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Jedrzej ŚNIATYCKI

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ORBITS OF FAMILIES OF VECTOR FIELDS ON SUBCARTESIAN SPACES

by Jędrzej ŚNIATYCKI

Contents.

1. Introduction.

This work is motivated by the program of Poisson reduction of Hamiltonian systems. Under the assumption that the action of the symmetry group G on the phase space P of the system is proper, the orbit space $S = P/G$ is stratified by orbit type ([9]). For a Hamiltonian system, each stratum of S is singularly foliated by symplectic leaves ([5], [21]). The orbit space S has a differential structure $C^\infty(S)$ given by push-forwards to S of

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G-invariant smooth functions on P ([17], [8]). Moreover, $C^\infty(S)$ has the structure of a Poisson algebra, usually called the reduced Poisson algebra. Following the approach initiated by Sjamaar and Lerman ([21]), we want to describe strata of the stratification of S as well as leaves of the singular foliation directly in terms of the reduced Poisson algebra $C^\infty(S)$.

The essential property of a smooth stratified space needed here is the fact that it is a smooth subcartesian space. The notion of a subcartesian space was introduced by Aronszajn ([1]), and subsequently developed by Aronszajn and Szeptycki ([2], [3]), and by Marshall ([13], [14]). Related notions were independently introduced and studied by Spallek ([24], [25]).

A smooth subcartesian space is a diffferential space in the sense of Sikorski ([18], [19], [20]), that is locally diffeomorphic to a subset of a Cartesian space $\mathbb{R}^n$. Hence, we can use the differential space approach and study properties of a subcartesian space in terms of its ring of globally defined smooth functions.

In this paper, we generalize to smooth subcartesian spaces the theorem of Sussmann on orbits of families of vector fields on manifolds ([27]), and investigate its applications. In order to do this, we must first extend to subcartesian spaces the results on the relationship between derivations and local one-parameter local groups of diffeomorphisms of locally semi-algebraic differential spaces obtained in [23].

Let S be a smooth subcartesian space, and $X : C^\infty(S) \to C^\infty(S) : h \mapsto X \cdot h$ be a derivation of $C^\infty(S)$. A curve $c : I \to S$, where I is an interval in $\mathbb{R}$, is an integral curve of X if

$$\frac{d}{dt}h(c(t)) = (X \cdot h)(c(t)) \ \text{ for all } \quad h \in C^\infty(S), \ t \in I.$$

We show that, for every derivation X of $C^\infty(S)$ and every $x \in S$, there exists a unique maximal integral curve of X passing through x.

We define a vector field on a smooth subcartesian space S to be derivation that generates a local one-parameter group of local diffeomorphisms of S. Let $\mathcal{F}$ be a family of vector fields on S. An orbit of $\mathcal{F}$ through a point $x \in S$ is the maximal set of points in S which can be joined to x by piecewise smooth integral curves of vector fields in $\mathcal{F}$. In other words, $y \in S$ belongs to the orbit of $\mathcal{F}$ through x if there exist a positive integer m, vector fields $X^1, \ldots, X^m \in \mathcal{F}$ and $(t_1, \ldots, t_m) \in \mathbb{R}^m$ such that

$$y = (\varphi_{t_m}^{X^m} \circ \cdots \circ \varphi_{t_1}^{X^1})(x).$$

We introduce the notion of a locally complete family of vector fields. A family $\mathcal{F}$ is locally complete if, for every $X, Y \in \mathcal{F}$, $t \in \mathbb{R}$, and

$x \in S$, for which the push-forward $\varphi_{t*}^X Y(x)$ is defined, there exist an open neighbourhood U of x and a vector field $Z \in \mathcal{F}$ such that the restriction of $\varphi_{t*}^X Y$ of U coincides with the restriction of Z to U. In particular, a family consisting of a single vector field X on S is locally complete because, for every $t \in \mathbb{R}$, $\varphi_{t*}^X X$ coincides with the restriction of X to the domain of $\varphi_{t*}^X X$. Thus, the notion of local completeness of a family of vector fields is unrelated to completeness of vector fields constituting the family.

MAIN THEOREM. — *Each orbit of a locally complete family of vector fields on a smooth subcartesian space S is a smooth manifold, and its inclusion into S is smooth.*

We refer to the partition of S by orbits of $\mathcal{F}$ as the singular foliation of S defined by $\mathcal{F}$, and to orbits of $\mathcal{F}$ as leaves of the singular foliation. In the case when S is a smooth manifold, and $\mathcal{F}$ is a locally complete family of smooth vector fields on S, orbits of $\mathcal{F}$ give rise to a singular foliation of S in the sense of Stefan ([26]). Stefan's definition of a singular foliation of a smooth manifold contains a condition of local triviality, similar to a local triviality of a stratification (see Section 6). Orbits of a locally complete family of vector fields on a subcartesian space need not satisfy an obvious extension of Stefan's condition.

We show that the family $\mathcal{X}(S)$ of all vector fields on a subcartesian space S is locally complete. The singular foliation of S defined by $\mathcal{X}(S)$ is minimal in the sense that, for every family $\mathcal{F}$ of vector fields on S, orbits of $\mathcal{F}$ are contained in orbits of $\mathcal{X}(S)$. In particular, the restriction of $\mathcal{F}$ to each orbit M of $\mathcal{X}(S)$ is a family $\mathcal{F}_M$ of vector fields on M, and orbits of $\mathcal{F}$ contained in M are orbits of $\mathcal{F}_M$.

We show that smooth stratified spaces are subcartesian. All stratified spaces considered here are assumed to be smooth. A stratified space S is locally trivial if it is locally diffeomorphic to the product of a stratified space and a cone ([16]). We introduce the notion of a strongly stratified vector field on a stratified space, and prove that the family of all strongly stratified vector fields on a locally trivial stratified space S is locally complete and that its orbits are strata of S. Hence, each stratum of S is contained in an orbit of the family $\mathcal{X}(S)$ of all vector fields on S. Moreover, we show that if S is a locally trivial stratified space then orbits of the family $\mathcal{X}(S)$ of all vector fields on S also give rise to a stratification of S. If the original stratification of S is minimal, then it coincides with the stratification by orbits of $\mathcal{X}(S)$. These results on stratified spaces are applied to describe singular Poisson reduction of Hamiltonian systems.

We discuss also subcartesian Poisson spaces and almost complex spaces. A combination of these two structures gives rise to a generalization to subcartesian spaces of stratified Kähler spaces studied by Huebschmann [11].

2. Differential spaces.

We begin with a review of elements of the theory of differential spaces ([20]). Results stated here will be used in our study of vector fields on subcartesian spaces.

A differential structure on a topological space R is a family of functions $C^\infty(R)$ satisfying the following conditions:

2.1. The family

$$\{f^{-1}((a,b)) \mid f \in C^\infty(R),\, a, b \in \mathbb{R}\}$$

is a sub-basis for the topology of R.

2.2. If $f_1, \ldots, f_n \in C^\infty(R)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \ldots, f_n) \in C^\infty(R)$.

2.3. If $f : R \to \mathbb{R}$ is such that, for every $x \in R$, there exist an open neighbourhood U_x of x and a function $f_x \in C^\infty(R)$ satisfying

$$f_x \mid U_x = f \mid U_x,$$

then $f \in C^\infty(R)$. Here the vertical bar $\mid$ denotes the restriction.

A differential space is a topological space endowed with a differential structure. Clearly, smooth manifolds are differential spaces.

LEMMA 1. — *For every open subset U of a differential space R and every $x \in U$, there exists $f \in C^\infty(R)$ satisfying $f \mid V = 1$ for some neighbourhood V of x contained in U, and $f \mid W = 0$ for some open subset W of R such that $U \cup W = R$.*

Proof follows ref. [20]. — Let U be open in R and $x \in U$. It follows from condition 2.1 that there exist a map $\varphi = (f_1, \ldots, f_n) : R \to \mathbb{R}^n$, with $f_1, \ldots, f_n \in C^\infty(R)$, and an open set $\widetilde{U} \subseteq \mathbb{R}^n$ such that $x \in \varphi^{-1}(\widetilde{U}) \subseteq U$. Since $\varphi(x) \in \widetilde{U} \subseteq \mathbb{R}^n$, there exists $F \in C^\infty(\mathbb{R}^n)$ such that $F \mid \widetilde{V} = 1$ for some neigbourhood $\widetilde{V}$ of $\varphi(x)$ in $\mathbb{R}^n$ contained in $\widetilde{U}$, and $F \mid \widetilde{W} = 0$ for some open set $\widetilde{W}$ in R^n such that $\widetilde{U} \cup \widetilde{W} = \mathbb{R}^n$. Since φ is continuous, $V = \varphi^{-1}(\widetilde{V})$ and $W = \varphi^{-1}(\widetilde{W})$ are open in V. Moreover, $\varphi^{-1}(\widetilde{U}) \subseteq U$ and $\widetilde{U} \cup \widetilde{W} = \mathbb{R}^n$ imply that $U \cup W = R$. By condition 2.2, $f = F(f_1, \ldots, f_n) \in C^\infty(R)$. Furthermore, $f \mid V = F \circ \varphi \mid V = F \mid \varphi(V) = F \mid \widetilde{V} = 1$. Similarly, $f \mid W = F \mid \widetilde{W} = 0$, which completes the proof. □

A continuous map $\varphi : S \to R$ between differential spaces S and R is smooth if $\varphi^* f = f \circ \varphi \in C^\infty(S)$ for every $f \in C^\infty(R)$. A homeomorphism $\varphi : S \to R$ is called a diffeomorphism if φ and φ^{-1} are smooth.

If R is a differential space with differential structure $C^\infty(R)$ and S is a subset of R, then we can define a differential structure $C^\infty(S)$ on S as follows. A function $f : S \to \mathbb{R}$ is in $C^\infty(S)$ if and only if, for every $x \in S$, there is an open neighborhood U of x in R and a function $f_x \in C^\infty(R)$ such that $f|(S \cap U) = f_x|(S \cap U)$. The differential structure $C^\infty(S)$ described above is the smallest differential structure on S such that the inclusion map $\iota : S \to R$ is smooth. We shall refer to S with the differential structure $C^\infty(S)$ described above as a differential subspace of R. If S is a closed subset of R, then the differential structure $C^\infty(S)$ described above consists of restrictions to S of functions in $C^\infty(R)$.

A differential space R is said to be locally diffeomorphic to a differential space S if, for every $x \in R$, there exists a neighbourhood U of x diffeomorphic to an open subset V of S. More precisely, we require that the differential subspace U of R be diffeomorphic to the differential subspace V of S. A differential space R is a smooth manifold of dimension n if and only if it is locally diffeomorphic to $\mathbb{R}^n$.

Let R be a differential space with a differential structure $C^\infty(R)$. A derivation on $C^\infty(R)$ is a linear map $X : C^\infty(R) \to C^\infty(R) : f \mapsto X \cdot f$ satisfying Leibniz' rule

$$X \cdot (f_1 f_2) = (X \cdot f_1) f_2 + f_1 (X \cdot f_2). \tag{1}$$

We denote the space of derivations of $C^\infty(R)$ by $\operatorname{Der} C^\infty(R)$. It has the structure of a Lie algebra with the Lie bracket $[X_1, X_2]$ defined by

$$[X_1, X_2] \cdot f = X_1 \cdot (X_2 \cdot f) - X_2 \cdot (X_1 \cdot f)$$

for every $X_1, X_2 \in \operatorname{Der} C^\infty(R)$ and $f \in C^\infty(R)$.

LEMMA 2. — *If $f \in C^\infty(R)$ is a constant function, then $X \cdot f = 0$ for all $X \in \operatorname{Der} C^\infty(R)$.*

Proof. — If $f \in C^\infty(R)$ is identically zero, then $f^2 = f = 0$, and Leibniz' rule implies that $X\cdot f = X\cdot f^2 = 2f(X\cdot f) = 0$ for every $X \in \operatorname{Der} C^\infty(R)$. Similarly, if f is a non-zero constant function, that is $f(x) = c \neq 0$ for all $x \in R$, then $f^2 = cf$, and the linearity of derivations implies that $X\cdot f^2 = X\cdot(cf) = c(X\cdot f)$. On the other hand, Leibniz' rule implies that $X\cdot f^2 = 2f(X\cdot f) = 2c(X\cdot f)$. Hence $c(X\cdot f) = 2c(X\cdot f)$. Since $c \neq 0$, it follows that $X\cdot f = 0$. □

LEMMA 3. — *If $f \in C^\infty(R)$ vanishes identically in an open set $U \subseteq R$, then $(X\cdot f) \mid U = 0$ for all $X \in \operatorname{Der} C^\infty(R)$.*

Proof. — If $f \in C^\infty(R)$ vanishes identically in an open set $U \subseteq R$, then for each $x \in U$, there exists by Lemma 1 a function $h \in C^\infty(R)$ such that $h(x) = 1$ and $hf = 0$. Therefore, $0 = X\cdot(hf) = h(X\cdot f) + f(X\cdot h)$ for every smooth derivation X. Evaluating this identity at x, we get $(X\cdot f)(x) = 0$ because $f(x) = 0$. Hence, $(X\cdot f) \mid U = 0$. □

LEMMA 4. — *Let U be open in R, and X_U a smooth derivation of $C^\infty(U)$. For each $x \in U$, there exist an open neighbourhood V of x contained in U, and $X \in \operatorname{Der} C^\infty(R)$ such that*

$$(X\cdot h) \mid V = (X_U\cdot (h \mid U)) \mid V \quad \textit{for all} \quad h \in C^\infty(R).$$

Proof. — Let U be an open neighbourhood of x_0 in R , and X_U a smooth derivation of $C^\infty(U)$. There exist open sets V and W in R such that $x_0 \in V \subseteq \bar{V} \subseteq W \subseteq \bar{W} \subseteq U$. Let $f \in C^\infty(R)$ be such that $f \mid \bar{V} = 1$ and $f \mid S\backslash W = 0$. Then $(f \mid U)X_U$ is a derivation of $C^\infty(U)$ which vanishes on $U\backslash W$. Hence, it extends to a smooth derivation X of $C^\infty(R)$ such that, for every $h \in C^\infty(R)$, $(X\cdot h) \mid V = (X_U\cdot (h \mid U)) \mid V$. □

A local diffeomorphism φ of R to itself is a diffeomorphism $\varphi : U \to V$, where U and V are open differential subspaces of R. For each $f \in C^\infty(R)$, the restriction of f to V is in $C^\infty(V)$, and $\varphi^* f = f \circ \varphi$ is in $C^\infty(U)$. If $\varphi^* f$ coincides with the restriction of f to U, we say that f is φ-invariant, and write $\varphi^* f = f$. For each $X \in \operatorname{Der}(C^\infty(R))$, the restriction of X to U is in $\operatorname{Der}(C^\infty(U))$, and the push-forward $\varphi_* X$ of X by φ is a derivation of $C^\infty(V)$ such that

$$(\varphi_* X)\cdot (f \mid V) = \varphi^{-1*}(X\cdot(\varphi^* f)) \quad \text{for all} \quad f \in C^\infty(R). \tag{2}$$

Since all functions in $C^\infty(V)$ locally coincide with restrictions to V of functions in $C^\infty(R)$, equation (2) determines $\varphi_* X$ uniquely. If $\varphi_* X$ coincides with the restriction of X to V, we say that X is φ-invariant and write $\varphi_* X = X$.

3. Subcartesian spaces.

Subcartesian spaces were introduced by Aronszajn ([1]), and developed in [3] and [13]. They are Hausdorff differential spaces locally diffeomorphic to a differential subspace of a Cartesian space. In other words, a subcartesian space is a Hausdorff differential space S that can be covered by open sets, each of which is diffeomorphic to a differential subspace of a Cartesian space. In the remainder of this paper, we restrict our considerations to differential spaces that are Hausdorff, second countable and paracompact.

In this section, we describe properties of differential subspaces of $\mathbb{R}^n$ which extend to subcartesian spaces. In the remainder of this section, R denotes a differential subspace of $\mathbb{R}^n$, considered as a differential space endowed with the standard differential structure $C^\infty(\mathbb{R}^n)$. In other words, a function $f : R \to \mathbb{R}$ is in $C^\infty(R)$ if, for every $x \in R$, there exist an open set U in $\mathbb{R}^n$ and $f_U \in C^\infty(\mathbb{R}^n)$ such that $f \mid U \cap R = f_U \mid U \cap R$.

LEMMA 5. — *Let W be an open subset of $R \subseteq \mathbb{R}^n$, and $f_W \in C^\infty(W)$. For every $x \in W$ there exist a function $f \in C^\infty(R)$ and a neighbourhood V of x contained in W such that $f \mid V = f_W \mid V$.*

Proof. — The proof is an immediate consequence of the definition of a differential subspace. □

For every differential space S, each $X \in \operatorname{Der} C^\infty(S)$ and every $x \in S$, we denote by $X(x) : C^\infty(S) \to \mathbb{R}$ the composition of the derivation X with the evaluation at x. In other words, $X(x) \cdot f = (X \cdot f)(x)$ for all $f \in C^\infty(S)$. We use the notation

$$\operatorname{Der}_x C^\infty(S) = \{X(x) \mid X \in \operatorname{Der} C^\infty(S)\}, \tag{3}$$

and refer to elements of $\operatorname{Der}_x C^\infty(S)$ as derivations of $C^\infty(S)$ at x.

LEMMA 6. — *Let R be a differential subspace of $\mathbb{R}^n$ and X a derivation of $C^\infty(R)$. For every $x \in R$, there is $\widetilde{X}(x) \in \operatorname{Der}_x C^\infty(\mathbb{R}^n)$ such that*

$$\widetilde{X}(x) \cdot f = X(x) \cdot (f \mid R) \quad \textit{for all} \quad f \in C^\infty(\mathbb{R}^n). \tag{4}$$

Proof. — Let $\widetilde{X}(x) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ be given by equation (4). It is a linear map satisfying Leibniz' rule $\widetilde{X}(x)\cdot(fh) = f(x)(\widetilde{X}(x)\cdot h) + h(x)(\widetilde{X}(x)\cdot f)$. Hence, $\widetilde{X}(x) \in T_x\mathbb{R}^n$, and it extends to a smooth vector field $\widetilde{X}$ on $\mathbb{R}^n$ so that $\widetilde{X}(x)$ is the value of $\widetilde{X}$ at $x \in \mathbb{R}^n$. Since vector fields on $\mathbb{R}^n$ are derivations of $C^\infty(\mathbb{R}^n)$, it follows that $\widetilde{X}(x) \in \mathrm{Der}_x\, C^\infty(\mathbb{R}^n)$. □

PROPOSITION 1. — *Let R be a differential subspace of $\mathbb{R}^n$ and X a derivation of $C^\infty(R)$. For every $f_1, \dots, f_m \in C^\infty(R)$ and every $F \in C^\infty(\mathbb{R}^m)$,*

$$X\cdot F(f_1,\dots,f_m) = \sum_{i=1}^{m} \partial_i F(f_1,\dots,f_m)(X\cdot f_i). \tag{5}$$

Proof. — Let $f_1, \dots, f_m \in C^\infty(R)$ and $x \in R$. We denote by $p_1, \dots, p_n : \mathbb{R}^n \to \mathbb{R}$ the coordinate functions on $\mathbb{R}^n$. There exist a neighbourhood U of $x \in \mathbb{R}^n$ and functions $F_1, \dots, F_m \in C^\infty(\mathbb{R}^n)$ such that $f_i \mid U \cap R = F_i(p_1, \dots, p_n) \mid U \cap R$. Hence, for every $F \in C^\infty(\mathbb{R}^m)$,

$$F(f_1,\dots,f_m) \mid U\cap R = F(F_1(p_1 \dots, p_n), \dots, F_m(p_1,\dots,p_n)) \mid U \cap R.$$

By Lemma 4, there exists $\widetilde{X}(x) \in \mathrm{Der}_x\, C^\infty(\mathbb{R}^n)$ such that equation (4) is satisfied. Hence,

$$\begin{aligned}
X(x)\cdot F(f_1,\dots,f_m) &= X(x)\cdot (F(F_1(p_1 \dots, p_n), \dots, F_m(p_1,\dots,p_n)) \mid R) \\
&= \widetilde{X}(x)\cdot (F(F_1(p_1 \dots, p_n), \dots, F_m(p_1,\dots,p_n))) \\
&= \sum_{i=1}^{n} (\partial_i F(F_1(p_1 \dots, p_n), \dots, F_m(p_1,\dots,p_n)))(x) \\
&\qquad\qquad \cdot (\widetilde{X}(x)\cdot F_i(p_1 \dots, p_n)) \\
&= \sum_{i=1}^{n} (\partial_i F(F_1(p_1 \dots, p_n), \dots, F_m(p_1,\dots,p_n)))(x) \\
&\qquad\qquad \cdot (X(x)\cdot (F_i(p_1 \dots, p_n) \mid R)) \\
&= \sum_{i=1}^{n} (\partial_i F(f_1,\dots,f_m))(x)(X(x)\cdot f_i) \\
&= \sum_{i=1}^{n} (\partial_i F(f_1,\dots,f_m))(x)(X\cdot f_i)(x).
\end{aligned}$$

This holds for every $x \in R$, which implies equation (5). □

PROPOSITION 2. — *Let R be a differential subspace of $\mathbb{R}^n$. For every $X \in \operatorname{Der} C^\infty(R)$ and each $x \in R$, there exists an open neighbourhood U of x in $\mathbb{R}^n$ and $\widetilde{X} \in \operatorname{Der} C^\infty(U)$ such that*

$$(\widetilde{X} \cdot (f \mid U)) \mid U \cap R = (X \cdot (f \mid R)) \mid U \cap R \quad \textit{for all} \quad f \in C^\infty(\mathbb{R}^n).$$

Proof. — Let $h_1, \ldots, h_n$ be the restrictions to R of Cartesian coordinates $p_1, \ldots, p_n$ on $\mathbb{R}^n$. For every $X \in \operatorname{Der}(C^\infty(R))$, the functions $X \cdot h_1, \ldots, X \cdot h_n$ are in $C^\infty(R)$. Hence, for every $x \in R$, there exist an open neighbourhood U of x in $\mathbb{R}^n$ and functions $f_1, \ldots, f_n \in C^\infty(\mathbb{R}^n)$ such that $(X \cdot h_i) \mid U \cap R = f_i \mid U \cap R$ for $i = 1, \ldots, n$.

Let $f \in C^\infty(\mathbb{R}^n)$. Then $f \mid U \cap R = f(h_1, \ldots, h_n) \mid U \cap R$ and

$$\begin{aligned}(X \cdot (f \mid R)) \mid U \cap R &= (X \cdot (f(h_1, \ldots, h_n))) \mid U \cap R \\ &= \sum_{i=1}^{n} (\partial_i f(h_1, \ldots, h_m) \mid U \cap R)(X \cdot h_i) \mid U \cap R \\ &= \sum_{i=1}^{n} ((\partial_i f) \mid U \cap R)(f_i \mid U \cap R).\end{aligned}$$

Consider the vector field

$$\widetilde{X} = f_1 \partial_1 + \cdots + f_n \partial_n \tag{6}$$

on U. Since U is open in $\mathbb{R}^n$, $\widetilde{X}$ is a derivation of $C^\infty(U)$. Moreover,

$$(X \cdot (f \mid R)) \mid U \cap R = (\widetilde{X} \cdot (f \mid U)) \mid U \cap R$$

for all $f \in C^\infty(\mathbb{R}^n)$. □

It follows from Proposition 2 that every derivation X of $C^\infty(R)$ can be locally extended to a derivation of $C^\infty(\mathbb{R}^n)$. Clearly, this extension need not be unique. Moreover, If $\widetilde{X}$ is a smooth vector field on $\mathbb{R}^n$, and $f \in C^\infty(\mathbb{R}^n)$, then the restriction of $\widetilde{X} \cdot f$ to R need not be determined by the restriction of f to R. Hence, not every vector field on $\mathbb{R}^n$ restricts to a derivation of $C^\infty(R)$.

LEMMA 7. — *Let U be an open subset of $R \subseteq \mathbb{R}^n$, and X_U a smooth derivation of $C^\infty(U)$. For each $x \in U$, there exist an open neighbourhood V of x contained in U, and $X \in \operatorname{Der} C^\infty(R)$ such that*

$$(X \cdot h) \mid V = (X_U \cdot (h \mid U)) \mid V \quad \textit{for all} \quad h \in C^\infty(R).$$

Proof. — By Lemma 1, there exist open sets V and W in R, such that $x \in V \subseteq U$ and $U \cup W = R$, and $f \in C^\infty(R)$ satisfying $f \mid V = 1$ and $f \mid W = 0$. For every $h \in C^\infty(R)$, let X be given by

$$(X\cdot h) \mid U = (f \mid U) X_U\cdot (h \mid U)$$
$$(X\cdot h) \mid W = 0.$$

In $U \cap W$ we have

$$((f \mid U) X_U\cdot (h \mid U)) \mid U \cap W = (f \mid U \cap W)(X_U\cdot (h \mid U)) \mid U \cap W = 0$$

because $f \mid W = 0$. Hence, X is well defined. Moreover, $X \in \operatorname{Der} C^\infty(R)$, and $f \mid V = 1$ implies $(X\cdot h) \mid V = (X_U\cdot (h \mid U)) \mid V$ for all $h \in C^\infty(R)$. □

LEMMA 8. — *Let R be a differential subspace of $\mathbb{R}^n$. If U and V are open subsets of $\mathbb{R}^n$ and $\varphi : U \to V$ is a diffeomorphism such that $\varphi(U \cap R) = V \cap R$, then the restriction of φ to $U \cap R$ is a diffeomorphism of $U \cap R$ onto $V \cap R$.*

Proof. — By assumption, R is a topological subspace of $\mathbb{R}^n$, the mapping $\varphi : U \to V$ is a homeomorphism, and $\varphi(U \cap R) = V \cap R$. Hence, for every open subset W of $\mathbb{R}^n$, $\varphi^{-1}(W \cap (V \cap R))$ is open in $U \cap R$, and $\varphi(W \cap (U \cap R))$ is open in $V \cap R$. Thus, φ induces a homeomorphism $\psi : U \cap R \to V \cap R$.

Moreover, φ induces a diffeomorphism of open differential subspaces U and V of $\mathbb{R}^n$. We want to show that $f \in C^\infty(V \cap R)$ implies $\psi^* f \in C^\infty(U \cap R)$. Given $x \in U \cap R$, let $y = \psi(x) \in V \cap R$. Since R is a differential subspace of $\mathbb{R}^n$ and $f \in C^\infty(V \cap R)$, there exist a neighbourhood W of $x \in V$ and a function $f_W \in C^\infty(V)$ such that $f \mid W \cap R = f_W \mid W \cap R$. Moreover, $\varphi^{-1}(W)$ is a neighbourhood of x in U, $\varphi^* f_W$ is in $C^\infty(U)$, and

$$\begin{aligned}
\psi^* f \mid (\varphi^{-1}(W) \cap R) &= f \circ \psi \mid (\varphi^{-1}(W) \cap R) = f \circ (\varphi \mid (\varphi^{-1}(W) \cap R)) \\
&= f \mid (W \cap R) = f_W \mid (W \cap R) \\
&= f_W \circ (\varphi \mid (\varphi^{-1}(W) \cap R)) = f_W \circ \varphi \mid (\varphi^{-1}(W) \cap R) \\
&= \varphi^* f_W \mid (\varphi^{-1}(W) \cap R).
\end{aligned}$$

Thus, for every $x \in U \cap R$, there exist a neighbourhood $\varphi^{-1}(W)$ of x in U and a function $\varphi^* f_W$ in $C^\infty(U)$ such that $\psi^* f \mid (\varphi^{-1}(W) \cap R) = \varphi^* f_W \mid (\varphi^{-1}(W) \cap R)$. This implies that $\psi^* f \in C^\infty(U \cap R)$.

It follows that ψ is smooth. In a similar manner we can prove that ψ^{-1} is smooth. Hence, ψ is a diffeomorphism. □

Let S be a subcartesian space. It can be covered by open sets, each of which is diffeomorphic to a differential subspace R of $\mathbb{R}^n$. All the properties of differential subspaces of $\mathbb{R}^n$ discussed in Lemmata 4 through 8 and Propositions 1 and 2 are local. Hence they extend to subcartesian spaces.

4. Families of vector fields.

In this section, we discuss properties of vector fields on subcartesian spaces.

In the category of smooth manifolds, translations along integral curves of a smooth vector field give rise to local diffeomorphisms. In the category of differential spaces, not all derivations generate local diffeomorphisms. We reserve the term vector field for a derivation that generates a local one-parameter group of local diffeomorphisms.

Let I be an interval in $\mathbb{R}$. A smooth curve $c : I \to S$ on a differential space S is an integral curve of $X \in \mathrm{Der}(C^\infty(S))$ if

$$(X\cdot f)(c(t)) = \frac{d}{dt} f(c(t))$$

for every $f \in C^\infty(S)$ and $t \in I$. If $0 \in I$ and $c(0) = x$, we refer to c as an integral curve of X through x.

THEOREM 1. — *Assume that S is a subcartesian space. For every $x \in S$ and every $X \in \mathrm{Der}(C^\infty(S))$ there exists a unique maximal integral curve $c : I \to S$ through x.*

Proof outline (a detailed proof is given in [23]). — Since S is a subcartesian space, given $x \in S$, there exists a neighbourhood V of x in S diffeomorphic to a differential subspace R of $\mathbb{R}^n$. In order to simplify the notation, we use the diffeomorphism between R and V to identify them, and write $V = R$. By Proposition 2, there exists an extension of X to a smooth vector field $\widetilde{X}$ on an open neighbourhood U of x in $\mathbb{R}^n$ given by equation (6).

Given $y \in R \subseteq \mathbb{R}^n$, consider an integral curve $\widetilde{c} : \widetilde{I} \to \mathbb{R}^n$ of $\widetilde{X}$ such that $\widetilde{c}(0) = y$. Let I be the connected component of $\widetilde{c}^{-1}(R)$ containing 0, and $c : I \to R$ the curve in R obtained by the restriction of $\widetilde{c}$ to I. Then, $c(0) = y$. Moreover, for each $t \in I$, and $f \in C^\infty(S)$ there exist a neigbourhood U of $\widetilde{c}(t)$ in $\mathbb{R}^n$ and a function $F \in C^\infty(\mathbb{R}^n)$ such that $f \mid R \cap U = F \mid R \cap U$. Hence, by Proposition 2,

$$\frac{d}{dt} f(c(t)) = \frac{d}{dt} F(\widetilde{c}(t)) = (\widetilde{X} \cdot F)(\widetilde{c}(t)) = (X \cdot f)(c(t)).$$

This implies that $c : I \to R$ is an integral curve of X through y. Since I is a connected subset of $\mathbb{R}$, it is an interval. Local uniqueness of c (up to an extension of the domain) follows from the local uniqueness of solutions of differential equations on $\mathbb{R}^n$.

The above argument gives existence and local uniqueness of integral curves of derivations of $C^\infty(S)$. The usual technique of patching local solutions, and the fact that the union of intervals with pairwise non-empty intersection is an interval, lead to the global uniqueness of integral curves of derivations on a subcartesian space S. □

Let X be a derivation of $C^\infty(S)$. We denote by $\varphi_t(x)$, the point on the maximal integral curve of X through x corresponding to the value t of the parameter. Given $x \in \mathbb{R}^n$, $\varphi_t(x)$ is defined for t in an interval I_x containing zero, and $\varphi_0(x) = x$. If t, s and $t+s$ are in I_x, $s \in I_{\varphi_t(x)}$ and $t \in I_{\varphi_s(x)}$, then $\varphi_{t+s}(x) = \varphi_s(\varphi_t(x)) = \varphi_t(\varphi_s(x))$. In the case when S is a manifold, the map φ_t is a diffeomorphism of a neighbourhood of x in S onto a neighbourhood of $\varphi_t(x)$ in S. For a subcartesian space S, the map φ_t might fail to be a local diffeomorphism. We adopt the following

DEFINITION OF A VECTOR FIELD. — *Let S be a subcartesian space. A derivation X of $C^\infty(S)$ is a vector field on S if, for every $x \in S$, there exists an open neighbourhood U of x in S, and $\varepsilon > 0$ such that, for every $t \in (-\varepsilon, \varepsilon)$, the map φ_t is defined on U, and its restriction to U is a diffeomorphism from U onto an open subset of S.*

Example 1. — Consider $S = [0, \infty) \subseteq \mathbb{R}$ with the structure of a differential subspace of $\mathbb{R}$. Let $(X \cdot f) = f'(x)$ for every $f \in C^\infty([0,\infty))$ and $x \in [0, \infty)$. Note that the derivative at $x = 0$ is is the right derivative; it is uniquely defined by $f(x)$ for $x \geqslant 0$. For this X, the map φ_t is given by $\varphi_t(x) = x + t$ whenever x and $x + t$ are in $[0, \infty)$. In particular, for every neighbourhood U of 0 in $[0, \infty)$ there exists $\delta > 0$ such that $[0, \delta) \subseteq U$. Moreover, φ_t maps $[0, \delta)$ onto $[t, \delta + t)$ which is not a neighbourhood of $t = \varphi_t(0)$ in $[0, \infty)$. Hence, the derivation X is not a vector field on $[0, \infty)$. On the other hand, for every $f \in C^\infty[0, \infty)$ such that $f(0) = 0$, the derivation fX is a vector field, because 0 is a fixed point of fX.

We say that a subcartesian space S is locally closed if every point of S has a neighbourhood diffeomorphic to the intersection of an open and a closed subset of a Cartesian space. There is a simple criterion characterizing vector fields on a locally closed subcartesian space; namely,

PROPOSITION 3. — *Let S be a locally closed subcartesian space. A derivation X of $C^\infty(S)$ is a vector field on S if the domain of every maximal integral curve of X is open in $\mathbb{R}$.*

Proof. — Consider first the case when S is a relatively closed differential subspace of $\mathbb{R}^n$. In other words, $S = U \cap C$, where U is open and C is closed in $\mathbb{R}^n$. By Proposition 2, we may assume that X on S extends to a vector field $\widetilde{X}$ on U. We denote by $\widetilde{\varphi}_t$ the local one-parameter group of local diffeomorphisms of U generated by $\widetilde{X}$. By Theorem 1, for every $x \in S$, there is a maximal interval $I_x \in \mathbb{R}$ such that $\varphi_t(x) = \widetilde{\varphi}_t(x) \in S$ for all $t \in I_x$.

Given $x \in S \subseteq U$, there exist $\varepsilon > 0$ and a neighbourhood U' of x in U such that such that, for every $t \in (-\varepsilon, \varepsilon)$, the map $\widetilde{\varphi}_t$ is defined on U', and its restriction to U' is a diffeomorphism from U' onto an open subset of U. In view of Lemma 8, it suffices to show that there exist $\delta \in (0, \varepsilon]$ and a neighbourhood U'' of x in U' such that $\widetilde{\varphi}_t$ maps $U'' \cap C$ to $\widetilde{\varphi}_t(U'') \cap C$ for all $t \in (-\delta, \delta)$. Suppose that there are no U'' and δ satisfying this condition. This means that, for every neighbourhood U'' of x in U, and every $\delta \in (0, \varepsilon]$, there exist a point $y \in U'' \cap C$ and $s \in (-\delta, \delta)$ such that $\widetilde{\varphi}_s(y) \notin \widetilde{\varphi}_s(U'') \cap C$. Since $\widetilde{\varphi}_t(y) \in \widetilde{\varphi}_t(U'')$ for every $t \in (-\varepsilon, \varepsilon)$, it follows that $\widetilde{\varphi}_s(y) \notin C$. Hence, s is not in the domain I_y the maximal integral curve of X through y. If $s > 0$, let u be the infimum of the set $\{t \in [0, s] \mid \widetilde{\varphi}_t(y) \notin C\}$. Then, $\varphi_t(y) \in C$ for all $t \in [0, u)$. Since $\varphi_t(y)$ is continuous in t and C is closed, it follows that $\varphi_u(y) \in C$. Moreover, for every $v > u$, there exists $t \in (u, v)$ such that $\varphi_t(y) \notin C$. It implies that $[0, \infty) \cap I_y = [0, u]$. Hence, the domain I_y of the maximal integral curve of X through y is not open in $\mathbb{R}$, contrary to the assumption of the theorem. Hence, the case $s > 0$ is excluded. Similarly, we can show that the case $s < 0$ is inconsistent with the assumption that the domains of all maximal integral curves of X are open.

We have shown that there exist $\delta \in (0, \varepsilon]$ and a neighbourhood U'' of x in U' such that $\widetilde{\varphi}_t$ maps $U'' \cap C$ to $\widetilde{\varphi}_t(U'') \cap C$ for all $t \in (-\delta, \delta)$. This implies that $\varphi_t(z) = \widetilde{\varphi}_t(z)$ is defined for every $t \in (-\delta, \delta)$, and each $z \in U''$. By Lemma 8, it follows that, φ_t restricted to $U'' \cap S$ is a diffeomorphism onto $\varphi_t(U'') \cap S$. Since this holds for every $x \in S$, we conclude that φ_t is a local one-parameter group of local diffeomorphisms of S. Hence, X is a vector field on S.

Consider now a derivation X on a locally closed subcartesian space S such that the domains of all maximal integral curves of X are open. For each $x \in S$, we denote by $\varphi_t(x)$ the point on the maximal integral curve of X through x corresponding to the value t of the parameter. The function $t \mapsto \varphi_t(x)$ is defined on an interval I_x in $\mathbb{R}$, which is open by hypothesis.

For every $x \in S$ there exist a neighbourhood W of x in S and a diffeomorphism χ of W onto a locally closed subspace $U \cap C$ of $\mathbb{R}^n$. By the first part of the proof, the push-forward of X by χ is a vector field on $U \cap C$. Since χ is a diffeomorphism, it follows that there exist a neighbourhood W' of x in $W \subseteq S$ and $\varepsilon > 0$ such that, for every $t \in (-\varepsilon, \varepsilon)$, the map φ_t is defined on W', and its restriction to W' is a diffeomorphism from W' onto an open subset of $W \subseteq S$. Hence, X is a vector field on S. □

The following example shows that the assumption that S is locally closed is essential in Proposition 2.

Example 2. — The set
$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1)^2 < 1 \text{ or } x_2 = 0\}$$
is not locally closed. The vector field $\widetilde{X} = \frac{\partial}{\partial x_1}$ on $\mathbb{R}^2$ restricts to a derivation X of $C^\infty(S)$. For every $x = (x_1, x_2) \in S$, $\widetilde{\varphi}_t(x) = (x_1 + t, x_2)$ for all $t \in \mathbb{R}$. Its restriction to S induces φ_t given by $\varphi_t(x_1, x_2) = (x_1 + t, x_2)$ for $t \in (-x_1 - \sqrt{1 - (x_2 - 1)^2}, -x_1 + \sqrt{1 - (x_2 - 1)^2})$ if $x_2 > 0$, and for $t \in \mathbb{R}$ if $x_2 = 0$. Hence, all integral curves of X have open domains. Nevertheless, φ_t fails to be a local one-parameter local group of diffeomorphisms of S.

Let $\mathcal{F}$ be a family of vector fields on a subcartesian space S. For each $X \in \mathcal{F}$, we denote by φ_t^X the local one-parameter group of local diffeomorphisms of S generated by X. We say that the family $\mathcal{F}$ is locally complete if, for every $X, Y \in \mathcal{F}$, $t \in \mathbb{R}$ and $x \in S$, for which $\varphi_{t*}^X Y(x)$ is defined, there exist an open neighbourhood U of x and $Z \in \mathcal{F}$ such that $\varphi_{t*}^X Y \mid U = Z \mid U$.

For example, a family consisting of a single vector field X is locally complete because $\varphi_{t*}^X X(x) = X(x)$ at all points $x \in S$ for which $\varphi_t(x)$ is defined.

THEOREM 2. — *The family $\mathcal{X}(S)$ of all vector fields on a subcartesian space S is locally complete.*

Proof. — For $X \in \mathcal{X}(S)$, let φ_t^X denote the local one-parameter group of local diffeomorphisms of S generated by X. For a given $t \in \mathbb{R}$ let U be a neighbourhood of $x \in S$ such that φ_t^X maps U diffeomorphically onto an open subset V of S. For each $Y \in \mathcal{X}(S)$, $\varphi_{t*}^X Y$ is in $\mathrm{Der}(C^\infty(V))$. If φ_s^Y denotes the local one-parameter group of diffeomorphisms of Y, then

$$\begin{aligned}\frac{d}{ds} f(\varphi_t^X(\varphi_s^Y(x)) &= \frac{d}{ds}(\varphi_t^{X*} f)(\varphi_s^Y(x)) = (Y \cdot (\varphi_t^{X*} f))(\varphi_s^Y(x)) \\ &= (\varphi_t^{X*}(\varphi_{t*}^X Y \cdot f))(\varphi_s^Y(x)) = (\varphi_{t*}^X Y \cdot f)(\varphi_t^X(\varphi_s^Y(x)))\end{aligned}$$

for every $f \in C^\infty(V)$ and $x \in V$ and $s \in I_y$ such that $\varphi_s^Y(x) \in U$. Hence, $s \mapsto \varphi_t^X(\varphi_s^Y(x))$ is an integral curve of $\varphi_{t*}^X Y$ through $\varphi_t^X(x)$. Since Y is a vector field, for every $x \in S$, there exist an open neighbourhood W of x in S, and $\varepsilon > 0$ such that, for every $s \in (-\varepsilon, \varepsilon)$, the map φ_s^Y is defined on W, and its restriction to W is a diffeomorphism from W onto an open subset of S. We can choose W and ε so that φ_s^Y maps W into U for all $s \in (-\varepsilon, \varepsilon)$. Since φ_t^X maps W diffeomorphically onto V, it follows that $\varphi_t^X \circ \varphi_s^Y$ restricted to W maps W difeomorphically onto $\varphi_t^X(\varphi_s^Y(W))$, which is open in V for all $s \in (-\varepsilon, \varepsilon)$. Hence $\varphi_{t*}^X Y$ is a vector field on V.

For every $x \in V$, there exists an open neighbourhood W of x such that $\bar{W} \subset V$. Let $f \in C^\infty(V)$ be such that $f(x) = 1$ and f vanishes identically on $V \backslash W$. Then $f\varphi_{t*}Y$ is a vector field on V vanishing on $V \backslash W$, and it extends to a vector field Z on S. Hence, $(\varphi_{t*}Y)(x) = f(x)(\varphi_{t*}Y)(x) = Z(x)$ and $Z \in \mathcal{X}(S)$.

The above argument is valid for every X and Y in $\mathcal{X}(S)$. Hence, $\mathcal{X}(S)$ is a locally complete family of vector fields. □

5. Orbits and integral manifolds.

In this section we prove that orbits of families of vector fields on a subcartesian space S are manifolds. This is an extension of the results of Sussmann ([27]), to the category of subcartesian spaces.

Let $\mathcal{F}$ be a family of vector fields on a subcartesian space S. For each $X \in \mathcal{F}$ we denote by φ_t^X the local one-parameter group of diffeomorphisms of S generated by X. The family $\mathcal{F}$ gives rise to an equivalence relation $\sim$ on S defined as follows: $x \sim y$ if there exist vector fields $X^1, \ldots, X^n \in \mathcal{F}$ and $t_1, \ldots, t_n \in \mathbb{R}$ such that

$$y = (\varphi_{t_n}^{X^n} \circ \cdots \circ \varphi_{t_1}^{X^1})(x). \tag{7}$$

In other words, $x \sim y$ if there exists a piecewise smooth curve c in S, with tangent vectors given by restrictions to c of vector fields in $\mathcal{F}$, which joins x and y. The equivalence class of this relation containing x is called the orbit of $\mathcal{F}$ through x. The aim of this section is to prove that orbits of locally complete families of vector fields on a subcartesian space S, are manifolds and give rise to a singular foliation of S. This is an extension of the results of Sussmann ([27]), to the category of differential spaces.

Following Sussmann's notation, we write $\xi = (X^1, \dots, X^m)$, $T = (t_1, \dots, t_m)$ and

$$\xi_T(x) = (\varphi_{t_m}^{X^m} \circ \cdots \circ \varphi_{t_1}^{X^1})(x).$$

The expression for $\xi_T(x)$ is defined for all (T, x) in an open subset $\Omega(\xi)$ of $\mathbb{R}^m \times S$. Let $\Omega_T(\xi)$ denote the set of all $x \in S$ such that $(T, x) \in \Omega(\xi)$. In other words, $\Omega_T(\xi)$ is the set of all x for which $\xi_T(x)$ is defined. Moreover, we denote by $\Omega_{\xi,x} \subseteq \mathbb{R}^m$ the set of $T \in \mathbb{R}^m$ such that $\xi_T(x)$ is defined.

We now assume that S is a subset of $\mathbb{R}^n$. For each $x \in S \subseteq \mathbb{R}^n$ and $\xi = (X^1, \dots, X^m) \in \mathcal{F}^m$, let

$$\rho_{\xi,x} : \Omega_{\xi,x} \to \mathbb{R}^n : T \mapsto \iota_S \circ \xi_T(x),$$

where $\iota_S : S \to \mathbb{R}^n$ is the inclusion map. If M is the orbit of $\mathcal{F}$ through x, considered as a subset of $\mathbb{R}^n$, then it is the union of all the images of all the mappings $\rho_{\xi,x}$, as m varies over the set $\mathbb{N}$ of natural numbers and ξ varies over $\mathcal{F}^m$. We topologize M by the strongest topology $\mathcal{T}$ which makes all the maps $\rho_{\xi,x}$ continuous. Since each $\rho_{\xi,x} : \Omega_{\xi,x} \to \mathbb{R}^n$ is continuous, it follows that the topology of M as a subspace of $\mathbb{R}^n$ is coarser than the topology $\mathcal{T}$. Hence, the inclusion of M into $\mathbb{R}^n$ is continuous with respect to the topology $\mathcal{T}$. In particular, M is Hausdorff. Since all the sets $\Omega_{\xi,x}$ are connected it follows that M is connected. The proof that the topology $\mathcal{T}$ of M defined above is independent of the choice of $x \in M$ is exactly the same as in [27].

If S is a subcartesian space, then it can be covered by a family $\{U_\alpha\}_{\alpha \in A}$ of open subsets, each of which is diffeomorphic to a subset of $\mathbb{R}^k$. The argument given above can be repeated in each U_α leading to a topology $\mathcal{T}_\alpha$ in $M_\alpha = U_\alpha \cap M$. For $\alpha, \beta \in A$, the topologies $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are the same when restricted to $M_\alpha \cap M_\beta$. We define the topology of M so that, for each $\alpha \in A$, the induced topology in M_α is $\mathcal{T}_\alpha$.

Suppose now that $\mathcal{F}$ is a family of vector fields on S. For each $x \in S$, let $D_{\mathcal{F}_x}$ be the linear span of $\mathcal{F}_x = \{X(x) \mid X \in \mathcal{F}\}$. Suppose there is a neighbourhood of $x \in S$ diffeomorphic to a subset of $\mathbb{R}^n$. Then, $\dim D_{\mathcal{F}_x} \leqslant n$.

LEMMA 9. — *For a locally complete family $\mathcal{F}$ of vector fields on a subcartesian space S, $\dim D_{\mathcal{F}_x}$ is constant on orbits of $\mathcal{F}$.*

Proof. — Given $x \in S$, let $\dim D_{\mathcal{F}_x} = k$, and $X^1, \dots, X^k \in \mathcal{F}$ be such that $\{X^1(x), \dots, X^k(x)\}$ is a basis in $D_{\mathcal{F}_x}$. Since the family $\mathcal{F}$ is locally complete, for every $X \in \mathcal{F}$, and $t \in \Omega_{\{X\},x}$, $\varphi^X_{t*}X^1(\varphi^X_t(x)), \dots, \varphi^X_{t*}X^k(\varphi^X_t(x))$ are in $D_{\mathcal{F}_{\varphi^X_t(x)}}$ and are linearly independent because φ^X_t is a local diffeomorphism. Hence, $\dim D_{\mathcal{F}_x} \leqslant \dim D_{\mathcal{F}_{\varphi^X_t(x)}}$. Using φ^X_{-t}, we can show that $\dim D_{\mathcal{F}_{\varphi^X_t(x)}} \leqslant \dim D_{\mathcal{F}_x}$. Hence, $\dim D_{\mathcal{F}_x} = \dim D_{\mathcal{F}_{\varphi^X_t(x)}}$ for every $X \in F$. Repeating this argument along $\xi_T(x) = (\varphi^{X^m}_{t_m} \circ \dots \circ \varphi^{X^1}_{t_1})(x)$, we conclude that $\dim D_{\mathcal{F}_y} = \dim D_{\mathcal{F}_x}$ for every y on the orbit of F though x. □

In analogy with standard terminology, we shall use the term integral manifold of $D_{\mathcal{F}}$ for a connected manifold M contained in S, such that its inclusion into S is smooth and, for every $x \in M$, $T_xM = D_{\mathcal{F}_x}$.

THEOREM 3. — *Let $\mathcal{F}$ be a locally complete family of vector fields on a subcartesian space S. Each orbit M of $\mathcal{F}$, with the topology $\mathcal{T}$ introduced above, admits a unique manifold structure such that the inclusion map $\iota_{MS} : M \hookrightarrow S$ is smooth. In terms of this manifold structure, M is an integral manifold of $D_{\mathcal{F}}$.*

Proof. — Let M be an orbit of $\mathcal{F}$. Since $\mathcal{F}$ is locally complete, for each $z \in M$, the dimension $m = \dim D_{\mathcal{F}_z}$ is independent of z, and there exist m vector fields $X^1, \dots, X^m$ in $\mathcal{F}$ that are linearly independent in an open neighbourhood V of z in S. Without loss of generality, we may assume that V is a subset of $\mathbb{R}^n$. By Proposition 2, the restrictions of $X^1, \dots, X^m$ to vector fields on V extend to vector fields $\widetilde{X}^1, \dots, \widetilde{X}^m$ on a neighbourhood U of z in $\mathbb{R}^n$. Without loss of generality, we may assume that they are linearly independent on U.

Given $x \in V \cap U \subseteq \mathbb{R}^n$, let $\xi = (X^1, \dots, X^m)$ be such that $X^1(x), \dots, X^m(x)$ form a basis of $D_{\mathcal{F}_x}$, $T = (t_1, \dots, t_m) \in \Omega_{\xi,x}$ and

$$\widetilde{\rho}_{\xi,x} : \Omega_{\xi,x} \to U \subseteq \mathbb{R}^n : T \mapsto (\varphi^{\widetilde{X}^m}_{t_m} \circ \dots \circ \varphi^{\widetilde{X}^1}_{t_1})(x).$$

For each $i = 1, \dots, m$,

$$u\frac{d}{dt}\varphi^{\widetilde{X}^i}_{t_i}(x) = u\widetilde{X}^i(\varphi^{\widetilde{X}^i}_t(x)).$$

Hence,

$$
\begin{aligned}
&T\widetilde{\rho}_{\xi,x}(T)(u_1,\dots,u_m)\\
&= u_1\frac{d}{dt_1}(\varphi_{t_m}^{\widetilde{X}^m}\circ\cdots\circ\varphi_{t_1}^{\widetilde{X}^1})(x)+\cdots+u_m\frac{d}{dt_m}(\varphi_{t_m}^{\widetilde{X}^m}\circ\cdots\circ\varphi_{t_1}^{\widetilde{X}^1})(x)\\
&= u_1(\varphi_{t_m}^{\widetilde{X}^m})_*\cdots(\varphi_{t_2}^{\widetilde{X}^2})_*\widetilde{X}(\varphi_{t_1}^{\widetilde{X}^1}(x))+\cdots+u_m\widetilde{X}^m(\varphi_{t_m}^{\widetilde{X}^m}(\cdots(\varphi_{t_1}^{\widetilde{X}^1}(x)))).
\end{aligned}
$$

In particular,

$$
T\widetilde{\rho}_{\xi,x}(0)(u_1,\dots,u_m)=u_1\widetilde{X}^1(x)+\cdots+u_m\widetilde{X}^m(x). \tag{8}
$$

Since the vectors $\widetilde{X}^1(x),\dots,\widetilde{X}^m(x)$ are linearly independent, it follows that $T\widetilde{\rho}_{\xi,x}(0):\mathbb{R}^m\to\mathbb{R}^n$ is one to one. Hence, there exists an open neighbourhood $W_{\xi,x}$ of 0 in $\mathbb{R}^m$ such that the restriction $\widetilde{\rho}_{\xi,x}\mid W_{\xi,x}$ of $\widetilde{\rho}_{\xi,x}$ to $W_{\xi,x}$ is an immersion of $W_{\xi,x}$ into $U\subseteq\mathbb{R}^n$. Therefore, $M_{\xi,x}=\widetilde{\rho}_{\xi,x}(W_{\xi,x})$ is an immersed submanifold of $U\subseteq\mathbb{R}^n$. Moreover, there exists a smooth map $\mu_{\xi,x}:M_{\xi,x}\to W_{\xi,x}$ such that $\mu_{\xi,x}\circ(\widetilde{\rho}_{\xi,x}\mid W_{\xi,x})=\mathit{identity}$. Every point $y\in M_{\xi,x}$ is of the form

$$
y=(\varphi_{t_m}^{\widetilde{X}^m}\circ\cdots\circ\varphi_{t_1}^{\widetilde{X}^1})(x)
$$

for some $T=(t_1,\dots,t_m)\in W_{\xi,x}$. Since $\widetilde{X}^1,\dots,\widetilde{X}^m$ are extensions to $U\subseteq\mathbb{R}^n$ of the restrictions to V of vector fields $X^1,\dots,X^m$ on S, it follows that $\varphi_{t_1}^{\widetilde{X}^1},\dots,\varphi_{t_n}^{\widetilde{X}^m}$ preserve V. Moreover, $x\in V$ so that

$$
y=(\varphi_{t_m}^{X^m}\circ\cdots\circ\varphi_{t_1}^{X^1})(x)\in V\subseteq S.
$$

Hence, $M_{\xi,x}$ is contained in $V\subseteq S$.

Let $\iota_{MV}:M_{\xi,x}\hookrightarrow V$ be the inclusion map. We want to show that it is smooth. Let $\iota_M:M_{\xi,x}\to U\subseteq\mathbb{R}^n$ and $\iota_V:V\to U\subseteq\mathbb{R}^n$ be the inclusion maps. Then $\iota_M=\iota_V\circ\iota_{MV}$ and $\iota_M^*f=\iota_{MV}^*\circ\iota_V^*f$ for every function $f\in C^\infty(U)$. Since $M_{\xi,x}$ is an immersed submanifold of U, it follows that $\iota_M^*f\in C^\infty(M_{\xi,x})$ for all $f\in C^\infty(U)$. Similarly, V is a differential subspace of U so that $\iota_V^*f\in C^\infty(V)$ for all $f\in C^\infty(U)$. Moreover, every $f_V\in C^\infty(V)$ is locally of the form ι_V^*f for some $f\in C^\infty(U)$. Since differentiability is a local property, it follows that $\iota_{MV}^*f_V\in C^\infty(M_{\xi,x})$ for every $f_V\in C^\infty(V)$. Hence, $\iota_{MV}:M_{\xi,x}\hookrightarrow V$ is smooth.

Thus, for every open set V in S, that is diffeomorphic to a subset of $\mathbb{R}^n$, each $x\in V$, and every $\xi=(X^1,\dots,X^m)$ such that $X^1(x),\dots,X^m(x)$ form a basis of $D_{\mathcal{F}_x}$, we have a manifold $M_{\xi,x}$ contained in V such that the inclusion map $\iota_{MV}:M_{\xi,x}\hookrightarrow V$ is smooth. Since V is open in S, the inclusion of $M_{\xi,x}$ into S is smooth.

Suppose that $M_{\xi_1,x_1} \cap M_{\xi_2,x_2} \neq \emptyset$. If $y \in M_{\xi_1,x_1} \cap M_{\xi_2,x_2}$, then $y = \widetilde{\rho}_{\xi_1,x_1}(T_1) = \widetilde{\rho}_{\xi_2,x_2}(T_2)$ for $T_1 \in V_{\xi_1,x_1}$ and $T_2 \in V_{\xi_2,x_2}$. Hence,

$$T_2 = \mu_{\xi_2,x_2}((\widetilde{\rho}_{\xi_1,x_1}(T_1))).$$

Since $\widetilde{\rho}_{\xi_1,x_1}$ and μ_{ξ_2,x_2} are smooth, it follows that the identity map on $M_{\xi_1,x_1} \cap M_{\xi_2,x_2}$ is a diffeomorphism of the differential structures on $M_{\xi_1,x_1} \cap M_{\xi_2,x_2}$ induced by the inclusions into M_{ξ_1,x_1} and M_{ξ_2,x_2}, respectively. Therefore $M_{\xi_1,x_1} \cup M_{\xi_2,x_2}$ is a manifold contained in S and the inclusion of $M_{\xi_1,x_1} \cup M_{\xi_2,x_2}$ into S is smooth.

Since $M = \bigcup_{\xi,x} M_{\xi,x}$, the above argument shows that M is a manifold contained in S such that the inclusion map $M \hookrightarrow S$ is smooth. Moreover, the manifold topology of M agrees with the topology $\mathcal{T}$ discussed above. Finally, equation (8) implies that M is an integral manifold of $D_{\mathcal{F}}$. □

We see from Theorem 3 that a locally complete family $\mathcal{F}$ of vector fields on a subcartesian space S gives rise to a partition of S by orbits of $\mathcal{F}$. We shall refer to such a partition as a singular foliation of S.

THEOREM 4. — *Orbits of the family $\mathcal{X}(S)$ of all vector fields on a subcartesian space S are manifolds. For every family $\mathcal{F}$ of vector fields on S, orbits of $\mathcal{F}$ are contained in orbits of $\mathcal{X}(S)$.*

Proof. — We have shown in Theorem 2 that the family $\mathcal{X}(S)$ of all vector fields on S is locally complete. Hence, it gives rise to a partition of S by manifolds. If $\mathcal{F}$ is a family of vector fields on S, then $\mathcal{F} \subseteq \mathcal{X}(S)$, and every orbit of $\mathcal{F}$ is contained in an orbit of $\mathcal{X}(S)$. □

Theorem 4 asserts that the singular foliation of a subcartesian space S by orbits of the family $\mathcal{X}(S)$ of all vector fields on S is coarsest within the class of singular foliations given by orbits of locally complete families of vector fields. The following example shows that there may be partitions of a differential space into manifolds which are coarser than the singular foliation by orbits of $\mathcal{X}(S)$.

Example 3. — Let $S = M_1 \cup M_2 \cup M_3 \cup M_4$, where

$$M_1 = \{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y = \sin(x^{-1})\},$$
$$M_2 = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ and } -1 < y < 1\},$$

$M_3 = \{(0,-1)\}$ and $M_4 = \{(0,1)\}$. Clearly, M_1 and M_2 are manifolds of dimension 1, while M_3 and M_4 are manifolds of dimension 0. However, for every $(0,y) \in M_2$, a vector $u \in T_{(0,y)}M_2$ can be extended to a vector field on S only if $u = 0$. Hence, orbits of the minimal singular foliation of S are M_1 and singletons $\{(0, y)\}$ for $-1 \leqslant y \leqslant 1$.

Having established the existence of the singular foliation of S by orbits of $\mathcal{X}(S)$, we can study arbitrary families of vector fields.

THEOREM 5. — *Let $\mathcal{F}$ be a family of vector fields on a subcartesian space S. For every $x \in S$, the orbit N of F through x is a manifold such that the inclusion map $\iota_{NS} : N \hookrightarrow S$ is smooth.*

Proof. — Since $\mathcal{F} \subseteq \mathcal{X}(S)$, the accessible set N of $D_{\mathcal{F}}$ through x is contained in the orbit M of $\mathcal{X}(S)$ through x. Let $D_{\mathcal{F}} \mid M$ be the restriction of $D_{\mathcal{F}}$ to the points of M. It is a generalized distribution on M, and N is an accessible set of $D_{\mathcal{F}}|M$. It follows from Sussmann's theorem ([27]), that N is a manifold and the inclusion map $\iota_{NM} : N \to M$ is smooth. Since the inclusion $\iota_{MS} : M \to S$ is smooth, it follows that $\iota_{NS} = \iota_{MS} \circ \iota_{NM} : N \hookrightarrow S$ is smooth. □

6. Stratified spaces.

In this section we show that smooth stratifications are subcartesian spaces. This enables us to use the results of the preceding sections in discussing stratified spaces. For a comprehensive study of stratified spaces see [10], [16] and the references quoted there.

Let S be a paracompact Hausdorff space. A stratification of S is given by a locally finite partition of S into locally closed subspaces $M \subseteq S$, called strata, satisfying the following conditions:

MANIFOLD CONDITION. — *Every stratum M of S is a smooth manifold in the induced topology.*

FRONTIER CONDITION. — *If M and N are strata of S such that the closure $\bar{N}$ of N has a non-empty intersection with M, then $M \subset \bar{N}$.*

A smooth chart on a stratified space S is a homeomorphism φ of an open set $U \subseteq S$ to a subspace $\varphi(U)$ of $\mathbb{R}^n$ such that, for every stratum M of S, the image $\varphi(U \cap M)$ is a smooth submanifold of $\mathbb{R}^n$ and the restriction $\varphi \mid U \cap M : U \cap M \to \varphi(U \cap M)$ is smooth. As in the case of manifolds, one introduces the notion of compatibility of smooth charts, and the notion of a maximal atlas of compatible smooth charts on S . A smooth structure on S is given by a maximal atlas of smooth charts on S. A continuous function $f : S \to \mathbb{R}$ is smooth if, for every $x \in S$ and every chart $\varphi : U \to \mathbb{R}^n$ with $x \in U$, there exist a neighbourhood U_x of x contained in U and a smooth function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f \mid U_x = g \circ \varphi \mid U_x$. For details see [16], sec. 1.3.

Stratifications can be ordered by inclusion. If we have two stratifications of the same space S, we say that the first stratification is smaller than the second if every stratum of the second stratification is contained in a stratum of the first one. For a stratified space S, there exists a minimal stratification of S. Some authors reserve the term stratification for a minimal stratification.

THEOREM 6. — *A smooth stratified space is a subcartesian space.*

Proof. — Let S be a smooth stratified space and $C^\infty(S)$ the space of smooth functions on S defined above. First, we need to show that the family $C^\infty(S)$ satisfies the conditions given at the beginning of Section 2.

A family $\{W_\alpha\}_{\alpha \in A}$ of open sets on S is a subbasis for the topology of S if, for each $x \in S$ and each open neighbourhood V of x in S, there exist $\alpha_1, \ldots, \alpha_p \in A$ such that $x \in W_{\alpha_1} \cap \cdots \cap W_{\alpha_p} \subseteq V$. Given $x \in S$, there exists a chart φ on S with domain U containing x. If V is a neighbourhood of x in S, then the restriction of φ to $V \cap U$ is a homeomorphism on a set $\varphi(V \cap U)$ in $\mathbb{R}^n$ containing $\varphi(x)$. There exist an open neighbourhood W of x in $V \cap U$ such that $\varphi(x) \in \varphi(W) \subseteq \overline{\varphi(W)} \subseteq \varphi(V \cap U)$ and a function $f \in C^\infty(S)$ such that $f \mid W = 1$ and $f \mid S \backslash (V \cap U) = 0$. Hence, $x \in f^{-1}((0,2)) \subseteq V$. This implies that condition 2.1 is satisfied.

Suppose that $f_1, \ldots, f_n \in C^\infty(S)$ and $F : C^\infty(\mathbb{R}^m)$. We want to show that $F(f_1, \ldots, f_m) \in C^\infty(S)$. For every $x \in S$ and every chart $\varphi : U \to \mathbb{R}^n$ with $x \in U$, there exist a neighbourhood U_x of x contained in U and smooth functions $g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ such that $f_1 \mid U_x = g_i \circ \varphi \mid U_x$ for $i = 1, \ldots, m$. Hence,

$$\begin{aligned} F(f_1, \ldots, f_m) \mid U_x &= F(f_1 \mid U_x, \ldots, f_m \mid U_x) \\ &= F(g_1 \circ \varphi \mid U_x, \ldots, g_m \circ \varphi \mid U_x) \\ &= F(g_1, \ldots, g_n) \mid \circ \varphi \mid U_x \end{aligned}$$

and condition 2.2 is satisfied.

In order to prove condition 2.3, consider $f : S \to \mathbb{R}$ such that, for every $x \in S$, there exist an open neighbourhood W_x of x and a function

$f_x \in C^\infty(S)$ satisfying

$$f_x \mid W_x = f \mid W_x. \tag{9}$$

Given $x \in S$, let $f_x \in C^\infty(S)$ and an open neighbourhood W_x be such that equation (9) is satisfied. Let $\varphi : U \to \mathbb{R}^n$ be a chart such that $x \in U$. There exist an open neighbourhood U_x of x contained in $W_x \cap U$ and a smooth function $g_x : \mathbb{R}^n \to \mathbb{R}$ such that $f_x \mid U_x = g_x \circ \varphi \mid U_x$. Since $U_x \subseteq W_x \cap U$, it follows from equation (9) that $f \mid U_x = g_x \circ \varphi \mid U_x$. This holds for every $x \in S$, which implies that $f \in C^\infty(S)$.

We have shown that smooth functions on S satisfy the conditions for a differential structure on S. Thus, S is a differential space. Local charts are local diffeomorphisms of S onto subsets of $\mathbb{R}^n$. This implies that S is a subcartesian space. □

A stratified space S is said to be topologically locally trivial if, for every $x \in S$, there exist an open neighbourhood U of x in S, a stratified space F with a distinguished point $o \in F$ such that the singleton $\{o\}$ is a stratum of F, and a homeomorphism $\varphi : U \to (M \cap U) \times F$, where M is the stratum of S containing x, such that φ induces smooth diffeomorphisms of the corresponding strata, and $\varphi(y) = (y, o)$ for every $y \in M \cap U$. The stratified space F is called the typical fibre over x. Since we are dealing here with the C^∞ category, we shall say that a smooth stratified space S is locally trivial if it is topologically locally trivial and, for each $x \in S$, the typical fibre F over S is smooth and the homeomorphism $\varphi : U \to (M \cap U) \times F$ is a diffeomorphism of differential spaces. In [8] we have shown that the orbit space of a proper action is locally trivial.

The stratified tangent bundle T^sS of a stratified space S is the union of tangent bundle spaces TM of all strata M of S. We denote by $\tau : T^sS \to S$ the projection map such that for every $x \in S$, $\tau^{-1}(x) = T_xM$, where M is the stratum containing x. For each chart φ on S, with domain U and range $V \subseteq \mathbb{R}^n$, one sets $T^sU = \tau^{-1}(U)$ and defines $T\varphi : T^sU \to T^sV \subseteq \mathbb{R}^{2n}$ by requiring that $(T\varphi) \mid TM \cap T^sU = T(\varphi \mid M \cap U)$ for all strata M of S. One supplies T^sS with the coarsest topology such that all $T^sU \subseteq T^sS$ are open and all $T\varphi$ are continuous, see [16]. A stratified vector field on S is a continuous section X of τ such that, for every stratum M of S, the restriction $X \mid M$ is a smooth vector field on M.

Let S be a smooth stratified space. By Theorem 6, it is a subcartesian space. The above definition of a stratified vector field does not ensure that it generates local one-parameter groups of local diffeomorphisms of

S. Conversely, one often uses the term stratification for a partition of a smooth manifold S which satisfies the Manifold Condition and the Frontier Condition. In this case, there exist vector fields on S, that generate local one-parameter groups of local diffeomorphisms of S, but are not stratified in the sense given above. In this paper, we shall use the term strongly stratified vector field on S for a vector field X on S that generates a local one-parameter group of local diffeomorphisms of S, and is such that, for every stratum M of S, X restricts to a smooth vector field on M. Thus, a strongly stratified vector field on S generates a local one-parameter group of local diffeomorphisms of S that preserves the stratification structure of S.

LEMMA 10. — *Let S be a locally trivial stratified space, and X_M a smooth vector field on a stratum M of S. For every $x \in M$, there exist a neighbourhood W of x in M and a strongly stratified vector field X on S such that $X_M \mid W = X \mid W$.*

Proof. — Let φ_t be the local one-parameter group of local diffeomorphisms of M generated by X_M. Since S is locally trivial, there exist a neighbourhood U of x in S, a smooth stratified space F and a diffeomorphism $\varphi : U \to (M \cap U) \times F$ such that φ induces smooth diffeomorphisms of the corresponding strata, and $\varphi(y) = (y, o)$ for every $y \in M \cap U$. Each stratum of $(M \cap U) \times F$ is of the form $(M \cap U) \times N$, where N is a stratum of F. Let X_U be a stratified vector field on $(M \cap U) \times F$ such that, for every stratum N of F,

$$X_U \mid (M \cap U) \times N = (X_M \mid M \cap U, 0).$$

It is a derivation generating a local one-parameter group of local diffeomorphisms ψ_t of $(M \cap U) \times F$ such that, for every $(y, z) \in (M \cap U) \times F$, $\psi_t(y, z)$ is defined whenever $\varphi_t(y)$ is defined, is contained in $M \cap U$, and $\psi_t(y, z) = (\varphi_t(y), z)$. Hence, X_U is a vector field on $(M \cap U) \times F$.

Let V_1 and V_2 be neighbourhoods of x in S such that $\bar{V}_1 \subseteq V_2 \subseteq \bar{V}_2 \subseteq U$. There exists a function $f \in C^\infty(S)$ such that $f \mid \bar{V}_1 = 1$ and $f \mid S \backslash V_2 = 0$. Let X be a stratified vector field on S such that

$$X(x') = f(x')(T\varphi^{-1}(X_U(\varphi(x')))) \quad \text{for} \quad x' \in U,$$
$$X(x') = 0 \quad \text{for} \quad x' \in S \backslash U.$$

Since X_U is a vector field on $(M \cap U) \times F$, it follows that X is a vector field on S. Let $W = V_1 \cap M$. Since $f \mid W = 1$, it follows that $X \mid W = X_M \mid W$, which completes the proof. □

Let $\mathcal{X}_s(S)$ denote the family of all strongly stratified vector fields on a smooth stratified space S.

LEMMA 11. — *The family $\mathcal{X}_s(S)$ of all strongly stratified vector fields on a locally trivial stratified space S is locally complete.*

Proof. — For $X \in \mathcal{X}_s(S)$, let φ_t denote the local one-parameter group of local diffeomorphisms of S generated by X. Suppose U is the domain of φ_t and V is its range. In other words φ_t maps U diffeomorphically onto V. In Lemma 8 we have shown that, for each $Y \in \mathcal{X}_s(S)$, $\varphi_{t*}Y$ is a vector field on V. By Lemma 1, for every $x \in V$, there exist an open neighbourhood W of x in V such that $\bar{W} \subseteq V$ and a function $f \in C^\infty(S)$ such that $f \mid W = 1$ and $f \mid S\backslash\bar{V} = 0$. Hence, there is a vector field Z on S such that $Z \mid V = (f \mid V)\varphi_{t*}Y$ and $Z \mid S\backslash\bar{V} = 0$. In particular, $Z \mid W = \varphi_{t*}Y \mid W$.

For every stratum M of S, the restriction of Y to M is tangent to M. Moreover, $X \in \mathcal{X}_c(S)$ implies that φ_{t*} preserves M. Hence, $\varphi_{t*}Y$ restricted to $V \cap M$ is tangent to $V \cap M$. Since $Z \mid V = \varphi_{t*}Y \mid V$, it follows that $Z \mid V \cap M$ is tangent to $V \cap M$. On the other hand $Z \mid S\backslash\bar{V} = 0$, which implies that $Z \mid (S\backslash\bar{V}) \cap M = 0$ is tangent to $(S\backslash\bar{V}) \cap M$. Hence, $Z \mid M$ is tangent to M. This ensures that Z is a strongly stratified vector field on S.

The argument above is valid for every X and Y in $\mathcal{X}_s(S)$. Hence, $\mathcal{X}_s(S)$ is a locally complete family of vector fields on S. □

THEOREM 7. — *Strata of a locally trivial stratified space S are orbits of the family $\mathcal{X}_s(S)$ of strongly stratified vector fields.*

Proof. — By Lemma 11, the family $\mathcal{X}_s(S)$ of all strongly stratified vector fields on S is complete. Hence, its orbits give rise to a singular foliation of S. By definition, for each stratum M of S and every $X \in \mathcal{X}_s(S)$, the restriction of X to M is tangent to M. Hence, orbits of $X_s(S)$ are contained in strata of S.

Let x and y be in the same stratum M of S. Since M is connected, there exists a piecewise smooth curve c in M joining x to y. In other words, there exist vector fields $X_M^1, \ldots, X_M^\ell$ on M such that $y = \varphi_{t_1}^\ell \circ \cdots \circ \varphi_{t_\ell}^1(x)$, where φ_t^i is the local one-parameter group of local diffeomorphisms of M generated by X_M^i for $i = 1, \ldots, \ell$. Let $x_t^i = \varphi_t^i \circ \varphi_{t_{i-1}}^{i-1} \circ \cdots \circ \varphi_{t_1}^1$ for every $i = 1, \ldots, \ell$ and $t \in [0, t_i]$. There exist a neighbourhood W_t^i of x_t^i in M and a vector field X_t^i on S such that $X_t^i \mid W_t^i = X_M^i \mid W_t^i$. The family

$\{W_t^i \mid i = 1, \ldots, \ell,\ t \in [0, t_i]\}$ gives a covering of the curve c joining x to y. Since the range of c is compact, there exists a finite subcovering $\{W_{t_j}^i \mid i = 1, \ldots, \ell, j = 1, \ldots, n_i\}$ covering c. Hence, c is a piecewise integral curve of the vector fields $X_{t_j}^i$, $i = 1, \ldots, \ell$, $j = 1, \ldots, n_i$, on S. This implies that M is an $\mathcal{X}(S)$ orbit. □

THEOREM 8. — *Let S be a locally trivial stratified space. Then the singular foliation of S by orbits of the family $\mathcal{X}(S)$ of all vector fields on S is a smooth stratification.*

Proof. — By Theorem 7, strata of the stratification of S are orbits of a family of vector fields on S. Hence, each stratum of the stratification of S is contained in an orbit of the family $\mathcal{X}(S)$ of all vector fields on S. Thus, every orbit of $\mathcal{X}(S)$ is the union of strata. Since strata of S form a locally finite partition of S, it follows that the singular foliation of S by orbits of $\mathcal{X}(S)$ is also locally finite.

Next, we show that orbits of $\mathcal{X}(S)$ are locally closed. Let P be an orbit of $X(S)$ through $x \in S$, and M the stratum of S containing x. Let W_0 be a neighbourhood of x in S which intersects a finite number of strata $M_1, \ldots, M_n$ of S. In other words,

$$W_0 = \bigcup_{i=1}^{n} W_0 \cap M_i.$$

Then

$$U = W_0 \cap P = \bigcup_{i=1}^{n} W_0 \cap M_i \cap P = \bigcup_{M_i \subseteq P} W_0 \cap M_i$$

is a neigbourhood of x in P. Each $W_0 \cap M_i$ is an open subset of M_i. Since M_i is locally closed, we can choose W_0 sufficiently small so that there exists a closed set V_i in S such that $W_0 \cap M_i = W_0 \cap V_i$. Hence,

$$U = \bigcup_{M_i \subseteq P} W_0 \cap M_i = \bigcup_{M_i \subseteq P} W_0 \cap V_i = W_0 \cap \Big(\bigcup_{M_i \subseteq P} V_i \Big),$$

and

$$V = \bigcup_{M_i \subseteq P} V_i$$

is closed as a finite union of closed sets. This shows that P is locally closed.

It remains to verify the Frontier Condition. Let P and Q be orbits of $\mathcal{X}(S)$ such that there exists a point $x \in \bar{P} \cap Q$, where $\bar{P}$ is the closure of P. We want to show that $Q \subseteq \bar{P}$. Suppose that Q is not contained in $\bar{P}$. Then there exists a point $y \in Q$ such that $y \notin \bar{P}$. Since Q is an orbit of $\mathcal{X}(S)$, there exists a piecewise smooth curve $\gamma : [0,1] \to Q$ such that $x = \gamma(0)$ and $y = \gamma(1)$. Let

$$\tau = \sup\{t \in [0,1] \mid \gamma(s) \in \bar{P} \text{ for all } s \leqslant t\}.$$

Since $y = \gamma(1) \notin \bar{P}$, it follows that $0 \leqslant \tau < 1$. Then $z = \gamma(\tau) \in \bar{P}$ and, for every $\varepsilon > 0$, there exists $t \in (0, \varepsilon)$ such that $\gamma(\tau + t) \notin \bar{P}$. Since γ is piecewise smooth, there exists a vector field X on S such that

$$X(z) = \lim_{t \to 0+} \dot{\gamma}(\tau + t) \neq 0.$$

Let φ_t^X denote the local one-parameter group of local diffeomorphisms of S generated by X. For sufficiently small $t_0 > 0$, we can choose a neighbourhood U of z in S such that φ_t^X is defined on U for all $0 \leqslant t \leqslant t_0$. Let $t \in (0, t_0)$ be such that $\gamma(\tau + t) \notin \bar{P}$. We have

$$\gamma(\tau + t) = \varphi_t^X(\gamma(\tau)) = \varphi_t^X(z).$$

Since $z \in \bar{P}$, it follows that $U \cap P \neq \emptyset$. Since P is an orbit of $\mathcal{X}(S)$, it follows that $\varphi_s^X(U \cap P) \subseteq P$ for all $0 \leqslant s \leqslant t$. Moreover, φ_t^X is a diffeomorphism of U on its image mapping $U \cap P$ onto $\varphi_t^X(U \cap P)$. Hence

$$\gamma(\tau + t) = \varphi_t^X(z) \in \varphi_t^X(\bar{P} \cap U) \subseteq \overline{\varphi_t^X(U \cap P)} \subseteq \bar{P},$$

which contradicts the assumption that $\gamma(\tau + t) \notin \bar{P}$. This implies that $Q \subseteq \bar{P}$, which completes the proof that the singular foliation of S by orbits of the family $\mathcal{X}(S)$ of all vector fields on S is a stratification.

We still need to show that the stratification of S by orbits of the family $\mathcal{X}(S)$ of all vector fields on S is smooth. By assumption, S is a smooth stratified space, hence its smooth structure is determined by a maximal atlas of compatible smooth charts on S. Let φ be a chart of this atlas with domain U and range $\varphi(U) \subseteq \mathbb{R}^n$. For each orbit M of $\mathcal{X}(S)$, the intersection $M \cap U$ is a manifold contained in U, and $\varphi(M \cap U)$ is a manifold contained in $\varphi(U)$ because φ is a diffeomorphism. Suppose that $f : \varphi(U \cap M) \to \mathbb{R}$ is smooth. Since M is locally closed, it follows that $\varphi(U \cap M)$ is a locally closed subset of $\mathbb{R}^n$. Hence, for every $y \in \varphi(U \cap M)$, there exist a neighbourhood V of y in $\mathbb{R}^n$ and $\widetilde{f} \in C^\infty(\mathbb{R}^n)$ such that $f \mid V \cap \varphi(U \cap M) = \widetilde{f} \mid V \cap (\varphi(U \cap M)$. Since φ is a smooth map of U into $\mathbb{R}^n$, $\varphi^* \widetilde{f} = \widetilde{f} \circ \varphi \in C^\infty(U)$. Moreover,

$$\begin{aligned}(\varphi \mid U \cap M)^* f \mid \varphi^{-1}(V) &= f \circ \varphi \mid \varphi^{-1}(V \cap \varphi(U \cap M)) \\ &= f \mid V \cap \varphi(U \cap M) \\ &= \widetilde{f} \mid V \cap (\varphi(U \cap M) \\ &= \widetilde{f} \circ \varphi \mid \varphi^{-1}(V \cap \varphi(U \cap M)) \\ &= \widetilde{f} \circ \varphi \mid \varphi^{-1}(V) \cap (U \cap M).\end{aligned}$$

Since this is valid for every $y \in \varphi(U \cap M)$, it follows that $\varphi \mid U \cap M : U \cap M \to \varphi(U \cap M)$ is smooth. This holds for every chart φ and every orbit M of $\mathcal{X}(S)$. Hence, the stratification of S by orbits of $\mathcal{X}(S)$ is smooth with respect to the atlas defining the smooth structure on S. □

Remark. — In Theorem 8, we have assumed that the original stratification of S is locally trivial. We do not know if the stratification of S given by orbits of $\mathcal{X}(S)$ is locally trivial.

7. Poisson reduction.

We can now return to the problem of Poisson reduction of a proper symplectic action

$$\Phi : G \times P \to P : (g, p) \mapsto \Phi(g, p) \equiv \Phi_g(p) \equiv gp$$

of a Lie group G on a symplectic manifold (P, ω), which has motivated this work. Here, ω is a symplectic form on P and, for every $g \in G$, $\Phi_g^* \omega = \omega$.

For every $p \in P$, the isotropy group G_p of p is given by

$$G_p = \{g \in G \mid \Phi_g(p) = p\}.$$

Since the action Φ is proper, all isotropy groups are compact. For every compact subgroup K of G, the set

$$P_K = \{p \in P \mid G_p = K\}$$

of points of isotropy type K, and the set

$$P_{(K)} = \{p \in P \mid G_p \text{ is conjugate to } K\}$$

of points of orbit type K are local manifolds. Thus means that connected components of P_K and $P_{(K)}$ are submanifolds of P ([4]).

Let $S = P/G$ denote the orbit space of the action Φ and $\rho : P \to S$ the orbit the orbit map associating, to each $p \in P$, the orbit $Gp = \{\Phi_g(p) \mid$

$g \in G\}$ of G through p. The orbit space S is stratified by orbit type ([9]). In other words, strata of S are connected components of $\rho(P_{(K)})$, as K varies over compact subgroups of G for which $P_{(K)} \neq \emptyset$. The space S is a differential space with a differential structure $C^\infty(S)$, introduced in [17], which consists of push-forwards to S of G-invariant smooth functions on P, it is locally trivial and minimal ([6], [7]). Minimality of the stratification implies that its strata are orbits of the family $\mathcal{X}(S)$ of all vector fields on S.

For each $f \in C^\infty(P)$, the Hamiltonian vector field of f is the vector field X_f on P defined by

$$X_f \lrcorner\, \omega = df, \tag{10}$$

where $\lrcorner$ denotes the left interior product of vector fields and forms. The Poisson bracket of $f_1, f_2 \in C^\infty(P)$ is given by

$$\{f_1, f_2\} = X_{f_1} \cdot f_2.$$

It is antisymmetric, satisfies the Jacobi identity

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0,$$

and the Leibniz rule

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\},$$

for every $f_1, f_2, f_3 \in C^\infty(P)$.

Since the action Φ of G on P preserves the symplectic form ω, the induced action of G on $C^\infty(P)$ preserves the Poisson bracket. Hence, the space $C^\infty(P)^G$ of G-invariant functions in $C^\infty(P)$ is a Poisson sub-algebra of $C^\infty(P)$. The differential structure

$$C^\infty(S) = \{h : S \to \mathbb{R} \mid \rho^* h \in C^\infty(P)\}$$

is isomorphic to $C^\infty(P)^G$. This implies that the Poisson bracket on $C^\infty(P)^G$ induces a Poisson bracket on $C^\infty(S)$ such that

$$\rho^*\{h_1, h_2\} = \{\rho^* h_1, \rho^* h_2\} \tag{11}$$

for every $h_1, h_2 \in C^\infty(S)$.

For every $h \in C^\infty(S)$, we denote by X_h the derivation of $C^\infty(S)$ given by

$$X_h \cdot f = \{h, f\} \quad \text{for all} \quad f \in C^\infty(S).$$

Sjamaar and Lerman showed that, for every $x \in S$, there exists a unique maximal integral curve γ of X_h through x ([21]). For every $f \in C^\infty(S)$,

$$\rho^*(X_h \cdot f) = X_{\rho^* h} \cdot \rho^* f,$$

where $X_{\rho^* h}$ is the Hamiltonian vector field of $\rho^* h \in C^\infty(P)$ defined by equation (10). Since $\rho^* h$ is G-invariant, the one-parameter local group $\varphi_t^{X_{\rho^* h}}$ of local diffeomorphisms of P generated by $X_{\rho^* h}$ commutes with the action of G. If $p \in \rho^{-1}(x) \subseteq P$, then $\gamma(t) = \rho \circ \varphi_t^{X_{\rho^* h}}(x)$. Hence, translations along integral curves of X_h give rise to a local one-parameter group $\varphi_t^{X_h}$ of local diffeomorphisms of S such that $\rho \circ \varphi_t^{X_{\rho^* h}} = \varphi_t^{X_h} \circ \rho$. This implies that the derivation X_h of $C^\infty(S)$ is a vector field on S in the sense of the definition adopted in Section 4. We shall refer to X_h as the Hamiltonian vector field on S corresponding to $h \in C^\infty(S)$.

Hamiltonian vector fields on (P, ω) preserve the symplectic form ω. Hence, they preserve the Poisson bracket on (P, ω). In other words,

$$(\varphi_{-t}^{X_h})^* \{(\varphi_t^{X_h}) f_1, (\varphi_t^{X_h}) f_2\} = \{f_1, f_2\} \quad \text{for all} \quad f_1, f_2, h \in C^\infty(P).$$

Restricting this equality to G-invariant functions, and taking into account the definition of the Poisson bracket on S, equation (11), we obtain

$$(12) \qquad (\varphi_{-t}^{X_h})^* \{(\varphi_t^{X_h}) f_1, (\varphi_t^{X_h}) f_2\} = \{f_1, f_2\} \quad \text{for all} \quad f_1, f_2, h \in C^\infty(S).$$

Let $\mathcal{H}(S)$ denote the family of all Hamiltonian vector fields on S.

PROPOSITION 4. — *The family $\mathcal{H}(S)$ is locally complete.*

Proof. — For $X_f \in \mathcal{H}(S)$, let φ_t denote the local one-parameter group of local diffeomorphisms of S generated by X_f. Suppose U is the domain of φ_t and V is its range. In other words φ_t maps U diffeomorphically onto V. For each $X_h \in \mathcal{H}(S)$, $\varphi_{t*} X_h$ is in $\mathrm{Der}(C^\infty(V))$. We have shown in the proof of Theorem 3 that $\varphi_{t*} X_h$ is a vector field on V.

It follows from equation (2) that, for each $k \in C^\infty(S)$,

$$(\varphi_{t*} X_h) \cdot (k \mid V) = \varphi_{-t}^* (X_h \cdot (\varphi_t^* k)) = \varphi_{-t}^* \{h, \varphi_t^* k\} = \{\varphi_{-t}^* h, k \mid V\},$$

where we have taken into account equation (12). Hence, $(\varphi_{t*} X_h)$ is the inner derivation of $C^\infty(V)$ corresponding to the restriction of $\varphi_{-t}^* h = h \circ \varphi_{-t}$ to V.

For every $x \in V$, there exists an open neighbourhood W of x such that $\bar{W} \subset V$. Let $\tilde{h} \in C^\infty(S)$ be such that $\tilde{h} \mid W = \varphi_{-t}^* h \mid W$. It follows that $(\varphi_{t*} X_h) \mid W = X_{\tilde{h}} \mid W$. Hence, $\mathcal{H}(S)$ is complete. □

Proposition 4 and Theorem 3 imply that orbits of the family $\mathcal{H}(S)$ of Hamiltonian vector fields on S give rise to a singular foliation of S. Moreover, local flows of Hamiltonian vector fields of G-invariant functions on P preserve local manifolds P_K ([8]). Hence, every Hamiltonian vector field X_h on S, corresponding to $h \in C^\infty(S)$, is strongly stratified with respect to the stratification of S by orbit type. Therefore, orbits of $\mathcal{H}(S)$ are contained in strata of the stratification of S by orbit type. Moreover, each orbit of $\mathcal{H}(S)$ is a symplectic manifold ([12] p. 130).

8. Subcartesian Poisson spaces.

In this section we generalize the notion of a Poisson manifold to a subcartesian space.

Let S be a subcartesian space. It will be called a Poisson space if its differential structure $C^\infty(S)$ has a Poisson algebra structure and inner derivations are vector fields on S. We denote by $\{f_1, f_2\}$ the Poisson bracket of f_1 and f_2 in $C^\infty(S)$, and assume that the map

$$C^\infty(S) \times C^\infty(S) \to C^\infty(S) : (f_1, f_2) \mapsto \{f_1, f_2\}$$

is bilinear, antisymmetric and satisfies both the Jacobi identity

(13)
$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\} \quad \text{for all} \quad f_1, f_2, f_3 \in C^\infty(S),$$

and the derivation condition

$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + \{f_1, f_3\} f_2 \quad \text{for all} \quad f_1, f_2, f_3 \in C^\infty(S). \tag{14}$$

Let

$$\mathcal{H}(S) = \{X_f : C^\infty(S) \to C^\infty(S) : h \to X_f \cdot h = \{f, h\}\}. \tag{15}$$

It follows from equation (14) that each $X_f \in \mathcal{H}(S)$ is a derivation of $C^\infty(S)$. Since derivations on a subcartesian space needs not be vector fields, we make an additional assumption that, for each $f \in C^\infty(S)$, the derivation $X_f \in \mathcal{H}(S)$ is a vector field on S. The vector field X_f is called the Hamiltonian vector field of f. The Jacobi identity implies that $\mathcal{H}(S)$ is a Lie algebra subalgebra of $\mathrm{Der}\, C^\infty(S)$. The Lie bracket on $\mathcal{H}(S)$ satisfies the identity

$$[X_{f_1}, X_{f_2}] = X_{\{f_1, f_2\}} \quad \text{for all} \quad f_1, f_2 \in C^\infty(S).$$

We shall refer to $\mathcal{H}(S)$ as the family of Hamiltonian vector fields on S.

LEMMA 12. — *For each open subset U of a Poisson space S, the Poisson bracket on $C^\infty(S)$ induces a Poisson bracket on $C^\infty(U)$.*

Proof. — Let $h_1, h_2 \in C^\infty(U)$. Since U is open in S, for each point $x \in U$, there exist a neighbourhood U_x of x in U and functions $f_{1,x}$, $f_{2,x} \in C^\infty(S)$ such that $h_1 \mid U_x = f_{1,x} \mid U_x$ and $h_2 \mid U_x = f_{2,x} \mid U_x$. We define

$$\{h_1, h_2\}(x) = \{f_{1,x}, f_{2,x}\}(x), \tag{16}$$

where the right hand side is the value at $x \in U_x \subseteq S$ of the Poisson bracket of functions in $C^\infty(S)$. We have to show that the right hand side of equation (16) is independent of the choice of U_x and $f_{1,x}$ and $f_{2,x}$. Let U'_x be another open neighbourhood of x in U and $f'_{1,x}, f'_{2,x} \in C^\infty(S)$ such that $h_1 \mid U'_x = f'_{1,x} \mid U'_x$ and $h_2 \mid U'_x = f'_{2,x} \mid U'_x$. Then,

$$f_{1,x} \mid U_x \cap U'_x = f'_{1,x} \mid U_x \cap U'_x \text{ and } f_{2,x} \mid U_x \cap U'_x = f'_{2,x} \mid U_x \cap U'_x. \tag{17}$$

Hence, $k_{1,x} = f'_{1,x} - f_{1,x}$ and $k_{2,x} = f'_{2,x} - f_{2,x}$ vanish on $U_x \cap U'_x$. Moreover,

$$\{f'_{1,x}, f'_{2,x}\}(x) = \{f_{1,x} + k_{1,x}, f_{2,x} + k_{2,x}\}(x) = \{f_{1,x}, f_{2,x}\}(x)$$

because $\{f_{1,x}, k_{2,x}\}$, $\{f_{2,x}, k_{1,x}\}$ and $\{k_{1,x}, k_{2,x}\}$ vanish on $U_x \cap U_{x'}$. This proves that the Poisson bracket on $C^\infty(U)$ is well defined by equation (16). Moreover, it is bilinear, antisymmetric and satisfies equations (13) and (14) because the Poisson bracket on $C^\infty(S)$ has these properties. □

LEMMA 13. — *The Poisson bracket $\{f_1, f_2\}$ on $C^\infty(S)$ is invariant under the local one-parameter groups of local diffeomorphisms of S generated by Hamiltonian vector fields.*

Proof. — For $X_f \in \mathcal{H}(S)$, let φ_t denote the local one-parameter group of local diffeomorphisms of S generated by X. Suppose U is the domain of φ_t and V is its range. In other words φ_t maps U diffeomorphically onto V. We want to show that

$$\varphi^*_{-t}\{\varphi^*_t f_1, \varphi^*_t f_2\} = \{f_1, f_2\} \quad \text{for all} \quad f_1, f_2 \in C^\infty(S). \tag{18}$$

Differentiating the left-hand side with respect to t we get

$$\begin{aligned}
&\frac{d}{dt}\varphi^*_{-t}\{\varphi^*_t f_1, \varphi^*_t f_2\} \\
&\quad = -\varphi^*_{-t}(X\cdot \{\varphi^*_t f_1, \varphi^*_t f_2\}) + \varphi^*_{-t}\{X\cdot (\varphi^*_t f_1), \varphi^*_t f_2\} \\
&\qquad\qquad + \varphi^*_{-t}\{\varphi^*_t f_1, X\cdot (\varphi^*_t f_2)\} \\
&\quad = \varphi^*_{-t}(-\{\{f, \{\varphi^*_t f_1, \varphi^*_t f_2\}\} + \{\{f, \varphi^*_t f_1\}, \varphi^*_t f_2\} + \{\varphi^*_t f_1, \{f, \varphi^*_t f_2\}\}) \\
&\quad = 0
\end{aligned}$$

because of Jacobi identity (13). Since $\varphi_0^* f_i = f_i$, it follows that

$$\varphi_{-t}^*\{\varphi_t^* f_1, \varphi_t^* f_2\} = \varphi_{-0}^*\{\varphi_0^* f_1, \varphi_0^* f_2\} = \{f_1, f_2\},$$

which completes the proof. □

LEMMA 14. — *Let S be a subcartesian Poisson space. The family $\mathcal{H}(S)$ of Hamiltonian vector fields is locally complete.*

Proof of this lemma is identical to the proof of Proposition 4.

THEOREM 9. — *Let S be a subcartesian Poisson space. Orbits of the family $\mathcal{X}(S)$ of all vector fields on S are Poisson manifolds. Orbits of the family $\mathcal{H}(S)$ of Hamiltonian vector fields are symplectic manifolds. For each Poisson leaf M, orbits of $\mathcal{H}(S)$ contained in M give rise to a singular foliation of M by symplectic leaves.*

Proof. — Let M be an orbit of $\mathcal{X}(S)$. By Theorem 4, it is a smooth manifold. Let $C^\infty(M)$ be the space of smooth functions on M defined in terms of the manifold structure of M. Let $\mathcal{T}$ denote the manifold topology of M described in Section 6. A function $h : M \to \mathbb{R}$ is in $C^\infty(M)$ if and only if, for each $x \in M$, there exists $V \in \mathcal{T}$ such that $x \in V$ and there exists a function $f_x \in C^\infty(S)$ such that $h \mid V = f_x \mid V$.

For $h_1, h_2 \in C^\infty(M)$ and $x \in M$, let $V \in \mathcal{T}$, and $f_{1,x}$, $f_{2,x} \in C^\infty(S)$ be such that $h_i \mid V = f_{i,x} \mid V$ for $i = 1, 2$. We define $\{h_1, h_2\}_M$ by the requirement that

$$\{h_1, h_2\}_M \mid V = \{f_{1,x}, f_{2,x}\} \mid V. \tag{19}$$

We have to show that the right-hand side of equation (19) is independent of the choice of $f_{1,x}$, and $f_{2,x}$ in $C^\infty(S)$. Suppose that $f'_{1,x}$, $f'_{2,x} \in C^\infty(S)$ satisfy the condition $h_i \mid V = f'_{i,x} \mid V$ for $i = 1, 2$. Let $k_{i,x} = (f'_{i,x} - f_{i,x})$. Then, $k_{i,x} \mid V = 0$, and

$$\begin{aligned}
&\{f'_{1,x}, f'_{2,x}\} \mid V \\
&\quad = \{f_{1,x} + k_{1,x}, f_{2,x} + k_{2,x}\} \mid V \\
&\quad = (\{f_{1,x}, f_{2,x}\} + \{f_{1,x}, k_{2,x}\} + \{k_{1,x}, f_{2,x}\} + \{k_{1,x}, k_{2,x}\}) \mid V \\
&\quad = (\{f_{1,x}, f_{2,x}\} + (X_{f_{1,x}} \cdot k_{2,x}) - (X_{f_{2,x}} \cdot k_{1,x}) + (X_{k_{1,x}} \cdot k_{2,x})) \mid V.
\end{aligned}$$

Since M is an orbit of $\mathcal{X}(S)$, V is an open subset of M, and $k_{1,x} \mid V = 0$ and $k_{2,x} \mid V = 0$, it follows that $(X \cdot k_{1,x}) \mid V = (X \cdot k_{2,x}) \mid V = 0$ for all $X \in \mathcal{X}(S)$. Hence, $\{f'_{1,x}, f'_{2,x}\} \mid V = \{f_{1,x}, f_{2,x}\} \mid V = 0$ and $\{h_1, h_2\}_M$ is well defined.

It follows from equation (19) that $\{h_1, h_2\}_M \in C^\infty(M)$. The Poisson bracket properties of $\{\cdot\,,\cdot\}_M$ follow from the corresponding properties of the Poisson bracket on $C^\infty(S)$.

By Lemma 14, the family $\mathcal{H}(S)$ of Hamiltonian vector fields on S is complete. Hence, its orbits give rise to a singular foliation of S. Theorem 4 implies that each orbit of $\mathcal{H}(S)$ is contained in an orbit of $\mathcal{X}(S)$.

We have shown that each orbit M of $\mathcal{X}(S)$ is a Poisson manifold. Orbits of $\mathcal{H}(S)$ contained in M coincide with orbits of the family of Hamiltonian vector fields on M, which gives rise to a foliation of M by symplectic leaves of M ([12] p. 130). $\square$

Let S be a Poisson space, and G be a connected Lie group with Lie algebra $\mathfrak{g}$. We denote by

$$\Phi : G \times S \to S : (g, x) \mapsto \Phi(g, x) \equiv \Phi_g(x) = gx$$

an action of G on S. We assume that this action is smooth, which implies that, for each $g \in G$, the map $\Phi_g : S \to S$ is a diffeomorphism. Moreover, we assume that, for every $g \in G$,

$$\Phi_g^* : C^\infty(S) \to C^\infty(S) : f \mapsto \Phi_g^*(f) = f \circ \Phi_g$$

is an automorphism of the Poisson algebra structure of $C^\infty(S)$. In other words,

$$\Phi_g^*\{f_1, f_2\} = \{\Phi_g^* f_1, \Phi_g^* f_2\} \text{ for all } g \in G \text{ and } f_1, f_2 \in C^\infty(S).$$

Finally, we assume that the action Φ is proper.

For each $\xi \in \mathfrak{g}$, we denote by X^ξ the vector field on S generating the action on S of the one-parameter subgroup $\exp t\xi$ of G. Clearly, $X^\xi \in X(S)$ for all $\xi \in \mathfrak{g}$. Since G is connected, its action on S is generated by the action of all one-parameter subgroups. Hence, each Poisson manifold of S is invariant under the action of G on S. We denote by $\Phi^M : G \times M \to M$ the induced action of G on a Poisson manifold M. The assumptions on the action of G on S imply that the action Φ^M is smooth and proper. Moreover, it preserves the Poisson algebra structure of $C^\infty(M)$.

Let $R_M = M/G$ denote the space of G-orbits in M and $\rho_M : M \to R_M$ the orbit map. Since M is a manifold, and the action of G on M is proper, it follows that R_M is a stratified space which can be covered by open sets, each of which is diffeomorphic to an open subset of a semi-algebraic set ([8], p. 727). Hence, R_M is a subcartesian space.

The differential structure $C^\infty(R_M)$ is isomorphic to the ring $C^\infty(M)^G$ of G-invariant smooth functions on M. Since the action of G on M preserves the Poisson bracket $\{\cdot,\cdot\}_M$ on $C^\infty(M)$, it follows that $C^\infty(M)^G$ is a Poisson subalgebra of $C^\infty(M)$. Hence, $C^\infty(R_M)$ inherits a Poisson bracket $\{\cdot,\cdot\}_{R_M}$.

For $h \in C^\infty(R_M)$, let $f = \rho_M^* h \in C^\infty(M)^G$. Let X_h be the derivation of $C^\infty(R_M)$ given by $X_h \cdot h' = \{h, h'\}_{R_M}$ for all $h' \in C^\infty(R_M)$. Similarly, X_f is the derivation of $C^\infty(M)$ given by $X_f \cdot f' = \{f, f'\}_M$ for all $f' \in C^\infty(M)$. The vector field X_f generates a one-parameter group φ_t of local diffeomorphisms of M which commutes with the action of G on M. Hence, φ_t induces a local one-parameter group of local diffeomorphisms ψ_t of the orbit space $R_M = M/G$ such that $\psi_t \circ \rho_M = \rho_M \circ \varphi_t$. The local group ψ_t is generated by X_h. Hence, X_h is a vector field on R_M.

It follows from the above discussion that the orbit space R_M is a subcartesian Poisson space. Hence, we can apply the results of Theorem 9.

PROPOSITION 5. — *Let M be a Poisson manifold, and $R_M = M/G$ be the orbit space of a properly acting Lie symmetry group G of the Poisson structure on M. Then R_M is a subcartesian Poisson space. It is a stratified space. Strata of R_M are orbits of the family $\mathcal{X}(R_M)$ of all vector fields on R_M. Each stratum is a Poisson manifold. The singular foliation of R_M by orbits of the Lie algebra $\mathcal{H}(R_M)$ of Hamiltonian vector fields of $C^\infty(R_M)$ gives rise to a refinement of the stratification of R_M by symplectic manifolds.*

Proof. — It follows from Theorem 9 that orbits of the family $\mathcal{X}(R_M)$ of all vector fields on R_M are Poisson manifolds. Stratification structure of R_M and its smoothly local triviality are consequences of the properness of the action of G on M ([9], [6], [7]). It follows from Theorem 7 and Theorem 9 that strata of R_M are Poisson manifolds. Moreover, Theorem 9 implies that orbits of the family $\mathcal{H}(R_M)$ of Hamiltonian vector fields are symplectic manifolds. The restriction of the singular foliation of R_M by symplectic manifolds to each stratum of R_M gives rise to a singular foliation of the stratum by symplectic manifolds. □

Let $R = S/G$ be the space of G-orbits in S and $\rho : S \to R$ the orbit map. It is a differential space with differential structure $C^\infty(R)$ isomorphic to the ring $C^\infty(S)^G$ of G-invariant smooth functions on S. The Poisson algebra structure of $C^\infty(S)$ induces a Poisson structure on $C^\infty(R)$. It follows from Corollary 3 and the discussion preceding it, that R is singularly foliated by Poisson manifolds, and each Poisson leaf is singularly foliated by symplectic leaves. We do not know if R is a subcartesian space. Hence, we cannot assert that Poisson leaves of R are orbits of the family $\mathcal{X}(R)$ of all vector fields on R, or that symplectic leaves of R are orbits of the family $\mathcal{H}(R)$ of Hamiltonian vector fields.

9. Almost complex structures.

In this section, we discuss almost complex structures defined on complete families of vector fields on subcartesian spaces. We assume here that the subcartesian space under consideration is paracompact. By a theorem of Marshall, this assumption ensures the existence of partitions of unity ([13]).

Let $\mathcal{F} = \{X^\alpha\}_{\alpha \in A}$ be a complete family of vector fields on a paracompact subcartesian space S. We denote by $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ the submodule of derivations of $C^\infty(S)$ consisting of locally finite sums $\Sigma_\alpha f_\alpha X^\alpha$, where $f_\alpha \in C^\infty(S)$, $X^\alpha \in \mathcal{F}$ and, for every $x \in S$, there is an open neighbourhood U of x in S such that $f_\alpha X^\alpha \mid U = 0$ for almost all α. Abusing somewhat the common terminology, we shall refer to $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ as the module of derivations generated by $\mathcal{F}$.

PROPOSITION 6. — *For every complete family $\mathcal{F}$ of vector fields on a paracompact subcartesian space S, the module $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ of derivations generated by $\mathcal{F}$ is a Lie subalgebra of the Lie algebra of all derivations of $C^\infty(S)$.*

Proof. — Recall that the completeness of $\mathcal{F} = \{X^\alpha\}_{\alpha \in A}$ implies that, for every α, β, t, and x for which $\varphi^\alpha_{t*} X^\beta(x)$ is defined, there exist an open neighbourhood U of x and $\gamma \in A$ such that $\varphi^\alpha_{t*} X^\beta \mid U = X^\gamma \mid U$. Hence, there exists $Z_U^{\alpha\beta} \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ such that $[X^\alpha, X^\beta] \mid U = Z_U^{\alpha\beta} \mid U$. In this way we get an open cover $\mathcal{U} = \{U\}$ of S. By shrinking open sets U, if necessary, we may assume that the covering $\mathcal{U} = \{U\}$ is locally finite. Since S is paracompact, there exists a partition of unity $\{f_U\}_{U \in \mathcal{U}}$ subordinate to this covering ([13]). Hence, $[X^\alpha, X^\beta] \mid U = \Sigma_U f_U (Z_U^{\alpha\beta} \mid U)$, where the sum on the right-hand side is locally finite. This implies that $[X^\alpha, X^\beta] \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$.

If $X = \Sigma_\alpha f_\alpha X^\alpha$ and $Y = \Sigma_\beta h_\beta X^\beta$ are in $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$, then

$$\begin{aligned}[X, Y] &= [\Sigma_\alpha f_\alpha X^\alpha, \Sigma_\beta h_\beta X^\beta] \\ &= \Sigma_{\alpha\beta}(f_\alpha h_\beta [X^\alpha, X^\beta] + f_\alpha (X^\alpha \cdot h_\beta X^\beta - h_\beta (X^\beta \cdot f_\alpha) X^\alpha).\end{aligned}$$

Since the sum is locally finite, it implies that $[X, Y] \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$. □

An almost complex structure on a complete family $\mathcal{F}$ of vector fields on S is a $C^\infty(S)$ module automorphism $J : \mathrm{Der}_{\mathcal{F}}(C^\infty(S)) \to \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ such that $J^2 = -1$. Since $J : \mathrm{Der}_{\mathcal{F}}(C^\infty(S)) \to \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ is a $C^\infty(S)$ module automorphism, it implies that, for each orbit M of $\mathcal{F}$, it gives rise to a linear map $J_M : TM \to TM$. Moreover, $J^2 = -1$ implies that $J_M^2 = -1$. Hence, an almost complex structure on a complete family $\mathcal{F}$ of vector fields on S induces an almost complex structure on each orbit of $\mathcal{F}$.

Since $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ is a Lie algebra, we may consider the torsion N of J defined as follows. For $X, Y \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$, let

$$N(X,Y) = 2\{[JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]\}. \tag{20}$$

LEMMA 15. — *The torsion of J is a skew symmetric bilinear mapping $N : \mathrm{Der}_{\mathcal{F}}(C^\infty(S)) \otimes \mathrm{Der}_{\mathcal{F}}(C^\infty(S)) \to \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ such that $N(fX, hY) = fhN(X,Y)$ for all X, Y in $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ and $f, h \in C^\infty(S)$.*

Proof. — Skew symmetry and bilinearity of N are self-evident. For every $X, Y \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$ and $f, h \in C^\infty(S)$:

$$\begin{aligned}
[JfX, JhY] &= fh[JX, JY] + f((JX)\cdot h)JY - h((JY)\cdot f)JX \\
J[fX, JhY] &= fhJ[X, JY] - f(X\cdot h)Y - h((JY)\cdot f)JX. \\
J[JfX, hY] &= fhJ[JX, Y] + f((JX)\cdot h)JY + h(Y\cdot f)X \\
[fX, hY] &= fh[X,Y] + f(X\cdot h)Y - h(Y\cdot f)X.
\end{aligned}$$

Hence, $N(fX, hY) = fhN(X,Y)$, which completes the proof. □

The almost complex structure J has eigenvalues $\pm i$ because $J^2 = -1$. Eigenspaces of J are contained in the complexification $\mathrm{Der}_{\mathcal{F}}(C^\infty(S)) \otimes \mathbb{C}$ of $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$. For every $X \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))$,

$$\begin{aligned}
J(X - iJX) &= JX - iJ^2X = JX + iX = i(X - iJX), \\
J(X + iJX) &= JX + iJ^2X = JX - iX = -i(X + iJX).
\end{aligned}$$

Hence

$$\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^{\pm} = \{X - (\pm i)JX \mid X \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))\}$$

are eigenspaces of J corresponding to eigenvalues $\pm i$, respectively.

LEMMA 16. — *Eigenspaces $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^{\pm}$ of J are closed under the Lie bracket if and only if the torsion N of J vanishes.*

Proof. — For every X and Y in $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))$, we have

$$\begin{aligned}
&[X-(\pm i)JX, Y-(\pm i)JY]\\
&= [X,Y]-(\pm i)[JX,Y]-(\pm i)[X,JY]+(\pm i)^2[JX,JY]\\
&= [X,Y]-(\pm i)[JX,Y]-(\pm i)[X,JY]-[JX,JY]\\
&= -N(X,Y)-J[JX,Y]-J[X,JY]-(\pm i)[JX,Y]-(\pm i)[X,JY]\\
&= -N(X,Y)-(\pm i)\{([JX,Y]+[X,JY])-(\pm iJ)([JX,Y]+[X,JY])\}.
\end{aligned}$$

Hence, $[X-(\pm i)JX, Y-(\pm i)JY] \in \mathrm{Der}_{\mathcal{F}}(C^\infty(S))^{\pm}$ if and only if $N(X,Y)=0$. □

THEOREM 10. — *Let J be an almost complex structure on a complete family $\mathcal{F}$ of vector fields on a paracompact subcartesian space S. Every orbit M of $\mathcal{F}$ admits a complex analytic structure such that $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^+ \mid M$ spans the distribution of holomorphic directions on $TM \otimes \mathbb{C}$ if and only if the torsion N of J vanishes.*

Proof. — For each orbit M of $\mathcal{F}$, the restriction of N to M is the torsion tensor N_M of the almost complex structure J_M on M. Suppose that M admits a complex analytic structure such that $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^+ \mid M$ spans the distribution of holomorphic directions on $TM\otimes\mathbb{C}$. For a complex manifold M, the distribution of holomorphic directions on $TM \otimes \mathbb{C}$ is involutive. Hence, $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^+ \mid M$ is closed under the Lie bracket of vector fields on $TM\otimes\mathbb{C}$. This implies that $N_M = 0$. Therefore, if every orbit M of $\mathcal{F}$ admits a complex analytic structure such that $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^+ \mid M$ spans the distribution of holomorphic directions on $TM \otimes \mathbb{C}$, then the torsion N vanishes.

Suppose now that $N=0$. Then $N_M = 0$ for every orbit M of $\mathcal{F}$. It follows that the almost complex structure J_M on M is integrable. By the Newlander-Nirenberg theorem, there exists a complex analytic structure on M such that $\mathrm{Der}_{\mathcal{F}}(C^\infty(S))^+ \mid M$ spans the distribution of holomorphic directions on $TM \otimes \mathbb{C}$ ([15]). Naively speaking, one could say that this complex structure is obtained by patching local diffeomorphisms from $\mathbb{C}^n$ to M in a manner similar to that used in the proof of Theorem 3. □

Assuming that the torsion tensor N of J vanishes, we are going to characterize smooth functions on S which are holomorphic on each orbit of $\mathcal{F}$. Let $C^\infty(S)^{\mathbb{C}} = C^\infty(S)\otimes\mathbb{C}$ be the complexification of $C^\infty(S)$. Each function in $C^\infty(S)^{\mathbb{C}}$ is of the form $f+ih$, where $f,h \in C^\infty(S)$. Such a function is holomorphic on each orbit of $\mathcal{F}$ if it is annihilated by derivations in $\mathrm{Der}_{\mathcal{F}}(C^{\infty}(S))^{-}$. In other words, a function $f + ih$ is holomorphic on orbits of $\mathcal{F}$ if it satisfies the differential equation $(X + iJX)(f + ih) = 0$ for each $X \in \mathcal{F}$. Separating real and imaginary parts we get a singularly foliated version of Cauchy-Riemann equations

$$X\cdot f - (JX)\cdot h = 0 \text{ and } (JX)\cdot f + X\cdot h = 0 \quad \text{for all} \quad X \in \mathcal{F}.$$

It should be noted that these equations may have very few solutions which are globally defined. This is why one usually employs sheaves in the study of holomorphic functions ([24]).

Combining the results of the last two sections, we can describe subcartesian Poisson-Kähler spaces.

Example 3. — Let S be a subcartesian Poisson space. It follows from the definition of Hamiltonian vector fields on S that the Poisson bracket $\{f, h\}$ on $C^{\infty}(S)$ satisfies the relations

$$X_f\cdot h = \{f, h\} = -\{h, f\} = -X_h\cdot f.$$

Hence, there exists a skew symmetric form Ω on $H(S)$ with values in $C^{\infty}(S)$ such that

$$\Omega(X_f, X_h) = \{f, h\}, \forall f, h \in C^{\infty}(S).$$

By Theorem 9, every orbit M of $\mathcal{H}(S)$ is a symplectic manifold. The symplectic form Ω_M of M is given by the restriction of Ω to M. Let $J : \mathrm{Der}_{\mathcal{H}(S)} C^{\infty}(S) \to \mathrm{Der}_{\mathcal{H}(S)} C^{\infty}(S)$ be an almost complex structure on $\mathcal{H}(X)$ such that

$$\Omega(JX, JY) = \Omega(Y, X), \forall X, Y \in \mathrm{Der}_{\mathcal{H}(S)} C^{\infty}(S),$$

for all $X \in \mathrm{Der}_{\mathcal{H}(S)} C^{\infty}(S)$. Define

$$g(X, Y) = \Omega(JX, Y)$$

for all $X, Y \in \mathrm{Der}_{\mathcal{H}(S)} C^{\infty}(S)$. It is symmetric because

$$\begin{aligned} g(Y, X) &= \Phi(JY, X) = \Phi(J^2Y, JX) = -\Phi(Y, JX) = \Phi(JX, Y) \\ &= g(X, Y). \end{aligned}$$

If $g(X, Y)$ is positive definite, that is $g(X, X)(x) = 0$ only if $X(x) = 0$, then its restriction to every orbit M of $H(S)$ defines a Riemannian metric g_M on M. For every $X, Y \in H(S)$ and $x \in M$, $g_M(X(x), Y(x)) = \Omega_M(JX(x), Y(x))$ and the form Ω_M is closed. Hence, M is an almost Kähler manifold. If the torsion N of J vanishes, then every orbit M of $\mathcal{H}(S)$ is a Kähler manifold. This example is a generalization to subcartesian spaces of Kähler stratified spaces studied by Huebschmann ([11]).

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Jędrzej ŚNIATYCKI,
University of Calgary
Department of Mathematics and Statistics
Calgary, Alberta (Canada).
sniat@math.ucalgary.ca