

Assume that the base locus of $mL - (m-1)D$ does not contain any component of D. We can of course always easily reduce to this case, by diminishing the relevant d_j's, without modifying the conclusion.

We now reproduce, in its great lines, the proof of the Corollary 5.2 in [Vi83]. Blowing up Y if needed, we can assume that M is a line bundle, and that $mL = mM + E + (m-1)D$, for E an effective divisor on Y, not containing any component of D, such that $E + D$ has a support of normal crossings. By hypothesis, NM is generated by global sections over a nonempty Zariski open subset of Z.

Observe next that, by an easy lemma, if V is a subsheaf bundle of $mK_{Y/Z} + D - E$, and $E' \leqslant E$ an effective divisor, then $g_*(V) = g_*(V(E'))$. Restrict indeed to a generic fibre of g. Then a section of $V(E')$, being a section of $mK_{Y/Z} + D$, must vanish at the generic point of E to the appropriate order.

We first treat the case in which $m > d_j, \forall j \in J$. The Corollary 5.1 of [Vi83] then applies without any change and shows that the subsheaf

$$g_*(K_{Y/Z} + L^{(m-1)}) \subset g_*(K_{Y/Z} + (m-1)L)$$

is weakly positive, where

$$L^{(m-1)} := (m-1)L - [((m-1)/m)(E+(m-1)D)],$$

the integral part of a $\mathbb{Q}$-divisor being computed just by taking the integral part of the coefficients, componentwise.

From the constructions made, we see that

$$g_*(K_{Y/Z} + L^{(m-1)}) \subset g_*(K_{Y/Z} + (m-1)L - [(m-1)^2/mD]),$$

and that they coincide over the generic point of Z. Indeed, we compute right below that the sheaf on the right is nothing, but $mK_{Y/Z} + D$. Thus the observation above applies, with $V = g_*(K_{Y/Z} + L^{(m-1)})$, and $E' := [(m-1)/mE]$ to give the conclusion.

The sheaf on the right is thus weakly positive, too. Computing, we get

$$\begin{aligned} K_{Y/Z} + (m-1)L - [(m-1)^2/mD] \\ &= K_{Y/Z} + (m-1)L - \sum_{j\in J}[(m-1)^2 m_j/m]D_j \\ &= mK_{Y/Z} + \sum_{j\in J}((m-1)m_j - (m-2)m_j + [m_j/m])D_j \\ &= mK_{Y/Z} + D, \end{aligned}$$

as desired. This ends the proof of the special case considered.

When $d_j = m$ for certain components D_j of D, so that $D = D^- + m\Delta$, each component D_j of D^- having $d_j < m$, one just needs to replace $K_{Y/Z}$ by $K_{Y/Z} + \Delta$ in the above proof, using the fact that $g_*(K_{Y/Z} + \Delta)$ is weakly positive on Z, by [Ka81] Theorem 32 (which can be proved also using the above result of Deligne in the Kähler case). □

5. Geometric consequences of additivity.

This fifth chapter contains the geometric consequences of the orbifold additivity result (Theorem 4.2). We show that manifolds with $\kappa = 0$ are special (5.1), the Albanese map of a special manifold is surjective, connected, and has no multiple fibres in codimension one (5.3). We also show (5.5 (resp. 5.7)), that the essential dimension $\mathrm{ess}(X) := \dim(C(X))$ of X is equal to its dimension $\dim(X)$ (resp. to $\dim(X) - 1$) iff X is of general type (resp. iff X has a fibration of general type with generic fibre a curve either rational or elliptic).

These are two special cases of the "decomposition theorem", which states that the core of X is a fibration of general type, if X is not special. This theorem is established in (5.8). We also show that the decomposition theorem implies the invariance of the essential dimension under finite étale covers, in particular that such covers of a special manifold are still special. We finally observe (see subsection 5.6) that the decomposition theorem asserts that any nonspecial X has a (unique) fibration both special and of general type, and a unique maximum Bogomolov sheaf.

We then show, as another application of 4.2, the second construction of the core "from below", as the "highest general type" fibration on a given X. This proof is much shorter, but less geometric, than the one given in Section 3.

We then show, although it is not used in the present paper, that the Stein factorised product of two fibrations of general type is itself of general type [1]

5.1. Varieties with κ=0.

The second fundamental example of special manifold is given by the following theorem.

THEOREM 5.1.— *Assume $X \in \mathcal{C}$ has $\kappa(X) = 0$. Then X is special.*

Proof.— This is an immediate application of Theorem 4.2. Let indeed $f : X \to Y$ be a fibration of general type. Then the results apply and give $\kappa(X) \geqslant \kappa(X_y) + \dim(Y)$. But $\dim(Y) > 0$, by hypothesis, and $\kappa(X_y) \geqslant 0$. Contradiction. □

[1] Maybe surprisingly, this does not seem to have been noticed, or even conjectured since [Bo79] that one could so naturally define a fibre product of two Bogomolov sheaves, although the techniques used in the present paper are available since two decades. This is probably because the question is immediate if one starts from the geometry of orbifolds and general type fibrations, but hidden if one starts from the sheaves, or even from the associated fibrations, without the geometric leading thread of multiple fibres.

Remark 5.2. — This theorem together with Theorem 2.27 shows that X has no Bogomolov sheaf if $\kappa(X) = 0$. Actually, it is expected that $\kappa^+(X) = 0$ if $\kappa(X) = 0$ (see [Ca95] for details and definitions). This equality is shown for X projective, or Kähler, using Yau's results, if $c_1(X) = 0$.

5.2. The Albanese map.

The following result uses indirectly only the easiest part of the Additivity Theorem 4.2. It nevertheless seems to belong to this section.

PROPOSITION 5.3. — *Let $X \in \mathcal{C}$ be special. Let $\alpha_X : X \to \mathrm{Alb}(X)$ be the Albanese map of X. Then α_X is surjective, connected, and has no multiple fibres in codimension one, that is $\Delta(\alpha_X)$ is empty.*

If X is only w-special (see Section 9.4 below for this notion), then α_X is surjective and connected.

Proof. — Assume $\alpha := \alpha_X$ is not onto. Let $Z \subset \mathrm{Alb}(X)$ be its image. After [Ue75], (10.9), there exists a fibration $g : Z \to W$ with W of general type. Let $\psi : X \to W'$ and $\sigma : W' \to W$ be the Stein factorisation of $\phi \circ \alpha = \sigma \circ \psi$. Then W' is of general type, too, since σ is finite. This contradicts the fact that X is special. Thus α is onto, and $Z = \mathrm{Alb}(X)$.

Let now $\alpha = \beta \circ \alpha'$ be the Stein factorisation of α, with $\alpha' : X \to A'$ connected and $\beta : A' \to \mathrm{Alb}(X)$ finite. A slight variation of the arguments of [K-V80] shows that if β is ramified, there exists a fibration $\phi : A' \to W'$, with W' of general type. Considering $\phi \circ \alpha'$, we get as above a contradiction to the fact that X is special. Thus β is unramified, hence isomorphic, by the universal property of α, which is thus connected and onto.

Assume now that the fibration $\alpha : X \to \mathrm{Alb}(X)$ has $\Delta(\alpha)$ nonempty. Let Δ be any component of $\Delta(\alpha)$. There exists a connected submersive quotient map $q : \mathrm{Alb}(X) \to A := \mathrm{Alb}(X)/B$, for some subtorus B of $\mathrm{Alb}(X)$, such that $\Delta = q^{-1}(D)$, for some big $\mathbb{Q}$-divisor $D \subset A$.

Consider the fibration $f := q \circ \alpha : X \to A$. The support of $\Delta(f) \subset A$ obviously contains D. We conclude from 1.14 that f is of general type, since K_A is trivial and D is big on A. Again this contradicts the assumption of X being special. And concludes the proof if X is special.

If X is only w-special, the first two steps of the proof apply without any change to give the last assertion. □

Question 5.4. — Let $X \in \mathcal{C}$ be a special manifold. Are then the generic fibres of α_X special?

One can easily show that this question has a positive answer if so does the conjecture $C^{orb}_{n,m}$.

5.3. Varieties of general type.

Although the results of this section are easy consequences of the main result 5.8 below, we give a direct proof.

THEOREM 5.5. — *For any manifold $X \in \mathcal{C}$, we have* $\mathrm{ess}(X) = \dim(X)$ *if and only if X is of general type.*

In particular, X is generically covered by a nontrivial family of special submanifolds, the fibres of c_X, if X is not of general type. The case when $\kappa(X) \geqslant 0$ is clear, by Theorem 5.1 applied to the fibres of the Iitaka-Moishezon fibration of X. So the result applies nontrivially only when $\kappa(X) = -\infty$, and its proof gives the following.

THEOREM 5.6. — *Let $X \in \mathcal{C}$ be such that $\kappa(X) = -\infty$ and* $\dim(X) > 0$. *Then $c_X : X \dashrightarrow C(X)$ has a general fibre F which is special of positive dimension, with $\kappa(F) = -\infty$.*

The proofs are easy applications of Theorem 4.2.

Proof of 5.5. — Assume first that X is of general type. So the identity map id_X of X is a special fibration of general type. It is thus the core by 3.12.

Conversely, assume that $\mathrm{ess}(X) = \dim(X)$, or what is the same, that $c_X = id_X$, and that $n := \dim(X) > 0$, so that X is not special.

There exists then a fibration of general type $f : X \to Y$, with $m := \dim(Y) > 0$. We proceed by induction on $n > 0$. So the assertion holds for manifolds of dimension smaller than n.

If F is a general fibre of f, then F is of general type. Indeed, otherwise, we could construct a relative core for f, by Proposition 2.8 in [Ca04], and deduce a nontrivial fibration on X with general fibre special, contradicting our initial hypothesis.

Thus F is of general type. But now f is a fibration of general type with general fibre itself of general type. By Theorem 4.2, we get that X is of general type, as claimed. □

Proof of 5.6. — We proceed by induction on $n := \dim(X) > 0$, assuming that X is not special. There exists then a fibration of general type $f : X \dashrightarrow Y$, with $\dim(Y) > 0$ and a factorisation $f = c_X \circ g$, for a certain fibration $g : C(X) \dashrightarrow Y$. We get first from the preceding result that $\dim(X_y) > 0$, otherwise $\kappa(X) = n$. Assume that $\kappa(X_y) \geqslant 0$. From Theorem 4.2 again, we learn that $\kappa(X) = -\infty \geqslant \dim(Y) + \kappa(X_y) \geqslant \dim(Y) \geqslant 0$. Contradiction. Thus $\kappa(X_y) = -\infty$. Because of the functoriality properties of the core, the restriction of c_X to the general fibre X_y of f is the core of X_y. Thus, by the induction hypothesis, $\kappa(F) = -\infty$, F being the general fibre of the core of X. □

In a similar way, we can get a very simple description of the next step, $\mathrm{ess}(X) = \dim(X) - 1$. (See Section 6.5 for the general version of the next Theorem 5.7).

THEOREM 5.7. — *Let $X \in \mathcal{C}$ be a manifold of dimension $n > 0$. Then $\mathrm{ess}(X) = n - 1$ if and only if one of the following two cases occurs.*

(a) $\kappa(X) = n - 1$ and J_X is a fibration of general type.

(b) The rational quotient $R(X)$ of X is of dimension $n-1$ and of general type.

Proof. — The fact that cases (a), (b) imply that $\mathrm{ess}(X) = n - 1$ was already shown in 3.16. So assume $c_X : X \dashrightarrow C(X)$ has $n-1$-dimensional image. Let F be the generic fibre of c_X. It is a special curve, hence rational or elliptic.

If F is rational then $C(X) = R(X)$, by the fact that r_X dominates c_X (see 3.24). Thus $R(X)$ is $n-1$-dimensional. Moreover, we get from 3.9 that $R(X)$ is Moishezon, and so also X is Moishezon, by [Ca85].

Because $C(R(X)) = C(X)$, by Theorem 3.26, which applies because X is Moishezon, we conclude that $c_{R(X)} = id_{R(X)}$, and so by the preceding Theorem 5.5, we infer that $R(X)$ is of general type, as claimed.

Next, if F is elliptic, then $c_X = J_X$, because J_X dominates c_X, and is obviously dominated by c_X if $\kappa(X) \geqslant 0$. We show the result by induction on $n \geqslant 2$, the case of curves being obvious.

Thus X is non-special, by assumption. Let $f : X \dashrightarrow Y$ be a fibration of general type, and let F be its general fibre. By 2.6, we have a factorisation $f = c_X \circ g$, for a certain fibration $g : C(X) \dashrightarrow Y$.

Moreover, because of the functoriality properties of the core, the restriction of c_X to the general fibre X_y of f is the core of X_y. The induction hypothesis thus applies to X_y, and we get that $\kappa(X_y) = d - 1$ if $d := \dim(X_y)$, and also that J_{X_y}, which is the restriction of c_X to X_y, is a fibration of general type.

We are thus in position of applying Theorem 4.2, since f is of general type. We get first that $\kappa(X) = \kappa(X_y) + \dim(Y) = n - 1$.

Then, because the general fibres X_y of f have nonnegative Kodaira dimension, we get also from Theorem 4.2 that J_X is of general type, since so is its restriction to the general fibre of f. □

5.4. The decomposition theorem.

This is the following asssertion, which motivates most of the present paper.

THEOREM 5.8. — *Let X be non special. Then c_X is a fibration of general type.*

Roughly speaking, this means that X "decomposes" into its "special part", the fibres of c_X, and its "core" or "essential part", the orbifold $(C(X)/\Delta(c_X))$, which is either a point, or of general type. Hence the name. The Decomposition theorem can also be restated in the following form. Any $X \in \mathcal{C}$ has a fibration both special and of general type. This fibration is unique, and it is the core of X (apply 3.12 to see the last two assertions).

Proof of Theorem 5.8. — Proceed by induction on $d := \mathrm{ess}(X)$. When $d = 0$, the result trivially holds. So assume it does when $\mathrm{ess}(F) < d$. Assume that $\mathrm{ess}(X) = d > 0$. By assumption, X is not special. So let $f : X \dashrightarrow Y$ be of general type. We have, by Theorem 2.6, a factorisation $\psi : C(X) \dashrightarrow Y$ of $f = \psi \circ c_X$.

By the characteristic property of the core, the restriction of c_X to the general fibre X_y of f is the core of X_y. Otherwise X would contain special subvarieties strictly larger than the generic fibre of c_X, a contradiction.

By the induction hypothesis, we conclude that the restriction of c_X to the general fibre X_y of f is a fibration of general type.

We conclude from Theorem 4.2 that c_X itself is a fibration of general type, as claimed. □

Remark 5.9. — We already showed some cases of the preceding Theorem 5.8.

(1) Up to dimension 3 as a byproduct of the description of the core given in low dimensions.

(2) When X is of general type, and when $\mathrm{ess}(X) = dim(X)$. In this last case, showing that X is of general type is actually the same as showing the Decomposition theorem in this case. The crucial step was actually applying C_{gt}^{orb}.

(3) Exactly the same remarks apply to the classification of the cases in which $\mathrm{ess}(X) = n - 1$.

(4) We shall show below two consequences of the Decomposition Theorem, that $C(X)$ is Moishezon, and meromorphic multisections of c_X are of general type.

THEOREM 5.10. — *Let $X \in \mathcal{C}$, and let $a_X : X \to Alg(X)$ be its algebraic reduction. There exists a factorisation $\phi : Alg(X) \dashrightarrow C(X)$ of $c_X = \phi \circ a_X$. Alternatively, $C(X)$ is Moishezon.*

Proof. — By Theorem 2.39, the general fibre of a_X is special. □

Let us now establish the following weak version of the Decomposition Theorem (2.11 shows that it is a consequence of this result).

PROPOSITION 5.11. — *Let $j : Z \dashrightarrow X$ be a meromorphic map such that $c_X \circ j : Z \dashrightarrow C(X)$ is surjective and generically finite, $c_X : X \to C(X)$ being, as usual, the core of $X \in \mathcal{C}$. Then Z is a variety of general type.*

Proof. — By the functoriality of the core, and the surjectivity of $c_X \circ j$, we get the existence of a factorisation $c_j : C(Z) \dashrightarrow C(X)$. Because $c_X \circ j = c_j \circ c_Z$ is generically finite, so is c_Z. The claim thus follows from Theorem 5.5. □

5.5. Finite étale covers.

Let us indicate that the Decomposition Theorem implies the invariance of $\mathrm{ess}(X)$ under finite étale covers.

THEOREM 5.12. — *Let $X \in \mathcal{C}$ be smooth and let $u : X' \to X$ be a finite étale cover. Let $c_u : C(X') \dashrightarrow C(X)$ be the induced map. Then c_u is* *generically finite. In particular,* ess(X) *is invariant under finite étale covers, and a finite étale cover of a special manifold is special.*

Remark that the proof of this seemingly easy statement requires here fairly deep tools. It would be interesting to know if there is an easy proof of it. (Notice that the proof shows that c_u is just the finite part of the Stein factorisation of $c_X \circ u$).

Proof. — We can assume that u is Galois, of group G. Due to its uniqueness, the map $c_{X'}$ is G-equivariant for an appropriate action of G on $C(X')$. Let $h : C(X') \dashrightarrow C'(X)$ be the G-quotient. We have natural maps $c'_X : X \dashrightarrow C'(X)$, by G-invariance, and $v : C'(X) \dashrightarrow C(X)$, since the fibres of c'_X, as images of those of $c_{X'}$ by u are special. Moreover, $c_X = v \circ c'_X$, by the general properties of c_X.

By the Decomposition Theorem, $c_{X'}$ is a fibration of general type. And so c'_X is also of general type, by 1.8. From 2.6, we infer the existence of a factorisation $w : C(X) \dashrightarrow C'(X)$ such that $c'_X = w \circ c_X$. Thus $C'(X) = C(X)$, and $h = c_u$ is a finite map. □

5.6. Essential and Bogomolov dimensions.

Definition 5.13. — *For* $X \in \mathcal{C}$, *let*

$$B(X) := \max\{p > 0, \text{ such that there exists a Bogomolov sheaf } F \subset \Omega^p_X\},$$

if there is no Bogomolov sheaf on X, *define* $B(X) := 0$.

From 2.27, we see that $B(X) = 0$ if and only if ess$(X) = 0$. In general, we have the following.

Corollary 5.14. — *For any* $X \in \mathcal{C}$, *we have* ess$(X) = B(X)$.

Proof. — Let F be any Bogomolov sheaf of dimension $p > 0$ on X. The associated fibration is thus of general type, and so dominated by c_X. Thus ess$(X) \geqslant p$. Conversely, because c_X is a fibration of general type, the Bogomolov sheaf associated to it has $p =$ ess(X). Hence the equality. □

Let us say that if F, G are Bogomolov sheaves on X that F *dominates* G if the fibration f of general type associated to F dominates the fibration g of general type associated to G.

Remark 5.15. — The Decomposition Theorem may then be restated in the following form. For any $X \in \mathcal{C}$, $\mathrm{ess}(X) = B(X)$, there exists on X a unique maximum Bogomolov sheaf, and it is defined by the core of X.

5.7. Construction of the core as the highest general type fibration.

We sketch here a second, shorter, construction of the core. It is more abstract and less geometric, avoiding the consideration of chains of special varieties.

THEOREM 5.16. — *Let $X \in \mathcal{C}$. Then X admits a fibration both of general type, and special. This fibration is unique up to equivalence. It is the core of X.*

Proof. — Everything is clear from 2.6, once the existence is known. We proceed by induction on $\dim(X) = n$. If X is special, we are finished. Otherwise, there exists a fibration $f : X \dashrightarrow Y$ of general type, and with $\dim(Y) > 0$ maximum among such fibrations.

If the general fibres of f are special, we are again finished. Otherwise, by the uniqueness of the core, we can then construct a relative core, which is a factorisation $f = g \circ h$, by fibrations $h : X \dashrightarrow Z$ and $g : Z \dashrightarrow Y$, in such a way that the restriction $h_y : X_y \dashrightarrow Z_y$ of h to the general fibre X_y of f is the core of X_y, and thus a fibration of general type, by the induction hypothesis, since $\dim(X_y) = \dim(X) - \dim(Y) < \dim(X)$).

But now, we see from 4.6 that h itself is of general type, because $g \circ h$ is, and also h_y. But $\dim(Z) > \dim(Y)$, and this contradicts the maximality of $\dim(Y)$. The fibres of f are thus special. □

Remark 5.17. — This proof was given in the very first version of this paper, under the hypothesis that 4.6 should be true. Remark also that property (2) of 3.3 can also be easily be proved, using 2.7.

Remark 5.18. — If $\mathrm{Ess}(X)$ is the essential algebra of X (see 3.5), its Kodaira dimension is thus equal to: $\mathrm{ess}(X) := \dim(C(X))$. We shall later conjecture (see 6.23) that the graded algebra $\mathrm{Ess}(X)$ is finitely generated and invariant by deformation of X.

5.8. Fibre products.

We answer here the question 2.33, although it is not used in the present paper.

THEOREM 5.19. — *Let $X \in \mathcal{C}$, and $f : X \dashrightarrow Y$, $g : X \dashrightarrow Z$ be two fibrations of general type. Let $h : X \dashrightarrow W$ be the connected part of the Stein factorisation of the product map $f \times g : X \dashrightarrow Y \times Z$. Then h is a fibration of general type.*

Proof. — This is an immediate generalisation of the argument in the classical case, when Y,Z are themselves of general type. Indeed, let $g' : W \dashrightarrow Z$ and $f' : W \dashrightarrow Y$ the fibrations such that $f = f' \circ h$ and $g = g' \circ h$. By the equality, deduced from 4.2,

$$\kappa(W, h) = \kappa(W_z, g'_z) + \kappa(Z, g = g' \circ h) = \kappa(W_z, g'_z) + \dim(Z),$$

we just need to check that the restriction of $g' : X_z \dashrightarrow W_z$ is of general type, for z general in Z. Observe the statement is bimeromorphically invariant. We thus assume that Z is projective (it is certainly Moishezon, since g is of general type). Take $S \subset Z$ be an intersection of generic hyperpane sections such that $(g')^{-1}(S) := T \subset W$ is mapped surjectively and generically finitely onto Y by f'. Let $V := h^{-1}(T) \subset X$. The restriction$f_V : V \dashrightarrow Y$ of f to V has thus the restriction $h_V : V \dashrightarrow T$ of h to V as Stein factorisation. It is thus a fibration of general type, by 2.10.

If $g' : T \dashrightarrow S$ has positive-dimensional fibres, the conclusion follows from 2.13. If not, this means that g dominates f, in which case the assertion is trivial. □

6. The decomposition of the core.

This sixth chapter establishes the canonical and functorial decomposition of the core as a tower of fibrations with orbifold fibres either κ-rationally generated, or with $\kappa = 0$.

Orbifold Iitaka-Moishezon fibrations [2] are defined without difficulty as in the non-orbifold context, by simply considering the orbifold canoni-

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[2] Usually called Iitaka fibration. The construction obtained independently by B. Moishezon [Mois] remained apparently unnoticed.

cal bundle (see Section 6.3). We denote by $J_{(Y/D)}$ the Iitaka-Moishezon fibration of the orbifold (Y/D).

The orbifold rational quotients have a more involved construction. In order to avoid any conjectural statement, we are lead to give a definition based on a refined Kodaira dimension similar to that introduced in [Ca95], and not on the geometry of rational curves. But once this is done, the decomposition is easily seen to exist and to be uniquely defined at each step.

To make this definition, we first start in 6.1 by defining and comparing some orbifold variants of the notion of rational connectedness. One of them, called κ-rational generatedness, turns out to give a good substitute of the notion of rational quotient in the orbifold context. Under the Conjecture 6.5 that orbifolds with negative Kodaira dimension should be uniruled by orbifold rational curves, κ-rational connectedness is equivalent to rational-generatedness and to compound rational-connectedness (see 6.11). Notice however that [G-H-S01] fails in the orbifold context, as shown by Example 6.13. So that in order to restore some version of rational-connectedness, one had to rely on a weaker notion than orbifold rational curves.

In Section 6.2, the orbifold rational quotient $r_{(Y/D)}$ of an orbifold (Y/D) is defined. It is, roughly speaking, the unique fibration f on (Y/D) with general orbifold fibres κ-rationally generated, and orbifold base of non-negative Kodaira dimension.

In Section 6.4, we show that the orbifold rational quotient r_f and Iitaka-Moishezon fibration J_f, when applied to the orbifold base of a *special* fibration $f : X \dashrightarrow Y$, produce again special fibrations $r_f \circ f$ and $J_f \circ f$.

This preservation property immediately leads in Section 6.5 to the expected decomposition of the core in the form $c_X = (J \circ r)^n$ of iterated orbifold rational quotients and Iitaka fibrations. This result is crucial, because it shows that special manifolds are exactly the compounds of $\kappa = 0$ and κ-rationally generated orbifolds. And specialness simply appears as the orbifold combination of the first two terms of the fundamental trichotomy of algebraic geometry.

Finally, in Subsection 6.6, we conjecture the deformation invariance of all the invariants introduced in the course of this decomposition. In particular, $\mathrm{ess}(X)$ and the *essential algebra* should be invariant under deformation, and deformations of special manifolds should remain special.

6.1. Orbifold rational connectedness.

We consider in this section an orbifold (Y/D) with $Y \in \mathcal{C}$ compact smooth and D an orbifold divisor of normal crossings on Y. We define several variants of rational connectedness, directly adapted from the non-orbifold case.

If $g : (Y/D) \to Z$ is a fibration, we define its orbifold base $(Z/\Delta(g, D))$ as in Section 1.6. If g is only meromorphic, we replace (Y/D) by a terminal modification $\mu : (Y'/D') \to (Y/D)$ such that $g' := g \circ \mu$ is holomorphic, and then define the orbifold base of g to be that of g'. We then define $\kappa(Z, g, D)$ as the minimum of the Kodaira dimensions of the orbifold bases so obtained.

The orbifold general fibre of $g : (Y/D) \to Z$ is defined as the orbifold (Y_z/D_z), with Y_z the general fibre of g, and $D_z := i^*(D)$, where $i : Y_z \to Y$ is the inclusion map.

An *orbifold rational curve* on (Y/D) is an irreducible rational curve C on Y such that $\kappa(\hat{C}, K_{\hat{C}} + \Theta(n^*(\Theta^{-1}(D)))) = -\infty$, where $n : \hat{C} \to C$ is the normalisation map, and Θ is the operation which sends any effective integral divisor $D^* = \sum_{j \in J} m_j D_j$, where the m_j's are positive integers and the D_j's prime distinct divisors, to the orbifold divisor $D := \Theta(D^*) := \sum_{j \in J}(1 - 1/m_j)D_j$. Of course, Θ^{-1} is the inverse operation, sending an orbifold divisor D to its associated integral D^*. Notice that $n^*(D) \neq \Theta(n^*(\Theta^{-1}(D)))$, unless C meets $|D|$ transversally.

DEFINITION 6.1.— *We say that (Y/D) is:*

(1) *Uniruled if it is covered by a family of curves, whose generic member is an orbifold rational curve,*

(2) *Rationally connected if two generic points of Y can be joined by a chain of orbifold rational curves,*

(3) *Rationally generated if, for any fibration $g : (Y/D) \dashrightarrow Z$, the orbifold base of some, or any, holomorphic model of g is uniruled (taking $g = id_Y$: (Y/D) itself is then uniruled),*

(4) *κ-rationally generated if, for any fibration $g : (Y/D) \dashrightarrow Z$, $\kappa(Z, g, D) = -\infty$.*

We note, as usual, for short: R.C (resp. R.G; resp. k-R.G) for rationally connected (resp. rationally generated; resp. κ-rationally generated). Notice that R.G was introduced in [Ca95] in the non-orbifold context, and that the notion of κ-rational generatedness is directly inspired from the definition of κ^+ in the same paper. Actually, one can even define:

DEFINITION 6.2. — *For (Y/D) as above, define $\kappa_+(Y/D) := \max\{\kappa(Z,g,D)\}$, where g runs through all fibrations $g : (Y/D) \dashrightarrow Z$ as above.*

Then κ-rational generatedness is defined by the equality $\kappa_+ = -\infty$.

Remark 6.3. — One can also introduce in the orbifold case many other invariants, based on the vanishing of holomorphic "covariant" tensors, which conjecturally characterise rational generatedness. We do not do this here, because one needs first to define these notions in the orbifold case. So we introduced only the simplest one, κ_+.

One could, for example, define $\kappa_{++}(Y/D) := \max\{\kappa(Y,F)\}$, where F ranges over all rank one coherent subsheaves of $\Omega^p_Y(log(D)), p > 0$, arbitrary. See Section 2.9 for this notion. Another (presumably equal) variant of orbifold rational connectedness would be defined by $\kappa_{++}(Y/D) = -\infty$.

Standard arguments that we do not need to reproduce easily show the following.

PROPOSITION 6.4. — *For any orbifold (Y/D) as above, one has:*

(1) $\kappa(Y/D) = -\infty$ if (Y/D) is uniruled.

(2) (Y/D) is k-R.G if it is R.G.

(3) (Y/D) is R.G if it is R.C.

The reverse implications to (1) and (2) above depend on a positive answer to the following fundamental conjecture.

CONJECTURE 6.5 (($\kappa = -\infty$)-Conjecture). — *Assume $\kappa(Y/D) = -\infty$. Then (Y/D) is uniruled.*

If this conjecture has a positive answer, then R.G and k-R.G are equivalent properties.

Remark 6.6. — Conjecture 6.5 seems especially delicate, if true, in the orbifold context. It is presently open even in the surface case $\dim(Y) = 2$, and seems to present great subtlety. It is similar to the case handled in [K-MK97], of log-Del Pezzo surfaces with a reduced boundary D. But their methods should apply to treat the above surface case.

Weaker conjectures relate to algebraicity properties of $\kappa = -\infty$ or rationally generated orbifolds.

For example: assume that (Y/D) is rationally generated, with $Y \in \mathcal{C}$. Is then Y Moishezon?

In the same vein, assume that $\kappa(Y/D) = -\infty$. Is then Y covered by a family of non-trivial subvarieties? (Here non-triviality means that the subvarieties have a dimension different from 0 and $\dim(Y)$).

LEMMA 6.7. — *Let $r : (Y/D) \dashrightarrow Z$ be a fibration. Assume that:*

(1) $\kappa_+(Y_z/D_z) = -\infty$, for a general orbifold fibre of r, and:

(2) $\kappa_+(Z'/D') = -\infty$, if (Z'/D') is the orbifold base of $g' : Y' \to Z'$, for some holomorphic representative g' of g.

Then $\kappa_+(Y/D) = -\infty$.

Proof. — Assume not. There would exist a fibration $h : (Y/D) \dashrightarrow V$ such that the orbifold base of any representative of h had a non-negative Kodaira dimension. Because the orbifold fibres of r have $\kappa_+ = -\infty$, this would imply the existence of a factorisation $\phi : Z \dashrightarrow V$ of $g = \phi \circ g$, by Lemma 6.9 below. But then we have a contradiction to the fact that $\kappa_+(Z'/D') = -\infty$, by Lemma 6.10.

LEMMA 6.8. — *Let $g : (Y/D) \to Z$ be a fibration, and $j : Z' \to Y$ be such that $g \circ j : Z' \to Y$ is generically finite and surjective. Assume that j is in general position with respect to g, which means that the intersection of $j(Z')$ and E is transverse at its generic point, for each component E of $g^{-1}(\Delta(g, D))$. (This condition is satisfied for the generic fibre Z' of any fibration $h : Y \to T$, if $\dim(T) > 0$). Then $\kappa(Z'/D') \geqslant \kappa(Y/\Delta(g, D))$, if $D' := j^*(D)$.*

Proof. — By the transversality assumption, we have, above the generic point of any irreducible divisor L of Z:

$$K_{Z'} \geqslant (g \circ j)^*(K_Z + \sum_{j \in J}(1 - 1/m_j)L_j),$$

if $g^*(L) = \sum_{j\in J} m_j L_j + R$, in the usual notations. If we write, for each j, the coefficient (possibly 0) of L_j as $(1 - 1/n_j)$, the coefficient c_j of L in $\Delta(g, D)$ is then defined by $c_j := 1 - 1/m$, with $m := inf_{j\in J}\{m_j n_j\}$. An easy computation, based on the inequality $m_j(1 - 1/m_j) + (1 - 1/n_j) = m_j(1 - 1/m_j n_j) \geqslant m_j(1 - 1/m)$ shows that:

$$K_{Z'} + D' \geqslant (g \circ j)^*(K_Z + \Delta(g, D))$$

above the generic point of L. The assertion is then obvious. □

LEMMA 6.9.— *Let $g : (Y/D) \to Z$ and $h : (Y/D) \to T$ be fibrations such that $\kappa(Z/\Delta(g, D)) \geqslant 0$. Then $\kappa(Y_t/D_t) \geqslant 0$, if (Y_t/D_t) is the orbifold general fibre of h.*

Proof.— Assume first that T is projective. By taking intersections of ample divisors in T, and their inverse images in Y, we can reduce to the case where the resulting $Y' \subset Y$ is mapped generically finitely onto Z by g. By the preceding lemma, we conclude that $\kappa(Y'/D') \geqslant 0$. From the easy addition theorem, we conclude that the generic orbifold fibre of h has $\kappa \geqslant 0$. The non-projective case can be obtained by using the arguments of 2.15. □

LEMMA 6.10.— *Let $f : (Y/D) \to Z$ and $g : Z \to T$ be two fibrations. Then $\Delta((g \circ f), D) = \Delta(g, \Delta(f, D))$, on suitable bimeromorphic models of f and g.*

Proof.— It is exactly the same as that of 1.33. □

An immediate consequence of 6.7 (see [Ca95] for the non-orbifold version, with the same proof) is the following.

COROLLARY 6.11.— *Assume Conjecture 6.5 holds. For any orbifold (Y/D), the following three properties are equivalent:*

(1) *(Y/D) is k-R.G.*

(2) *(Y/D) is R.G.*

(3) *(Y/D) is compound rationally connected, which means that there exists a sequence of fibrations $r_i : Y_i \to Y_{i+1}$ $(i = 0, 1, ..., m)$ such that $Y_0 = (Y/D)$, Y_m is a point, and the orbifold fibres of each of the r_i is R.C, the orbifold structure on Y_{i+1} being defined inductively as the base orbifold of $r_i : (Y_i/D_i) \to Y_{i+1}$.*

Remark 6.12. — One knows from [G-H-S01] that the reverse implication to 6.4 (3) holds in the non-orbifold context. So it is natural to ask if it still holds in the presence of orbifolds divisors. This is *not* the case, as the simple example below shows.

Example 6.13. — Let $Y = \mathbb{P}^1 \times \mathbb{P}^2$, let $D = \sum_{j=1}^{3}(1-1/m_i)D_j$, where $m_1 > 3, m_2 = m_3 = 1/2$, and each D_j is a generic divisor of bidegree $(1, d)$, $d \geqslant 3$. Let $g : Y \to Z = \mathbb{P}^1$ be the first projection. Its orbifold fibres are all orbifold rational curves (because $1/m + 1/2 + 1/2 < 2$). Elementary arguments however show that every orbifold rational curve on (Y/D) is a fibre of g, although $\kappa_+(Y/D) = -\infty$.

Remark 6.14. — Another natural question in this context is whether the "glueing Lemma" of [K-M-M92] extends to this context. For example, can one add to an orbifold rational curve C' in (Y/D) sufficiently many "free" orbifold rational curves if (Y/D) is uniruled, in such a way that the union deforms to an irreducible orbifold rational curve?

We shall now justify the introduction of the notion of κ-generatedness, by showing that it permits to define the notion of κ-rational quotient (even in the orbifold case) without solving Conjecture 6.5 above.

6.2. Orbifold κ-rational quotient.

We are now in position to define orbifold κ-rational quotients.

THEOREM 6.15. — *Let (Y/D) be an orbifold, with $Y \in \mathcal{C}$, as always. There exists then, up to equivalence, a unique fibration $r_{Y/D}$: $Y \dashrightarrow R(Y/D) = Z$ such that:*

1. *the general orbifold fibre $(Y_r/D_r), r \in R(Y/D)$ of $r_{Y/D}$ has $\kappa_+(Y_r/D_r) = -\infty$,*
2. *$\kappa(R(Y/D), r_{Y/D}, D) \geqslant 0$.*

We call $r_{Y/D}$ the orbifold κ-rational quotient of (Y/D).

Proof. — Uniqueness. Assume there exists another fibration h : $Y \dashrightarrow Z'$ with the same properties. The fibres of $r_{Y/D}$ have $\kappa_+ = -\infty$, and so must be mapped to points by h. Thus Z dominates h. And conversely. Thus $h = r_{Y/D}$.

For the existence, let $g : (Y/D) \dashrightarrow Z$ have general fibres with $\kappa_+ = -\infty$, with $\dim(Z)$ minimal among g's with the preceding property.

If $\kappa(Z'/D_{g'}) \geqslant 0$, where $D_{g'}$ is the orbifold base of g', an arbitrary holomorphic representative of g, we are finished. Otherwise, arguing by induction on $\dim(Y)$, there exists a fibration $s : (Z/D_g) \dashrightarrow V$ such that $\dim(V) < \dim(Z)$, and such that the general orbifold fibre of s has $\kappa_+ = -\infty$. But then, from the above Lemma 6.7, we conclude that the general orbifold fibre of $s \circ g$ has also $\kappa_+ = -\infty$, thus contradicting the minimality of $\dim(Z)$. □

6.3. Orbifold Iitaka-Moishezon fibration.

The Iitaka-Moishezon fibration of an orbifold (Y/D) as in Section 6.1 above can be defined without difficulty by considering the linear system $|mK_{Y/D}|$ for m sufficiently big and divisible. We shall then denote by $J_{Y/D} : (Y/D) \dashrightarrow J(Y/D)$ the resulting Iitaka-Moishezon fibration. By the usual properties of Iitaka fibrations, one has $\kappa(Y_s/D_s) = 0$, for its general orbifold fibre.

6.4. Special fibrations.

We describe in this and the next section an alternative construction of the core of an arbitrary X, along the program described in the introduction.

The idea is quite simple. Assume we have a *special* fibration $f : X \dashrightarrow Y$. We try to understand the obstructions to constructing a *non-trivial* $h : Y \dashrightarrow Z$ such that $g := h \circ f : X \dashrightarrow Z$ is still special, this in terms of the invariant $\kappa(Y, f)$.

If f is of general type, h does not exist, because then f is the core of X, by 3.12. We shall see that this is in fact the only obstruction.

The other possibilities can actually be reduced, as in the classical Iitaka-Moishezon classification Program, to the cases $\kappa = 0$, or $-\infty$, using the orbifold Iitaka fibrations and κ-rational quotients introduced above.

So, we consider in the sequel a *special* fibration $f : X \dashrightarrow Y$, with $X \in \mathcal{C}$. For a suitable admissible model of f, let the orbifold (Y/D), be defined by $D := \Delta(f)$. Thus $\kappa(Y/D) = \kappa(Y, f)$.

Let $r_f : Y \dashrightarrow R(Y/D) = Z$ be its orbifold κ-rational quotient. We thus have $\kappa(Z/D_Z) \geqslant 0$, if (Z/D_Z) is the orbifold base of r_f. We call r_f the *rational reduction of* f. It is the identity map of Y if $\kappa(Y, f) \geqslant 0$.

We are thus in position to define $J_f := J_{(Z/D_Z)} : (Z/D_Z) \dashrightarrow J(Z/D_Z)$, the Iitaka fibration of (Z/D_Z). We call J_f the *Iitaka reduction of* f.

Notice that the general orbifold fibre of r_f (resp. J_f) has $\kappa_+ = -\infty$ (resp. $\kappa = 0$), and that $\dim(J(Z/D_Z)) = \kappa(Z/D_Z)$.

We next define $s_f := J_f \circ r_f : (Y/D) \dashrightarrow J(Z/D_Z)$, and call it the *special reduction of* f.

As a consequence of the Theorem 6.16 below, we also define $rs_f := r_{s_f \circ f}$, and call it the *reduced special reduction of* f.

If the context is clear, we write r, J, $J \circ r$ instead of r_f, J_f, $S_f = J_f \circ r_f$.

THEOREM 6.16. — *Let $f : X \dashrightarrow Y$ be a special fibration, with $X \in \mathcal{C}$. Then X is special if either*

(i) *$\kappa(Y, f) = 0$ or,*

(ii) *$\kappa_+(Y/D) = -\infty$, with $D = \Delta(f)$, and f admissible.*

Proof. — We first establish (i). This follows from 4.2 and its corollaries. Assume indeed X were not special. By 2.6, there would exists a fibration $g : Y \dashrightarrow Z$ such that $g \circ f : X \dashrightarrow Z$ were of general type. But then, we would have $\kappa(Y_z, f_z) \geqslant 0$, because $\kappa(Y, f) = 0$, by the easy addition theorem and thus $0 = \kappa(Y, f) = \kappa(Y_z, f_z) + \dim(Z) \geqslant \dim(Z) > 0$. Contradiction, and (i) is true.

To prove (ii), simply assume X were not special. Then there would exist $h : X \dashrightarrow V$ of general type. By 2.6, we had a factorisation $\phi : Y \dashrightarrow V$ of $h = \phi \circ f$. But then, on suitable models, we had $\Delta(h = \phi \circ f) = \Delta(\phi, D)$, contradicting $\kappa_+(Y/D) = -\infty$. □

COROLLARY 6.17. — *Let $f : X \dashrightarrow Y$ be a special fibration, with $X \in \mathcal{C}$. Then*

(i) *$r_f \circ f$ is a special fibration,*

(ii) *$s_f \circ f = J_f \circ r_f \circ f$ is a special fibration,*

(iii) *rs_f is a well-defined fibration, up to equivalence.*

Proof. — This is simply because the restriction of f to the general orbifold fibre of $r_f \circ f$ has orbifold base the corresponding fibre of r_f, which is an orbifold with $\kappa_+ = -\infty$, by construction. Then Theorem 6.16 (ii) applies to give the conclusion. The same argument shows (ii), replacing f by $r_f \circ f$, r_f by J_f, and Theorem 6.16 (ii) by Theorem 6.16 (i). Then

(iii) follows, because $g := s_f \circ f$ is a special fibration, so that $r_g = r_{s_f \circ f}$ is well-defined. $\square$

6.5. The decomposition of the core.

Let $f : X \dashrightarrow Y$ be a special fibration, with $X \in \mathcal{C}$. Define inductively, for $k \geqslant 0$, a descending sequence of special fibrations $s_f^k : X \dashrightarrow S^k(f)$ and $rs_f^k : X \dashrightarrow RS^k(f)$ by the following.

(i) $s_f^0 := f$, and $rs_f^0 := r_f \circ f$.

(ii) $s_f^{k+1} := s_{s_f^k} \circ s_f^k$, and $rs_f^k := rs_{s_f^k} \circ s_f^k$.

We call s_f^k (resp. rs_f^k) *the k^{th}-special reduction of f* (resp. *the k^{th}-reduced special reduction of f*).

When $f = id_X$, which is indeed a special fibration, we write $s_X^k = (J \circ r)^k$, $S^k(X)$, $rs_X^k = r \circ (J \circ r)^k$ and $RS^k(X)$ instead of s_f^k, $S^k(f)$, rs_f^k and $RS^k(f)$, and speak of the *special reductions of X*.

From this we immediately deduce the following theorem, by applying the above iteration starting with $f := id_X$, the identity map of X, thus getting the sequence of k^{th}-special reductions of X.

THEOREM 6.18.— *Let $X \in \mathcal{C}$ and $n := \dim(X)$. Then $(J \circ r)^n = s_X^n = c_X$, the core of X.*

Of course, in general, the sequence of fibrations s_X^k will be stationary before its n^{th}-term is reached.

DEFINITION 6.19.— *For $X \in \mathcal{C}$ and $k \geqslant 0$, define $s^k(X) := \dim(S^k(X))$, and $rs^k(X) := \dim(RS^k(X))$. Define also $\lambda(X)$ as the smallest $k \geqslant 0$ such that $s^k(X) = s^{k+1}(X)$.*

These are new bimeromorphic invariants of X. We call $s^k(X)$ the *k^{th}-special dimension* of X, $rs^k(X)$ the *reduced k^{th}-special dimension of X*, and $\lambda(X)$ the *special length* of X. Standard easy arguments give the following proposition.

PROPOSITION 6.20.— *Let $X \in \mathcal{C}$. The following holds.*

(1) *The invariants $s^k(X)$ and $rs^k(X)$ are invariant under finite étale covers $u : X' \to X$ of X. The same holds for the associated fibrations, that is $s_{X'}^k$ (resp. $rs_{X'}^k$) is the (connected part of the) Stein factorisation of $s_X^k \circ u$ (resp. $rs_X^k \circ u$).*

(2) *The fibrations s^k_X and rs^k_X are functorial in X, i.e. a dominant meromorphic map $g : X \dashrightarrow X'$ induces natural dominant meromorphic maps between the corresponding S^k's and RS^k's.*

One has next the following "strictness" property for the core decomposition, which we quote without proof.

THEOREM 6.21.— *Let $X \in \mathcal{C}$, and let $k \geqslant j \geqslant 0$ be integers. Let S (resp. R) be the general fibre of s^k_X (resp. rs^k_X). The restrictions of s^j_X and rs^j_X to S and R coincide respectively with s^j_S, rs^j_S, s^j_R and rs^j_R.*

Remark 6.22.— The sequence of $(2n+2)$ integers

$$n = s^0 \geqslant rs^0 \geqslant s^1 \geqslant rs^1 \geqslant ... \geqslant s^k \geqslant rs^k \geqslant ... \geqslant s^n \geqslant rs^n \geqslant 0$$

partitions the class of n-dimensional manifolds in $\mathcal{C}$ into a certain finite number $c(n)$ of classes, according to the number of steps and the dimensions of the steps needed to decompose the core in orbifold rational quotients and Iitaka fibrations. One has for example $c(0) = 1$, $c(1) = 3$, $c(2) = 8$ and $c(3) = 21$, by direct listing. The sequence $c(n)$ satifies $c(n+1) = 4c(n) - 4c(n-1) + c(n-2)$. From which one gets, for any $n \geqslant 0$, $c(n) = (\alpha^{n+1} - \beta^{n+1})/\sqrt{5}$, with $\alpha := (3+\sqrt{5}/2)$, and $\beta := (3-\sqrt{5}/2)$.

6.6. Deformation invariance.

We next state the following conjecture, for $X \in \mathcal{C}$ smooth.

CONJECTURE 6.23 (Deformation Conjecture).— *The integers $\lambda(X)$, $s^k(X)$ and $rs^k(X)$, for X compact smooth and Kähler, and all $k \geqslant 0$, are deformation invariants of X. Moreover, the corresponding fibrations s^k_X and rs^k_X vary holomorphically with X, on suitable models (ie: the (bimeromorphically well-defined) family of fibres of c_X should deform with X).*

In particular, ess(X) *is a deformation invariant of X, and the class of special manifolds is stable under deformation.*

We state separately the next conjecture:

CONJECTURE 6.24 (Deformation and finiteness Conjecture).— *For X compact, smooth and Kähler, the graded Essential Algebra $Ess(X)$ of*

X (see 3.5) should be finitely generated and invariant by deformation of $X \in \mathcal{C}$.

Even weaker versions of the forelast statement are difficult: is $\mathrm{ess}(X)$ upper or lower semi-continuous under smooth deformations? Is it lower semi-continuous under degenerations?

The following is important and maybe accessible: is a degeneration of special manifolds still special, in the sense that all of its irreducible components are special?

The preceding conjecture extends the classical conjecture concerning the deformation invariance of the Kodaira dimension for compact Kähler manifolds. Notice that this is the case for $k = 0$, and in the projective case, and deformation among projective manifolds, for s^1_X, by the invariance of plurigenera, due to Y.T. Siu ([Si98],[Si02]).

Of course, one could still extend the preceding conjectures to the category of orbifolds, appropriately defined.

7. The fundamental group.

In this §7, we consider the fundamental group, and conjecture ("abelianity conjecture" 7.1) that a special manifold should have an almost abelian fundamental group. This conjecture is supported and motivated by the fact that rationally connected manifolds are simply connected, that this conjecture is standard for the case $\kappa = 0$, and the preceding conjectural orbifold decomposition of any special manifold as a tower of fibrations with fibres of one of these two types (in a slightly generalised sense). This conjecture seems to sum up all conjectures with the same conclusion.

We show that this conjecture is true for linear and torsionfree solvable representations of the fundamental group, as an immediate consequence of previous results by various authors. As usual, we extend this conjecture to the orbifold case, anyway necessary to solve the non-orbifold one in higher dimensions.

7.1. The abelianity conjecture.

Our considerations here are guided by the following conjecture.

CONJECTURE 7.1. — *Let X be special. Then $\pi_1(X)$ is virtually abelian.*

Recall that a group is said to be *virtually abelian* (one also says almost abelian) if it has a subgroup of finite index which is abelian.

This conjecture is motivated by the fact that it should be true for klt orbifolds either rationally connected or with $\kappa = 0$, and the fact that a special manifold should be (after Section 6.5) a tower of orbifold fibrations with orbifold fibres in the preceding two classes.

Examples 7.2. —

(1) *Curves.* A curve is special iff its fundamental group is virtually abelian.

(2) *Rationally connected manifolds.* They have trivial fundamental group and are special ([Ca94], [Ca95], [K-M-M92]).

More generally, if r_X is the rational quotient of X (see 3.4), then it induces an isomorphism between the fundamental groups of X and $R(X)$ by [Ko93]. So that X and $R(X)$ are simultaneously special and have simultaneously virtually abelian fundamental groups. The above conjecture should thus essentially reduces to the case where $\kappa(X) \geqslant 0$ if the uniruledness conjecture is true.

(3) *Manifolds with $a(X) = 0$, or with $\kappa(X) = 0$* are special, and standard conjectures say they should have a virtually abelian fundamental group. These conjectures are thus special cases of conjecture I above.

Notice that Conjecture 7.1 is established for manifolds with $c_1(X) = 0$, which are indeed special, and have almost abelian fundamental group, by the Calabi-Yau theorem. This provides strong support for and motivates the case where $\kappa = 0$. See [Ca95] for another example of result supporting Conjecture 7.1 in the case $a(X) = 0$.

(4) *Surfaces and Threefolds.* By 3.33 and 3.39, Conjecture 7.1 holds for Kähler surfaces and projective threefolds with $\kappa \neq 2$, except maybe for the ones with $\kappa = 2$, and having a model of the Iitaka-Moishezon fibration with orbifold base a log-Enriques or log-Fano normal surface. In this case, the conjecture is open because of an insufficient knowledge of these orbifolds.

(5) *Orbifold version.* This case shows clearly that the solution of Conjecture 7.1 above should be extended to the orbifold situation. Recall that the fundamental group of (Y/Δ) is defined as the quotient of the fundamental group of the complement of Δ in Y, modulo the normal

subgroup generated by elements of the form $\lambda_j^{m_j}$, where λ_j is a small loop winding once aroud Δ_j, if $\Delta := \sum_{j\in j}(1-1/m_j)\Delta_j$. So the right form of the Abelianity conjecture is the following.

CONJECTURE 7.3. — *Let (Y,Δ) be an orbifold, with $Y \in \mathcal{C}$ smooth, and Δ of normal crossings. The fundamental group of (Y/Δ) is trivial if (Y/Δ) is κ-rationally generated, and almost abelian if (Y/Δ) is special.*

Notice that a positive answer for surfaces solves the remaining cases of the abelianity conjecture in dimension three.

(6) *Manifolds with $-K_X$ nef.* These are conjectured to have a virtually abelian fundamental group. This conjecture is thus also a consequence of Conjecture 7.1 under a positive answer to the question asked above of whether such manifolds are special. The main results concerning this conjecture are in [D-P-S93], [D-P-S93], [Pa98] and [Zh96].

(7) *Kähler manifolds X covered by $\mathbb{C}^d$.* These are special, by the results of the next Section 8.2, and S. Iitaka conjectured them to be covered by complex tori. Easy arguments show that this happens precisely if $\pi_1(X)$ is virtually abelian. So this conjecture is also a special case of the abelianity conjecture above.

7.2. Linear and solvable quotients.

Recall first (5.3) the following theorem.

THEOREM 7.4. — *Let $X \in \mathcal{C}$ be a special manifold. Let $\alpha_X : X \to \mathrm{Alb}(X)$ be the Albanese map of X. Then it is surjective, connected. Moreover, α_X has no multiple fibres in codimension one.*

Remark 7.5. — Due to 5.1, this extends and sharpens slightly a result of [Ka81].

Remark 7.6. — Assume that Conjecture 7.1 holds, or alternatively, that X is special and has almost abelian fundamental group. Easy standard arguments then show, using the surjectivity of the Albanese map and its absence of multiple fibres in codimension one, that the universal cover of X is holomorphically convex, in accordance with the conjecture of Shafarevitch, and that its Remmert reduction makes it proper over an affine complex space, which is the universal cover of its Albanese variety.

Now using [Ca01] (see also [A-N99], [Be89] and [Sim93]), we get the following result.

THEOREM 7.7. — *If X is special and if $\mu : \pi_1(X) \to G$ is a surjective morphism of groups, with G solvable and torsionfree, then G is virtually abelian. We say that torsionfree solvable quotients of $\pi_1(X)$ are almost abelian.*

Proof. — This is an easy consequence of 4.2', which says that the Albanese map of X is surjective and connected, and of corollary 4.2' of [Ca01], which says that if the Albanese map of X' is surjective for any finite étale cover X' of X, then the conclusion of 7.7 is true for X. □

Remark this applies in particular when $a(X) = 0$ (it is shown by a different method in [Ca95]), and when $\kappa(X) = 0$.

Now, using a result of [Zu97], itself based on results of N. Mok [Mo92] (see also [Si82]), and 7.7, we get the following theorem.

THEOREM 7.8. — *Let X be special, and let $\rho : \pi_1(X) \to Gl(N, \mathbb{C})$ be a linear representation. Then $G :=$ Image(ρ) is virtually abelian. We say that linear quotients of $\pi_1(X)$ are almost abelian.*

So, counter-examples to Conjecture 7.1, if any, can't be detected by linear representations of their fundamental groups.

Proof. — Let G' be the Zariski closure of G. Replacing X by a suitable étale cover, we may assume that G' is connected. We have an exact sequence of groups

$$1 \longrightarrow R \longrightarrow G' \longrightarrow S \longrightarrow 1,$$

with S semi-simple and R solvable. Consider the representation $\rho' = \sigma \circ \rho : \pi_1(X) \to S$, where $\sigma : G \to S$ is the above quotient. By [Zu97], thm. 5, p.105, we get that S is trivial (i.e. reduced to the unit element). Thus $G \subset R$ is solvable, and can be assumed to be torsionfree, passing to a suitable finite étale cover, by A. Selberg's theorem. The conclusion now follows from 7.7. □

Remark 7.9. — The above results 7.8 and 7.7 hold, more generally, for w-special manifolds (see 9.4 below).

COROLLARY 7.10. — *Conjecture 7.1 holds when $\pi_1(X)$ is linear (i.e. has a faithfull linear representation).*

An easy consequence of 7.8 is the following.

THEOREM 7.11. — *Let $X \in \mathcal{C}$ be covered by $\mathbb{C}^d$, with $d := \dim(X)$. Assume that $\pi_1(X)$ is either solvable, or linear. Then X is covered by a complex torus (i.e. S. Iitaka's Conjecture then holds for X).*

Proof. — We know that X is special by 8.2 below (one can also get directly from [K-O75] that X is w-special, which is sufficient for our purpose). Thus $\pi_1(X)$ is virtually abelian by 7.7 or 7.8 above. Replacing X by a suitable unramified cover, the conclusion thus follows from the next Lemma 7.12. □

LEMMA 7.12. — *Let $X \in \mathcal{C}$ be covered by $\mathbb{C}^d$, with $d := \dim(X)$. Assume that $\pi_1(X)$ is abelian. Then X is a complex torus.*

Proof. — Let $\alpha : X \to A$ be the Albanese map of X. By 4.2', it is surjective and connected. Let F be a generic fibre of α. It is sufficient to show that $\dim(F) = 0$, because X is covered by $\mathbb{C}^d$. Indeed, α is then birational. And so X contains a rational curve if α is not isomorphic. But this rational curve would lift after normalisation to the universal cover of X. A contradiction. Assume $\dim(F) > 0$. Then $\mathrm{image}(\pi_1(F) \to \pi_1(X))$ would be infinite, again because $\mathbb{C}^d$ does not contain positive dimensional compact subvarieties. Because $\pi_1(X)$ is abelian, this implies by dualising that the restriction map $H^0(X, \Omega^1_X) \to H^0(F, \Omega^1_F)$ is nonzero. A contradiction to the fact that F is a fibre of α. □

8. An orbifold generalisation of Kobayashi-Ochiai's extension theorem.

In this section, we establish and apply an orbifold version (see (8.2)) of the famous extension theorem of Kobayashi-Ochiai ([K-O75]), which asserts that a nondegenerate (i.e. somewhere submersive) meromorphic map ψ from a dense Zariski open subset U of a complex manifold V to a variety of general type Y extends meromorphically to V.

Our version says this still holds true even if Y is not assumed to be of general type, provided ψ factorises as $\psi = f \circ \phi$, with $f : X \dashrightarrow Y$ a fibration of general type, and $\phi : U \dashrightarrow X$ meromorphic.

The proof is an orbifold version of the proof of Kobayashi-Ochiai. This result implies among other more general results that a manifold X is special if there is a meromorphic nondegenerate map $\phi : \mathbb{C}^n \dashrightarrow X$. If ϕ

has dense image in the metric topology, the Kobayashi pseudometric of X vanishes identically, and we get a case in which conjecture III_H of §9 holds.

8.1. Statements.

The setting considered in all of this Section 8 is the following.

Notation 8.1. — Let V be a connected complex manifold, D a reduced divisor on V, and $U := V - D$. Let $\phi : U \dashrightarrow X$ be a meromorphic map, with $X \in \mathcal{C}$. We always assume $\psi := f \circ \phi : U \dashrightarrow Y$ is nondegenerate (i.e. has maximal rank $p = \dim(Y)$ somewhere). Let $f : X \dashrightarrow Y$ be a fibration, with $p := \dim(Y)$.

Our main result is the following theorem.

THEOREM 8.2. — *Let V, U, X, Y, ϕ, f be as above. Assume that f is a fibration of general type. Then:*

(a) *ψ extends meromorphically to $\psi' : V \dashrightarrow Y$,*

(b) *for any $m > 0$ sufficiently divisible and $s \in H^0(Y, m(K_Y + \Delta(f)))$, $\psi^*(s)$ extends to $(\psi')^*(s) \in H^0(V, (\Omega_V^p)^{\otimes m}((m-1)D))$.*

Remark 8.3. — The result of Kobayashi-Ochiai is the special case where $X = Y$, so that X is of general type.

The proof will be given in the next sections. It is just an orbifold modification of the original proof of [K-O75]. We shall first give some applications of this result, which are criteria of a transcendental nature ensuring that certain varieties are special.

DEFINITION 8.4. — *Let $U = V - D$ be the complement of a reduced divisor D in the connected complex manifold V. Say that the pair (V, D) is log-special if for $p > 0$, there is no rank one coherent subsheaf $L \subset (\Omega_V^p)$ and no $m > 0$ such that the complete linear system defined by $H^0(V, mL + (m-1)D)$ is of general type (i.e. defines a meromorphic map of maximal rank p).*

As a first application, we immediately get from 8.2 the following corollary.

COROLLARY 8.5. — *Let $U = V - D$ be the complement of the divisor D in the connected complex manifold V. Let $\phi : U \dashrightarrow X$ be nondegenerate. Assume that the pair (V, D) is log-special. Then X is special.*

We now consider the case when $V = X$ and D is empty. So the extension part of 8.2 is certainly trivial. Observe that the pair $(V, D) := (X, \varnothing)$ is log-special precisely if there is no Bogomolov sheaf on X. We thus recover one direction of theorem 2.27.

COROLLARY 8.6. — *Assume that X has no Bogomolov sheaf. Then X is special.*

To get some concrete applications, we take more restrictive conditions than log-specialness.

DEFINITION 8.7. — *Let $U = V - D$ be the complement of a reduced divisor D in the connected complex manifold V. Say that the pair (V, D) is log-RC if $H^0(V, (\Omega^p_V)^{\otimes m}((m-1)D))$ vanishes for any $m > 0$ (log-RC is for log-rationally connected).*

Of course, if (V, D) is log-RC, it is log-special. If V is projective rationally connected, the pair $(V, \varnothing)$ is then log-RC.

One easily checks the following (shown in a generalised form below).

Example 8.8. — The pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$ is log-RC.

From 8.2, we get:

COROLLARY 8.9. — *Assume that the pair (V, D) is log-RC. Then X is special, in the situation of 8.1 and 8.2.*

Specialising from Example 8.8, we get the following definition.

DEFINITION 8.10. — *Say that X is $\mathbb{C}^d$-dominable (resp. covered by $\mathbb{C}^d$) if there exists $\phi : \mathbb{C}^d \dashrightarrow X$, nondegenerate (resp. if the universal cover of X is isomorphic to $\mathbb{C}^d$).*

The terminology $\mathbb{C}^d$-dominable is from [B-L00]. From 8.9, we then deduce:

COROLLARY 8.11. — *Let $X \in \mathcal{C}$ be $\mathcal{C}^d$-dominable. Then X is special. In particular, if X is covered by $\mathbb{C}^d$, then X is special.*

Remark that manifolds $X \in \mathcal{C}$ covered by $\mathcal{C}^d$ are conjectured by S.Iitaka to be covered by a torus (i.e. to have a finite étale cover which is a torus). This is shown to be true for surfaces in [Ii73], and for threefolds in [C-Z99]. For X projective of arbitrary dimension, this follows (see [C-Z99]) from the standard conjectures of the Minimal Model Program. See 7.11 for another case where this conjecture is known to hold true.

We now give a geometric criterion for the pair (V, D) to be log-RC. This criterion may explain the name log-RC, and shows that being log-RC may be seen as a transcendental analogue of rational connectedness.

DEFINITION 8.12. — *We say that the pair (V, D) is H'-special if there exists a meromorphic map $\phi : U \dashrightarrow X$, where $U = W \times \mathbb{C}$, with W quasi-projective, and moreover such that ϕ is holomorphic and constant at the generic point of $W \times \{0\}$ (i.e. $\phi(w, 0) = a$, for some fixed $a \in X$ and any $w \in W$).*

Example 8.13. — The pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$ is H'-special. Blow-up a point outside of $\mathbb{P}^{n-1}$ to see this.

PROPOSITION 8.14. — *The pair (V, D) is log-RC if it is H'-special.*

The easy but lengthy proof is deferred to Section 8.8 below. Although the following clearly results from the preceding observations, we state it separately.

COROLLARY 8.15. — *Let $X \in \mathcal{C}$ be H'-special. Then X is special.*

Remark 8.16. —

(1) The preceding corollary will be linked with hyperbolicity considerations in the next section (see 9.3 below, where the notion of strong-hyperbolicity specialness is introduced).

(2) Due to 8.2, and in this situation, it is tempting to ask if X is special when V itself is special.

This is not true, because transcendental holomorphic maps do not preserve multiplicities along D. The following example shows this.

Example 8.17. — Let B, E be elliptic curves, and $a_i \in B$ be distinct points, $i = 1, \dots, 4$. Let B^* be the complement of these points in B. Let $V := E \times B$ and $U := E \times B^* \subset V$. Let now $m \geqslant 2$ be an integer. Let now X be obtained from V by applying to V the four logarithmic transformations of multiplicity m respectively to its fibres $E \times \{a_i\}$ over B. Then X is projective, and contains U as a Zariski open subset. However, V is special, while X is not.

So the specialness of V does not imply that of X, even when ϕ is the embedding of a Zariski open subset of both V and X. Except when X is a curve, as observed by M. Zaidenberg (this is an immediate consequence of [K-O75])).

8.2. Sketch of proof of the Kobayashi-Ochiai's extension theorem.

We shall very briefly first sketch the proof of the classical result of Kobayashi-Ochiai, and then indicate in Section 8.6 the changes needed to obtain the orbifold version.

We consider here only the extension part of the statement (i.e. the fact that ψ extends meromorphically to V). The second part, the control of the order of poles of $(\psi')^*(s)$, is shown in Section 8.7 by a lemma of independent interest.

Recall we want to show that if $\psi : U := (V - D) \dashrightarrow Y$ is nondegenerate, then ψ extends meromorphically to V, provided Y is of general type. The idea is thus that maps to general type varieties do not have essential singularities. This is analogous to the fact that a bounded function on the unit disc minus the origin extends across the origin. Now general type manifolds behave like manifolds with universal cover a bounded domain, in the sense that they admit pseudo-volume forms with negative Ricci form.

The proof (as explained in [Ko98], Chap. 7) goes in several steps.

(1) It is sufficient to deal with the special case $V = \mathbb{D}^p$, $D = \{0\} \times \mathbb{D}^{p-1}$, so that $U = \mathbb{D}^p - \{0\} \times \mathbb{D}^{p-1} := \mathbb{D}^{(p)}$. This is simply because meromorphic maps to a projective variety (or even a Kähler manifold) extend meromorphically across codimension 2 or more analytic subsets. The case where $\dim(V) > p$ can be reduced to the equidimensional case (see [Ko98], p.374-75).

Notations. For future reference, we denote by $\mathbb{D}(r)$ the disk of radius r centered at 0 in $\mathbb{C}$, by $\mathbb{D} := \mathbb{D}(1)$ the unit disk; $\mathbb{D}^p(r)$ and $\mathbb{D}^{(p)}(r)$ are constructed as $\mathbb{D}^p$ and $\mathbb{D}^{(p)}$ above, replacing $\mathbb{D}$ by $\mathbb{D}(r)$.

(2) Because Y is of general type, it has a pseudo-volume form w of Ricci curvature negative, bounded from above by a constant $-1/C < 0$. This form is constructed out of a basis of a linear subsystem L of mK_Y giving an embedding of Y into some projective space, once its fixed components have been removed. Another property used in the proof is that if w_j is the pseudo-volume form associated to a single element of the basis of L above, the quotient w_j/w is a smooth function on Y. This function is thus bounded. These notions and constructions are explained in more detail in Section 8.4 below.

(3) An elementary lemma (see 8.22 below), together with the argument exposed in (5) below, show that the special case **(1)** above is true, if one shows that w being the preceding pseudo-volume form on Y, then the integral of $\psi^*(w)$ converges on $\mathbb{D}^{(p)}(1/2)$.

(4) By the celebrated Ahlfors-Schwarz lemma (see 8.21 below), we have the estimate $w \leqslant C\beta$, if β is the homogeneous volume form on $\mathbb{D}^{(p)}$, with constant Ricci curvature -1, and C is the positive constant of **(2)** above. One concludes from the elementary fact that β has finite volume on $\mathbb{D}^{(p)}(1/2)$.

(5) Let us now conclude the proof in the equidimensional case. The forms $w_j := \psi^*(s_j)$ of the linear system $\psi^*(mK_Y)$ extend meromorphically to $\mathbb{D}^p$, by the steps **(2,3,4)**. But the map ψ is defined by the N-tuple of maps $(w_0 : w_1 : \cdots : w_N)$ with values in $\mathbb{P}^N$. The map ψ itself thus extends meromorphically to $\mathbb{D}^p$, as claimed.

We shall refer to [Ko98] for the details not given here. For the main points **(2,3,4)**, we shall introduce the basic definitions and properties in the Sections 8.4 and 8.6 below. Finally, the proof of the orbifold version will be given in Section 8.6.

The proof of point (b) in the statement of Theorem 8.2 will be given in the separate Section 8.7 below. Although it can be deduced from [Ko98], which contains a similar result in implicit form, we prefer to give an independent, more general statement.

8.3. Pseudo-volume and ricci forms.

We need to recall some classical notions. See, for example, [Ko98], Chapters 1 and 7 for more details.

Let M be a complex manifold. We say that v is a *pseudo-volume form* (with holomorphic degeneracies) on M if it is a form of type (p,p) on M, with $p := \dim(M)$, such that locally, in any coordinate system $(x_1, \ldots, x_p) = (x)$, one has $v{=}|a|^2 \cdot T \cdot vol_{(x)}$, with $vol_{(x)}{:=}(i^{p^2} d(x) \wedge \overline{d(x)})$, a is a holomorphic nonzero function, and T a smooth (i.e. $\mathcal{C}^\infty$) everywhere positive function locally defined on M.

For such a form v, its *Ricci form* is the real $(1,1)$-form

$$\mathrm{Ricci}(v) := -dd^c(\log T) = -i\partial\overline{\partial}(\log T),$$

and its *Ricci function* is

$$K_v := (-\mathrm{Ricci}(v))^m/v.$$

Both are well-defined independently of local coordinates, because changing charts multiplies T by the square of the modulus of the Jacobian, the log of which has vanishing dd^c.

From the definitions, one gets also immediately the functoriality of these notions. If $f : N \to M$ is a dominant holomorphic map, with $p = \dim(N)$, and if $v' := f^*(v)$, then the definitions apply to v', and one has both $\mathrm{Ricci}(v') = f^*(\mathrm{Ricci}(v))$ and $K_{v'} = f^*(K_v)$.

Remark 8.18. — The Ricci form is thus the special case for $-K_M$ of the notion of curvature form of a singular hermitian metric on a line bundle, neglecting the singular part (cohomologically significant, however) of the curvature current.

Assume that v has holomorphic degeneracies. Then we say that v has *negatively bounded Ricci curvature* if $(-\mathrm{Ricci}(v))$ is an everywhere positive or zero $(1,1)$-form, and if $-K_v \geqslant C$ for some constant $C > 0$, everywhere on M.

Observe that K_v is then nonpositive from the condition on $\mathrm{Ricci}(v)$, and that K_v may take the value $(-\infty)$ at points where a vanishes. Note that if M is compact and $-\mathrm{Ricci}(v)$ everywhere positive, then v has negatively bounded Ricci curvature.

Observe too that v' has negatively bounded Ricci curvature if and only if so has v, for $v' = f^*(v)$ as above.

Example 8.19. — We give the examples used in the proof. As said above, these are bounded domains.

(1) The unit disc. Take, in the linear coordinates,

$$v := idx \wedge \overline{dx}/(1 - |x|^2)^2.$$

Then $\mathrm{Ricci}(v) = -v$, and $K_v = -1$.

(2) The unit disc with origin removed $\mathbb{D}^*$. Take, in the linear coordinates,

$$v := idx \wedge \overline{dx}/(|x|)^2(\log(|x|)^2)^2.$$

Once again $\mathrm{Ricci}(v) = -v$, and $K_v = -1$. Actually the forms in the preceding two examples correspond under the universal covering map from $\mathbb{D}$ to $\mathbb{D}^*$. This can be checked explicitely using the Poincaré upper half plane as an intermediate step. An important property used in the proof is that $\int_{\mathbb{D}^*(r)} v < +\infty$ if $r < 1$. Of course, the domain over which the integral is taken is the disc of radius r with origin deleted.

(3) $M = \mathbb{D}^{(p)}$. Take, in the linear coordinates,

$$v := i^{p^2} dx \wedge \overline{dx}/(|x_1|)^2(\log(|x_1|)^2)^2 \prod_{2\leqslant j\leqslant p} (1 - |x_j|^2)^2.$$

One again has $\mathrm{Ricci}(v) = -v$, and $K_v = -1$. Again, one has $\int_{\mathbb{D}^{(p)}(r)} v < +\infty$.

The above volume forms have thus negatively bounded Ricci function. We give in the next section the example used in the proof in an essential way.

8.4. Pseudo-volume forms on varieties of general type.

Let Y be a p-dimensional manifold of general type, that is with K_Y big. From Kodaira's lemma, for m large enough, we can write $mK_Y = H + A$, with H effective and A very ample. Let L be a free linear subsystem of the complete linear system $|A|$, such that the associated regular map $\Lambda : Y \to \mathbb{P}^N$ ($N = \dim L$) is an embedding. Let $(s) := (s_0, s_1, \ldots, s_N)$ be a complex basis of $H + L$, and h a section of $\mathcal{O}_Y(H)$ vanishing exactly on H. Define

$$w := \Big(\sum_{j\in\{0,\ldots,N\}} i^{mp^2} s_j \wedge \overline{s_j}\Big)^{1/m}.$$

Then w is a pseudo-volume form with holomorphic degeneracies along H, since it is written in local coordinates as $w = |a|^2 w_1$, with $w_1 = (\sum_{j\in\{0,\ldots,N\}} |t_j|^2)^{1/m}\mathrm{vol}_{(x)}$, if a is a local equation for H, and $s_j = t_j(\mathrm{vol}_{(x)})^{\otimes m}, \forall j$, using the notation introduced in Section 8.2 above.

Notice that L being free, w_1 is a smooth volume form on Y. We now compute its Ricci form.

By its very definition, we have $\mathrm{Ricci}(w) = -\Lambda^*(\Theta)$, where Θ is, up to a positive constant, the curvature form of the Fubini-Study metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$.

This is in accordance with Remark 8.18: the Ricci form of w is just $-\Lambda^*(\Theta)$, since the singular part of the metric is given by h.

The Ricci form of w is thus negative everywhere. As a consequence, $K_w := (\mathrm{Ricci}(w)^p/v)$ is negative everywhere and $-\infty$ on H. By the compactness of Y, we conclude that this function is negatively bounded by some constant $-1/C$, for some $C > 0$.

Remark 8.20. — For $j \in \{0, \ldots, N\}$, define $w_j := (i^{mp^2} s_j \wedge \overline{s_j})^{1/m}$. This is still a pseudo-volume form with holomorphic degeneracies on Y. The function w_j/w is obviously smooth on Y, and thus bounded. There exists $B > 0$ such that $w_j \leqslant Bw, \forall j \in \{0, \ldots, N\}$.

8.5. The lemma of Ahlfors-Schwarz.

Despite its simplicity, it is a very powerful tool to obtain bounds on the growth of pseudo-volume forms. We give the simplest version, sufficient for our applications.

LEMMA 8.21. — *Let v be a pseudo-volume form with holomorphic degeneracies on $\mathbb{D}^{(p)}$. Assume that $K_v \leqslant -1/C < 0$ everywhere on $\mathbb{D}^{(p)}$. Then $v \leqslant \beta$, where β is the volume form on $\mathbb{D}^{(p)}$ with constant Ricci function -1 defined in Example 8.19 (3) above.*

Sketch of proof. See [Ko98], (2.4.14) for details. One can reduce, by introducing the volume forms relative to polyradii $r < 1$, to the case when the quotient smooth function $k := v/\beta$ has a maximum at an interior point b of the closure of $\mathbb{D}^{(p)}$. Obviously, v does not vanish at b. The real $(1,1)$-form $dd^c(\log k) = i\partial\overline{\partial}(\log k)$ is then nonnegative at b. By functoriality, it is sufficient to check this on a complex parametrised curve. But then, $i\partial\overline{\partial}(\log k) = (\partial^2(\log k)/\partial s^2 + \partial^2(\log k)/\partial t^2) ds \wedge dt$ in real coordinates $x = s + it$, and the claim follows.

Rescaling v, we assume that $C = 1$. Now, $i\partial\overline{\partial}(\log k) = i\partial\overline{\partial}(\log V) - i\partial\overline{\partial}(\log B) = \mathrm{Ricci}(\beta) - \mathrm{Ricci}(v) \leqslant 0$, if V and B are the functions such that

$v = V \cdot vol_{(x)}$ and $\beta = B \cdot vol_{(x)}$. We get Ricci($\beta$) $\leqslant$ Ricci(v). Hence also $-v \geqslant K_v \cdot v = \text{Ricci}(\beta)^{\wedge p} \leqslant \text{Ricci}(v)^{\wedge p} = K_\beta \cdot \beta = -\beta$, and the conclusion follows. □

We now state an extension criterion which is used crucially in the proof.

PROPOSITION 8.22.— *Let* $s \in H^0(\mathbb{D}^{(p)}, mK_{\mathbb{D}^{(p)}})$ *be such that* $\int_{\mathbb{D}^{(p)}(r)} (i^{mp^2} s \wedge \overline{s})^{1/m} < +\infty$, *for some* $r < 1$. *Then* s *extends to a meromorphic section of* $mK_{\mathbb{D}^p}$. *Moreover, the poles of this extension are of order at most* $m-1$ *along* $\{0\} \times \mathbb{D}^{p-1}$.

The proof is given in [Ko98], (7.5.7-8). The proof reduces to the case when $p = 1$ and the elementary fact that a holomorphic function on the punctured unit disc which is $\mathrm{L}_{(2/m)}$ on some $\mathbb{D}(r), r < 1$ extends meromorphically with a pole of order at most $m-1$.

8.6. Proof of the orbifold version.

Setting. We consider in this section the following data: $f : X \dashrightarrow Y$ is a fibration of general type, $\phi : U \dashrightarrow X$ is a meromorphic map which is dominant (i.e. submersive at some point), X and V are connected complex manifolds, with X compact, and $j : U \subset V$ is a Zariski dense open subset, complement of some divisor D of V. Let $\psi := f \circ \phi : U \dashrightarrow Y$. Blowing-up X and Y if needed, we shall assume that f is holomorphic, that Y is projective, and that f is high (see 1.31). We let $p > 0$ be the dimension of Y, and $\Delta := \Delta(f)$ be the multiplicity divisor for f, as defined in 1.1.4. Our purpose here is to establish that ψ extends meromorphically to V.

PROPOSITION 8.23.— *Let* $m > 0$ *be an integer such that* $m\Delta$ *is Cartier. Then:*

(1) f *defines an injection of sheaves* $f^* : \mathcal{O}_Y(m(K_Y+\Delta)) \to (\Omega^p_X)^{\otimes m}$, *after suitable modifications of* X *and* Y.

(2) In particular, let $s \in H^0(Y, m(K_Y+\Delta))$. *Then* $f^*(s) \in H^0(X, (\Omega^p_X)^{\otimes m})$. *Recall* Y *has dimension* p.

Proof.— Just as in the proof of 4.17, because we have an injection of sheaves (also denoted f^*) $f^* : \mathcal{O}_Y(m.(K_Y)) \to (\Omega^p_X)^{\otimes m}$, it is sufficient to check the above injection outside a codimension two subvariety of X, provided f is “high”.

So we first assume that we are near, in the analytic topology, the generic point of Δ_i. But the assertion follows here from an easy local computation. Indeed m_i divides $m = m'm_i$ by assumption, and we can choose local coordinates $(x) = (x_1, ..., x_d)$ near the generic point of the divisor $D_i \subset X$, a component of $f^{-1}(\Delta_i)$ mapped surjectively to Δ_i by f in such a way that in suitable local coordinates $(y) := (y_1, ..., y_p)$, near the corresponding generic point of Δ_i, one has $f(x) = (y) = (y_1, ..., y_p)$, with $y_1 = (x_1)^{m_i+m''}$, and $y_j = x_j$ for $j > 1$, with m'' some appropriate nonnegative integer. Local equations for D_i and Δ_i being respectively $x_1{=}0$ and $y_1{=}0$.

Thus we get a local generator of $f^*(\mathcal{O}_Y(m(K_Y + \Delta)))$ in the form $f^*(d(y)/y_1^{1-1/m_i})^{\otimes m}$, and this expression is equal to

$$f^*(d(y)^{\otimes m}/y_1^{m-m'}) = x_1^{(m_i+m''-1)m-(m_i+m'')(m-m')} d(x)^{\otimes m} = x_1^{m'm''} d(x)^{\otimes m},$$

up to a nonzero constant factor, where $d(y) := dy_1 \wedge \ldots \wedge dy_p$ and $d(x) := dx_1 \wedge \ldots \wedge dx_p$. Hence the claim, since the exponent of x_1 in the last expression is obviously nonnegative.

We thus see that f^* is well-defined outside the finite union B of all divisors of X which are mapped to codimension two or more analytic subsets of Y. By a suitable composition of blow-ups $u : X' {\to} X$ and $v : Y' {\to} Y$, we get $f' : X' {\to} Y'$ holomorphic such that $f \circ u = v \circ f'$, and moreover such that the strict transform B' of B in X' has all of its irreducible components mapped onto a divisor of Y' by f', depending on the component, of course.

Then $g := v^{-1} \circ f = f' \circ u^{-1} : X \dashrightarrow Y'$ is holomorphic outside a codimension two or more analytic subset A of X, the indeterminacy locus of u^{-1}.

But then $g^*(\mathcal{O}_{Y'}(m(K_{Y'} + \Delta(f')))$ injects into $(\Omega^p_X)^{\otimes m}$ over $X - A$. By Hartog's theorem, for example, this injection extends through A. Now composing with $u^* : (\Omega^p_X)^{\otimes m} {\to} (\Omega^p_{X'})^{\otimes m}$, we see that $(f')^*$ injects $\mathcal{O}_{Y'}(m(K_{Y'} + \Delta(f'))$ into $(\Omega^p_{X'})^{\otimes m}$, as asserted.

The second assertion is an immediate consequence of the first. □

The first part of this argument (in simpler form) applies to give the following proposition.

PROPOSITION 8.24.— *Let $u : Y' \to Y$ be a Δ-nice covering, in the sense of 4.16. But we choose u such that it ramifies at order exactly m_j above each component Δ_j of $\Delta(f)$. Then:*

(1) u defines an injection of sheaves $u^* : \mathcal{O}_Y(m(K_Y + \Delta)) \to \mathcal{O}_{Y'}(mK_{Y'})$.

(2) In particular, let $s \in H^0(Y, m(K_Y+\Delta))$. Then $u^*(s) \in H^0(Y', mK_{Y'})$.

We now come to the crucial point of our modification of the proof of the extension theorem of Kobayashi-Ochiai. In a first step, we proceed as in Section 8.4, by the construction of a pseudo-volume form on Y with meromorphic - not holomorphic - degeneracies.

Notations 8.25.— Let $\phi : \mathbb{D}^{(p)} \dashrightarrow X$ be such that $\psi := f \circ \phi : \mathbb{D}^{(p)} \dashrightarrow Y$ is nondegenerate. We need to extend ψ meromorphically to $\mathbb{D}^p$. Since $K_Y + \Delta$ is big on Y, we can write, by Kodaira's lemma $m(K + \Delta) = H + A$, with A very ample and H effective on Y, m being choosen sufficiently divisible (by the l.c.m of the m_i's) and sufficiently big. Let $h \in H^0(Y, \mathcal{O}_Y(H))$ be an element defining H. Let $\delta \in H^0(Y, \mathcal{O}_Y(m\Delta))$ be a section vanishing exactly on $m\Delta$, with the right multiplicities. For $\sigma \in |A|$, the complete linear system defined by H, we denote by $s := h\sigma \in H^0(Y, m(K_Y + \Delta))$.

Let now $\sigma_0, \dots, \sigma_N$ a basis of $H^0(Y, A)$ and let also $s_j := h\sigma_j$, for $j \in J := \{0, ..., N\}$. Let further $t_j := s_j/\delta, \forall j$. These are meromorphic sections of mK_Y.

Define $v := (\sum_{j=0}^{j=N} (i^{p^2 m} t_j \wedge \bar{t_j}))^{1/m}$. Thus v is a global pseudo-volume form with meromorphic degeneracies on Y.

Notice however that, by Proposition 8.23 above, $f^*(t_j)$ is, for any j, a holomorphic section of $(\Omega^p_X)^{\otimes m}$. This is simply because in local coordinates on Y, we can write $t_j = (T_j/d)dy$, for some holomorphic function T_j, where d (resp. dy) is a local generator of the ideal defining $m\Delta$ (resp. K_Y). (I thank M. Paun for this important observation). The assertion thus follows from 8.23 and the fact that dy/d is a local section of $m(K_Y + \Delta)$.

In particular, $\psi^*(t_j)$ is a holomorphic section of mK_M, with $M = \mathbb{D}^{(p)}$, and so $w := \psi^*(v)$ is naturally a pseudo-volume form with holomorphic degeneracies on M.

For the same reason, $f^*(s_j)$ is, for any j, a holomorphic section of $(\Omega^p_X)^{\otimes m}$.

Exactly in the same way we defined $w := \psi^*(v)$, we can define a pseudo-volume form with holomorphic degeneracies $\bar{w}$ on M by $\bar{w} :=$

$(\sum_{j=0}^{j=N}(i^{p^2m}\psi^*(s_j)\wedge\overline{\psi^*(s_j)})^{1/m}$. Notice that here, however, there is no pseudo-volume form $\bar{v}$ on Y such that $\bar{w}=\psi^*(\bar{v})$.

The crucial point is now the following lemma.

LEMMA 8.26. — Ricci$(\bar{w})$ = Ricci$(w) = \psi^*(-\Lambda^*(\Theta))$, *if* $\Lambda : Y \to \mathbb{P}^N$ *is the regular map defined by the complete linear system* $|A|$ *on* Y, *and* Θ *the curvature form of the Fubini-Study metric on* $\mathbb{P}^N$. *In particular,* Ricci(w) *is thus negative everywhere, and there exists* $C>0$ *such that* $K_{\bar{w}} \leqslant -1/C$ *everywhere on* $\mathbb{D}^{(p)}$.

Proof. — Write again $t_j = (T_j/d)dy$, as above. We deduce that $v = |h/d|^{2/m}V \cdot \mathrm{vol}_{(y)}$, if h is a local equation for H, and $\mathrm{vol}_{(y)}$ a local volume form on Y. Here $V = (\sum_{j\in}|T_j|^2)^{1/m}$.

Now, from 8.23 again, we see that $\psi^*(\mathrm{vol}_{(y)}/|d|^{2/m}) = |G|^2\mathrm{vol}_{(M)}$, for a certain holomorphic function G defined on $\psi^{-1}(U)$, for $U \subset Y$, the open subset where the said trivialisations and charts are defined. Recall that $-i\partial\bar{\partial}V = -\Lambda^*(\Theta)$, by 8.4 above. A computation of the Ricci curvature then gives the first assertion for w.

The assertion for $\bar{w}$ is deduced from the fact that $\bar{w} = \psi^*(|d|^{2/m})w$, so that these two pseudo-volume forms have the same Ricci-form. We then get: $K_w = \psi^*(-\Lambda^*(\Theta))^{\wedge p}/\psi^*(v) = \psi^*(-\Lambda^*(\Theta))^{\wedge p}/\psi^*(|hG^m|^{2/m}V)\mathrm{vol}_M$ And so, $K_{\bar{w}} = \psi^*(|d|^{-2/m}K_w) \leqslant (-1/C)(1/\psi^*(|h|^{2/m}V))$, because V is differentiable, and h locally bounded from above everywhere on Y.

To express things differently, one can compute the Ricci function of $\bar{w}$ symbolically, as if it were of the form $\psi^*(\bar{v})$, with
$\bar{v} := (\sum_{j=0}^{j=N}(i^{p^2m}s_j\wedge\overline{s_j}))^{1/m}$. □

By the argument given in Section 8.2, to show 8.2, it is sufficient to show that each of the $\psi^*(s_j)$'s extend meromorphically to $\mathbb{D}^p$, and this is true if the integral of w over $\mathbb{D}^{(p)}(r)$ is convergent, for some $0<r<1$.

To show this convergence, we simply need, by the Ahlfors-Schwarz Lemma, to show that w has everywhere negatively bouded Ricci function, by some constant $-1/C<0$. But this exactly what the preceding Lemma 8.26 claims. This concludes the proof of the first assertion of 8.2.

Remark 8.27. — Observe that the preceding proof shows with minor adaptations, using 8.24, the following orbifold version of Theorem 8.2. If (Y/Δ) is an orbifold of general type, Y smooth and Δ supported on a normal crossings divisor, then any nondegenerate meromorphic map

$\phi : U \dashrightarrow (Y/\Delta)$ extends meromorphically to V. Here, to say that ϕ is a meromorphic map to the orbifold (Y/Δ) means that any point $u \in U$ at which ϕ is holomorphic, with $y := \phi(u)$ a smooth point of the support of Δ, lying on Δ_i, then ϕ lifts locally around u to a holomorphic map to the ramified cover of Y near y, ramifying at order exactly m_i above Δ_i.

8.7. Extension of pluri-canonical neromorphic forms.

We now want to control the order of the poles along D of the meromorphic extension to V of $\psi^*(s) \in H^0(U, (\Omega^p_U)^{\otimes m})$, for $s \in H^0(Y, m(K_Y + \Delta))$. More precisely, notations being as above:

PROPOSITION 8.28. — *For any $m > 0$ sufficiently divisible and s any element of $H^0(Y, m(K_Y + \Delta(f)))$, $\psi^*(s)$ extends to $(\psi')^*(s) \in H^0(V, (\Omega^p_V)^{\otimes m}((m-1)D))$.*

Proof. — Recall that the support of Δ is of normal crossings. We can assume that ψ' is holomorphic, because the assertion holds for V if it does for any of its blow-ups. Recall that s can be written in local coordinates as follows, s being a section of $m(K_Y + \Delta(f))$, $s = (d(y)/(y_1^{1-1/m_1} \dots y_p^{1-1/m_p}))^{\otimes m}$, if Δ has local equation the denominator of this expression. Some of the m_i may be one, and so the corresponding terms don't contribute. Let now $(v) = (v_1, \dots, v_n)$ be local coordinates near a generic point of some component D' of D, such that D' has local equation $w := v_1 = 0$. We can assume that near this point the map ψ' is given by $\psi'(v) = (v_1^{r_1} Y_1, \dots, v_1^{r_p} Y_p)$, for integers r_i and nowhere vanishing functions Y_i of (v). Computing, we get, letting r be $r_1 + \dots + r_p$, $(\psi')^*(s) = (w^{r-1}\Theta/w^{r-s}Y')^{\otimes m}$, for some holomorphic p-form Θ on V, and Y' a nowhere vanishing function on V, the product of the Y_i's, and $s := r_1/m_1 + \dots + r_p/m_p$. This shows the proposition, because sm is a positive integer, since each m_i divides m. □

8.8. Proof that H'-specialness implies log-rational connectedness.

We prove here Proposition 8.14. We use the notations and setting of 8.1, 8.12 and 8.7.

Proof. — We just need to show that any element s' of $H^0(V, (\Omega^p_V)^{\otimes m}((m-1)D))$ vanishes, provided it is on U of the form $\phi^*(s'')$

for some $s'' \in H^0(X, (\Omega^p_X)^{\otimes m})$. For this, we shall just show that its restriction to any $F := \{w\} \times \mathbb{P}^1(\mathbb{C}) \subset V$ has to vanish for any $w \in W$. The argument is similar to, and motivated by the one showing that sections of $(\Omega^p_X)^{\otimes m}$ vanish if X is rationally connected (see [Ca95], for example).

Let Ω^q denote the restriction of Ω^q_V to F, for $q \geqslant 0$, and let N be the (trivial) dual of the normal bundle of F in V.

From the exact sequence $1 \to N \to \Omega^1 \to \Omega^1_F \to 1$, we get, for $q > 0$:

$$(@) 1 \to \wedge^q N \to \Omega^q \to \Omega^1_F \otimes \wedge^{q-1} N \to 1.$$

Notice that all these sequences are split. We now fix $q = p > 0$, and denote respectively by $B := \wedge^p N$ the kernel, and $A := \Omega^1_F \otimes \wedge^{p-1} N$ the quotient of the corresponding exact sequence (@).

From this same split exact sequence, we get now for $(\Omega^p)^{\otimes m}$ a decreasing (split) filtration by subbundles:

$$\{0\} := W_{m+1} \subset W_1 \subset .... \subset W_0 := (\Omega^p)^{\otimes m},$$

with successive quotients $W_0/W_1 = A^{\otimes m}, \ldots, W_k/W_{k+1} = A^{\otimes(m-k)} \otimes B^k$, $\ldots, W_m = B^{\otimes m}$. Notice that, for $k = 0, \ldots, m$, the bundle $A^{\otimes(m-k)} \otimes B^k$ is isomorphic to $\mathcal{O}_F(-2(m-k))$ tensorised by a trivial vector bundle N_k on F.

We now take local coordinates $((w), t)$ on V near $(w, 0)$, where $(w) := (w_1, \ldots, w_{n-1})$ are local coordinates on W, and t is a a local coordinate near $0 \in \mathbb{P}^1$. The map ϕ takes the form $\phi((w), t) = (t\phi_1, \ldots, t\phi_d)$, for holomorphic functions ϕ_i of $((w), t)$, with $d := \dim(X)$, in local coordinates near $a \in X$, since $\phi(W \times \{0\}) = \{a\}$, by our H'-speciality assumption.

Locally, if σ'' is a p-form on X near a, then $\sigma' := \phi^*(\sigma'')$ is written $\sigma' = dt \wedge \alpha + t\beta$, for α and β forms in the dw_i's. If now σ'' is a p-form, then $\sigma' = t^{p-1} dt \wedge \alpha + t^p \beta$, with the same conditions.

We now consider the case when $s' = \phi^*(s'')$ for some $s'' \in H^0(X, (\Omega^p_X)^{\otimes m})$. From the preceding remarks, we can write $s'' = s''_0 + \cdots + s''_k + \cdots + s''_m$, where each of the s''_k is the tensor product of $m-k$ terms of the form $t^{p-1} dt \wedge \alpha$, and of k terms of the form $t^p\beta$. So that, for each k, s''_k is actually the piece of degree k of s'' for the above graduation, and so turns out to be a section of the bundle $A^{m-k} \otimes B^k$ defined above, and vanishing at 0 to order $(m-k)(p-1) + kp = m(p-1) + k$.

Now, we assumed that s'' has a pole of order at most $m-1$ at the point at infinity of F. Thus s''_k defines a meromorphic section of $\mathcal{O}_F(-2(m-k)) \otimes N_k$, with N_k a trivial bundle, this section vanishing at 0 to order $m(p-1)+k$ at least, and having a pole of order $m-1$ at most at infinity.

This section thus has to vanish, because otherwise the number of poles should be the opposite of the degree. But here $(m-1)-m(p-1)-k < 2(m-k)$ for any integers $m, p > 0$ and $0 \leqslant k \leqslant m$. This concludes the proof of 8.14. □

9. Relationships with arithmetics and hyperbolicity.

The next Section 9.1 formulates and discusses two conjectures concerning the Kobayashi pseudometric of a compact Kähler manifold X. The first one (9.2) states that special manifolds are *exactly* the ones having zero Kobayashi pseudometric d_X. The second one (9.15) states that for any X, d_X is the lift by the core $c_X : X \dashrightarrow C(X)$ of the orbifold Kobayashi pseudometric of the base orbifold of c_X, defined in 9.10, and that this orbifold pseudometric is a metric outside some algebraic subset S of $C(X)$. Recall that the orbifold base of c_X is of general type (if not a point), so this second part of the Conjecture 9.15 is the orbifold version of Lang's hyperbolicity conjecture.

The next Section 9.2 is the exact analog for arithmetic geometry. First, special projective manifolds are conjectured to be *exactly* the ones which have a "potentially dense" set of rational points; then nonspecial projective manifolds X are conjectured to have their rational points mapped by the core to a proper algebraic subset of $C(X)$, this for any field of definition (finitely generated over the rationals). This naturally extends Lang's arithmetic conjecture to any orbifold X. As observed by P. Eyssidieux, one may also consider the function field versions (see [Ca01]).

We give a very brief discussion, here, because we simply refer only to known results by many authors.

We end this section by a brief discussion and comparison of specialness with two of its variants, for $X \in \mathcal{C}$: the first one (weak-specialness) means that no finite étale cover of X has a fibration onto a variety of general type; the second one (*gcd*-specialness) is defined exactly as specialness, except that multiplicity of a (pure-dimensional) fibre is classically defined using the *gcd* of the multiplicities of its components, and not their infimum. These notions lead to interesting and natural comparison problems, especially in hyperbolicity and arithmetics.

9.1. The Kobayashi pseudometric.

We refer to [Ko98] for a systematic treatment of the following notions, and to [Ca01] for a more detailed discussion.

Let X be a complex manifold. Let d_X be the *Kobayashi pseudometric* on X, which is the largest pseudometric d on X such that $h^*(d) \leqslant p_{\mathbb{D}}$ for any holomorphic map $h : \mathbb{D} \to X$, where $\mathbb{D}$ is the unit disc in $\mathcal{C}$ and $p_{\mathbb{D}}$ is the Poincaré metric on $\mathbb{D}$. Recall that a pseudometric on X is a map $d : X \times X \to [o, +\infty)$ satisfying all the axioms of a distance, except that $d(x, y)$ may be zero, even if $x \neq y$. For example, it is easy to check that: $d_{\mathbb{C}^n} \equiv 0$, and using the Ahlfors-Schwarz lemma, that: $p_{\mathbb{D}} = d_{\mathbb{D}}$.

A fundamental property of d, immediate from its definition, is that holomorphic maps are distance decreasing: if $h : X \to Y$ is any holomorphic map, then $h^*(d_Y) \leqslant d_X$. Hence if $h : \mathbb{C} \to X$ is holomorphic, then d_X vanishes on the (metric) closure H of the image of h. Thus d_X will vanish identically if, for example, $H = X$, or if any two points of X can be connected by such a chain of H_i's. Rational and elliptic curves, or complex tori, for example, have thus vanishing d.

9.1.1. Hyperbolically special manifolds.

DEFINITION 9.1. — *The manifold $X \in \mathcal{C}$ is said to be Hyperbolically Special (H-special, for short) if d_X vanishes identically on $X \times X$.*

For example, rationally connected manifolds and complex tori are H-special.

CONJECTURE 9.2 (III_H-conjecture). — *$X \in \mathcal{C}$ is special if and only if X is H-special.*

This conjecture holds for curves, rationally connected manifolds, and surfaces not of general type. A crucial class for which it is open is the class of manifolds with $\kappa = 0$, or even with $c_1(X) = 0$. The case of Calabi-Yau manifolds seems to be especially difficult. See [Vo02] for an approach to the measure-hyperbolic aspect.

Let us mention two properties shared in common by the classes of special and H-special manifolds: the surjectivity of the Albanese map (5.3 and 9.24); manifolds connected by chains of subvarieties in either class are in the class.

9.1.2. Geometric variants of H-specialness.

The vanishing of the Kobayashi pseudometric of $X \in \mathcal{C}$ should be interpreted as the existence of "many" or "large" entire curves in X (i.e. nonconstant holomorphic maps $h : \mathbb{C} \to X$).

We now define a class of manifolds by a geometric condition of this type, strong enough to show specialness.

DEFINITION 9.3. — *Let X be a compact complex connected manifold. Recall (8.12) we said that X is H'-special if there exists a meromorphic nondegenerate map $\psi : W \times \mathbb{C} \dashrightarrow X$ such that:*

(1) W is quasi-projective,

(2) ψ is constant along $W \times \{0\}$ (i.e. takes on a constant value $\psi(w) = a \in X$ at the generic point w of $W \times \{0\}$).

We say that X is strongly H-special (SH-special, for short) if the map ψ can be chosen such that, moreover:

(3) the image of ψ is dense in X (for the metric topology, of course).

Notice that $X \in \mathcal{C}$ is H-special if it is SH-special. Possibly the H'-specialness and SH-specialness might be equivalent properties. Examples of SH-manifolds are complex tori, rationally connected manifolds, $\mathbb{C}^n$-dominable manifolds, and manifolds covered by $\mathbb{C}^n$.

Remark 9.4. — Rephrased in our terminology, the main result of [B-L00] says that $\mathbb{C}^2$-dominability, H'-specialness, and SH-specialness are equivalent properties for projective surfaces.

This may however be a low-dimensional phenomenon, due to the fact that rationality and rational-connectedness are equivalent in dimension two. See below for questions naturally arising in this context.

Let us ask:

Question 9.5. — Let X be a projective rationally connected threefold:

(1) Is X $\mathbb{C}^3$-dominable?

(2) If X is $\mathbb{C}^3$-dominable, is it unirational?

The answers are not necessarily expexted to be affirmative. For example, there is no obvious reason that a general three-dimensional quartic-which is not expected to be unirational-should be $\mathbb{C}^3$-dominable. Another interesting example is K. Ueno's threefold U: it is rationally connected and $\mathbb{C}^3$-dominable, but it is unknown if it is unirational. Recall ([Ue75], 11.7.1, p. 137) that U is the quotient of E^3, E the Gauss elliptic curve, by the cyclic group of order four generated by the multiplication by $\sqrt{-1}$ on each factor.

In Section 8.2, we proved the following result, as a consequence of an orbifold version of Kobayashi-Ochiai extension theorem.

Theorem 9.6. — *Let X be H'-special. Then X is special. Thus, in particular, X is special if it is in $\mathcal{C}$, and either SH-special, or $\mathbb{C}^n$-dominable, or covered by $\mathbb{C}^n$, with $n := \dim(X)$.*

A positive answer to the following (difficult) question would then establish that H-special manifolds are special (i.e. one half of conjecture III_H).

Question 9.7. — If X is H-special, is it H'-special?

9.1.3. Fundamental group and H-specialness.

By combining the abelianity Conjecture 7.1 and Conjecture III_H (9.2), we obtain some conjectures about the fundamental group of H-special manifolds.

Conjecture 9.8. — *Let $h : \mathbb{C} \to X$ be a holomorphic map. Assume that $X \in \mathcal{C}$, and that h has metrically dense image. Then $\pi_1(X)$ is almost abelian.*

One might even just assume that the image of h is Zariski dense. The gap between this conjecture and the proved results can be measured by the fact that the much easier conjecture of S. Iitaka (which is equivalent to the particular case when X is covered by $\mathbb{C}^n$ above) is still open.

9.1.4. The orbifold Kobayashi pseudometric.

The conjectural description of the Kobayashi pseudometric for an arbitrary X in $\mathcal{C}$ using Conjecture III_H starts with the following remark.

Remark 9.9. — Let $c_X : X \to C(X)$ be the core of $X \in \mathcal{C}$. If Conjecture 9.2 holds, then d_X vanishes on every fibre of c_X, by the metric continuity of d_X, and there exists a unique pseudometric δ_X on $C(X)$ such that $d_X = (c_X)^*(\delta_X)$. We shall now give a conjectural description of δ_X as the orbifold Kobayashi pseudometric on $(C(X)/\Delta(c_X))$, which needs first to be defined.

DEFINITION 9.10. — *Let (Y/D) be an orbifold. An orbifold map $h : \mathbb{D} \to (Y/D)$ is a holomorphic map $h : \mathbb{D} \to Y$ such that, for any $z \in \mathbb{D}$ such that $a = h(z)$ is a smooth point $a \in D_j$ of the support of D, the i^{th}-derivative $h^{(i)}(z)$ belongs to the tangent space $T_a(D_j)$ for $i = 0, 1, \dots, m_j - 1$, if $D := \sum_{k\in K}(1 - 1/m_k)D_k$.*

Define now $d_{(Y/D)}$ as the largest pseudometric d on Y such that $h^*(d) \leqslant d_{\mathbb{D}}$, for any orbifold map $h : \mathbb{D} \to (Y/D)$.

An orbifold will be, of course, said to be H-special if its orbifold Kobayashi pseudometric vanishes. The orbifold version of conjecture III_H now becomes (with the same name):

CONJECTURE 9.11 (III_H-conjecture). — *Let (Y/D) be an orbifold, with $Y \in \mathcal{C}$ smooth, and the support of D of normal crossings. Then (Y/D) is special if and only if it is H-special.*

A relative version of Conjecture III_H is:

CONJECTURE 9.12. — *Let $f : X \to Y$ be an admissible model of a special fibration defined on $X \in \mathcal{C}$. Then $d_X = f^*(d_{(Y/\Delta(f))})$. In particular, $\delta_X = d_{(C(X)/\Delta(c_X))}$ is the Kobayashi pseudometric of the base orbifold of c_X. Recall that δ_X was defined in Remark 9.9 above.*

This conjecture can be proved in the next two particular cases.

THEOREM 9.13. — *Let X be smooth and projective. Let $r_X : X \to R(X)$ be its rational quotient (see 3.23). Then $d_X = (r_X)^*(d_{R(X)})$. (No orbifold structure appears, here, because r_X has no multiple fibre in codimension one, by 3.29).*

Proof. — Clearly, there exists a pseudometric d on $Y := R(X)$ such that $d_X = r_X^*(d)$. We have $d \geqslant d_Y$, by the decreasing property of d_X under holomorphic maps. When Y is a curve, the result holds true, because r_X has then a section, by [G-H-S01]. The general case follows from the algebraic approximation result of [D-L-S94]. □

One can also rephrase in our terminology the main results of [B-L00] and [Lu01] to give another case of a positive answer to the second part of conjecture IV_H.

THEOREM 9.14. — *Let X be smooth and projective. Let $f : X \to C$ be a fibration over a curve, with generic fibre an abelian variety. Then $d_X = (r_X)^*(d_{C/\Delta(f)})$.*

9.1.5. The core and the Kobayashi pseudometric.

We can now give a conjectural qualitative description of the Kobayashi pseudometric on an arbitrary $X \in \mathcal{C}$, using the core of X.

CONJECTURE 9.15 (IV_H-conjecture). — *Let (Y/D) be an orbifold of general type, with Y smooth and D of normal crossings. Then $d_{(Y/D)}$ is a metric outside some proper algebraic subset $S_D \subset Y$.*

This is thus simply the orbifold extension of Lang's hyperbolicity conjecture (see [La86] and [La91]).

Notice that, combined with Conjecture 9.12, it gives the following simple description of d_X for arbitrary manifolds in $\mathcal{C}$, namely:

CONJECTURE 9.16. — *Let $X \in \mathcal{C}$. Then $d_X = c_X^*(\delta_X)$, where $\delta_X = d_{(C(X)/\Delta(c_X))}$ is the Kobayashi pseudometric of the base orbifold of c_X. And δ_X is a metric outside some proper algebraic subet S of $C(X)$.*

Remark 9.17. — One can also consider in Conjectures III_H and IV_H the p-dimensional Eisenman length function on Ω_X^p, for arbitrary $p > 0$. It is then natural to conjecture that this length function vanishes identically on X if and only if $p > \text{ess}(X)$, and is obtained as the pull-back by c_X of the appropriately defined orbifold Eisenman length function on $(C(X)/\Delta(c_X))$ if $p \leqslant \text{ess}(X) = \dim(C(X))$. And that this orbifold length function should then be nonzero outside some proper algebraic subset of $C(X)$.

9.2. Arithmetics.

We assume now that X is a projective complex manifold, and that $K \subset \mathbb{C}$ is a field of definition of X, finitely generated over $\mathbb{Q}$. We denote by $X(K)$ the set of K-rational points of X.

PROPOSITION 9.18. — *The fibrations c_X and J_X are defined over K.*

Proof. — This follows from standard arguments on Galois group operations, as explained to me by F. Bogomolov. See [Ca01]. We actually do not use it here.

DEFINITION 9.19. — *The complex projective manifold X is said to be arithmetically special (A-special, for short) if $X(K')$ is Zariski dense in X, for some finite extension K'/K. This notion is usually called "potential density". The above terminology gives a more unified point of view.*

One could of course replace above the Zariski topology by the metric topology. It is then a difficult question whether the two notions of aritmetical specialness would coincide.

CONJECTURE 9.20 (III_A-conjecture). — *The complex projective manifold X is special if and only if it is A-special.*

This conjecture is presently known to hold for curves and Abelian varieties, but not even for rationally connected manifolds, although the analogous property in the complex function field case is known to be true (see [Ca01] for a more detailed discussion).

CONJECTURE 9.21 (IV_A-conjecture). — *Let $c_X : X \to C(X)$ be the core of a nonspecial projective manifold X, defined over K, a subfield of $\mathbb{C}$ finitely generated over the rationals. Let K'/K be any finite extension. There exists a proper algebraic subset $S \subset C(X)$ such that $c_X(X(K')) \subset S$.*

Remark 9.22. — It is actually in this situation natural to define $(Y/D)(K)$, the set of K-rational points of an orbifold (Y/D) defined over K, and to conjecture that this set is contained in a proper algebraic subset of Y if (Y/D) is of general type. And moreover that $c_X(X(K))$ is contained in $(C(X)/\Delta(c_X))$ if c_X is the core of X, any complex projective manifold defined over K. E. Peyre explained me how to define $(Y/D)(K)$ ([Pe01]). The definition is inspired from the function field case, discussed in [Ca01].

We end this short discussion with the particular case of subvarieties of complex tori, a small class of varieties for which the preceding conjectures have been essentially solved.

9.3. Subvarieties of complex tori.

THEOREM 9.23. — *Let W be a subvariety of a complex torus T. Assume that W is of general type. Then,*

(1) d_W is a metric on W if T is a simple torus.

(2) $W(K)$ is almost contained *in $S(W)$, if K is a field of definition of W, where $S(W)$ is the projective vatiety which is the union of maximal complex subtori of T contained in W. Being almost contained in $S(W)$ means that the set of points of $W(K)$ not belonging to $S(W)$ is finite.*

The above hypothesis that T is a simple torus above is certainly unnecessary: the conclusion should then be that d_W is a metric on $(W - S(W))$. This should be proved by refining the proof of (1).

Proof. — (1) follows from the theorem of Bloch-Ochiai([Bl26], [Oc77]) asserting that there is no nonconstant map from the complex line to W. This implies that d_W is a metric ([Ko98], [Ko76]). The statement (2), which contains Mordell's conjecture, is simply the main result of [Fa94]. □

We showed in 5.3 that the Albanese map of a special manifold is, among other properties, surjective. Due to III_H- and III_A-conjectures, this should imply that the same holds for H-special and A-special manifolds. Indeed:

COROLLARY 9.24. — *Let $\alpha_X : X \to \mathrm{Alb}(X)$ be the Albanese map of a manifold $X \in \mathcal{C}$. Then α_X is surjective if either*

(1) X is H-special, and Alb(X) *is a simple torus, or*

(2) X is A-special.

Proof. — Assume not. Let then $u : Z := \alpha_X(X) \to W$ be the Ueno reduction of $Z \subset \mathrm{Alb}(X)$. Then W is a subvariety of general type and positive dimension of some quotient torus T of Alb(X). The conclusion thus follows from 9.23, just above.

Remark 9.25. — Of course, due to conjectures III_H and III_A, one concludes that α_X should be connected and without multiple fibres in codimension one if X is either H- or A-special. The connected part of the assertion should follow from extending 9.23 to the case of varieties W of general type and generically finite over some complex torus. It is possible that the techniques of Noguchi-Winkelmann might give the answer for multiple fibres, in the case of H-special manifolds.

We now define two other notions of specialness and core, which need to be compared with the one defined and used above.

9.4. Weakly-special manifolds.

DEFINITION 9.26. — *$X \in \mathcal{C}$ is said to be weakly-special (w-special for short), if there is no pair (u, f') in which $u : X' \to X$ is a finite étale covering, and $f' : X' \dashrightarrow Y'$ is a surjective meromorphic map onto a manifold Y' of positive dimension and of general type in the usual sense.*

From 1.8, we get the following proposition.

PROPOSITION 9.27. — *If X is special, it is w-special.*

Proof. — It is the same as in 2.33. Assume by contradiction that a triple (u, X', f') as in the preceding definition exists. We can assume the cover u to be Galois, of group G. If f' is G-equivariant, then f' descends to $f : X \dashrightarrow Y := (Y'/G)$, and f is of general type by 1.8. Otherwise, just replace f' by the least upper bound f" of the finite family $(f' \circ g)_{g \in G}$. Now f" is G-equivariant, and maps X' to Y", which is of general type, by 2.30. □

Example 9.28. — All the examples listed in 2.3 are thus w-special.

Notice that, contrary to the case of special manifolds, it is obvious that any finite étale cover of a w-special manifold is again w-special. For surfaces, the converse is also true.

PROPOSITION 9.29. — *A weakly special surface is special.*

Proof. — The list of special surfaces given in 3.33 shows the claim, once one observes that for an elliptic fibration $f : X \to C$ from a surface X, the absence of multiple fibres implies the existence of a reduced component in each fibre, which follows, for example, from the list of singular fibres given in [B-P-V84]. □

The same assertion for threefolds is not true, as was asked and expected in [Ca01]. The following example is due to Bogomolov-Tschinkel.

THEOREM 9.30 ([B-T02]). — *There exists a simply-connected projective threefold X which is weakly-special, but not special.*

Their construction can be briefly sketched as follows: $X = B \times_C S$, where B is an elliptic surface with $\kappa(B) = 1$, equipped with a second non-elliptic fibration $f : B \to C = \mathbb{P}^1$ having a fibre F such that $B - F$ is simply-connected, while $g : S \to C$ is another simply-connected elliptic surface such that $g : S \to C$ has some multiple fibre. The elliptic fibration $\phi : X \to B$ (which is the core of X) is thus of general type, although its base B is a special surface with $\kappa = 1$. Because X is simply-connected, and does not map to a surface or curve of general type, it is W-special. □

9.5. The related conjecture of Abramovich and Colliot-Thélène.

It has been conjectured (see [H-T00]) by D. Abramovich and J.L. Colliot-Thélène that weakly-special manifolds defined over number fields are potentially dense (i.e. A-special).

The above Example 9.30 of Bogomolov-Tschinkel thus shows that their conjecture is not compatible with the Conjecture III_H (9.20) above: their conjecture claims that the above X is A-special, if X and ϕ are defined over a number field K, which can easily be realised, while 9.20 claims that, for any finite extension K'/K, $X(K')$ does not map to a Zariski dense subset of $B(K')$. So let us ask explicitely:

Question 9.31. — Let X be a Bogomolov-Tschinkel threefold (as in 9.30 above), defined over a number field K. Is then X A-special?

The natural approach to this question is to show that, if (B/D) is the base orbifold of the core $\phi : X \to B$ of X, then $(B/D)(K')$ is contained in some proper algebraic subset $S \subset B$, this for any number field $K', K \subset K'$. A simpler question is asked in 9.35, below.

One may, of course, also consider the (much easier) hyperbolic version of 9.31.

Another variant of the core has been constructed by D. Abramovich in [Ab97], motivated by Harris's conjecture that varieties with no map onto a variety of general type should be potentially dense, if defined over a number field. Abramovich construction enjoys many nice properties, but is very unstable with respect to finite étale covers. This conjecture of Harris has been thus disproved in [CT-S-S97]. I thank J.L. Colliot-Thélène and D. Abramovich for learning me the references above.

Recall from Proposition 5.3, a property common to special and weakly-special manifolds:

PROPOSITION 9.32. — *Let X be w-special. The Albanese map of X is then surjective and connected.*

Observe that, in contrast to the case X special, this result does not assert that the Albanese map of X is multiplicity free if X is w-special.

9.6. Classical *gcd*-multiplicities.

We can still introduce a third notion of special manifold, which interpolates between the preceding two ones (weakly-special, and special). It is based on the classical definition of multiple fibres, using *gcd* instead of *inf*. We shall be brief on this. The first version of the present paper was written in terms of these *gcd*-multiplicities. After reading it, S. Lu also observed independently that one could, without changing proofs or statements of this first version, replace these *gcd*-multiplicities by the *inf*-multiplicities used in the present text.

DEFINITION 9.33. — *Let $f : X \to Y$ be a fibration, and let $\Delta \subset Y$ be an irreducible reduced divisor. Let $f^*(\Delta) =: (\sum_{j\in J} m_j D_j) + R$, where $f(D_j) = \Delta, \forall j$, while $f(R)$ has codimension two or more in Y. Define the gcd-multiplicities m^- as $m^-(\Delta, f) := gcd(m_j, j \in J)$. Obviously, m^- divides the multiplicity used in the present text.*

Now using this definition of multiplicity, one can define, exactly as we did, with the same proofs, the notion of base orbifold and Kodaira dimension of a fibration. This leads to the *gcd*-versions of fibration of general type, and special manifold. The *gcd*-core can be constructed also with the same properties. One property only is possibly lost: *gcd*-fibrations of general type are no longer naturally in bijective correspondance with Bogomolov sheaves.

On the other hand, the additivity result 4.2 and its proof, even becoming slightly simpler at a point, remain. So it may happen that the *gcd*-core is still a fibration of *gcd*-general type.

Let us notice the following obvious points: a special manifold is *gcd*-special, because a fibration of general type is of *gcd*-general type. A *gcd*-special manifold is w-special, by the *gcd*-version of 1.8. Observe that for surfaces, the three notions coincide, because the two extreme do, by the previous section.

At this point, one may wonder:

(a) Are the two theories actually different?

(b) If yes, which one is the "right" one?

Concerning the second question (b) above, assuming a positive answer to (a): it seems that the version given here (with inf, not *gcd*) is the right one. This is supported by the correspondance with Bogomolov sheaves, which provides a direct link between fibrations of inf-general type and geometric positivity properties of cotangent sheaves. Moreover, the inf-notion is obviously better suited to the study of Kobayashi pseudo-metric.

The only feature in favour of the *gcd*-theory is that it is more closely related to the fundamental group. This property however does not seem, by far, compensate the other advantages of the inf-notion.

Concerning (a), although no counterexamples are known, it seems plausible that the three notions are distinct.

The first possible example would be a *gcd*-special, but not special threefold. Observe that the example of Bogomolov-Tschinkel (9.30) is w-special, but not *gcd*-special.

There are two possible sources of threefolds which may be *gcd*-special, but not special: fibrations onto a curve with generic fibre either $K3$ or special surfaces with $\kappa(F) = 1$. Such a fibration should be then of general type, but not of *gcd*-general type. The construction of such examples would then give a negative answer to the following question, which has a positive answer when F is an abelian or rational surface, for example:

Question 9.34.— Let $f : X \to C$ be a fibration of a threefold X to a curve C, with generic fibre F a special surface. Does one have $\Delta(f) = \Delta^-(f)$?

We now state separately Conjectures IV_H and IV_A in the special case that the base is a curve. Because this case seems accessible by extending the existing techniques, used to solve the *gcd*-version, already known.

PROPOSITION 9.35.— *Let $f : X \to C$ be a fibration of $X \in \mathcal{C}$ onto a curve C. Assume that f is of general type. Let $h : \mathbb{C} \to S$ be a holomorphic map. Then $f \circ h$ is constant.*

Proof. — This is clear, by Liouville's theorem and uniformisation, if the genus g of C is two or more. When $g < 2$, this is an immediate application of Nevanlinna theory, using [Nog76]. I thank J. Noguchi for this reference. □

PROPOSITION 9.36. — *Let f be as above, of general type. $X(K)$ contained in finitely many of the fibres of f, if K is a field of definition for f, and finitely generated over $\mathbb{Q}$.*

An affirmative answer can be deduced from the orbifold version of Falting's theorem ([D-G95]).

Remark 9.37. — An interesting case is that of general type fibrations $f : S \to \mathbb{P}^1$ with smooth fibres of general type, and S a simply-connected surface. They provide new test cases for Lang's conjectures.

Concerning the easier *gcd*-version:

PROPOSITION 9.38. — *Let $f : X \to C$ be a fibration of X, projective, onto a curve C. Assume that f is of gcd-general type.*

(1) Let $h : \mathbb{C} \to X$ be a holomorphic map. Then: $f \circ h$ constant.

(2) $X(K)$ is contained in finitely many of the fibres of f, if K is a field of definition for f, and finitely generated over $\mathbb{Q}$.

Proof. — There is a finite étale cover X' of X which maps onto a curve of general type. The conclusion now follows for (1) from uniformisation for curves, and for (2) from [Fa94] and the theorem of Chevalley-Weil (see [H-S00], exercise C7, p.292) □

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