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APPLICATIONS OF SPINOR CLASS FIELDS: EMBEDDINGS OF ORDERS AND QUATERNIONIC LATTICES

by Luis ARENAS-CARMONA (*)

1. Introduction.

Chevalley [6] studied the following question:

Let D be a central simple algebra over a number field k, let L be a maximal abelian subalgebra of D, and let $\mathfrak{L}$ be an order in L. Does every maximal order of D contain an isomorphic copy of $\mathfrak{L}$?

When the answer to the above question was negative, Chevalley asked:

If Z is the set of isomorphism classes of maximal orders of D, and X the subset of classes of orders containing a copy of $\mathfrak{L}$, what are the possible values of the ratio of their cardinalities $\rho = \sharp X/\sharp Z$?

When $\mathfrak{L} = \mathcal{O}_L$ is the ring of integers of a field L and $D = \mathbb{M}_n(k)$ is a matrix algebra, Chevalley proved that $\rho = [\Sigma \cap L : k]^{-1}$ where Σ is the Hilbert class field of k ([6], p. 26).

Chinburg and Friedman [7] answered Chevalley's questions when D is a quaternion algebra and $\mathfrak{L}$ is an arbitrary order in L. Their result,

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when $\mathfrak{L} = \mathcal{O}_L$, again implies $\rho = [\Sigma \cap L : k]^{-1}$, where in this case Σ is a multiple quadratic extension contained in the wide Hilbert class field of k and defined in terms of the ramification of the quaternion algebra. These results suggest that a similar theorem might hold for a general central simple algebra. In this paper we prove this for algebras that are everywhere locally either matrix or division algebras, a condition that trivially holds for global matrix algebras or quaternion algebras.

THEOREM 1. — *Let D be a central simple algebra over k of dimension $[D : k] = n^2 > 4$. Assume that D is locally either a matrix algebra or a division algebra at every finite place. Then there exists an Abelian extension Σ/k with the following property: For any field extension L/k of maximal dimension that embeds into D, exactly $[\Sigma \cap L : k]^{-1}$ of the conjugacy classes of maximal orders of D contain a copy of the ring $\mathcal{O}_L$ of integers of L.*

Our proof makes essential use of the spinor genus of a maximal order $\mathfrak{D}$. It also applies to quaternion algebras, but the result involves spinor genera when the algebra is totally definite. The field Σ in Theorem 1 is a spinor class field defined in terms of local (spinor) norms. We prove the existence of a bijective map f that associates an element of the Galois group $\mathrm{Gal}(\Sigma/\mathrm{k})$ to each conjugacy class of maximal orders of D. This correspondence is canonical up to the choice of a single class of maximal orders containing a copy of $\mathcal{O}_L$. The set of all conjugacy classes of orders which contain a copy of $\mathcal{O}_L$ is mapped by f to the subgroup $\mathrm{Gal}(\Sigma/\Sigma \cap \mathrm{L})$.

Our proof of Theorem 1 begins by generalizing the definition of spinor class fields given by Estes and Hsia [8]. Hsia studied the problem of whether a quadratic form with coefficients in the ring of integers $\mathcal{O}_k$ of a number field k is represented by another quadratic form of the same kind ([9], [10]). Clearly, a necessary assumption is that this hold at all completions of k. Hsia carried out his study in the language of genera and spinor genera of lattices ([15], p. 297). In this language, he calculates $\rho = \sharp X/\sharp Z$, where Z is the set of spinor genera in the genus of an integral quadratic lattice Λ and X is the subset of spinor genera containing a form that represents a fixed lattice M. Hsia's main results are as follows:

- If $\dim \Lambda \geqslant \dim M + 3$, then ρ is 0 or 1.
- If $\dim \Lambda = \dim M + 2$, then ρ is 0, 1, or 1/2.
- If $\dim \Lambda \leqslant \dim M + 1$, then ρ is 0 or 2^{-t} for some integer $t \geqslant 0$.

In this paper we extend Hsia's results to skew-Hermitian $\mathfrak{D}$-lattices. Thus, let D be a quaternion division algebra over k with standard involution

$q \mapsto \bar{q}$, and let $\mathfrak{D}$ be a maximal order in D. A skew-Hermitian form is a (-1)-Hermitian form in the sense of [19], p. 236. A skew-Hermitian space over D is a pair (V, h) where V is a D-module and h is a skew-Hermitian form on V. A skew-Hermitian $\mathfrak{D}$-lattice is a lattice Λ in V ([15], p. 209) satisfying $\mathfrak{D}\Lambda = \Lambda$. The concepts of genera and spinor genera are also defined for these lattices [4].

In Section 4.1, we prove

THEOREM 2. — *Let (V, h) be a non-degenerate skew-Hermitian space over a quaternion algebra D over a number field k. Let Λ be a $\mathfrak{D}$-lattice in V of maximal rank, and let M be an arbitrary $\mathfrak{D}$-lattice in V. Let $W = kM$ and assume that Λ represents M. Let Z be the set of spinor genera in the genus of Λ, let X be the subset of spinor genera containing a form that represents M, and let $\rho = \sharp X/\sharp Z$. Then, ρ is* 0 *or* 1 *provided* $\dim_D V \geqslant \dim_D W + 2$. *If* $\dim_D V = \dim_D W + 1$, *then ρ is* 0, 1, *or* 1/2. *If $\dim_D V = \dim_D W$, then ρ is 0 or 2^{-t} for some integer $t \geqslant 0$, except possibly if $W = V$ and if there exists a dyadic place v, ramified for D, such that v^2 divides* 2 *and every Jordan component either of Λ_v or of M_v is non-diagonalizable.*

In the course of the proof of Theorem 2 we show that the set of spinor genera in the genus of Λ has a group structure. The subset of spinor genera that represents M is a subgroup, except possibly in the case mentioned in the last sentence of Theorem 2. Again, our proof establishes a bijection between the set Z and the Galois group $\mathrm{Gal}(\Sigma/\mathrm{k})$ for some spinor class field Σ/k. The subset X corresponds to the stabilizer of an intermediate extension $\Sigma_{\Lambda|M}$, the so-called relative spinor class field or representation field of Λ over M.

In this paper we define a concept of relative spinor class field that applies to semisimple algebraic groups whose fundamental group is the group of roots of unity μ_n. Examples of such groups are unitary groups of quaternionic skew-Hermitian forms and automorphism groups of central simple algebras. The field $\Sigma_{\Lambda|M}$, when it is well defined, plays a central role in the proof of Theorems 1 and 2. The idea of extending the concept of spinor genera to maximal orders in central simple algebras appears already in [5], even before spinor class fields were defined.

2. Preliminaries.

Throughout this article k denotes a number field. The set of both finite (or non-archimedean) and infinite (or archimedean) places ([15], p. 7) in k is denoted $\Pi(k)$.

All algebraic groups considered here are subgroups of the general linear group $\mathrm{GL}(V)$ of a finite dimensional k-vector space V. For any field extension E/k, the group of E-points of G is denoted G_E. For any place $v \in \Pi(k)$, the field k_v is the completion of k at v ([15], p. 11). We write G_v for G_{k_v}, with its natural topology induced from the topology of k_v. The same conventions apply to spaces and algebras. All spaces and algebras are assumed to be finite dimensional over k or k_v.

Let S be a finite subset of $\Pi(k)$ containing the infinite places. The set S is fixed throughout. Let $\mathcal{O}$ denote the set of S-integers ([17], p.11) of k. For the sake of generality, all results in what follows are stated in the context of S-integers. An S-lattice in the space V is an $\mathcal{O}$-module contained in a free module ([15], p. 209). An S-order in the algebra D is an S-lattice $\mathfrak{D}$ satisfying $1 \in \mathfrak{D}$, and $\mathfrak{D}\mathfrak{D} = \mathfrak{D}$. Let $v \in \Pi(k) - S$. If Λ denotes an S-lattice, then Λ_v denotes its closure in k_v. We say that Λ_v is the localization at v of Λ. Observe that $\mathcal{O}$ is itself an S-lattice. The localization $\mathcal{O}_v$ is the ring of integers of k_v.

If $G \subseteq \mathrm{GL}(V)$ is a linear algebraic group and Λ is an S-lattice in V, the stabilizer of Λ in G_k is denoted G_k^{Λ}. The definition of G_v^{Λ} is analogous.

Let Λ and Λ' be S-lattices of maximal rank in V. Then, $\Lambda_v = \Lambda'_v$ for all but a finite number of places v. If $\Lambda_v = \Lambda'_v$ for all $v \in \Pi(k) - S$, then $\Lambda = \Lambda'$. If, for every place $v \in \Pi(k) - S$, $\Lambda''(v)$ is a local lattice, and $\Lambda_v = \Lambda''(v)$ for all but a finite number of places v, then there exists a global S-lattice Λ'' such that $\Lambda''(v) = \Lambda''_v$ for all v ([15], §81:14).

For any algebraic group G, we denote by $G_{\mathbb{A}}$ its group of adelic points, which is the restricted topological product of the localizations G_v with respect to the lattice stabilizers G_v^{Λ} ([17], p. 249), where Λ is an S-lattice of maximal rank in V. This definition does not depend on Λ since any two such S-lattices are equal at almost all places. In particular, for the multiplicative group GL_1, we write $J_k = (\mathrm{GL}_1)_{\mathbb{A}}$, the idele group of k.

Let $\sigma \in G_{\mathbb{A}}$. Define $\sigma\Lambda$ as the S-lattice satisfying the local relations $(\sigma\Lambda)_v = \sigma_v\Lambda_v$. The G-genus of Λ is the orbit $G_{\mathbb{A}}\Lambda$. The G-class of Λ is the set of lattices $G_k\Lambda$. We omit the prefix G if it is clear from the context.

The stabilizer of Λ in $G_{\mathbb{A}}$ is denoted $G_{\mathbb{A}}^{\Lambda}$. The set of classes contained in a genus is in one-to-one correspondence with the set of double cosets

$$G_k \backslash G_{\mathbb{A}} / G_{\mathbb{A}}^{\Lambda}. \tag{1}$$

The cardinality of $G_k \backslash G_{\mathbb{A}} / G_{\mathbb{A}}^{\Lambda}$ is called the class number of Λ with respect to G.

Let $\bar{k}$ be the algebraic closure of k, and let $\mathcal{G} = \mathrm{Gal}(\bar{k}/k)$. For any semisimple algebraic group G with universal cover $\widetilde{G}$ and fundamental group F we have a short exact sequence

$$\{1\} \longrightarrow F_{\bar{k}} \longrightarrow \widetilde{G}_{\bar{k}} \longrightarrow G_{\bar{k}} \longrightarrow \{1\}.$$

This gives, by (1.11) in [17] a long exact sequence in cohomology

$$\{1\} \longrightarrow F_k \longrightarrow \widetilde{G}_k \longrightarrow G_k \longrightarrow H^1(\mathcal{G}, F_{\bar{k}}).$$

Assume henceforth that $F_{\bar{k}}$, as $\mathcal{G}$-module, is isomorphic to the group μ_n of n-roots of unity, for some n. Then $H^1(\mathcal{G}, F_{\bar{k}}) = k^*/(k^*)^n$ ([21], p. 83), where R^* denotes the group of units of the ring R. This defines a map $\theta : G_k \longrightarrow k^*/(k^*)^n$, the spinor norm on G_k. We say that G is a group with spinor norm. There is also a local spinor norm $\theta_v : G_v \longrightarrow k_v^*/(k_v^*)^n$ at any place v. It is known ([16], Lemma 13) that the sequence

$$\prod_{v \in \Pi(k)} \widetilde{G}_v \longrightarrow \prod_{v \in \Pi(k)} G_v \xrightarrow{\Pi_v \theta_v} \prod_{v \in \Pi(k)} (k_v^*)/(k_v^*)^n$$

can be restricted to the set of adelic points to get a sequence

$$\widetilde{G}_{\mathbb{A}} \xrightarrow{\Psi_{\mathbb{A}}} G_{\mathbb{A}} \xrightarrow{\Theta_{\mathbb{A}}} J_k / J_k^n. \tag{2}$$

Example 1.— Let D be a central simple algebra over k and let $G = D^*/k^*$ be the automorphism group of D. Then, $\widetilde{G}$ is the group $SL(D) = \{x \in D^* | N(x) = 1\}$ where N is the reduced norm ([13], pp. 21-22). Therefore, $F_{\bar{k}} = SL(D_{\bar{k}}) \cap \bar{k}^* = \mu_n$, where $n^2 = \dim_k D$.

Let $g \in G_k$. Then, g is the inner automorphism of D defined by $g(a) = bab^{-1}$, for some $b \in D^*$. A preimage of g in the universal cover $SL(D_{\bar{k}})$ is $\lambda^{-1}b$ where $\lambda^n = N(b)$. Hence, the image of g under the coborder map is the cocycle $\alpha_\sigma = \lambda^{-\sigma}\lambda$ which corresponds, via the isomorphism

$H^1(\mathcal{G},\mu_n) \cong k^*/(k^*)^n$ ([21], p. 83), to the class $\lambda^n(k^*)^n \in k^*/(k^*)^n$. It follows that $\theta(g) = N(b)(k^*)^n$.

Example 2. — Let B be a non-degenerate k-bilinear form on the space V. Let $G = O^+(B)$ be the special orthogonal group of B. Then, $F = \mu_2$ and G_k is generated by elements of the form $\tau_v\tau_w$, where v and w are in V_k and satisfy $B(v,v) \neq 0$, $B(w,w) \neq 0$ ([15], p. 102). Here, the symmetry τ_z is given by

$$\tau_z(x) = x - 2B(x,z)B(z,z)^{-1}z.$$

In this case, $\theta(\tau_v\tau_w) = B(v,v)B(w,w)(k^*)^2$ ([15], p. 137). This is the case studied in [10] and [11].

Example 3. — Let D be a quaternion algebra, and let (V,h) be a skew-Hermitian space over D. Let $G = U^+(h)$ be the special unitary group of h. Then, $F = \mu_2$ and G_k is generated by elements of the form $(s;\sigma)$, where $s \in V_k$, $\sigma - \bar{\sigma} = h(s,s) \neq 0$, and

$$(s;\sigma)(w) = w - h(w,s)\sigma^{-1}s.$$

In this case, $\theta\big((s;\sigma)\big) = N(\sigma)(k^*)^2$ ([4], p. 173).

In some cases, the spinor norm provides us with a method to compute the number of classes in the genus of a lattice. If we denote the kernel of the spinor norm by $G'_{\mathbb{A}}$, the quotient $G_{\mathbb{A}}/(G_kG^{\Lambda}_{\mathbb{A}}G'_{\mathbb{A}})$ is in one-to-one correspondence with the quotient

$$\Theta_{\mathbb{A}}(G_{\mathbb{A}})/\Big(\theta(G_k)\Theta_{\mathbb{A}}(G^{\Lambda}_{\mathbb{A}})\Big). \tag{3}$$

$G_kG'_{\mathbb{A}}$-orbits are called *spinor genera* ([15], §102:7). The strong approximation theorem [12] states that if G is simply connected, absolutely almost simple, and $G_S = \prod_{v\in S} G_v$ is not compact, then $\overline{G_SG_k} = G_{\mathbb{A}}$. If the universal cover $\widetilde{G}$ of G satisfies these conditions, and if G is a group with spinor norm, so that (2) holds, then

$$G'_{\mathbb{A}} = \Psi_{\mathbb{A}}(\widetilde{G}_{\mathbb{A}}) = \Psi_{\mathbb{A}}(\overline{\widetilde{G}_S\widetilde{G}_k}) \subseteq \overline{G_SG_k} \subseteq \overline{G^{\Lambda}_{\mathbb{A}}G_k} = G^{\Lambda}_{\mathbb{A}}G_k,$$

since $G^{\Lambda}_{\mathbb{A}}$ is compact and G_k is closed. It follows that every spinor genus in the genus of Λ contains exactly one class. In particular, the class number equals the cardinality of the group (3). This is the case when G is the automorphism group of a central simple algebra D of dimension at least 9, since D_v contains a matrix algebra at every infinite place v. Since all maximal orders in a local central simple algebra are conjugate ([17], p. 46), it follows that:

LEMMA 2.0.1. — *Let D be a central simple algebra of dimension at least 9. Then the set of conjugacy classes of maximal orders in D equals the set of spinor genera in the genus of any maximal order of D.*

This result will allow us to reduce the proof of Theorem 1 to Proposition 4.3.4, which is a statement about spinor genera.

We return to the general case of a group G with spinor norm. Let $p : k^* \longrightarrow k^*/(k^*)^n$ and $p_v : k_v^* \longrightarrow k_v^*/(k_v^*)^n$ be the natural projections. Let $X \subseteq G_k$, $X_v \subseteq G_v$. We define $\mathfrak{H}_k(X) = p^{-1}\big(\theta(X)\big)$, $\mathfrak{H}_v(X_v) = p_v^{-1}\big(\theta_v(X_v)\big)$. Analogously, if $P : J_k \longrightarrow J_k/J_k^n$ is the natural projection, and $Y \subseteq G_{\mathbb{A}}$, we define $\mathfrak{H}_{\mathbb{A}}(Y) = P^{-1}\big(\Theta_{\mathbb{A}}(Y)\big)$. The Hasse principle for $\widetilde{G}$ ([17], p. 286) readily implies that

$$\mathfrak{H}_k(G_k) = \mathfrak{H}_{\mathbb{A}}(G_{\mathbb{A}}) \cap k. \tag{4}$$

For any S-lattice Λ, we define

$$H_{\mathbb{A}}(\Lambda) = \mathfrak{H}_{\mathbb{A}}(G_{\mathbb{A}}^{\Lambda}), \qquad H_k(\Lambda) = \mathfrak{H}_k(G_k^{\Lambda}), \qquad H_v(\Lambda) = \mathfrak{H}_v(G_v^{\Lambda}). \tag{5}$$

Class field theory ([14]) yields a one-to-one correspondence between open subgroups H of J_k containing k^* and finite Abelian extensions L/k. If L and L' are the extensions corresponding to H and H', then $H \subseteq H'$ if and only if $L' \subseteq L$. We regard the ring of S-integers $\mathcal{O}$ as an S-lattice of rank 1. Let $J_{k,S} = (GL_1)_{\mathbb{A}}^{\mathcal{O}}$ be the group of S-integral ideles. The Hilbert S-class field $\mathcal{H}_S$ is the class field corresponding to $k^* J_{k,S}$. This field is unramified at all finite places and splits completely at finite places in S. If ∞ is the set of infinite places, $\mathcal{H}_\infty$ is the (wide) Hilbert class field of k ([14], p. 224). The class field corresponding to $k^* J_{k,\infty}^+$, where $J_{k,\infty}^+$ is the group of integral ideles that are positive at all real places, is called the strict Hilbert class field $\mathcal{H}_\infty^+$. It seems desirable to have a similar concept for a more general algebraic group. In order to do this, we must replace the set of double cosets (1), which in general has no additional structure, by a quotient of J_k as in (3). We want to do this in a way that allow us to use class field theory. This was done in ([11], p.4) for the orthogonal group of a quadratic form. This procedure can be generalized to other semisimple algebraic groups, as we show now.

In the rest of §2, let G be any semisimple group with fundamental group μ_n, i.e., a group with spinor norm. Let Λ be an S-lattice in V. Let $J_k^\dagger$ be the set of ideles that are positive at those infinite primes at which the spinor norm is not surjective. It follows from Theorem 1 on page 60 of [13] that the spinor norm is surjective at all finite places. Note that $\mathfrak{H}_{\mathbb{A}}(G_{\mathbb{A}}) = J_k^\dagger$. Also, by (4), $\mathfrak{H}_k(G_k) = J_k^\dagger \cap k^*$. Furthermore, because of the weak approximation theorem ([17], p. 14), we have $k^* J_k^\dagger = J_k$. It follows that

$$J_k/k^* = k^* J_k^\dagger/k^* \cong J_k^\dagger/(k^* \cap J_k^\dagger) = \mathfrak{H}_{\mathbb{A}}(G_{\mathbb{A}})/\mathfrak{H}_k(G_k).$$

Thus, for the set $H_{\mathbb{A}}(\Lambda)$ defined in (5), we have a canonical isomorphism

$$J_k/\left(k^* H_{\mathbb{A}}(\Lambda)\right) \cong \mathfrak{H}_{\mathbb{A}}(G_{\mathbb{A}})/\left(\mathfrak{H}_k(G_k) H_{\mathbb{A}}(\Lambda)\right).$$

The subgroup $k^* H_{\mathbb{A}}(\Lambda)$ on the left-hand side above corresponds, via class field theory, to an Abelian field extension Σ_Λ/k. This field Σ_Λ is, by definition, the spinor class field of Λ. For the case of a quadratic form, see [11].

Example 1 continued.— Let $\mathfrak{D}$ be a maximal S-order in a central simple k-algebra D. Let $G = D^*/k^*$ be the automorphism group of D. As all maximal S-orders are locally conjugate ([17], p. 46), $\Sigma = \Sigma_{\mathfrak{D}}$ depends only on D. We call $\Sigma_{\mathfrak{D}}$ the spinor class field of maximal S-orders of D, and denote it Σ_D. Let $v \in \Pi(k) - S$. If $D_v \cong \mathbb{M}_r(D_0)$, i.e., $r \times r$ matrices over a division algebra D_0 with maximal order $\mathfrak{D}_0$, then $\mathfrak{D}_v$ is conjugate to $\mathfrak{D}_1 = \mathbb{M}_r(\mathfrak{D}_0)$ and $G_v^{\mathfrak{D}_1} = (\mathfrak{D}_1^* D_0^*)/k_v^*$. It follows that $H_v(\mathfrak{D}) = \mathcal{O}_v^*(k_v^*)^r$. In particular, $H_{\mathbb{A}}(\mathfrak{D})$ contains $J_{k,\infty}^+$. Thus, $\Sigma_{\mathfrak{D}}$ is contained in the strict Hilbert S-class field of k. For example, if $D \cong \mathbb{M}_n(k)$ and $S = \infty$, then $\Sigma_{\mathfrak{D}}$ is the maximal subextension of exponent n of the wide Hilbert class field.

3. Representations of S-lattices.

Let V be a vector space, let $G \subseteq \mathrm{GL}(V)$ be a group with spinor norm, and let Λ be an S-lattice of maximal rank in V. By definition an S-lattice $M \subseteq V$ is G-represented by Λ if there exists an element g in G_k such that $gM \subseteq \Lambda$. If $\mathfrak{X}$ is a set of S-lattices, we say that M is G-represented by $\mathfrak{X}$ if it is represented by some element of $\mathfrak{X}$.

We call an adelic point $u = (u_\wp)_\wp \in G_{\mathbb{A}}$ a generator for $\Lambda|M$, if $M \subseteq u\Lambda$. Let $X_{\Lambda|M} \subseteq G_{\mathbb{A}}$ denote the set of such generators. In all that follows we assume $M \subseteq \Lambda$. The following lemmas are immediate from the definitions and generalize Lemmas 2.1-2.3 in [10]:

LEMMA 3.1.— *Let $g \in G_{\mathbb{A}}$. Then, M is G-represented by the spinor genus of $g\Lambda$ if and only if $g \in G_k G'_{\mathbb{A}} u$ for some $u \in X_{\Lambda|M}$.*

LEMMA 3.2.— *$X_{\Lambda|M} = X_{\Lambda|M} G^{\Lambda}_{\mathbb{A}} = G^M_{\mathbb{A}} X_{\Lambda|M}$, where $G^M_{\mathbb{A}}$ is the stabilizer of M. In particular, if Γ is the point-wise stabilizer of the subspace kM of V, then $X_{\Lambda|M} = \Gamma_{\mathbb{A}} X_{\Lambda|M}$.*

LEMMA 3.3.— *The set of spinor genera in the genus of Λ that G-represent M is in bijection with the image of $\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$ in the quotient $J_k/\,(k^* H_{\mathbb{A}}(\Lambda))$.*

Let $H^-(\Lambda|M) \subseteq J_k$ be the group generated by $k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$. Let $H_-(\Lambda|M)$ the maximal group H satisfying $Hk^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M}) = k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$. Let $\Sigma_-(\Lambda|M)$ be the class field corresponding to $H^-(\Lambda|M)$, and let $\Sigma^-(\Lambda|M)$ be the class field corresponding to $H_-(\Lambda|M)$. We call $\Sigma^-(\Lambda|M)$ the upper relative spinor class field and $\Sigma_-(\Lambda|M)$ the lower relative spinor class field. Observe that $\Sigma_-(\Lambda|M) \subseteq \Sigma^-(\Lambda|M)$. If $\Sigma_-(\Lambda|M) = \Sigma^-(\Lambda|M)$, we denote this field by $\Sigma_{\Lambda|M}$ and call it the relative spinor class field.

Notice that $\Sigma_{\Lambda|M}$ is defined if and only if $k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$ is a group. Whenever the relative spinor class field is defined, the fraction of the total number of spinor genera in the genus of Λ that G-represent M is

$$\frac{1}{[\Sigma_{\Lambda|M} : k]} = \frac{\left|\Big(k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})\Big)\Big/\Big(k^*H_{\mathbb{A}}(\Lambda)\Big)\right|}{\left|J_k\Big/\Big(k^*H_{\mathbb{A}}(\Lambda)\Big)\right|}$$

(Lemma 3.3). All indices above are bounded by the class number of G, which is finite ([17], p. 251). It was proved in [10] that the relative spinor class field is always defined if G is the orthogonal group of a quadratic form. It is not known whether this is the case for all groups G with spinor norm.

The spinor norm associates an element $\psi_\Lambda(t)$ in the quotient $J_k/\,(k^*H_{\mathbb{A}}(\Lambda))$ to every spinor genus t in the genus of Λ. If Λ' is another lattice in the genus of Λ we get $\psi_{\Lambda'}(t)\psi_\Lambda\,(\mathrm{spn}(\Lambda')) = \psi_\Lambda(t)$, where $\mathrm{spn}(\Lambda')$ denotes the spinor genus of Λ'. For an abelian extension Σ/k, let $x \mapsto (x, \Sigma/k)$

denote the Artin map for ideles ([14], p. 206). We obtain the following result:

Proposition 3.4. — *The map $f(t) = (\psi_\Lambda(t), \Sigma_\Lambda/k)$ associates an element of* $\mathrm{Gal}(\Sigma_\Lambda/k)$ *to every spinor genus in the genus of Λ. If $\Sigma_{\Lambda|M}$ is defined, then the set of spinor genera representing M is the pre-image of* $\mathrm{Gal}(\Sigma_\Lambda/\Sigma_{\Lambda|M})$ *under this map. In this case, a spinor genus t represents M if and only if $f(t)$ is trivial on $\Sigma_{\Lambda|M}$, and the map $t \mapsto (\psi_\Lambda(t), \Sigma_{\Lambda|M}/k)$ does not depend on the choice of the lattice Λ representing M.*

4. Applications.

4.1. Skew-Hermitian forms over quaternion division algebras.

Let (V, h) be a non-degenerate skew-Hermitian space of dimension at least 2 over a quaternion division algebra D. The form h is non-degenerate if for any non-zero $x \in V$ there exists $y \in V$, such that $h(x, y) \neq 0$. We assume this throughout this section. In all of this section $G = U^+(h)$ is the special unitary group of (V, h) ([17], p. 84). Classes, genera, and spinor genera (§2) of skew-Hermitian lattices are defined as classes, genera, and spinor genera with respect to this group. Let $\mathfrak{D}$ denote a maximal S-order of D, and let Λ be a $\mathfrak{D}$-lattice of maximal rank in V. Let M be an arbitrary $\mathfrak{D}$-lattice in V, and let $W = kM$. We assume that the restriction of h to W is non-degenerate. Let $W^\perp$ be the orthogonal complement of W ([19], p. 238) and let

$$\Gamma = \{g \in G | gw = w\ \forall w \in W\}, \qquad T = \{g \in G | gw = w\ \forall w \in W^\perp\}.$$

The group Γ can be identified with the unitary group of the restriction of h to the space $W^\perp$. Similarly, the group T can be identified with the unitary group of the restriction of h to W. It follows from Lemma 3.2 that

$$X_{\Lambda|M} = X_{\Lambda|M} G_{\mathbb{A}}^{\Lambda} = T_{\mathbb{A}}^{M} X_{\Lambda|M} = \Gamma_{\mathbb{A}} X_{\Lambda|M}. \tag{6}$$

As left vector spaces over D, the dimensions $\dim_D V$ and $\dim_D W$ are well defined ([18], Theorem 2.8.14).

PROPOSITION 4.1.1. — *If $\dim_D W \leqslant \dim_D V - 2$, then every spinor genus in the genus of Λ represents M. If $\dim_D W = \dim_D V - 1$, then either all the spinor genera in the genus of Λ represent M, or exactly half of them do.*

Proof. — If $\dim_D W^{\perp} > 1$, then Γ is a semisimple linear algebraic group ([17], p. 92). Hence the spinor norm on Γ is surjective at every finite place ([17], thm. 6.20). It follows from the construction of the universal cover for unitary groups of skew-Hermitian forms ([4], p. 173) that one can assume the universal cover of Γ to be contained in $\widetilde{G}$, so that the spinor norm on Γ is the restriction of the spinor norm on G. By weak approximation, $k^*\mathfrak{H}_{\mathbb{A}}(\Gamma_{\mathbb{A}}) = J_k$. It follows that $k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M}) = J_k$ and therefore $\Sigma_{\Lambda|M} = k$.

Now, assume $\dim_D W^{\perp} = 1$. Then, $W^{\perp} = Ds$, for some $s \in V$. Let $a = h(s,s)$, and let $K = k(a)$. Then, any element of Γ_v is of the form $(s;\sigma)$, where $\sigma \in K_v$ and $\sigma - \bar{\sigma} = a$. Any $b \in K_v$ is of the form $b = \lambda\sigma$, where $\lambda \in k_v$ and $\sigma - \bar{\sigma} = a$ (ex. 3). It follows that $\mathfrak{H}_{\mathbb{A}}(\Gamma_{\mathbb{A}}) = N_{K/k}(J_K)$. Since K/k is a quadratic extension, we have $[J_k : k^*N_{K/k}(J_K)] = 2$. It follows from (6) that either $k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M}) = k^*\mathfrak{H}_{\mathbb{A}}(\Gamma_{\mathbb{A}})$ or $k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M}) = J_k$. In either case $k^*\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$ is a group and $\Sigma_{\Lambda|M}$ is contained in the quadratic extension K. □

PROPOSITION 4.1.2. — *Assume that at every dyadic place $v \in S$, at least one of the following conditions hold:*

- *$\Lambda_v = \mathfrak{D}_v u_v \perp L_v$ for a vector $u_v \in V_v$ and a lattice $L_v \subseteq V_v$.*
- *$M_v = \mathfrak{D}_v u_v \perp L_v$ for a vector $u_v \in V_v$ and a lattice $L_v \subseteq V_v$.*
- *2 is a prime in $\mathcal{O}_v$.*
- *D_v splits, i.e., $D_v \cong \mathbb{M}_2(k_v)$.*

Then $G'_{\mathbb{A}}X_{\Lambda|M}$ is a group. In particular, the spinor class field is defined.

Recall that $\Lambda = M \perp L$ means that $\Lambda = M \oplus L$ and $h(m,l) = 0$ for $m \in M$ and $l \in L$.

Proof. — We need to see that $\mathfrak{H}_{\mathbb{A}}(X_{\Lambda|M})$ is a group. It suffices to work locally. At those places at which the quaternion algebra splits, we can use the isomorphisms given in [4] (Lemma 6, p. 175, and Theorem 7, p. 181) to reduce the problem to the quadratic forms case, which was already solved in [10] (p. 131). Therefore, we can assume that the quaternion algebra does not split at v.

By (6), it suffices to prove that either $[k_v^* : H_v(\Lambda)] \leqslant 2$ or $[k_v^* : H_v(M)] \leqslant 2$. It follows from the paragraph preceding (6) that if $\Lambda_v = \Lambda_1 \perp \Lambda_2$, then $H_v(\Lambda) \supseteq H_v(\Lambda_1)$. It is proved in [2] that we can always find such a decomposition where either $\Lambda_1 = \mathfrak{D}s$, for $s \in V$, or Λ_1 is an indecomposable lattice of rank 2. If $\Lambda_1 = \mathfrak{D}s$, for $s \in V$, then $H_v(\Lambda_1) = N_{E/k_v}(E^*)$, where $E = k_v(a)$ and $a = h(s,s)$. If Λ_1 is indecomposable of rank 2, and 2 is a prime in the ring of local integers $\mathcal{O}_v$, it is proved in [2] that $H_v(\Lambda_1) = k_v^*$. A similar argument holds for M. $\square$

Corollary 4.1.2.1. — *Let D be a quaternion division algebra over $\mathbb{Q}$, and let (V,h) be a skew-Hermitian space over D. Let $\mathfrak{D}$ be a maximal order of D, Λ a $\mathfrak{D}$-lattice in V, M a $\mathfrak{D}$-sublattice of Λ. Then, the spinor class field $\Sigma_{\Lambda|M}$ is defined.*

Proof of Theorem 2. — By Proposition 4.1.1 we can assume that $\dim_D V = \dim_D W$, i.e., $V = W$. We can assume also that we are not in the case mentioned in the last sentence of Theorem 2. This implies that at least one of the conditions in Proposition 4.1.2 holds and the spinor class field is defined. By the definition of the relative spinor class field (§3) the number of spinor genera representing M divided by the total number of spinor genera is $[\Sigma_{\Lambda|M} : k]^{-1}$. Since $\Sigma_{\Lambda|M} \subseteq \Sigma_\Lambda$ is an Abelian extension of exponent 2, then $[\Sigma_{\Lambda|M} : k]^{-1} = 2^{-t}$ for some non-negative integer t. $\square$

4.2. Maximal S-orders and S-suborders.

In the rest of this paper, D is a central simple algebra over k, $\mathfrak{D}$ is a maximal S-order in D, and $\Sigma = \Sigma_D$ is the spinor class field for maximal S-orders of D as defined in the last paragraph of §2. Let $\mathfrak{L}$ be an arbitrary S-suborder of $\mathfrak{D}$, not necessarily commutative yet, and let $L = k\mathfrak{L}$. The group of automorphisms of D is $G = PGL_1(D) = D^*/k^*$, where D^* acts on D by conjugation. The G-class of $\mathfrak{D}$ is its conjugacy class. The Conjugacy classes of maximal S-orders of D are classified by $G_k \backslash G_{\mathbb{A}} / G_{\mathbb{A}}^{\mathfrak{D}}$. This equals $G_{\mathbb{A}}/G_k G'_{\mathbb{A}} G_{\mathbb{A}}^{\mathfrak{D}}$, unless D_v is a quaternion division algebra at every infinite place v, since in any other case G_v is not compact at these places (§2). Except for the case mentioned above, the words spinor genus can be replaced by conjugacy class in all the results in below.

We apply the preceding theory to the lattices $\Lambda = \mathfrak{D}$ and $M = \mathfrak{L}$. Let $X = X_{\mathfrak{D}|\mathfrak{L}}$. Let $F = C_D(L)$, the centralizer of L in D, and let $\Gamma = F^*/k^*$.

By Lemma 3.2, $X = \Gamma_{\mathbb{A}} X G_{\mathbb{A}}^{\mathfrak{D}}$. Hence,

$$\mathfrak{H}_{\mathbb{A}}(X) = \mathfrak{H}_{\mathbb{A}}(\Gamma_{\mathbb{A}})\mathfrak{H}_{\mathbb{A}}(X) = N(F_{\mathbb{A}}^*)\mathfrak{H}_{\mathbb{A}}(X)H_{\mathbb{A}}(\mathfrak{D}). \tag{7}$$

A sufficient condition for every spinor genus of maximal S-orders to represent $\mathfrak{L}$ is $k^* H_{\mathbb{A}}(\mathfrak{D})N(F_{\mathbb{A}}^*) = J_k$. Since $1 \in X$, we have $X \supseteq \Gamma_{\mathbb{A}} G_{\mathbb{A}}^{\mathfrak{D}}$. Note that $\mathfrak{H}_{\mathbb{A}}(\Gamma_{\mathbb{A}} G_{\mathbb{A}}^{\mathfrak{D}})$ is a group, even though $\Gamma_{\mathbb{A}} G_{\mathbb{A}}^{\mathfrak{D}}$ might not be a group. Since Γ is the point-wise stabilizer of $\mathfrak{L}$, we obtain the following result:

PROPOSITION 4.2.1. — *Assume that for every $v \notin S$ and for every embedding $\phi : \mathfrak{L}_v \to \mathfrak{D}_v$ there exists an automorphism ρ of $\mathfrak{D}_v$, whose restriction to $\mathfrak{L}_v$ is ϕ. Then $X = \Gamma_{\mathbb{A}} G_{\mathbb{A}}^{\mathfrak{D}}$ and the relative spinor class field is defined.*

Example. — Let $\mathfrak{L} = \mathfrak{D}_0$ be an S-order of a central simple subalgebra D_0 of D. Let $n^2 = \dim_k D$, $m^2 = \dim_k D_0$. Let N_1 be the reduced norm on the central simple algebra $F = C_D(D_0)$. If $x \in F$, then $N(x) = N_1(x)^m$. It follows that at any local place v, $N(F_v^*) = (k_v^*)^m$. Therefore, $\Sigma^-(\mathfrak{D}|\mathfrak{L})$ is a subextension of Σ of exponent m. In particular, if the degree of the extension Σ/k is relatively prime to m, any spinor genus of maximal S-orders in D represents $\mathfrak{D}_0$. On the other hand, if D and D_0 are unramified outside of S and $\mathfrak{D}_0$ is a maximal S-order of D_0, then the sufficient condition of Proposition 4.2.1 is satisfied. We conclude then that the relative spinor class field is defined and equals the largest subextension of Σ of exponent m.

4.3. Rings of S-integers of maximal subfields.

Assume henceforth that L is a maximal subfield of D. Recall that $L = k\mathfrak{L}$, so that $\mathfrak{L}$ is an S-order (not necessarily maximal) in L. Let $n = [L : k]$, so that $\dim_k(D) = n^2$.

In this case the centralizer F of L equals L. Hence $N(F_{\mathbb{A}}^*) = N_{L/k}(J_L)$ and by (7) it follows that $\mathfrak{H}_{\mathbb{A}}(X)N_{L/k}(J_L)H_{\mathbb{A}}(\mathfrak{D}) = \mathfrak{H}_{\mathbb{A}}(X)$. From this and Proposition 4.2.1, we obtain the following result:

PROPOSITION 4.3.1. — *For any S-order $\mathfrak{L} \subseteq L \cap \mathfrak{D}$, the upper relative spinor class field $\Sigma^-(\mathfrak{D}|\mathfrak{L})$ is contained in $\Sigma \cap L$. Furthermore, assume that at every $v \notin S$, any local embedding of $\mathfrak{L}_v$ in $\mathfrak{D}_v$ can be extended to an automorphism of $\mathfrak{D}_v$. Then, the relative spinor class field is defined and equals $\Sigma \cap L$.*

COROLLARY 4.3.1.1. — *If $\Sigma \cap L = k$, then $\Sigma_{\mathfrak{D}|\mathfrak{L}}$ is defined and equals k.*

Remark 4.3.2. — Note that Σ/k is Galois and Abelian. Hence, the k-extension $\Sigma \cap L$ is independent of the embedding of L into an algebraic closure of k.

Next we study some conditions that guarantee the assumption of Proposition 4.3.1. Notice that we can work locally.

LEMMA 4.3.3. — *Assume that $\mathfrak{L}$ is the maximal S-order of L. If D_v is a division algebra or the matrix algebra $\mathbb{M}_n(k_v)$, then the assumption in Proposition 4.3.1 is satisfied at v.*

Proof. — If D_v is a division algebra, then $G_v^{\mathfrak{D}} = G_v$, and there is nothing to prove. Assume D_v is a matrix algebra. Replacing $\mathfrak{D}_v$ by a conjugate, we can assume $\mathfrak{D}_v = \mathbb{M}_n(\mathcal{O}_v)$. Let π be a uniformizing parameter of k_v. Take $u \in D_v^*$, such that $\mathfrak{L}_v \subseteq u\mathfrak{D}_v u^{-1}$. It suffices to prove that $u \in L_v^*\mathfrak{D}_v^*$. As $\mathfrak{L}$ is maximal, $\mathfrak{L}_v = L_v \cap \mathfrak{D}_v = L_v \cap u\mathfrak{D}_v u^{-1}$. Let $I = L_v \cap u\mathfrak{D}_v$. Then I is a $\mathfrak{L}_v$-module of maximal rank in L_v. Hence, $I = \lambda\mathfrak{L}_v$ for some $\lambda \in L_v^*$. Replacing u by $\lambda^{-1}u$, we can assume $I = \mathfrak{L}_v$. In particular, $u^{-1} \in \mathfrak{D}_v$, and $(\pi u)^{-1} \notin \mathfrak{D}_v$. We claim that $u \in \mathfrak{D}_v^*$.

By elementary divisors theory, we have $u = xzy$, where $x, y \in \mathfrak{D}_v^*$, and z is the diagonal matrix $\mathrm{diag}(\pi^{-r_1}, \ldots, \pi^{-r_n})$, for some $r_1 \geqslant \ldots \geqslant r_n = 0$. Replacing u by z, and $\mathfrak{L}_v$ by $x^{-1}\mathfrak{L}_v x$, we can assume $u = \mathrm{diag}(\pi^{-r_1}, \ldots, \pi^{-r_n})$. Assume $r_t \neq 0$ and $r_{t+1} = r_{t+2} = \ldots = r_n = 0$. The condition $\mathfrak{L}_v = L_v \cap u\mathfrak{D}_v u^{-1}$ shows that all elements of $\bar{\mathfrak{L}}_v = \mathfrak{L}_v/\pi\mathfrak{L}_v$ are of the form

$$t(A, B, D) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where $A \in \mathbb{M}_t(\mathcal{O}_v/\pi\mathcal{O}_v)$ and $D \in \mathbb{M}_{n-t}(\mathcal{O}_v/\pi\mathcal{O}_v)$. Now, let $l \in \mathfrak{L}_v$, $\bar{l} \in \bar{\mathfrak{L}}_v$ the image of l. Assume $\bar{l} = t(A, B, 0)$ with A or B not equal to 0. Then, $\pi^{-1}l \in u\mathfrak{D}_v$, but $\pi^{-1}l \notin \mathfrak{D}_v$. This contradicts the assumption $I = \mathfrak{L}_v$. We conclude that $D = 0$ implies $A = B = 0$ and the map sending $t(A, B, D)$ to D is injective on $\bar{\mathfrak{L}}_v$. It follows that $\bar{\mathfrak{L}}_v$ is isomorphic to a subalgebra of $\mathbb{M}_{n-t}(\mathcal{O}_v/\pi\mathcal{O}_v)$. However, $\mathfrak{L}_v = \mathcal{O}_v(\alpha)$ for some $\alpha \in \mathfrak{L}_v$, and the minimal polynomial of $\bar{\alpha}$, the image of α in $\bar{\mathfrak{L}}_v$, has degree n. This proves that $n = n - t$. Therefore, $u \in \mathfrak{D}_v^*$ as claimed. □

We say that D has partial ramification at a place v if D_v is neither a division algebra nor a matrix algebra. The following result is now immediate.

PROPOSITION 4.3.4. — *Assume D has no partial ramification outside of S. Assume that $\mathfrak{L}$ is the maximal S-order of a field L, and $\mathfrak{L} \subseteq \mathfrak{D}$. Then, the relative spinor class field $\Sigma_{\mathfrak{D}|\mathfrak{L}}$ is defined and $\Sigma_{\mathfrak{D}|\mathfrak{L}} = \Sigma \cap L$.*

COROLLARY 4.3.4.1. — *Assume that $\mathfrak{L}$ is the maximal S-order of L. Assume also that n is square-free, and that $\Sigma \supseteq L$. Then the spinor class field is defined and equals L.*

Proof. — Let $v \notin S$. Then, $L_v = \bigoplus_w L_w$, where the sum extends over all places w of L dividing v, and each L_w is a field. As $L \subseteq \Sigma$ is a Galois extension of k, all the extensions L_w/k_v have the same degree l_v.

Assume $D_v = \mathbb{M}_{f_v}(D_0)$, where D_0 is a division algebra. Let $e_v^2 = \dim_{k_v}(D_0)$, so that $n = f_v e_v$. Then, $H_v(\mathfrak{D}) = \mathcal{O}_{k_v}^*(k_v^*)^{f_v}$. On the other hand, $\Sigma \supseteq L$ implies

$$H_v(\mathfrak{D}) \subseteq k_v^* \cap \Big(k^* N_{L/k}(J_L)\Big) = \mathcal{O}_{k_v}^*(k_v^*)^{l_v}.$$

We conclude that $l_v|f_v$. As L_v embeds in D_v, we obtain $e_v|l_v$. Hence, $e_v|f_v$, i.e., $e_v^2|n$. We conclude that D does not ramify outside of S, and so Proposition 4.3.4 applies. □

If L is a sum of fields, $L = L_1 \oplus \ldots \oplus L_t$, Proposition 4.3.4 still holds, if we replace $L \cap \Sigma$ by $L_1 \cap \ldots \cap L_t \cap \Sigma$.

The hypotheses in Proposition 4.3.4 are trivially satisfied by matrix algebras and quaternion algebras. The first case was already known to Chevalley ([6]). For the second case, see [7]. Notice that, since we study spinor genus, we need not to require that G_S is non-compact (§2). This is equivalent to the *Eichler condition* in [7]. Friedman informed us that Schultze-Pillot has pointed out in [20] that the possible selectivity ratios 0, 1, and 1/2, for the quaternionic case follow from Theorem 2 in [9], which concerns representations of one quadratic form by another. Such direct connection will not work in the general case, showing the importance of a more general theory.

Proof of Theorem 1. — By Proposition 4.3.4 the spinor class field $\Sigma_{\mathfrak{D}|\mathfrak{L}}$ equals $\Sigma \cap L$. By the definition of the relative spinor class field (§3) the number of spinor genera representing $\mathfrak{L}$ divided by the total number of spinor genera is $[\Sigma \cap L : k]^{-1}$. Finally, by Lemma 2.0.1, every spinor genera contains exactly one class if $\dim_k D \geqslant 9$. The conclusion follows. □

We now exhibit a different condition that guarantees the assumption of Proposition 4.3.1.

PROPOSITION 4.3.5. — *Assume that L_v is a field, and that L_v/k_v is unramified. Then the assumption in Proposition 4.3.1 is satisfied at v.*

Proof. — If L_v is a field, and if L_v/k_v is unramified, then $\bar{\mathfrak{L}}_v$ is a field. Let D_0 be a division algebra such that $D_v = \mathbb{M}_r(D_0)$ and $\mathfrak{D}_v = \mathbb{M}_r(\mathfrak{D}_0)$ for the maximal order $\mathfrak{D}_0$ of D_0. Reasoning as in the proof of Lemma 4.3.3, we can assume that $u = \mathrm{diag}(i^{-r_1}, \dots, i^{-r_n})$, for a uniformizing parameter i of D_0. As $i \in G_v^{\mathfrak{D}}$, we can assume that, for some $t \in \{0, \dots, n-1\}$,

$$r_{t+1} = \dots = r_n = 0, \text{ and } r_s > 0 \text{ for } s \leqslant t.$$

As before, one obtains that the elements of $\bar{\mathfrak{L}}_v$ have a block of zeroes in the lower left corner. As $\bar{\mathfrak{L}}_v$ is a field, this gives a contradiction unless $t = 0$. Hence, $u \in G_v^{\mathfrak{D}}$. □

Example. — We now show that some condition must be required on either L_v or D_v to ensure the equality $\mathfrak{H}_{\mathbb{A}}(X) = \mathfrak{H}_{\mathbb{A}}(\Gamma_{\mathbb{A}} G_{\mathbb{A}}^{\mathfrak{D}})$ and therefore the conclusion of Proposition 4.3.1. In particular, the condition on D cannot be completely removed from Proposition 4.3.4.

Assume now that L_1 and L_2 are different unramified quadratic extensions of k. Assume that the uniformizing parameters π_v and π_w at $v, w \notin S$, generate J_k/H, where H is the idele class subgroup corresponding to $L = L_1 L_2$. Let B be a quaternion algebra that ramifies only at v and w. Then, it is not hard to show that $L \subseteq \Sigma$, and that L embeds into $D = \mathbb{M}_2(B)$. Let $\mathfrak{D}$ be a maximal order in D such that $\mathfrak{D}_v = \mathbb{M}_2(\mathfrak{B}_v)$, for a maximal order $\mathfrak{B}_v$ of B_v and the corresponding condition holds for w. Let E_v be the unique unramified quadratic extension of k_v. Then

$$L_v \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} E_v \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} E_v.$$

Let $\mathfrak{L}_v$ be the maximal order of L_v. It is not hard to check that the element $u \in G_{\mathbb{A}}$, such that $u_v = \mathrm{diag}(i_v, 1)$ for a uniformizing parameter i_v of B_v and $u_\wp = 1$ for $\wp \neq v$, is a generator for $\mathfrak{D}|\mathfrak{L}$. As $N(u_v) = -\pi_v$, it follows that $\Sigma^-(\mathfrak{D}|\mathfrak{L})$ splits completely at v. Analogously, we have that $\Sigma^-(\mathfrak{D}|\mathfrak{L})$ splits completely at w. It follows that $\Sigma_{\mathfrak{D}|\mathfrak{L}}$ is defined and equals k. However, $\Sigma \cap L = L$.

Finally, we show the necessity of the condition that $\mathfrak{L}$ is the maximal order of L in Propositions 4.3.4 and 4.3.5. For this, we show that $\Sigma^-(\mathfrak{D}|\mathfrak{L})$ equals k if $\mathfrak{L}$ is small enough. In fact:

PROPOSITION 4.3.6.— *Let $\mathfrak{L}'$ be the maximal order of L. Let v be a finite place, unramified for D, and let π_v be the uniformizing parameter at v. Assume that $\mathfrak{L}_v \subseteq \mathcal{O}_v + \pi_v \mathfrak{L}'_v$. Then v splits completely on $\Sigma^-(\mathfrak{D}|\mathfrak{L})$.*

To prove this result one shows, by a computation, that conjugating by the matrix $\mathrm{diag}(\pi_v, 1, \dots, 1)$ defines an element $g \in G_v$, which is a generator for $\mathfrak{D}_v|\mathfrak{L}_v$, and whose spinor norm is π_v.

COROLLARY 4.3.6.1.— *Let $\mathfrak{L}'$ be as in previous proposition. Assume that the finite places $v_1, \dots, v_t$ are unramified for D, and its images under the Artin map generates* $\mathrm{Gal}(\Sigma \cap L/k)$. *Let $\rho \in \mathcal{O}_k$, be such that $|\rho|_{v_s} < 1$ for $s = 1, \dots, t$. Then, for any order $\mathfrak{L}$ of L satisfying $\mathfrak{L} \subseteq \mathcal{O}_k + \rho \mathfrak{L}'_k$, the relative spinor class field $\Sigma_{\mathfrak{D}|\mathfrak{L}}$ is defined and equals k.*

Remark 4.3.7.— In case that $n = 2$, Propositions 4.3.3 and 4.3.6 suffice to compute all relative spinor class fields. Thus we recover the results in [7].

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