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PROPERTIES OF NON-HERMITIAN QUANTUM FIELD THEORIES

by Carl M. BENDER

1. Introduction.

The Hamiltonian of a physical system must embody the continuous symmetries and discrete symmetries of that system. There is little doubt that the continuous symmetry group of the universe is the proper Lorentz group. (The *proper* Lorentz group contains all those elements of the Lorentz group that are continuously connected to the identity.) This talk addresses the question of what the discrete symmetry of the universe is.

It is clear that the universe is *not* invariant under the full Lorentz group. Recall that the full Lorentz group is in four distinct parts: (1) the proper Lorentz group; (2) the elements of the proper Lorentz group multiplied by the parity reflection operator $\mathcal{P}$; (3) the elements of the proper Lorentz group multiplied by the time reflection operator $\mathcal{T}$; (4) the elements of the proper Lorentz group multiplied by the parity reflection and time reflection operators $\mathcal{PT}$. It has been experimentally verified that the universe does not exhibit parity reflection symmetry and it also does not exhibit time reversal symmetry. However, a famous theorem in the subject of quantum field theory (one of the few rigorous theorems in quantum field theory!) known as the $\mathcal{PCT}$ theorem, states that the universe *is* invariant under combined space-time reflection and particle-antiparticle interchange [1].

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The proof of the $\mathcal{PCT}$ theorem rests on several crucial assumptions, namely, that the Hamiltonian is Hermitian (so that the spectrum is real) and that the spectrum is bounded below. The existence of a real positive spectrum allows one to extend the Lorentz group to the *complex* Lorentz group. The complex Lorentz group consists of *two*, and not four, disconnected parts because in the complex Lorentz group there is a continuous path from the identity to the element $\mathcal{PT}$ that reflects space-time. Note that $\mathcal{PCT}$ symmetry is a much weaker condition than Hermiticity; one must assume that $H = H^\dagger$ in order to prove the $\mathcal{PCT}$ theorem, but we cannot conclude that $H = H^\dagger$ from $\mathcal{PCT}$ symmetry.

The hypothesis made in this talk is that the discrete symmetry of the universe is $\mathcal{PCT}$ symmetry. In this talk we only consider quantum theories in which particles are their own antiparticles; thus, we will assume that the symmetry of the universe is space-time reflection, or $\mathcal{PT}$ symmetry. We argue that space-time reflection symmetry ($\mathcal{PT}$ symmetry) is a simple and natural physical constraint on the Hamiltonian. Hermiticity symmetry $H = H^\dagger$ is a convenient mathematical condition, but one whose physical justification is remote and obscure. We will see that in many (but not all) cases the assumption of $\mathcal{PT}$ symmetry leads to a spectrum that is real and positive.

2. Origin of the idea.

In the late 1980s I coauthored a series of papers in which a technique was developed for solving nonlinear problems in classical and quantum mechanics and quantum field theory by expanding perturbatively in powers of a parameter that measures the nonlinearity of the problem. To illustrate, let us consider the Thomas-Fermi differential-equation boundary-value problem [2]

$$y''(x) = y^{3/2}/\sqrt{x}, \qquad y(0) = 1,\ y(\infty) = 0. \tag{1}$$

This is a difficult problem to solve numerically because there are instabilities and there is no analytical solution. Our approach to this problem is to introduce a small parameter ϵ in the *exponent*:

$$y''(x) = y\,(y/x)^{\epsilon}, \qquad y(0) = 1,\ y(\infty) = 0, \tag{2}$$

and to solve for $y(x)$ as a series in powers of ϵ:

$$y(x) = e^{-x} + \epsilon Y_1(x) + \epsilon^2 Y_2(x) + \epsilon^3 Y_3(x) + \cdots . \tag{3}$$

The advantage of this procedure is that, unlike many perturbation expansions, the perturbation expansion (3) has a nonzero radius of convergence. The solution to the original Thomas-Fermi boundary-value problem is obtained by setting $\epsilon = 1/2$ in (1) [2].

While I was visiting Saclay, Bessis told to me that he and Zinn-Justin had come across the complex non-Hermitian Hamiltonian

$$H = p^2 + ix^3, \tag{4}$$

whose spectrum appeared to be real and positive. To examine this surprising conjecture we used the perturbation method described above to calculate the eigenvalues of the class of quantum mechanical Hamiltonians

$$H = p^2 + x^2(ix)^\epsilon, \tag{5}$$

where ϵ is a real parameter. Using a variety of analytical and numerical methods we were able to establish with confidence [3], [4] that for the infinite class of Hamiltonians for which $0 \leqslant \epsilon$ the entire spectrum of H in (5) is real and positive (see Fig. 1). The Hamiltonian (4) considered by Bessis and Zinn-Justin is a special case corresponding to $\epsilon = 1$. This class of Hamiltonians includes the interesting special case $\epsilon = 2$ for which $H = p^2 - x^4$. It is most surprising that the spectrum of this Hamiltonian is real and positive even though it contains a wrong-sign potential.

These quantum mechanical models can be immediately extended to quantum field theory. For example, in a $-x^4$ theory the expectation value $\langle x \rangle$ is *not* zero. The corresponding result for a $-g\phi^4$ quantum field theory in D-dimensional Euclidean space is that the one-point Green's function $G_1 = \langle \phi \rangle$ is also nonzero. This finding may allow us to construct new models for the Higgs boson. We also examine bound states in a $-g\phi^4$ quantum field theory.

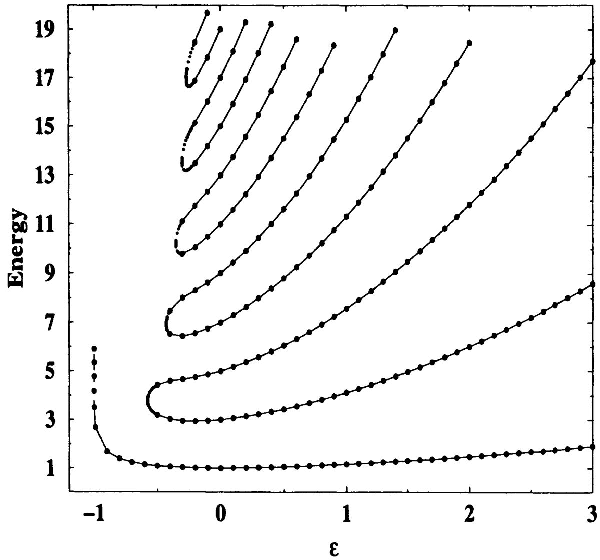

Figure 1. Energy levels of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon$ as a function of the parameter ϵ. There are three regions: When $\epsilon \geqslant 0$, the spectrum is real and positive and the energy levels rise with increasing ϵ. The lower bound of this region, $\epsilon = 0$, corresponds to the harmonic oscillator, whose energy levels are $E_n = 2n + 1$. When $-1 < \epsilon < 0$, there are a finite number of real positive eigenvalues and an infinite number of complex conjugate pairs of eigenvalues. As ϵ decreases from 0 to -1, the number of real eigenvalues decreases; when $\epsilon \leqslant -0.57793$, the only real eigenvalue is the ground-state energy. As ϵ approaches -1^+, the ground-state energy diverges. For $\epsilon \leqslant -1$ there are no real eigenvalues.

3. Positive spectrum for wrong sign potentials.

The spectrum shown in Figure 1 depends crucially on the boundary conditions for the Hamiltonian in (5). For example, let us consider the case $\epsilon = 2$ for which the Hamiltonian is

$$H = p^2 - gx^4, \tag{6}$$

where we have inserted a coupling constant $g > 0$. There are several ways to obtain the Hamiltonian H in (6). One way is to substitute $g = |g|e^{i\theta}$ into the Hamiltonian $H = p^2 + gx^4$ and to rotate from $\theta = 0$ to $\theta = \pi$. Under this rotation, the ground-state energy $E_0(g)$ becomes complex. Evidently, $E_0(g)$ is real and positive when $g > 0$ and complex when $g < 0$.[(1)] One can also obtain (6) as the limit of the Hamiltonian $H = p^2 + gx^2(ix)^\epsilon$ as $\epsilon : 0 \to 2$. Having studied Hamiltonians like that in (6) in great detail, we and others have shown that for $\epsilon \geqslant 0$ the spectra of such Hamiltonians are real, positive, and discrete. The spectrum of the limiting Hamiltonian (6) obtained in this manner is similar to that of the Hamiltonian $H = p^2 + gx^4$ ($g > 0$); it is entirely real, positive, and discrete. Very recently, the reality and positivity of the spectra have been established rigorously [5].

How can one Hamiltonian (6) possess two different spectra? The answer lies in the boundary conditions satisfied by the eigenfunctions $\psi_n(x)$. In the first case, in which $\theta = \arg g$ is rotated from 0 to π, $\psi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-4\pi/3 < \arg x < -\pi$. In the second case, in which α runs from 0 to 2, $\psi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-\pi < \arg x < -2\pi/3$. In this case the boundary conditions hold in wedges that are symmetric with respect to the imaginary axis; these boundary conditions enforce the $\mathcal{PT}$ symmetry of H and account for the reality of the spectrum.

4. One-point Green's function G_1.

There is another striking difference between the two theories corresponding to H in (6). The one-point Green's function $G_1(g)$ is given by

$$G_1(g) = \langle 0|x|0\rangle / \langle 0|0\rangle \equiv \int_C dx\, x\psi_0^2(x) \Big/ \int_C dx\, \psi_0^2(x), \tag{7}$$

where C is a contour that lies in the asymptotic wedges described above. The value of $G_1(g)$ for H in (6) depends on the limiting process by which we obtain H. If we substitute $g = g_0 e^{i\theta}$ into the Hamiltonian $H = p^2 + gx^4$ and rotate from $\theta = 0$ to $\theta = \pi$, we get $G_1(g) = 0$ for all g on the semicircle in the complex-g plane. Thus, this rotation in the complex-g plane preserves parity symmetry ($x \to -x$). However, if we define H in (6) by using the

(1) Rotating from $\theta = 0$ to $\theta = -\pi$, we obtain the same Hamiltonian as in (6) but the spectrum is the complex conjugate of the spectrum obtained when we rotate from $\theta = 0$ to $\theta = \pi$.

Hamiltonian $H = p^2 + gx^2(ix)^\epsilon$ and allowing ϵ run from 0 to 2, we find that $G_1(g) \neq 0$. Indeed, $G_1(g) \neq 0$ for *all* values of $\epsilon > 0$. Thus, in this theory $\mathcal{PT}$ symmetry (reflection about the imaginary axis, $x \to -x^*$) is preserved, but parity symmetry is permanently broken.

These two different results for $G_1(g)$ emphasize the importance of the boundary conditions in the integrals in (7) for determining the one-point Green's function. We are concerned in this talk with the theory that preserves $\mathcal{PT}$ symmetry. In this theory the energy spectrum is real and positive and $G_1(g)$ is nonzero.

We have extended these quantum-mechanical arguments to the quantum field theory whose D-dimensional Euclidean space Lagrangian is

$$\mathcal{L} = (\nabla\phi)^2/2 + m^2\phi^2/2 - g\phi^4/4. \tag{8}$$

What is remarkable about this "wrong-sign" field theory is that, when it is obtained using the $\mathcal{PT}$-symmetric limit, the energy spectrum is real and positive, and the one-point Green's function is nonzero. Furthermore, the field theory is renormalizable, and in four dimensions is asymptotically free (and thus nontrivial). Based on these features of the theory, we believe that the theory may provide a useful setting to describe the Higgs particle.

The one-point Green's function G_1 is a *complex* functional integral in Euclidean space: $G_1 = \int_C \mathcal{D}\phi\,\phi(0)e^{-L[\phi]}/\int_C \mathcal{D}\phi\, e^{-L[\phi]}$. Here, $L[\phi] = \int d^Dx\,\mathcal{L}$ and C is a contour in the complex-ϕ plane defined as follows: Functional integrals are infinite products of ordinary integrals, one integral for each lattice point in Euclidean space. For these ordinary integrals the contour of integration must lie within 45° wedges that lie in the lower-half plane and are centered about $-45°$ and $-135°$. In D-dimensional space we use $\epsilon = gm^{D-4}/4$ to represent the dimensionless coupling constant. The small-ϵ asymptotic behavior of G_1 is determined by a *soliton* (not an instanton). In general, G_1 has a *negative imaginary* value:

$$(9)\; D = 0 : G_1 \sim -\frac{i}{m}2^{-1/2}\epsilon^{-1/2}e^{-1/\epsilon}\ (\epsilon \to 0^+);$$
$$D = 1 : G_1 \sim -\frac{i}{\sqrt{m}}16\sqrt{\pi e}(2/\epsilon)^{2/3}e^{-16/(3\epsilon)}3^{-1/6}/\Gamma^2(1/3)\ (\epsilon \to 0^+).$$

In dimension D, $G_1 \sim e^{-4\Lambda[D]/\epsilon}$ as $\epsilon \to 0^+$, where $\Lambda[D]$ is determined by a spherically symmetric soliton. Numerical values of $\Lambda[D]$ for $0 \leqslant D \leqslant 4$ are given in Ref. [6].

5. Bound states.

A significant difference between the conventional Lagrangian

$$\mathcal{L} = (\nabla\phi)^2/2 + m^2\phi^2/2 + g\phi^4/4 \tag{10}$$

and the $\mathcal{PT}$-symmetric Lagrangian (8) is that when g is sufficiently small, the $\mathcal{PT}$-symmetric theory possesses bound states while the conventional theory does not. These bound states persist in the non-Hermitian $\mathcal{PT}$-symmetric $-g\phi^4$ quantum field theory for all dimensions $0 \leqslant D < 3$ but are not present in the conventional Hermitian $g\phi^4$ field theory.

We calculate the bound-state energies perturbatively. For the conventional Lagrangian (10) in one dimension (the anharmonic oscillator) the perturbation series for the kth energy level E_k begins

$$E_k \sim m[k + 1/2 + 3(2k^2 + 2k + 1)\epsilon/4 + \mathrm{O}(\epsilon^2)] \quad (\epsilon \to 0^+), \tag{11}$$

where $\epsilon = g/(4m^3)$. The *renormalized mass* M is the first excitation above the ground state:

$$M \equiv E_1 - E_0 \sim m[1 + 3\epsilon + \mathrm{O}(\epsilon^2)] \text{ as } \epsilon \to 0^+. \tag{12}$$

To determine if the two-particle state is bound, we examine the *second* excitation above the ground state. We define $B_2 \equiv E_2 - E_0 \sim m\left[2 + 9\epsilon + \mathrm{O}(\epsilon^2)\right]$ as $\epsilon \to 0^+$. If $B_2 < 2M$, then a two-particle bound state exists and the (negative) binding energy is $B_2 - 2M$. If $B_2 > 2M$, then the second excitation above the vacuum is interpreted as an unbound two-particle state. In the small-coupling regime, where perturbation theory is valid, the conventional anharmonic oscillator does not possess a bound state. Indeed, using WKB, variational methods, or numerical calculations one can show that there is no two-particle bound state for any $g > 0$. Thus, the gx^4 interaction represents a repulsive force.[2]

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[2] In general, a repulsive force in a quantum field theory is represented by an energy dependence in which the energy of a two-particle state decreases with separation. The conventional anharmonic oscillator Hamiltonian corresponds to a field theory in one space-time dimension where there cannot be any spatial dependence. The repulsive nature of the force is understood to mean that the energy B_2 needed to create two particles at a given time is more than twice the energy M needed to create one particle.

We obtain the perturbation series for L in (8) from the perturbation series for the conventional theory by replacing ϵ with $-\epsilon$. Thus, while the conventional anharmonic oscillator does not possess a two-particle bound state, the $\mathcal{PT}$-symmetric oscillator does indeed possess such a state. We give the binding energy of this state in units of the renormalized mass M and we define the *dimensionless* binding energy Δ_2 by

$$\Delta_2 \equiv (B_2 - 2M)/M \sim -3\epsilon + \mathrm{O}(\epsilon^2) \qquad (\epsilon \to 0^+). \tag{13}$$

This bound state evaporates when ϵ increases beyond $\epsilon = 0.0465\ldots$. As ϵ continues to grow, Δ_2 reaches a maximum of 0.427 at $\epsilon = 0.13$ and then approaches 0.28 as $\epsilon \to \infty$.

In the $\mathcal{PT}$-symmetric anharmonic oscillator, there are not only two-particle bound states for small coupling constant but also k-particle bound states for all $k \geqslant 2$. The dimensionless binding energies are $\Delta_k \equiv (B_k - kM)/M \sim -3k(k-1)\epsilon/2 + \mathrm{O}(\epsilon^2)$ as $\epsilon \to 0+$. Since the coefficient of ϵ is negative, the dimensionless binding energy becomes negative as ϵ increases from 0, and there is a k-particle bound state. The higher k-particle bound states cease to be bound for smaller values of ϵ; the binding energies Δ_3, Δ_4, Δ_5, and Δ_6 become positive as ϵ increases past 0.039, 0.034, 0.030, and 0.027 [7].

Figure 2 shows that for any value of ϵ there are always a finite number of bound states and an infinite number of unbound states. The number of bound states decreases with increasing ϵ until there are no bound states at all. Observe that there is a range of ϵ for which there are only two- and three-particle bound states. This situation is analogous to the physical world in which one observes only states of two and three bound quarks. In this range of ϵ if one has an initial state containing a number of particles (renormalized masses), these particles will clump together into bound states, releasing energy in the process. Depending on the value of ϵ, the final state will consist either of two- or of three-particle bound states, whichever is energetically favored. Note also that there is a special value of ϵ for which two- and three-particle bound states can exist in thermodynamic equilibrium.

These results generalize from quantum mechanics to the D-dimensional $\mathcal{PT}$-symmetric $-g\phi^4$ quantum field theory. There exists a bound state because *to leading order in the dimensionless coupling constant* ϵ the binding energy becomes negative as ϵ increases from 0. We calculate the bound-state energy by summing all "sausage-link" graphs and identifying

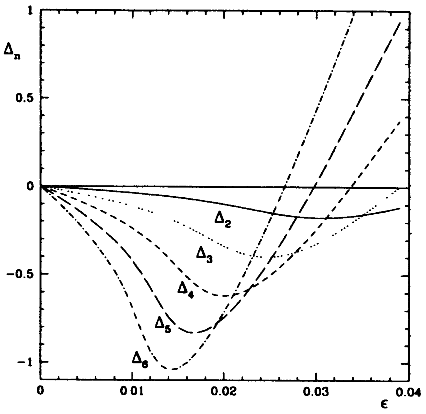

Figure 2. Dimensionless binding energies Δ_2, Δ_3, Δ_4, Δ_5, and Δ_6 for the two-particle, three-particle, four-particle, five-particle, and six-particle bound states plotted as functions of the dimensionless coupling constant ϵ. Note that the multiparticle bound states cease to be bound as ϵ increases past 0.0465, 0.039, 0.034, 0.030, *and* 0.027.

the bound-state pole. The dimensionless binding energy to leading order in ϵ is

$$\Delta_2 \sim -(4\pi)^{(D-1)/(D-3)}[3\Gamma(3/2 - D/2)]^{2/(3-D)}\epsilon^{2/(3-D)}, \tag{14}$$

which reduces to (13) at $D = 1$. Equation (14) holds for $0 \leqslant D < 3$ because we have performed mass renormalization (but not wave function or coupling-constant renormalization).

Let us compare a $g\phi^3$ theory with a $g\phi^4$ theory: A $g\phi^3$ theory represents an attractive force. The bound states that arise as a consequence of this force can be found by using the Bethe-Salpeter equation. However, the $g\phi^3$ field theory is unacceptable because the spectrum is not bounded below. If we replace g by ig, the spectrum becomes real and positive, but the force becomes repulsive and there are no bound states. The same is true for a two-scalar theory with interaction of the form $ig\phi^2\chi$ [8]. This latter theory is an acceptable model of scalar electrodynamics, but has no analog of positronium.

6. Two-point Green's function.

There are many other quantum field-theoretic results. For example, we have just completed a study of the two-point Green's function in quantum field theory [9]. A byproduct of this research shows that in a $\mathcal{PT}$-symmetric quantum theory the eigenstates of the Hamiltonian are *complete*.

7. New results.

I conclude by reporting a major breakthrough in $\mathcal{PT}$-symmetric quantum theory [10]. Subsequent to the presentation of this talk we have now been able to establish that $\mathcal{PT}$-symmetric quantum mechanics has an inner product that is associated with a *positive-definite* norm. Thus, a $\mathcal{PT}$-symmetric quantum theory is a fully consistent, unitary, probabilistic, physical quantum mechanical theory. We have found that every $\mathcal{PT}$-symmetric Hamiltonian has a symmetry $\mathcal{C}$ that has until now not been discovered. The linear operator $\mathcal{C}$ commutes with the $\mathcal{PT}$ operator and also with the Hamiltonian H. Also, $\mathcal{C}^2 = 1$, so the eigenvalues of $\mathcal{C}$ are ± 1. The positive-definite inner product is taken with respect to the $\mathcal{CPT}$ operator.

In summary, we have generalized the condition of Hermiticity in quantum mechanics to the statement of $\mathcal{CPT}$ invariance. In effect we have established the converse of the $\mathcal{CPT}$ theorem. If we assume that the Hamiltonian is symmetric and possesses space-time reflection symmetry, and that this symmetry is not spontaneously broken, then the Hamiltonian is Hermitian with respect to $\mathcal{CPT}$ conjugation. In effect, we are replacing the usual mathematical condition of Hermiticity, whose physical content is questionable, by the physical condition of space-time symmetry. This symmetry ensures the reality of the spectrum of the Hamiltonian in complex quantum theories.

Conventional Hermitian Hamiltonians and $\mathcal{PT}$-symmetric Hamiltonians have two important features in common, namely, symmetry and even-dimensionality. When a conventional Hermitian theory is formulated in real Hilbert space, Hamiltonians are required to be symmetric because they represent physical observables. In $\mathcal{PT}$-symmetric quantum theory we are extending this real formulation of quantum mechanics into the complex domain. However, we must retain the symmetry of Hamiltonians for the same reason as in conventional theory. Also, in the real formulation of quantum theory the dimensionality of the Hilbert space must be even. This is necessary in order to introduce a complex structure in the real Hilbert space. In the present theory we require the introduction of the $\mathcal{CPT}$ structure. From a physical point of view this is because half of the eigenstates — those having negative $\mathcal{PT}$ norm — might be interpreted as states representing antiparticles. Therefore, for each particle state there is a corresponding antiparticle state. These two states are always formed pairwise, in the sense that when $\mathcal{PT}$ symmetry is spontaneously broken, corresponding pairs of eigenstates and eigenvalues become complex conjugates of one another. This is because the secular equation for a $\mathcal{PT}$ symmetric Hamiltonian is always real [11].

In a conventional Hermitian quantum field theory the operators $\mathcal{C}$ and $\mathcal{P}$ commute, but in a $\mathcal{PT}$-symmetric quantum field theory these operators do not commute. As a consequence, it is not necessarily true that particles and antiparticles have the same energy eigenvalues. Recall that the condition of space-time reflection symmetry is weaker than the condition of Hermiticity, and therefore it is possible to consider new kinds of quantum field theories, whose self-interaction potential are, for example, $ig\phi^3$ or $-g\phi^4$, that have previously been thought to be unacceptable. A plausible signal of one of these new theories would be the observation of a particle and its corresponding antiparticle having different masses.

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