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THE SMALL SCHOTTKY-JUNG LOCUS IN POSITIVE CHARACTERISTICS DIFFERENT FROM TWO

by Fabrizio ANDREATTA

1. Introduction.

In this paper we investigate the problem of characterizing the ideal sheaf defining the locus of Jacobians of curves of genus g in the moduli space of principally polarized abelian varieties of dimension g over any field of characteristic different from 2. We use the so-called Schottky-Jung relations. In [vG], Thm 1.6, it is proven that over $\mathbb{C}$ the irreducible component of the locus defined by the Schottky-Jung relations containing the Jacobians consists only of Jacobians. See also [Do1] and [Do2], Ch. 2. The idea of B. van Geemen is to reduce the problem to the study of the Schottky-Jung relations induced to the boundary of the moduli space of principally polarized abelian varieties and to apply an induction procedure. We borrow and generalize this idea to prove the following theorem. Let g be an integer $\geqslant 2$. Let k be a field of characteristic different from 2. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. Let $\mathfrak{M}_g$ be the moduli space of smooth curves of genus g. Let $\mathfrak{J}_{g,k}$ be the closed subscheme of $\mathcal{A}_g \otimes k$ defined as the scheme theoretic image of $\mathfrak{M}_g \otimes k$ via the Torelli map. This is the map which associates to a curve its Jacobian. Let $\mathfrak{S}_{g,k}^{\text{small}}$ be the closed subscheme of $\mathcal{A}_g \otimes k$ defined by the Schottky-Jung relations over k; see 4.6.

THEOREM. — *The irreducible component of $\mathfrak{S}_{g,k}^{\text{small}}$ containing $\mathfrak{J}_{g,k}$ is equal to $\mathfrak{J}_{g,k}$.*

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The first novelty of our approach is that we work purely algebraically. In §2 we review the Prym construction for degenerating curves. In §3 we briefly recall the formalism of Mumford's theta functions. These are a valuable substitute for the classical theta functions when one does not work over $\mathbb{C}$. The main difficulty is to deduce the Schottky-Jung relations between suitable Mumford's theta functions from the Prym construction. We overcome the difficulty in §4 following closely an idea of Mumford's sketched in [Mu2]. The second novelty is the use in our context of the recent progress on toroidal compactifications of moduli spaces of polarized abelian varieties. We refer the reader to [FC] for the construction of such compactifications. In [FC] a description of degenerating semiabelian schemes over affine schemes, satisfying suitable conditions, is given in terms of so-called "degeneration data". It is also proven that the families described in this way cover the boundary of toroidal compactifications of moduli spaces of polarized abelian varieties (possibly with extra structure). The Fourier-Jacobi expansion of theta functions, as reviewed in §5, can be described in terms of degeneration data. Conversely, the degeneration data are encoded in the Fourier-Jacobi expansion of theta functions. This allows us to translate the Schottky-Jung relations, induced to the boundary, into restrictions on the degeneration data of semiabelian schemes satisfying these relations. The Fourier-Jacobi expansion enables us also to go deeper in the boundary than in [vG], where only rank 1 degenerations are used. The advantage is that we can focus on neighborhoods of points of the boundary whose abelian part is of low dimension, in our case equal to 2, and which consequently are easy to handle. This is the main ingredient in the proof of the extension of Van Geemen's theorem contained in §7.

Contents.

2. The Prym variety.

In this section we review the Prym construction. This appears in [Mu4] for covers of smooth curves (over a field) and in [Be] for admissible covers of semistable curves (over a field). We extend it to the case of families of semistable curves over an integral normal base scheme with smooth generic fibers. This appears partly in [Be], §6.2, with emphasis on the Prym variety itself. Since we are interested in the precise relation between the Prym variety and the Jacobian varieties it arises from, we give some details.

2.1. The setup.

Let S be an integral normal scheme with function field K. Assume that 2 is invertible in S. Let

$$\widetilde{f}: \widetilde{\mathcal{C}} \longrightarrow S$$

be a semistable curve over S i.e., a proper and flat morphism whose geometric fibers are reduced, connected, of dimension 1 and have only ordinary double points as singularities. Suppose that

1) $\widetilde{f}$ is smooth over K;
2) $\widetilde{\mathcal{C}}$ has genus $2g-1$ for some positive integer g;
3) there exists an automorphism ι of $\widetilde{\mathcal{C}}$ over S such that
   3.a) ι has order 2;
   3.b) ι acts freely on the smooth part $\widetilde{\mathcal{C}}^{\mathrm{sm}}$ of $\widetilde{f}$;
   3.c) for any geometric point s of S and any ordinary double point $c \in \widetilde{\mathcal{C}}_s$ fixed by ι, the branches of $\widetilde{\mathcal{C}}_s$ at c are not interchanged by ι_s.

Consider the quotient map

$$\pi: \widetilde{\mathcal{C}} \longrightarrow \mathcal{C} := \widetilde{\mathcal{C}}/\langle\iota\rangle.$$

Denote by

$$f: \mathcal{C} \longrightarrow S$$

the induced morphism. Then,

1) by [Be], Lemma 3.1, f defines a semistable curve over S;
2) the morphism π restricted to $\widetilde{\mathcal{C}}^{\mathrm{sm}}$ is finite and étale. In particular, f is smooth over K;

3) by the Hurwitz formula $\mathcal{C}$ has genus g.

Let S^o be the open subscheme of S, where $f:\mathcal{C} \to S$ is smooth.

2.2. *Remark.* — The morphism π is an *admissible covering* in the sense of [HM], p. 57. This notion appears already in [Be], p. 157. Its importance is due to the fact that such coverings arise in the process of compactifying the moduli space of étale, 2 : 1 coverings of smooth curves of genus g. See [Be], pp. 180, Part (b) and [HM], §4, Theorem 4.

2.3. The associated Picard varieties.

Consider the fppf sheaves

$$\mathcal{X} := \mathrm{Pic}^0(\mathcal{C}/S) \qquad \text{and} \qquad \mathcal{Y} := \mathrm{Pic}^0(\widetilde{\mathcal{C}}/S).$$

They are defined by the presheaves associating to an fppf cover $T \to S$ the abelian group of line bundles on $\mathcal{C} \times_S T$ (resp. on $\widetilde{\mathcal{C}} \times_S T$), which are fiberwise over T of multidegree 0, modulo the line bundles coming from T by pull-back. By [BLR], Theorem 9.4.1, they are represented by semiabelian schemes over S i.e., by smooth, separated, commutative group schemes over S such that the fiber over any geometric point of S is the extension of an abelian variety by a torus. Note that over the open subscheme S^o of S where f is smooth the semiabelian schemes

$$\mathcal{X} \times_S S^o = \mathrm{Pic}^0(\mathcal{C} \times_S S^o/S^o) \quad \text{and} \quad \mathcal{Y} \times_S S^o = \mathrm{Pic}^0(\widetilde{\mathcal{C}} \times_S S^o/S^o)$$

are abelian and are endowed with principal polarizations; see [BLR], Prop. 9.4.4. Finally, let

$$\pi^\star : \mathcal{X} \longrightarrow \mathcal{Y}$$

be the morphism induced by π by functoriality of Pic^0.

2.4. The theta torsors.

After possibly replacing S with an étale surjective cover, *choose* an effective Cartier divisor κ on $\mathcal{C} \times_S S^o$ relative to S^o whose square is isomorphic to the dualizing sheaf on $\mathcal{C} \times_S S^o$. Note that $\pi^*(\kappa)$ is an effective Cartier divisor. Its square is isomorphic to the dualizing sheaf on $\widetilde{\mathcal{C}} \times_S S^o$; cf. [Be], Lemma 3.2. Define the theta divisor

$$\theta_{\mathcal{X} \times_S S^o} \quad (\text{resp. } \theta_{\mathcal{Y} \times_S S^o})$$

on $\mathcal{X} \times_S S^o$ (resp. on $\mathcal{Y} \times_S S^o$) by translating the image of the $g-1$th symmetric power of $\mathcal{C} \times_S S^o$ in $\mathrm{Pic}^{g-1}\big(\mathcal{C} \times_S S^o/S^o\big)$ to $\mathcal{X} \times_S S^o = \mathrm{Pic}^0\big(\mathcal{C} \times_S S^o/S^o\big)$ (resp. the image of the $2g-2$th symmetric power of $\widetilde{\mathcal{C}} \times_S S^o$ in $\mathrm{Pic}^{2g-2}(\widetilde{\mathcal{C}} \times_S S^o/S^o)$ to $\mathcal{Y} \times_S S^o = \mathrm{Pic}^0(\widetilde{\mathcal{C}} \times_S S^o/S^o))$ via translation by $-\kappa$ (resp. by $-\pi^*(\kappa)$). The Riemann-Roch theorem implies that the divisors $\theta_{\mathcal{X}\times_S S^o}$ and $\theta_{\mathcal{Y}\times_S S^o}$ are symmetric. Moreover, they induce the given principal polarizations on $\mathcal{X} \times_S S^o$ and $\mathcal{Y} \times_S S^o$. Denote by

$$\mathcal{L}_{\mathcal{X}\times_S S^o} \quad (\text{resp. } \mathcal{L}_{\mathcal{Y}\times_S S^o})$$

the associated symmetric $\mathbb{G}_m$-bundle which we rigidify along the identity element of $\mathcal{X}\times_S S^o$ (resp. of $\mathcal{Y}\times_S S^o$). By [Br], Prop. 2.4, it uniquely defines a cubical $\mathbb{G}_m$-torsor over $\mathcal{X}\times_S S^o$ (resp. $\mathcal{Y}\times_S S^o$). By [MB], Theorem II.3.3 and [MB], Theorem II.1.1, there exists a unique cubical $\mathbb{G}_m$-torsor $\mathcal{L}_{\mathcal{X}}$ (resp. $\mathcal{L}_{\mathcal{Y}}$) on $\mathcal{X}$ (resp. on $\mathcal{Y}$) extending $\mathcal{L}_{\mathcal{X}\times_S S^o}$ (resp. $\mathcal{L}_{\mathcal{Y}\times_S S^o}$).

2.5. *Remark.* — The line bundles on a smooth curve whose square are isomorphic to the dualizing sheaf are called *theta characteristics.* Hence, the κ chosen in the previous section is a theta characteristic. By Riemann's singularity theorem, $\Gamma(\mathcal{C}\times_S S^o, \kappa)$ is a locally free sheaf of rank equal to the multiplicity of the divisor $\Theta_{\mathcal{X}\times_S S^o}$ at the identity. We choose κ so that the rank of $\Gamma(\kappa, \mathcal{C} \times_S S^o)$ is even i.e., κ is a so-called *even theta characteristic.*

2.6. *Remark.* — The discussion in 2.4 depends on the choice of the theta characteristic κ. The definition is needed to get the Schottky-Jung relations. In general, the dualizing sheaf of $\mathcal{C} \to S$ does not admit a theta characteristic. For example, this is the case if, for some geometric point $s \in S$, the pull-back of the dualizing sheaf of $\mathcal{C}_s$ to the normalization of some irreducible component of $\mathcal{C}_s$ has odd degree. This explains why we worked over the open subscheme S^o. On the other hand:

2.7. Proposition. — *Étale locally on S one may choose a line bundle $\tilde{L}$ (resp. L) over $\widetilde{\mathcal{C}}$ (resp. $\mathcal{C}$) such that the locus*

$$\theta'_{\mathcal{Y},\tilde{L}} := \{M \in \mathcal{Y} | \dim_{k(s)} \Gamma(\widetilde{\mathcal{C}}_s, \tilde{L} \otimes M) \geqslant 1 \ \ \forall s \in S\}$$

$$(\text{resp}\, \theta'_{\mathcal{X},L} := \{M \in \mathcal{X} | \dim_{k(s)} \Gamma(\mathcal{C}_s, L \otimes M) \geqslant 1 \ \ \forall s \in S\})$$

defines an effective Cartier divisor in $\mathcal{Y}$ (resp. $\mathcal{X}$) relative to S. Over S^o it coincides with a translation of $\theta_{\mathcal{Y}\times_S S^o}$ (resp. $\theta_{\mathcal{X}\times_S S^o}$).

Proof. — This is the main result of [Be], §2. See, in particular, [Be], Remark 2.4.

2.8. *Remark.* — For the reader not familiar with the concept of cubical torsors, we simply remark that they are a generalization of the concept of polarization in the context of abelian varieties to the category of semiabelian schemes. See [FC], Remark I.2.4 for a quick introduction and [Br], Chap. I–III or [MB], Chap. I for a full exposition of the notion of cubical $\mathbb{G}_m$-torsors.

2.9. Duals and polarizations.

By [FC], §II.2, there exist a semiabelian scheme $\mathcal{X}^t$ over S and a biextension

$$\mathcal{P}_{\mathcal{X}} \longrightarrow \mathcal{X}^t \times_S \mathcal{X}$$

extending the abelian scheme $\mathcal{X}_K^t$, dual to $\mathcal{X}_K$, and the Poincaré biextension over $\mathcal{X}_K^t \times_K \mathcal{X}_K$. Analogous definitions are given for $\mathcal{Y}$ instead of $\mathcal{X}$. Denote by

$$\lambda_{\mathcal{X}}: \mathcal{X} \xrightarrow{\sim} \mathcal{X}^t \quad \text{and} \quad \lambda_{\mathcal{Y}}: \mathcal{Y} \xrightarrow{\sim} \mathcal{Y}^t$$

the isomorphisms extending the principal polarizations defined over K. By [FC], Prop. I.2.7, there exists a unique homomorphism

$$\pi^{\star,t}: \mathcal{Y}^t \longrightarrow \mathcal{X}^t$$

extending the dual of $\pi_K^\star$.

2.10. The norm map.

Let T be a scheme over S. Let L be a line bundle on $\widetilde{\mathcal{C}} \times_S T$, fiberwise over T of multidegree 0. Let

$$\psi: L \otimes \iota^*(L) \cong \iota^*(L) \otimes L \cong \iota^*\big(L \otimes \iota^*(L)\big)$$

be the isomorphism induced by flipping the two factors. Define

$$\mathrm{Nm}(L) := L \otimes \iota^*(L)/\langle\psi\rangle.$$

By descent theory it is a line bundle on the open sublocus of $\mathcal{C} \times_S T$ where $\pi \times_S T$ is étale. A local computation around the points where π is not étale shows that $\mathrm{Nm}(L)$ is a line bundle.

2.11. PROPOSITION. — *The Norm map* $\mathrm{Nm}: \mathcal{Y} \longrightarrow \mathcal{X}$

1) *is a group homomorphism;*

2) *satisfies*

$$\mathrm{Nm} = \lambda_{\mathcal{X}}^{-1} \circ \pi^{\star,t} \circ \lambda_{\mathcal{Y}};$$

3) *satisfies*

$$\mathrm{Nm} \circ \pi^\star = [2]: \mathcal{X} \longrightarrow \mathcal{X};$$

4) *is smooth.*

Proof. — Claim (1) is clear. By [Mu4], p. 328, (i) the equality claimed in (2) holds over $\bar{K}$. Hence, Claim (2) holds. Claim (3) follows from a direct computation. By (3) we deduce that Nm is surjective both on the levels of varieties and tangent spaces. Hence, it is smooth.

2.12. The kernel of $\pi^\star$.

Let $s \in S$ be a geometric point. Denote by a subscript s the base change from S to s. Let $\eta \in \mathrm{Ker}(\pi_s^\star)$. Suppose that $\eta \neq 0$. Then, η is the class of a line bundle L on $\mathcal{C}_s$. Since $\mathrm{Nm}_s \circ \pi_s^\star = [2]$, we have that $L^2 \cong O_{\mathcal{C}_s}$. Choose such an isomorphism α. It defines the structure of a $O_{\mathcal{C}_s}$-algebra on the coherent $O_{\mathcal{C}_s}$-module $O_{\mathcal{C}_s} \oplus L$; see [Mu4], p. 326. The associated scheme

$$\phi: D = \mathrm{Spec}\big(O_{\mathcal{C}_s} \oplus L\big) \longrightarrow \mathcal{C}_s$$

is finite and étale of degree 2. The isomorphism α defines canonically an isomorphism $\beta: \phi^*(L) \to O_D$. Moreover, for any scheme T over s, any morphism $\psi: T \to \mathcal{C}_s$ and any isomorphism $\gamma: \psi^*(L) \cong O_T$, there is a unique morphism $\nu: T \to D$ of $\mathcal{C}_s$-schemes such that $\nu^*(\beta) = \gamma$. In particular, we get a morphism of $\mathcal{C}_s$-schemes

$$\widetilde{\mathcal{C}}_s \longrightarrow D_s,$$

which must be an isomorphism.

Vice versa, suppose that $\pi_s: \widetilde{\mathcal{C}}_s \to \mathcal{C}_s$ is étale. Then, $\pi_{s,*}\big(O_{\widetilde{\mathcal{C}}_s}\big)$ is a coherent $O_{\mathcal{C}_s}$-algebra, locally free of rank 2 as $O_{\mathcal{C}_s}$-module. The map $2^{-1}\,\mathrm{Trace}_{\widetilde{\mathcal{C}}_s/\mathcal{C}_s}$ defines a splitting

$$\pi_{s,*}\big(O_{\widetilde{\mathcal{C}}_s}\big) = O_{\mathcal{C}_s} \oplus L$$

as $O_{\mathcal{C}_s}$-modules. The algebra structure implies that L^2 is trivial. Since $\widetilde{\mathcal{C}}_s$ is connected, L itself is not trivial. We conclude:

2.13. PROPOSITION. — *Let*

$$H := \mathrm{Ker}\big(\pi^\star\big).$$

Then, H is a quasi-finite and flat group scheme over S. If $s \in S$ is a geometric point, we have

1) $H_s = \{0\}$ *if and only if π_s is not étale;*

2) *over the open subscheme S^o of S where π is étale, $H \cong \mathbb{Z}/2\mathbb{Z}$. We denote by η the generator.*

2.14. The Prym variety.

Define

$$i: \mathcal{P} \hookrightarrow \mathcal{Y}$$

as the inverse image via $\lambda_{\mathcal{Y}}$ of the identity component $\mathrm{Ker}\big(\pi^{\star,t}\big)^0$ of $\mathrm{Ker}\big(\pi^{\star,t}\big)$. This is called *the Prym variety* associated to the covering π. Since Nm, and consequently $\pi^{\star,t}$, is a smooth morphism, the scheme $\mathcal{P}$ is a semiabelian scheme of relative dimension $g-1$ over S. Compare with [Be], §6.2.

For any line bundle L on $\widetilde{\mathcal{C}}$, of multidegree 0 fiberwise over S, the line bundle $L^2 = \big(L \otimes \iota^*(L)\big) \otimes \big(L \otimes \iota^*(L^{-1})\big)$ is in the image of $\mathcal{X} \times_S \mathrm{Ker}\big(\pi^{\star,t}\big)$. Indeed, $L \otimes \iota^*(L) = \pi^\star\big(\mathrm{Nm}(L)\big)$ and $\mathrm{Nm}\big(L \otimes \iota^*(L^{-1})\big)$ is trivial. Hence, the morphism

$$\big(\mathcal{X}/H\big) \times_S \mathcal{P} \xrightarrow{\pi^\star \times i} \mathcal{Y}$$

is surjective fiberwise over S and, in particular, it is an isogeny.

The following proposition extends to the present case the well-known facts concerning the Prym construction for smooth curves over an algebraically closed field contained in [Mu4].

2.15. PROPOSITION. — *The following hold:*

1. *We have an equality of effective Cartier divisors*

$$i^{-1}(\theta_{\mathcal{Y}\times_S S^o}) = 2\theta_{\mathcal{P}\times_S S^o}$$

where $\theta_{\mathcal{P}\times_S S^o}$ is a symmetric divisor on $\mathcal{P}$ relative to S^o inducing a principal polarization on $\mathcal{P}\times_S S^o$. In particular, there is a symmetric cubical $\mathbb{G}_m$-torsor $\mathcal{L}_{\mathcal{P}}$ over $\mathcal{P}$ such that

$$i^*(\mathcal{L}_{\mathcal{Y}}) \cong \mathcal{L}_{\mathcal{P}}^2$$

as symmetric cubical $\mathbb{G}_m$-torsors over $\mathcal{P}$.

2. *We have an equality of effective Cartier divisors*

$$(\pi^\star)^{-1}(\theta_{\mathcal{Y}\times_S S^o}) = \theta_{\mathcal{X}\times_S S^o} + t_\eta^\star(\theta_{\mathcal{X}\times_S S^o}).$$

Here, t_η denotes the translation by η on $\mathcal{X}\times_S S^o$. In particular, $(\pi^\star)^*(\mathcal{L}_{\mathcal{Y}})$ is isomorphic to $\mathcal{L}_{\mathcal{X}}^2$ as symmetric cubical $\mathbb{G}_m$-torsor.

3. *We have a cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}\otimes\mathcal{L}_{\mathcal{P}}^2 & \longrightarrow & \mathcal{L}_{\mathcal{Y}} \\ \big\downarrow & & \big\downarrow \\ \big(\mathcal{X}/H\big)\times_S\mathcal{P} & \longrightarrow & \mathcal{Y}, \end{array}$$

where $\mathcal{M}$ is the symmetric cubical $\mathbb{G}_m$-torsor got by descending $\mathcal{L}^2_{\mathcal{X}}$ to $\mathcal{X}/H$.

4. *Let $H_K^{\perp}$ be the orthogonal of H_K inside $\mathcal{X}[2]_K$ with respect to the Weil pairing. The pullback to $H_K^{\perp}/H_K$ of the cubical torsor descended on $\mathcal{X}_K/H_K$ from $\mathcal{L}^2_{\mathcal{X}_K}$ is trivial. In particular, it induces a symplectic structure on $H_K^{\perp}/H_K$. There is an isomorphism*

$$\psi: \mathcal{P}_K[2] \longrightarrow H_K^{\perp}/H_K,$$

preserving the symplectic structures, such that the kernel of $\mathcal{X}_K/H_K \times_K \mathcal{P}_K \to \mathcal{Y}_K$ is the graph of ψ.

Proof. — To prove the statements in (1) and (2) concerning the divisors, we may assume that $S = \mathrm{Spec}(K)$ and K is algebraically closed. See [Mu4], Prop. p. 342 for (1). See [Mu4], Corollary 5.3 for (2). For the proof of (3) and (4) over K, see the arguments of [Mu4], p. 330. Let G be a semiabelian scheme over S. Since S is normal, the category of symmetric cubical $\mathbb{G}_m$-torsors over G is equivalent to the category of symmetric cubical $\mathbb{G}_m$-torsors over G_K [Br], Prop. 2.4. By the equivalence of categories just stated, statements (1), (2), and (3) follow since they hold over K.

2.16. *Remark.* — For a more geometric and explicit approach in the case of Mumford's curves see [vS].

3. Mumford's theta functions.

In this section we briefly review Mumford's construction of 2-adic theta functions as in [Mu1]. Since we are interested in degenerations of theta functions on S, we slightly generalize the approach of Mumford.

3.1. Some notations. — *Let A be a semiabelian scheme over S, abelian over K. Let $\mathcal{L}$ be a symmetric $\mathbb{G}_m$-torsor over A. Fix a rigidification $\varphi_0: L \times_A 0 \to \mathbb{A}^1_S$ of the associated line bundle L. By [Br], Prop. 2.4 we can consider $\mathcal{L}$ as a cubical $\mathbb{G}_m$-torsor over A.*

3.2. Definition. — *Let $K(\mathcal{L})$ be the Zariski closure in A of the kernel of the polarization $A_K \to A_K^t$ induced by $\mathcal{L}_K$. By [MB], Theorem IV.2.4, it is a group scheme quasi-finite and flat over S.*

3.3. Definition. — *Let n be an integer. Define (A_n, p_n) as the couple consisting of $A_n = A$ and the morphism*

$$p_n\colon A_n \xrightarrow{[2^n]} A \quad \text{if} \quad n \geqslant 0 \qquad p_n\colon A \xrightarrow{[2^{-n}]} A_n \quad \text{if} \quad n \leqslant 0.$$

The set $\{(A_n, p_n)\}_n$ is an inverse system via the natural morphisms

$$p_{n,m}\colon A_n \longrightarrow A_m$$

defined, for all integers m and n such that $m \leqslant n$, as multiplication by 2^{n-m}. Compare with [Mu1], Definition 7.1.

For any integer n define the fppf sheaf $A_n[2^\infty]$ as the direct limit

$$A_n[2^\infty] := \lim_{m \to \infty} A_n[2^m],$$

where $A_n[2^m]$ is the kernel of multiplication by 2^m.

Define the 2-divisible fppf *sheaf $V(\underline{A})$ as the projective limit over $n \in \mathbb{Z}$:*

$$V(\underline{A}) = \varprojlim A_n[2^\infty],$$

where the transition morphisms are induced from the morphisms $p_{n,m}$. For any integer n define the projection morphism

$$\underline{p}_n\colon V(\underline{A}) \longrightarrow A_n[2^\infty].$$

Define the lattice $T(A)$ inside $V(\underline{A})$ by

$$T(A) := \underline{p}_0^{-1}(0).$$

It acts by translation on $V(\underline{A})$. Compare with [Mu1], Definition 7.2.

Finally, define $\Gamma(\underline{L})$ as the direct limit over $n \in \mathbb{N}$:

$$\Gamma(\underline{L}) := \varinjlim \Gamma\big(A_n, p_n^*(L)\big).$$

The transition morphisms are defined by the pullbacks of sections via the morphisms $p_{n,m}^*$. Compare with [Mu1], Definition 7.3 and [Mu1], §7, pp. 108-109.

3.4. *Remark.* — We prefer to deal with $V(\underline{A})$ as an fppf sheaf since we are interested in degenerations of theta functions, while Mumford works over algebraically closed fields in order to be able to find a basis of the $\mathbb{Q}_2$-topological vector space $V(\underline{A})$.

3.5. Definition ([Mu1], Definition 7.4). — *Define the elements of the infinite theta group $\mathcal{G}(\underline{\mathcal{L}})$ as the pairs of elements $(x, \{\phi_n\}_{n \geqslant n_x})$ where*

- *x is an element of $V(\underline{A})$;*

- *let n_x be the smallest integer n for which $\underline{p}_n(x)$ belongs to the group $K\big(p_n^*(\mathcal{L})\big)$; see 3.6. The set $\{\phi_n\}_n$, with n integer such that $n \geqslant n_x$, is a compatible system of automorphisms of $p_n^*(\mathcal{L})$ lifting translation by $\underline{p}_n(x)$ on A_n.*

There is a natural action

$$\mathcal{G}(\underline{\mathcal{L}}) \times \Gamma(\underline{L}) \longrightarrow \Gamma(\underline{L}).$$

Moreover, $\mathcal{G}(\underline{\mathcal{L}})$ sits in an exact sequence

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathcal{G}(\underline{\mathcal{L}}) \longrightarrow V(\underline{A}) \longrightarrow 0$$

such that the torus $\mathbb{G}_{m,S}$ is in the center of $\mathcal{G}(\underline{\mathcal{L}})$. In particular, we get a skew symmetric pairing

$$e_{\mathcal{L}}(\_,\_)\colon V(\underline{A}) \times V(\underline{A}) \longrightarrow \mathbb{G}_{m,S}$$

as follows. If $x, y \in V(\underline{A})$ and $\bar{x}, \bar{y} \in \mathcal{G}(\underline{\mathcal{L}})$ are liftings of x, y, then

$$e_{\mathcal{L}}(x,y) := \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1}.$$

See [Mu1], §7, pp. 103–104.

Let v be a non-negative integer. Denote by $\mathcal{G}\big(p_v^*(\mathcal{L})\big)$ the finite theta group on A_v defined as in [MB], Theorem IV.2.4(iii). The following hold:

- *its fiber over K is the theta group of the line bundle associated to $p_v^*(\mathcal{L}_K)$ [MB], Theorem IV.2.4(ii).*

- *it acts on $p_v^*(\mathcal{L})$;*

- *it sits in an exact sequence:*

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathcal{G}\big(p_v^*(\mathcal{L})\big) \longrightarrow K\big(p_v^*(\mathcal{L})\big) \longrightarrow 0;$$

[MB], Theorem IV.2.4(iii);

- *by construction there is a natural surjective homomorphism*

$$\mathcal{G}(\underline{\mathcal{L}})|_{\underline{p}_v^{-1}\left(K(p_v^*(\mathcal{L}))\right)} \longrightarrow \mathcal{G}\big(p_v^*(\mathcal{L})\big)$$

of group schemes compatible with the induced action on $p_v^*(\mathcal{L})$;

- *the morphism*

$$\mathcal{G}(\underline{\mathcal{L}}) \longrightarrow \varprojlim \mathcal{G}\big(p_n^*(\mathcal{L})\big)$$

is an isomorphism. The limit on the RHS is taken over the non-negative integers n.

Define

$$\mathcal{K}_v := \mathrm{Ker}\Big(\mathcal{G}(\underline{\mathcal{L}})|_{\underline{p}_v^{-1}\left(K\left(p_v^*(\mathcal{L})\right)\right)} \longrightarrow \mathcal{G}\big(p_v^*(\mathcal{L})\big)\Big).$$

The group $\mathcal{K}_v$ *is the subgroup of elements of* $\mathcal{G}(\underline{\mathcal{L}})$ *acting trivially on* $\Gamma\big(A_v, p_v^*(L)\big)$.

3.6. *Remark* ([Mu1], "4 > 2" Lemma, Section 7). — If s is a positive integer $[s]^*(\mathcal{L})$ is isomorphic to $\mathcal{L}^{s^2}$ and $A[s^2]$ is contained in $K(\mathcal{L}^{s^2})$. Hence, the integer n_x of 3.5 exists.

3.7 Definition ([Mu1], Definition 7.5). — *The exact sequence defined in 3.5 for* $\mathcal{G}(\underline{\mathcal{L}})$ *admits a natural right splitting* σ *as sets defined as follows:*

Let $a = (a_n)_{n\in\mathbb{Z}}$ *be an element of* $V(\underline{A})$. *Let* n_0 *be an integer such that for any* $n \geqslant n_0$ *we have* $a_n = 2b_n$ *with* b_n *an element in* $K\big(p_n^*(\mathcal{L})\big)$. *For any* $n \geqslant n_0$ *choose* $c_n = (b_n, \phi_n)$ *in* $\mathcal{G}\big(p_n^*(\mathcal{L})\big)$ *above* b_n *such that* $\big(\psi_n\big)^{-1} \circ \phi_n \circ \psi_n = \phi_n^{-1}$ *where* ψ_n *is any isomorphism of* $p_n^*(\mathcal{L})$ *lifting* $[-1]$ *on* A_n. *Define*

$$\big(\sigma(a)\big) := \big(c_n^2\big)_n \in \varprojlim \mathcal{G}\big(p_n^*(\mathcal{L})\big).$$

It is an element of $\mathcal{G}(\underline{\mathcal{L}})$ *above* a.

3.8. *Remark* ([Mu1], Lemma 7.3). — If x and y are two elements of $V(\underline{A})$, then

$$\sigma(x+y) = e_{\mathcal{L}}\left(\frac{y}{2}, \frac{x}{2}\right)^2 \sigma(x)\,\sigma(y).$$

3.9. Definition ([Mu1], §8, p. 113). — *Define the theta morphism*

$$\begin{array}{ccc} \Gamma(\underline{L}) & \stackrel{\vartheta}{\longrightarrow} & \mathrm{Hom}_{\mathrm{fppf}}(V(\underline{A}), \mathbb{A}^1_S) \\ \beta & \longmapsto & \vartheta_\beta \end{array}$$

as follows:

If β *is an element of* $\Gamma(\underline{L})$ *and* $a = (a_n)_{n\in\mathbb{Z}}$ *is an element of* $V(\underline{A})$, *let*

$$\vartheta_\beta(a) = \varphi_0\big((\sigma(-a)(\beta))(0)\big),$$

where the action of $\sigma(-a)$ *is defined by the action of* $\mathcal{G}(\underline{\mathcal{L}})$ *on* $\Gamma(\underline{L})$ *(3.5), the map* φ_0 *is defined in 3.1.*

In particular, suppose that $a, x \in V(\underline{A})$. *By 3.8*

$$\big(\sigma(x) \circ \vartheta_\beta\big)(a) = e_{\mathcal{L}}\left(\frac{x}{2}, \frac{a}{2}\right)^2 \vartheta_\beta(a-x).$$

Compare with [Mu1], Property 8.I.

3.10. Definition ([Mu1], §2, p. 304). — *We define a quadratic character*

$$e_\star^{\mathcal{L}}\colon \frac{1}{2}T(A)/T(A) \longrightarrow \{\pm 1\}$$

as follows. Let $\phi\colon L \xrightarrow{\sim} [-1]^*(L)$ be an isomorphism normalized so that $\phi(0)\colon L(0) \to [-1]^*(L)(0) = L(0)$ is the identity; it exists since $\mathcal{L}$ is symmetric. Define

$$e_\star^{\mathcal{L}}\colon A[2] \longrightarrow \{\pm 1\}$$

by $\phi(x)\colon L(x) \xrightarrow{\sim} L(-x) = L(x)$ being multiplication by $e_\star^{\mathcal{L}}(x)$. Note that $\underline{p}_0$ identifies

$$\frac{1}{2}T(A)/T(A) \xrightarrow{\sim} A[2].$$

By [Mu1], Lemma 7.4 the character $e_\star^{\mathcal{L}}$ satisfies the following: if $x \in \mathcal{K}_0$ and $y \in T(A)$ is its image in $V(\underline{A})$, then

$$x = e_\star^{\mathcal{L}}\left(\frac{y}{2}\right)\sigma(y).$$

By the discussion of [Mu1], §7, p. 108, given the polarization on A defined by $\mathcal{L}$, the structure of $\mathcal{G}(\underline{\mathcal{L}})$ as abstract group and the section $\sigma\colon V(\underline{A}) \to \mathcal{G}(\underline{\mathcal{L}})$, there is a 1-1 correspondence between

1. *quadratic characters on $\frac{1}{2}T(A)/T(A)$;*
2. *line bundles on A defining the given polarization.*

3.11. Proposition. — *Let $g\colon B \to A$ be a morphism of semiabelian schemes. Let $\mathcal{L}$ and $\mathcal{M} \cong g^*(\mathcal{L})$ be symmetric, cubical $\mathbb{G}_m$-torsors with compatible rigidifications. Let β_A be an element of $\Gamma(\underline{L})$ and let $a = (a_n)$ be an element of $V(\underline{B})$. Then,*

$$\vartheta_{\beta_A}\big(g(a)\big) = \vartheta_{g^*(\beta_A)}(a).$$

Moreover,

$$e_{\mathcal{M}}(a,b) = e_{\mathcal{L}}(g(a), g(b))$$

for any a and b in $V(\underline{B})$ and, finally,

$$e_\star^{\mathcal{M}}(a) = e_\star^{\mathcal{L}}\big(g(a)\big)$$

for any a in $\frac{1}{2}T(B)$.

3.12. Assumption. — *Suppose from now until the end of the section that the base S is local with maximal ideal I and algebraically* *closed residue field. Suppose that A is an abelian scheme over S. Choose a compatible system of 2^nth roots of unity (ζ_{2^n}) in O_S for all non-negative integers n. Suppose that $\mathcal{L}$ has degree 1. Let β_A be a global section (unique up to constant) of the associated line bundle L. Write*

$$\vartheta_A := \vartheta_{\beta_A}.$$

We can apply Mumford's theory of theta functions as in [Mu1]. Mumford works over an algebraically closed field. With our assumptions we can extend his theory to our context.

3.13. Definition. — *Let g be the dimension of A. Define the bilinear pairing*

$$\begin{array}{rccc} \chi(-,-): & \mathbb{Q}_2^g \times \mathbb{Q}_2^g & \longrightarrow & \mathbb{G}_m(S) \\ & \big((a_1,\ldots,a_g),(b_1,\ldots,b_g)\big) & \longmapsto & \zeta_\gamma^\alpha \end{array}$$

as follows. Let $\langle a,b\rangle := a_1b_1+\cdots+a_gb_g$ and let v_2 be the 2-adic valuation. Put γ and α equal to 0 if $v_2\big(\langle a,b\rangle\big) \geqslant 0$ and

$$\gamma = 2^{-v_2(\langle a,b\rangle)} \quad \text{and} \quad \alpha = 2^{-v_2(\langle a,b\rangle)}\langle a,b\rangle$$

otherwise. Consider $\mathbb{Q}_2^g \times \mathbb{Q}_2^g$ as a constant $\mathbb{Q}_2$-vector space sheaf over S with the symplectic structure defined by

$$\begin{array}{rccc} e(\_\,\_): & (\mathbb{Q}_2^g \times \mathbb{Q}_2^g) \times (\mathbb{Q}_2^g \times \mathbb{Q}_2^g) & \longrightarrow & \mathbb{G}_m(S) \\ & \big((x_1,y_1),(x_2,y_2)\big) & \longmapsto & \chi(x_1,y_2)\chi(x_2,y_1)^{-1}. \end{array}$$

Define

$$\mathcal{G}_g := \mathbb{G}_{m,S} \times \mathbb{Q}_2^g \times \mathbb{Q}_2^g.$$

The symplectic structure on $\mathbb{Q}_2^g \times \mathbb{Q}_2^g$ defines a group structure on $\mathcal{G}_g$ and an exact sequence:

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathcal{G}_g \longrightarrow \mathbb{Q}_2^g \times \mathbb{Q}_2^g \longrightarrow 0,$$

cf. [Mu1], §7, p. 106.

Define a full theta structure on $\big(A,\mathcal{L}\big)$ to be an isomorphism as sheaves of $\mathbb{Q}_2$-vector spaces over S:

$$\overline{c}_A: V(\underline{A}) \longrightarrow \mathbb{Q}_2^g \times \mathbb{Q}_2^g$$

such that $\overline{c}_A$ respects the symplectic structures and sends $T(A)$ to $\mathbb{Z}_2^g \times \mathbb{Z}_2^g$. By 3.12 such isomorphisms exist. Fix one.

The isomorphism $\overline{c}_A$ defines an isomorphism of groups

$$c_A: \mathcal{G}(\underline{\mathcal{L}}) \longrightarrow \mathcal{G}_g;$$

see [Mu1], Definition 7.8 and the discussion after that. The isomorphism c_A is the unique one compatible with $\bar{c}_A$ and satisfying the following. Let a be in $V(\underline{A})$ and let $\bar{c}_A(a) = (x, l)$. Then

$$(1, x, l)/c_A\big(\sigma(a)\big) = \chi\Big(\frac{x}{2}, \frac{l}{2}\Big)^2.$$

As in [Mu1], Lemma 7.4, it follows that for all $a \in T(A)$, putting $\bar{c}_A(a) = (x, l)$,

$$\chi\Big(\frac{x}{2}, \frac{l}{2}\Big)^2 = e_\star^{\mathcal{L}}\Big(\frac{a}{2}\Big).$$

Define

$$\mathcal{H}_g := \mathrm{Hom}_{\mathrm{loc\,const}}(\mathbb{Q}_2^g, \mathbb{A}_S^1).$$

The group $\mathcal{G}_g$ (and, hence, $\mathcal{G}(\underline{\mathcal{L}})$ via c_A) acts on $\mathcal{H}_g$ as follows:

$$\begin{array}{ccc} \mathcal{G}_g \times \mathcal{H}_g & \longrightarrow & \mathcal{H}_g \\ \big((\alpha, x, l), \phi\big) & \longmapsto & \alpha\,\chi(l, \_)\,\phi(\_ + x). \end{array}$$

Recall that $\mathcal{H}_g$ is the unique admissible irreducible representation of weight 1 of $\mathcal{G}_g$ (and hence of $\mathcal{G}(\underline{\mathcal{L}})$) up to isomorphism. See [Mu1], Definition 7.9 and the theorem after it.

The description of the representation theory of the group $\mathcal{G}(\underline{\mathcal{L}})$ is given in [Mu1], §7, pp. 114–115, by the following diagram:

$$\begin{array}{ccc} \Gamma(\underline{L}) & \xrightarrow{\sim} & \mathcal{H}_g \\ & \searrow \vartheta & \Big\downarrow T \\ & & \mathrm{Hom}_{\mathrm{loc.const.}}(V(\underline{A}), K), \end{array}$$

where $\Gamma(\underline{L})$ is defined in 3.3, the map ϑ is defined in 3.9, the isomorphism is as admissible irreducible weight 1 representations of $\mathcal{G}(\underline{\mathcal{L}})$ and T is the unique morphism making the diagram commute.

3.14. Definition ([Mu1], Corollary p. 115). — *For all $a, b \in \mathbb{Q}_2^g$ define*

$$\delta\begin{bmatrix} a \\ b \end{bmatrix}(y) = \begin{cases} 0 & y \notin a + \mathbb{Z}_2^g \\ \chi(b, y) & y \in a + \mathbb{Z}_2^g. \end{cases}$$

The element $\delta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is in $\mathcal{H}_g$. Let $a' \in a + \mathbb{Z}_2^g$ and $b' \in b + \mathbb{Z}_2^g$, then

$$\delta\begin{bmatrix} a' \\ b' \end{bmatrix} = \chi(b' - b, a)\delta\begin{bmatrix} a \\ b \end{bmatrix}.$$

In particular, if $a, b \in (1/2^v)\mathbb{Z}_2^g$ for some non-negative integer v, then $\delta\left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right)$ does not depend on the choice of $a' \in a + 2^v\mathbb{Z}_2^g$ and of $b' \in b + 2^v\mathbb{Z}_2^g$. The elements

$$\vartheta_A\begin{bmatrix} a \\ b \end{bmatrix} := T\Big(\delta\begin{bmatrix} a \\ b \end{bmatrix}\Big),$$

where a and b are chosen representatives in $\mathbb{Q}_2^g$ of the cosets $\mathbb{Q}_2^g/\mathbb{Z}_2^g$, define a basis of $\vartheta\big(\Gamma(\underline{L})\big)$. Moreover,

$$\vartheta_A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vartheta_A;$$

the latter is defined in 3.12.

The group $\mathcal{G}(\underline{\mathcal{L}})$ permutes the elements of this basis (up to scalar). The action is described via the isomorphism c_A in 3.13 of $\mathcal{G}(\underline{\mathcal{L}})$ with $\mathcal{G}_g = \mathbb{G}_{m,S} \times \mathbb{Q}_2^g \times \mathbb{Q}_2^g$ by

$$(\alpha, x, l) \circ \delta \begin{bmatrix} a \\ b \end{bmatrix} = \alpha\, \chi(b, x) \delta \begin{bmatrix} a - x \\ b + l \end{bmatrix}.$$

3.15. *Remark.* — In particular, the set

$$\left\{ \delta \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \frac{1}{2^v}\mathbb{Z}_2^g/\mathbb{Z}_2^g \right\}$$

is the one fixed by the subgroup $\mathcal{K}_v$ of $\mathcal{G}(\underline{\mathcal{L}})$ defined in 3.5. Remark that

$$\mathcal{K}_v \cong \big\{(\alpha, x, l) : \alpha = 1,\ x, l \in 2^v\mathbb{Z}_2^g\big\}$$

via the isomorphism c_A of 3.13. Also

$$\mathcal{G}\big(p_v^*(\mathcal{L})\big) \xleftarrow{\sim} \mathcal{G}(\underline{\mathcal{L}})/\mathcal{K}_v \xrightarrow{\sim} \mathbb{G}_{m,S} \times \frac{1}{2^v}\mathbb{Z}_2^g/2^v\mathbb{Z}_2^g \times \frac{1}{2^v}\mathbb{Z}_2^g/2^v\mathbb{Z}_2^g.$$

Hence, a full theta structure defines for each integer v a finite theta structure on $\big(A, p_v^*(\mathcal{L})\big)$ of type $(2^{2v}, \ldots, 2^{2v}, 2^{2v+1}, \ldots, 2^{2v+1})$.

3.16. Theta characteristics.

Consider the following exact sequence:

$$0 \longrightarrow \mathbb{Z}_2^g \longrightarrow \tfrac{1}{2^v}\mathbb{Z}_2^g \xrightarrow{2^v} \mathbb{Z}_2^g/2^v\mathbb{Z}_2^g \longrightarrow 0.$$

The sequence admits a canonical right splitting h (as sets!)

$$h\colon \mathbb{Z}_2^g/2^v\mathbb{Z}_2^g \longrightarrow \frac{1}{2^v}\mathbb{Z}_2^g.$$

The image via h of the set $\mathbb{Z}_2^g/2^v\mathbb{Z}_2^g$ defines a choice of representatives of the cosets defined by $\frac{1}{2^v}\mathbb{Z}_2^g/\mathbb{Z}_2^g$. Call $\mathbb{Z}_2^g/2^v\mathbb{Z}_2^g \times \mathbb{Z}_2^g/2^v\mathbb{Z}_2^g$ the set of *theta characteristics of order* $\frac{1}{2^v}$. If (a, b) are $1/2^v$-theta characteristics let

$$\delta \begin{bmatrix} a \\ b \end{bmatrix} := \delta \begin{bmatrix} h(a) \\ h(b) \end{bmatrix}.$$

Finally call a $1/2^v$-theta characteristic (a, b) *even* if $\chi\big(h(a), h(b)\big) = 1$.

3.17. *Remark.* — If a and b are chosen theta characteristics and α and β are in $\mathbb{Z}_2^g$, it follows from 3.14

$$\delta \begin{bmatrix} h(a)+\alpha \\ h(b)+\beta \end{bmatrix} = \chi\big(\beta, h(a)\big)\, \delta \begin{bmatrix} a \\ b \end{bmatrix}.$$

The group $\mathcal{G}(p_v^*\mathcal{L})$, defined in 3.5 and identified in 3.15 with triples (α, x, l) with α in $\mathbb{G}_{m,S}$ and (x, l) in $\frac{1}{2^v}\mathbb{Z}_2^g/2^v\mathbb{Z}_2^g$, acts as follows:

$$\begin{aligned}(\alpha, x, l) \circ \vartheta_A \begin{bmatrix} a \\ b \end{bmatrix} \\ = \alpha\, \chi\big(h(b), x\big)\, \chi\big(h(b)+l-h(b+2^v l), h(a-2^v x)\big)\, \vartheta_A \begin{bmatrix} h(a-2^v x) \\ h(b+2^v l) \end{bmatrix}.\end{aligned}$$

3.18. *Remark.* — Let w be the element $\alpha\,\sigma(y)$ of $\mathcal{G}(\underline{\mathcal{L}})$ with y in $V(\underline{A})$ and σ defined as in 3.7. Due to 3.9

$$w \circ \vartheta_A = \alpha\, e_{\mathcal{L}}\Big(\frac{y}{2}, \text{-}\Big)\, \vartheta_A(\text{-} - y).$$

See 3.12 for the definition of ϑ_A. Hence,

$$\Big\{\sigma(y) \circ \vartheta_A \,:\, y \in \frac{1}{2^v} T(A)\Big\} \quad \text{and} \quad \Big\{\vartheta_A \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{Z}_2^g/2^v\mathbb{Z}_2^g\Big\}$$

span the same vector space $\vartheta\Big(\Gamma\big(A, p_v^*(L)\big)\Big)$. The two sets are translations by elements of $\mathcal{G}\big(p_v^*(\mathcal{L})\big)$ of the element $\vartheta_A\binom{0}{0} = \vartheta_A$.

3.19. Proposition. — *Consider two full theta structures $\overline{c}_A^1$ and $\overline{c}_A^2$ on $\big\{(A_n, p_n^*(L))\big\}_n$ as in 3.13. The two corresponding sets*

$$\Big\{\vartheta_A \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{Z}_2^g/2^v\mathbb{Z}_2^g\Big\}$$

are the same if and only the two theta structures of level

$$(2^{2v}, \ldots, 2^{2v}, 2^{2v+1}, \ldots, 2^{2v+1}),$$

induced on $\big(A, p_v^*(\mathcal{L})\big)$ by $\overline{c}_A^1$ and $\overline{c}_A^2$ as in 3.15, are the same.

4. The Schottky-Jung relations.

In this section we deduce the so-called Schottky-Jung relations for smooth curves; see 4.6. We follow the idea sketched in [Mu2]. We assume we are in the situation of §1 with $S = \mathrm{Spec}(K)$ and K is an algebraically closed field.

4.1. The setup for $\mathrm{Pic}^0(\mathcal{C}/S) = \mathcal{X}$.

The notations $\mathcal{X}$ and $\mathcal{L}_{\mathcal{X}}$ are as in 2.3 and in 2.4 with $S = \mathrm{Spec}(K)$. Since $\mathcal{L}_{\mathcal{X}}$ is of degree 1, the associated line bundle has a unique non-zero section $\beta_{\mathcal{X}}$ up to scalar. Choose a rigidification of the associated line bundle $L_{\mathcal{X}}$. We freely use the notation of §3. Fix a full theta structure

$$\overline{c}_{\mathcal{X}}: \big(V(\underline{\mathcal{X}}), e_{\mathcal{L}_{\mathcal{X}}}(\_,\_)\big) \longrightarrow \big(\mathbb{Q}_2^g \times \mathbb{Q}_2^g, e(\_,\_)\big)$$

and an element $\tilde{\eta}$ of $\frac{1}{2}T(\mathcal{X})$ such that

- $\underline{p}_0(\tilde{\eta}) = \eta$;
- $\overline{c}_{\mathcal{X}}(\tilde{\eta}) = (\tilde{\eta}_1, \tilde{\eta}_2) \in \big((0,\dots,0,1)\mathbb{Q}_2, (0,\dots,1)\mathbb{Q}_2\big)$;
- $\underline{p}_1(\tilde{\eta}_1, \tilde{\eta}_2) = h(\eta)$, where h is as in 3.16.

Identify $V(\underline{\mathcal{X}})$ with its image via

$$\begin{array}{ccc} V(\underline{\mathcal{X}}) & \hookrightarrow & V(\underline{\mathcal{X}}) \times V(\underline{\mathcal{X}}) \\ x & \longmapsto & (x, x+\tilde{\eta}). \end{array}$$

We have a surjective morphism

$$\Big(\mathcal{G}(\underline{\mathcal{L}_{\mathcal{X}}}) \times \mathcal{G}(\underline{\mathcal{L}_{\mathcal{X}}})\Big)|_{V(\underline{\mathcal{X}})} \longrightarrow \mathcal{G}\Big(\underline{\mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}})}\Big)$$

defined by

$$(\phi_1, \phi_2) \longmapsto \phi_1 \otimes t_\eta^*(\phi_2),$$

where $|_{V(\underline{\mathcal{X}})}$ stands for the restriction to $V(\underline{\mathcal{X}})$ via the identification above. Consider the abelian variety $\mathcal{X}$ with the $\mathbb{G}_m$-torsor $\mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}})$. Define

1. with the notation of 3.5
$$e_{\mathcal{X}}(\_,\_) := e_{\mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}})}(\_,\_) = e_{\mathcal{L}_{\mathcal{X}}}(\_,\_) e_{\mathcal{L}_{\mathcal{X}}}(\_ + \tilde{\eta}, \_ + \tilde{\eta});$$
2. with the notation of 3.10 and of 3.13
$$e_\star^{\mathcal{X}}(x) := e_\star^{\mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}})}(x) = \chi\big(\overline{c}_{\mathcal{X}}(\tilde{\eta})\big)^2 \chi\big(\overline{c}_{\mathcal{X}}(x)\big)^2 \chi\big(\overline{c}_{\mathcal{X}}(x+\tilde{\eta})\big)^2$$
for any x in $\frac{1}{2}T(\mathcal{X})/T(\mathcal{X})$.

4.2. *Remark.* — Recall that the definition of $e_\star^{\mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}})}$ uses a normalized isomorphism $\xi: \mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}}) \xrightarrow{\sim} [-1]^* \big(\mathcal{L}_{\mathcal{X}} \otimes t_\eta^*(\mathcal{L}_{\mathcal{X}})\big)$. The natural choice is that of

$$\xi := \chi\big(\overline{c}_{\mathcal{X}}(\tilde{\eta})\big)^2 \phi \otimes t_\eta^*(\phi),$$

where $\phi: \mathcal{L}_{\mathcal{X}} \xrightarrow{\sim} [-1]^* (\mathcal{L}_{\mathcal{X}})$ is a normalized isomorphism. Indeed,

$$\xi(0) = \chi\big(\overline{c}_{\mathcal{X}}(\tilde{\eta})\big)^2 \phi(0) \cdot \phi(\eta)$$

is the identity assuring that ξ is normalized.

4.3. The setup for the Prym variety $\mathcal{P}$.

The notations $\mathcal{P}$ and $\mathcal{L}_{\mathcal{P}}$ are as in 2.14 and in 2.15 with $S = \mathrm{Spec}(K)$. The same procedure of 4.1 is applied to $\mathcal{L}_{\mathcal{P}}$ and $\beta_{\mathcal{P}}$ and $\bar{c}_{\mathcal{P}}$. Considering the abelian variety $\mathcal{P}$ with the line bundle $\mathcal{L}^2_{\mathcal{P}}$, define:

1. $e_{\mathcal{P}}(\_,\_) := e_{\mathcal{L}^2_{\mathcal{P}}}(\_,\_) = e_{\mathcal{L}_{\mathcal{P}}}(\_,\_)e_{\mathcal{L}_{\mathcal{P}}}(\_,\_)$;
2. for any x in $\frac{1}{2}T(\mathcal{P})/T(\mathcal{P})$
$$e^{\mathcal{P}}_{\star}(x) := e^{\mathcal{L}^2_{\mathcal{P}}}_{\star}(x) = e^{\mathcal{L}_{\mathcal{P}}}_{\star}(x)\, e^{\mathcal{L}_{\mathcal{P}}}_{\star}(x) = \chi\big(\bar{c}_{\mathcal{P}}(x)\big)^4.$$

4.4. The setup for $\mathcal{Y} = \mathrm{Pic}^0(\widetilde{\mathcal{C}}/S)$.

The notations $\mathcal{Y}$ and $\mathcal{L}_{\mathcal{Y}}$ are as in 2.3 and in 2.4 with $S = \mathrm{Spec}(K)$. By 2.15, the abelian variety $\mathcal{Y}$ is a quotient of $\mathcal{X} \times \mathcal{P}$ and the $\mathbb{G}_m$-torsor $\mathcal{L}_{\mathcal{Y}}$ is $\big(\mathcal{L}_{\mathcal{X}} \otimes t^{\star}_{\eta}(\mathcal{L}_{\mathcal{X}})\big) \otimes \mathcal{L}^2_{\mathcal{P}}$ descended to $\mathcal{Y}$. Hence,

1.a) $V(\underline{\mathcal{Y}}) = V(\underline{\mathcal{X}}) \times V(\underline{\mathcal{P}})$;

1.b) the bilinear form $e_{\mathcal{L}_{\mathcal{Y}}}$ on $V(\underline{\mathcal{Y}})$ is defined by
$$e_{\mathcal{L}_{\mathcal{Y}}}\big((x,a),(y,b)\big) := e_{\mathcal{X}}(x,a)\; e_{\mathcal{P}}(y,b)$$
for all (x,y) and (a,b) in $V(\underline{\mathcal{X}}) \times V(\underline{\mathcal{P}})$.

There is a unique monomorphism
$$\psi\colon \frac{1}{2}T(\mathcal{P}) \longrightarrow \frac{1}{2}T(\mathcal{X})$$
such that

- $\bar{c}_{\mathcal{X}}\big(\psi(1/2T(\mathcal{P}))\big)$ is contained in the $\mathbb{Q}_2$-vector space $\mathbb{Q}_2^{g-1} \times \{0\} \times \mathbb{Q}_2^{g-1} \times \{0\}$;
- the restriction of $\bar{c}_{\mathcal{X}}$ to
$$(\psi \otimes \mathbb{Q}_2)\big(V(\underline{\mathcal{P}})\big) \xrightarrow{\sim} \mathbb{Q}_2^{g-1} \times \{0\} \times \mathbb{Q}_2^{g-1} \times \{0\} \cong \mathbb{Q}_2^{g-1} \times \mathbb{Q}_2^{g-1}$$
preserves the symplectic structures and coincides with $\bar{c}_{\mathcal{P}}$;
- the induced isomorphism
$$\mathcal{P}[2] \longleftarrow \underline{p}_0 \frac{1}{2}T(\mathcal{P})/T(\mathcal{P}) \xrightarrow{\sim} \tilde{\eta}^{\perp}/\tilde{\eta} \xleftarrow{\sim} H^{\perp}/H$$
coincides with the morphism defined in (5) of 2.15. Here, $\tilde{\eta}^{\perp}$ stands for the orthogonal in $V(\underline{\mathcal{X}})$ with respect to the pairing $e_{\mathcal{X}}$.

Then

2.a) $T(\mathcal{Y}) = \big(\frac{1}{2}\Delta\big(T(\mathcal{P})\big) \times \mathbb{Z}_2\tilde{\eta}\big) + \big(\tilde{\eta}^{\perp} \times T(\mathcal{P})\big)$, where Δ is $\psi \times \mathrm{id}$. This follows from 2.15;

2.b) with the identification above the map $e_\star^{\mathcal{L}_\mathcal{Y}}$ on
$$\frac{1}{2}T(\mathcal{Y}) \hookrightarrow \overline{c}_\mathcal{X}(V(\underline{\mathcal{X}})) \times \overline{c}_\mathcal{P}(V(\underline{\mathcal{P}})),$$
identified with its image, is defined by
$$\big((a,x),(b,y)\big) \longrightarrow \chi(\tilde{\eta}_1,\tilde{\eta}_2)^2\,\chi(a,x)^2\,\chi(a+\tilde{\eta}_1, x+\tilde{\eta}_2)^2\,\chi(b,y)^4.$$

4.5. Theta functions on $\mathrm{Pic}^0(\mathcal{C}/S)$, on $\mathrm{Pic}^0(\widetilde{\mathcal{C}}/S)$ and on the Prym variety $\mathcal{P}$.

Choose a non-zero section $\beta_\mathcal{Y}$ of $\mathcal{L}_\mathcal{Y}$. As in 3.12 we write $\vartheta_\mathcal{Y}$ instead of $\vartheta_{\beta_\mathcal{Y}}$. We use similar notations for the section $\beta_\mathcal{X}$ defined in 4.1 and the section $\beta_\mathcal{P}$ defined in 4.3.

By 2.15 and 3.11, there is a non-zero constant d_1 such that for any a in $V(\underline{\mathcal{P}})$ we have
$$\vartheta_\mathcal{Y}(a) = d_1\vartheta_\mathcal{P}(a)^2.$$
Analogously, there is a non-zero constant d_2 such that for any b in $V(\underline{\mathcal{X}})$ we have
$$\vartheta_\mathcal{Y}(b) = d_2\,\vartheta_\mathcal{X}(b)\,\vartheta_\mathcal{X}(b+\tilde{\eta}).$$
Equivalently, due to 3.14, we have
$$\vartheta_\mathcal{Y}\begin{bmatrix}0\\0\end{bmatrix}|_{V(\underline{\mathbb{P}})}(\_) = d_1\,\vartheta_\mathbb{P}\begin{bmatrix}0\\0\end{bmatrix}^2(\_)$$
and
$$\vartheta_\mathcal{Y}\begin{bmatrix}0\\0\end{bmatrix}|_{V(\underline{\mathcal{X}})}(\_) = d_1\,\vartheta_\mathcal{X}\begin{bmatrix}0\\0\end{bmatrix}(\_)\,\vartheta_\mathcal{X}\begin{bmatrix}0\\0\end{bmatrix}(\_+\tilde{\eta}).$$
Let c be an element of $(1/2)T(\mathcal{P})$ and let $\overline{c}_P(c) = (\alpha,\beta) \in \mathbb{Q}_2^{g-1} \times \mathbb{Q}_2^{g-1}$. Define $c' := \psi(c) \in (1/2)T(\mathcal{X})$. Then
$$\vartheta_\mathcal{Y}(-c) = d_2\,\vartheta_\mathcal{P}(-c)^2 \quad \text{and} \quad \vartheta_\mathcal{Y}(-c') = d_1\,\vartheta_\mathcal{X}(-c')\,\vartheta_\mathcal{X}(-c'+\tilde{\eta}).$$
By 3.15, $\big(1, c-c'\big) \circ \vartheta_\mathcal{Y} = \vartheta_\mathcal{Y}$. Hence
$$\vartheta_\mathcal{Y}(-c') = \big(\big(1,c-c'\big)\circ\vartheta_\mathcal{Y}\big)(-c'). \tag{4.5.1}$$
By 3.13
$$\big(1,c-c'\big) = \left(e_\star^{\mathcal{L}_\mathcal{Y}}\left(\frac{c-c'}{2}\right)\right)\sigma(c-c')$$

and

$$e_\star^{\mathcal{L}_\mathcal{Y}}\left(\frac{c-c'}{2}\right) = \chi(\tilde\eta_1,\tilde\eta_2)^4\chi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)^8 = 1.$$

Hence, by 3.9,

$$\big((1, c-c')\circ\vartheta_\mathcal{Y}\big)(-c') = e_{\mathcal{L}_\mathcal{Y}}\left(\frac{c-c'}{2}, -c'\right)\vartheta_\mathcal{Y}(-c). \tag{4.5.2}$$

Since

$$e_{\mathcal{L}_\mathcal{Y}}\left(\frac{c-c'}{2}, -c'\right) = e_\mathcal{X}\left(\frac{c'}{2}, c'\right) = 1,$$

we conclude by (4.5.1) and by (4.5.2) that

$$\vartheta_\mathcal{Y}(-c') = \vartheta_\mathcal{Y}(-c). \tag{4.5.3}$$

On the other hand we have

$$\begin{aligned}
&\text{by (3.10)} && \big((1,c')\circ\vartheta_\mathcal{X}\big)(0) = e_\star^{\mathcal{L}_\mathcal{X}}(c')\big(\sigma(c')\circ\vartheta_\mathcal{X}\big)(0)\\
&\text{by (3.13)} && \qquad = \chi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)^2\big(\sigma(c')\circ\vartheta_\mathcal{X}\big)(0)\\
&\text{by (3.9)} && \qquad = \chi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)^2\vartheta_\mathcal{X}(-c').
\end{aligned}$$

Analogously

$$\big((1, c'-\tilde\eta)\circ\vartheta_\mathcal{X}\big)(0) = \chi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)^2\chi\left(\frac{\tilde\eta_1}{2},\frac{\tilde\eta_2}{2}\right)^2\vartheta_\mathcal{X}(-c'+\tilde\eta)$$

and

$$\big((1,c)\circ\vartheta_\mathbb{P}\big)(0) = \chi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)^2\vartheta_\mathbb{P}(-c).$$

Hence, with the notation of 3.14, we get the following:

4.6. PROPOSITION. — *Let $\left[\begin{smallmatrix} a \\ b\end{smallmatrix}\right]$ be the set of $\frac{1}{2}$-theta characteristics of $\mathcal{P}$ as defined in 3.16. Identify them with $\frac{1}{2}$-theta characteristics $\left[\begin{smallmatrix} a & 0 \\ b & 0\end{smallmatrix}\right]$ on $\mathcal{X}$ via the map ψ defined in 4.4. Then*

$$\vartheta_\mathcal{X}\begin{bmatrix} a & 0\\ b & 0\end{bmatrix}(0)\,\vartheta_\mathcal{X}\left[\begin{pmatrix} a & 0\\ b & 0\end{pmatrix} + h(\eta)\right](0) = \frac{d_1}{d_2}\chi\left(\frac{\tilde\eta_1}{2},\frac{\tilde\eta_2}{2}\right)^2\vartheta_\mathcal{P}\begin{bmatrix} a\\ b\end{bmatrix}(0)^2.$$

4.7. COROLLARY. — *We have the following commutative diagram:*

$$\begin{array}{ccc}
\mathfrak{M}_{g,H}^{(4,8)} & \longrightarrow & \mathcal{A}_g^{(4,8),H}\\
\big\downarrow & & \big\downarrow{\scriptstyle \gamma_g^H}\\
\mathcal{A}_{g-1}^{(2,4)} & \xrightarrow{\vartheta_{2,g-1}} & \mathbb{P}^{N_{g-1}}.
\end{array}$$

Here

- $\mathbb{P}^{N_{g-1}}$ *is the projective space in the coordinates* $R_{\binom{a}{b}}$, *where* (a,b) *ranges among all the* $1/2$*-theta characteristics satisfying* $\chi(a,b)=1$;
- H *in* $\mathfrak{M}_{g,H}^{(4,8)}$ *stands for the Hurwitz data* $\pi\colon\widetilde{C}\to C$ *given by fixing the 2-torsion point* η *defined in 2.1;*
- H *in* $\mathcal{A}_g^{(4,8),H}$ *stands for the choice of the theta characteristic appearing in the definition of* γ_g^H;
- $\gamma_g^H\big([X]\big)=\Big(\dots,\vartheta_X\begin{bmatrix} a & 0\\ b & 0\end{bmatrix}(0)\,\vartheta_X\Big[\begin{pmatrix} a & 0\\ b & 0\end{pmatrix}+h(\eta)\Big](0),\dots\Big)$;
- $\vartheta_{2,g-1}([P])=\Big(\dots,\vartheta_P\begin{bmatrix} a\\ b\end{bmatrix}(0)^2,\dots\Big)$.

Proof. — Only the level structures remain to be justified. The use of the (4,8)-level structures is a consequence of 3.19. The (2,4)-level structure is motivated by the following remark.

4.8. *Remark.* — Let P be a g dimensional abelian variety and $\mathcal{L}$ an ample, degree 1 line bundle on P. We have a surjective homomorphism

$$\mathcal{G}(\underline{\mathcal{L}})\times\mathcal{G}(\underline{\mathcal{L}})\longrightarrow\mathcal{G}(\underline{\mathcal{L}}^2)$$

defined by $(\phi_1,\phi_2)\mapsto\phi_1\otimes\phi_2$.

Let $\bar{c}_P : V(\underline{P})\to\mathbf{Q}_2^{2g}$ be a full that structure on $(P,\mathcal{L})$. The vector space spanned by the theta functions

$$\left\{\vartheta_P\begin{bmatrix} x\\ l\end{bmatrix}^2 \;\Big|\; (x,l)\in\frac{1}{2}\mathbb{Z}_2^{2g}\right\}$$

is an irreductible weight one representation of the group $\mathcal{G}(\underline{\mathcal{L}}^2)$. As in 3.19, such representation defines a level (2,4)-theta structure on $(P,\mathcal{L})$.

Conversely given a level (2,4)-theta structure on $(P,\mathcal{L})$ the above set of theta functions is canonically defined.

5. Fourier-Jacobi expansion of the Schottky-Jung relations.

In this section we study the Schottky-Jung relations defined in 4.6 induced to the boundary. See the corollaries of 5.5, where we study the

Fourier-Jacobi expansion of theta functions. An account on Fourier-Jacobi or q-expansions can be found in [Ch], §III2, pp. 143–149, with the warning that we use a basis for the theta functions which is different from Chai's one. This forces us to go through the theory to see what happens in our case.

5.1. A review of some results of Chai.

Let ξ be a rational polyhedral cone contained in $\mathrm{Symm}^2(\mathbb{Z}^r)\otimes\mathbb{R}$; see [Ch], §III.2.4, pp. 146–147. Let v be a non-negative integer. Let $E := R[\![q^a]\!]$, where

- R is the universal deformation space of a principally polarized abelian variety with $(2^{2v},\dots,2^{2v},2^{2v+1},\dots,2^{2v+1})$-theta structure and r marked sections;
- a ranges among the quadratic forms on $(2^{-v}\mathbb{Z}^r)$ so that
  - the associated bilinear form lies in ξ;
  - the element $q_{i,i}^{\frac{1}{2^v}} := q^{\frac{e_i}{2^v}\otimes\frac{e_i}{2^v}}$, belongs to the maximal ideal of E;
  - the element $q_{i,j}^{\frac{1}{2^v}} = q_{j,i}^{\frac{1}{2^v}} := q^{\frac{e_i+e_j}{2^v}\otimes\frac{e_i+e_j}{2^v}}$ if $i\neq j$, is in the maximal ideal of E;

(Denote by $q^{z\otimes z}$ the associated bilinear form on $2^{-v}\mathbb{Z}^r$ with coefficients in E. It is is uniquely defined in terms of the elements $q_{i,j}^{\frac{1}{2^v}}$.)

Then, E is a noetherian excellent domain, complete with respect to an ideal I and with fraction field K. If v is positive, $\mathrm{Spf}(E)$ appears as a "chart" at the boundary of any toroidal compactification of the moduli stacks of abelian varieties with theta structure of level $(2^{2v},\dots,2^{2v},2^{2v+1},\dots,2^{2v+1})$ whose associated cone decomposition contains ξ.

5.1.1. *The degeneration data.*

Over $S=\mathrm{Spec}(E)$ we have, as in [Ch], §III.2, p. 143:

1. an abelian scheme B and a principal polarization $\lambda\colon B\to B^t$;
2. an extension G of B by a split torus T of rank r and character group $\mathbb{X}(T)$. Denote by $\pi\colon G\to B$ the projection map. Let
$$j\colon \mathbb{X}(T)\longrightarrow B$$
be the morphism such that $\lambda\circ j\colon\mathbb{X}(T)\to B^t$ defines G as an extension of B by T;

3. an isomorphism $Y = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r \cong \mathbb{X}(T)$ as constant group schemes over S and a group homomorphism
$$\iota\colon \frac{1}{2^v} Y_K \longrightarrow G(K)$$
such that the following diagram commutes:
$$\begin{array}{ccc} Y & \xrightarrow{\iota} & G(K) \\ \downarrow \wr & & \downarrow \pi \\ \mathbb{X}(T) & \xrightarrow{j} & B(S) = B(K); \end{array}$$
4. an ample line bundle L_B on B, rigidified above the identity element of B, inducing the polarization $-\lambda$.
5. a theta structure of level $(2^{2v}, \ldots, 2^{2v}, 2^{2v+1}, \ldots, 2^{2v+1})$ on $[2^v]^*(L_B)$;
6. a symmetric ample sheaf data $\alpha\colon \frac{1}{2^v}Y \to j^*\big(L_B\big)^{-1}(S)$ with respect to ι.

5.1.2. *The quotient.*

As in loc. cit., we use Mumford's construction to get the couple (X, L_X) consisting of

i. the semiabelian scheme X, "quotient" of G by Y;
ii. the ample symmetric line bundle L_X over X, "quotient" of $L_G := \pi^*\big(L_B\big)$ by Y with
   - a rigidification $\varphi_0\colon L_X \xrightarrow{\sim} \mathbb{A}^1_S$ above the identity element of X
   - a $(2^{2v}, \ldots, 2^{2v}, 2^{2v+1}, \ldots, 2^{2v+1})$-theta level structure on $[2^v]^*(L_X)$ over the non-empty open subset U of S where X is an abelian scheme.

Denote by $\mathcal{L}_B$ (resp. $\mathcal{L}_X$) the cubical $\mathbb{G}_m$-torsors on G (respectively X) associated to L_B (resp. L_X); see 2.8 for this notion.

5.1.3. *Properties of Mumford's uniformization.*

For later purposes we note the following. Define
$$G \longrightarrow G_i$$
as the push-forward of G by the character e_i for every $i = 1, \ldots, r$. Let $\iota_i\colon Y \to G_i(K)$ be the composition of ι and the natural map $G \to G_i$. Then,

- $G \times_S \partial S \cong X \times_S \partial S$, where ∂S is the complement of the scheme theoretic image of $\operatorname{Spec}(R[q^{a\otimes a}]_{a\in\mathbb{Z}^r}) \to S$;
- the divisor associated to the section $\iota_i(e_i)$ of the $\mathbb{G}_{m,S}$-bundle $j^*(G_i)$ coincides with the divisor $(q_{i,i}^2)$;
- if $i \neq j$, the divisor associated to the section $\iota_i(e_j)$ of the $\mathbb{G}_{m,S}$-bundle $j^*(G_i)$ coincides with the divisor $(q_{i,j}\, q_{i,i}^{-1}\, q_{j,j}^{-1})$.

5.2. Example: the case of Jacobians.

As an example, let $\mathcal{C}_0$ be an irreducible semistable curve over $k = \bar{k}$ with singular points $Q_1, \ldots, Q_r$. Let

- $V = \operatorname{Spec}(A)$ be the universal deformation space of $\mathcal{C}_0$;
- $\mathcal{C} \to V$ be the (algebraization of) the universal (formal) curve over V;
- for each $i = 1, \ldots, r$,
$$\mathcal{C}^{\setminus Q_i} \cong A[\![x, y]\!]/(xy - q_i)$$
be the completion of $\mathcal{C}$ at Q_i;
- $V' := \operatorname{Spec}(A/(q_1, \ldots, q_r))$;
- $\mathbb{D}_i$ (resp. $\mathbb{D}$) be the desingularization of $\mathcal{C}' := \mathcal{C} \times_V V'$ along the singular section of $\mathcal{C}_i$ extending Q_i for every $i = 1, \ldots, r$ (resp. $Q_1, \ldots, Q_r$);
- P_i and R_i be the sections of $\mathbb{D}_i$ over Q_i for every i;
- $Q\colon V \to \mathcal{C}$ be a smooth section;
- $X := \operatorname{Pic}^0(\mathcal{C}/V)$;
- L_X be the line bundle associated to the theta divisor on X i.e., the translation of the image of $\mathcal{C}^{g-1}$ in $\operatorname{Pic}^{g-1}(\mathcal{C}/S)$ by $-(g-1)Q$.

By [FC] the couple (X, L_X) can be "uniformized" i.e., obtained by pulling-back via $V \to S$ (more precisely, to a cover of V) Mumford's construction described in 5.1 for a suitable S (choosing a $(2^{2v}, \ldots, 2^{2v}, 2^{2v+1}, \ldots, 2^{2v+1})$-theta level structure on $[2^v]^*(L_X)$).

5.2.1. *Properties.*

The following is known [An]:

1. $G \times_V V' \cong \operatorname{Pic}^0(\mathcal{C}'/V')$;
2. $B \times_S V' \cong \operatorname{Pic}^0(\mathcal{D}/V')$;

3. $L_B \times_S V'$ is the line bundle associated to the theta divisor of $\mathrm{Pic}^0(\mathcal{D}/V')$ defined translating the image of the $(g-r-1)$th symmetric power of $\mathcal{D}$ in $\mathrm{Pic}^{g-r-1}(\mathcal{D}/V')$ by the point $-(g-r-1)Q \times_V V'$;
4. the character group $\mathbb{X}(T)$ is canonically isomorphic to $\mathbb{Z}Q_1 \oplus \cdots \oplus \mathbb{Z}Q_r$;
5. the map $j \times_S V'$ is defined (up to a non canonical choice of sign) by sending Q_i to $P_i - R_i$;
6. the divisor of V associated to the section $\iota_i(Q_i)$ of the $\mathbb{G}_m$-bundle $j^*(G_i)$ coincides with the divisor $q_i = 0$ with multiplicity 1;
7. if $i \neq j$, the element $q_{i,j}\, q_{i,i}^{-1}\, q_{j,j}^{-1}$ is invertible in V;
8. $G_i \times_S V' \cong \mathrm{Pic}^0(\mathcal{D}_i/V')$.

In particular, it follows from 5.1 that

6'. the element $\iota_i(Q_i)$ extends to an element of $G_i\left(V(q_i^{-1})\right)$;

7'. if $i \neq j$, the element $\iota_i(Q_j)$ extends to an element of $G_i(V)$.

Finally,

9. $\iota_i(Q_j) \times_V V'$ coincides with the difference of P_j and R_j (as sections of $\mathcal{D}_i \to V'$).

5.3. Relations between 2-divisible groups on G and on X via Mumford's construction.

We use the notations of 3.3 and of 5.1. As explained in [Ch], § III.5, pp. 69–70, we have, for any non-negative integer n, exact sequences:

$$0 \longrightarrow G[2^n] \longrightarrow X[2^n]$$

and

$$0 \longrightarrow G[2^n]_K \longrightarrow X[2^n]_K \longrightarrow \left(\mathbb{Z}\left[\frac{1}{2^n}\right]/\mathbb{Z}\right) \otimes_{\mathbb{Z}} Y_K \longrightarrow 0.$$

We have also the exact sequence

$$0 \longrightarrow T[2^n] \longrightarrow G[2^n] \longrightarrow B[2^n] \longrightarrow 0.$$

In the notations of 3.3 we get exact sequences:

$$0 \longrightarrow V(\underline{G}) \longrightarrow V(\underline{X})$$

and

$$0 \longrightarrow V(\underline{G}_K) \longrightarrow V(\underline{X}_K) \longrightarrow \mathbb{Q}_2 \otimes_{\mathbb{Z}} Y_K \longrightarrow 0$$

and

$$0 \longrightarrow V(\underline{T}) \longrightarrow V(\underline{G}) \longrightarrow V(\underline{B}) \longrightarrow 0.$$

By [MB], Theorem IV.2.4(iv) and (v), the morphisms appearing in these exact sequences are compatible with the symplectic pairings defined in 3.5. Define the étale sheaf

$$V(\underline{Y}_K) := \mathbb{Q}_2 \otimes_{\mathbb{Z}} Y_K.$$

Analogous sequences exist for the infinite theta groups associated to $\mathcal{L}_X$ over X, to $\mathcal{L}_B$ over B, to $\pi^*(\mathcal{L})$ over G and to the restriction of $\pi^*(\mathcal{L})$ to T.

For $i = 1, \ldots, 2g$, denote by e_i the standard vector of $\mathbb{Q}_2^g \times \mathbb{Q}_2^g$ with 1 at the ith component and zeroes at the other components. *Choose* a full theta structure

$$\bar{c}_{X_{\bar{K}}} \colon V(\underline{X}_{\bar{K}}) \longrightarrow \mathbb{Q}_2^g \times \mathbb{Q}_2^g$$

as in 3.13 such that

1) $\bar{c}_{T_{\bar{K}}} := \bar{c}_{X_{\bar{K}}}|_{\underline{T}_{\bar{K}}}$ identifies $V(\underline{T}_{\bar{K}}) \xrightarrow{\sim} \mathbb{Q}_2 \cdot e_{2g-r+1} \oplus \cdots \oplus \mathbb{Q}_2 \cdot e_{2g} \cong \mathbb{Q}_2^r$;
2) $\bar{c}_{Y_{\bar{K}}} := \bar{c}_{X_{\bar{K}}}|_{\underline{Y}_{\bar{K}}}$ identifies $V(\underline{Y}_{\bar{K}}) \xrightarrow{\sim} \mathbb{Q}_2 \cdot e_{g-r+1} \oplus \cdots \oplus \mathbb{Q}_2 \cdot e_g \cong \mathbb{Q}_2^r$;
3) $\bar{c}_{B_{\bar{K}}} := \bar{c}_{X_{\bar{K}}}|_{\underline{B}_{\bar{K}}}$ identifies $V(\underline{B}_{\bar{K}}) \xrightarrow{\sim} \mathbb{Q}_2 \cdot e_1 \oplus \cdots \oplus \mathbb{Q}_2 \cdot e_{g-r} \oplus \mathbb{Q}_2 \cdot e_{g+1} \oplus \cdots \oplus \mathbb{Q}_2 \cdot e_{2g-r} \cong \mathbb{Q}_2^{2g-2r}$.

In particular, this determines a splitting

$$V(\underline{X}_{\bar{K}}) \xrightarrow{\sim} V(\underline{B}_{\bar{K}}) \times V(\underline{Y}_{\bar{K}}) \times V(\underline{T}_{\bar{K}}).$$

Also, we get that any $1/2^n$-theta characteristic $(a'', b'') \in \mathbb{Z}_2^g/2^n\mathbb{Z}_2^g \times \mathbb{Z}_2^g/2^n\mathbb{Z}_2^g$ of $X_{\bar{K}}$ factors accordingly as

$$\big((a', a), (b', b)\big) \in \mathbb{Z}_2^{g-r}/2^n\mathbb{Z}_2^{g-r} \times \mathbb{Z}_2^r/2^n\mathbb{Z}_2^r \times \mathbb{Z}_2^{g-r}/2^n\mathbb{Z}_2^{g-r} \times \mathbb{Z}_2^r/2^n\mathbb{Z}_2^r.$$

5.4. DEFINITION. — *Let μ be an element of $V(\underline{X}_{\bar{K}})$. Call it a vanishing cycle if it belongs to $V(\underline{Y}_{\bar{K}})$. Call it orthogonal to a vanishing cycle if it belongs to $V(\underline{T}_{\bar{K}})$.*

5.5. The Fourier-Jacobi expansion of theta functions via Mumford's construction.

The notation is as in 5.1. By [Ch], Lemma III.2.3.1 and [FC], Theorem III.6.2, there is a 1-1 correspondence between

- the couples consisting of a section t of the line bundle $[2^v]^*(L_B)$ and an element $\bar{y}$ of $\frac{1}{2^v}Y/2^vY$
- the sections $\vartheta_{t,\bar{y}}$ of $[2^v]^*(L_X)$.

Similarly, one gets a description of the theta group of $[2^v]^*(L_{X_{\bar K}})$ (resp. its action on global sections) in terms of the theta group of $[2^v]^*(L_{B_{\bar K}})$ (resp. its action on the global sections). Define the isomorphism ϕ as the composite

$$\phi: \mathcal{G}([2^v]^*\mathcal{L}_{X_{\bar K}}) \xrightarrow{\sim} \mathbb{G}_{m,\bar K} \times B_{\bar K}[2^{2v}] \times (Y/2^{2v}Y) \times T_{\bar K}[2^{2v}]$$
$$\xrightarrow{\sim} \mathbb{G}_{m,\bar K} \times \frac{1}{2^v}\mathbb{Z}_2^{g-r}/2^v\mathbb{Z}_2^{g-r} \times \frac{1}{2^v}\mathbb{Z}_2^r/2^v\mathbb{Z}_2^r \times \frac{1}{2^v}\mathbb{Z}_2^{g-r}/2^v\mathbb{Z}_2^{g-r} \times \frac{1}{2^v}\mathbb{Z}_2^r/2^v\mathbb{Z}_2^r.$$

Let $w'' := (\alpha, a', a, b', b)$ be an element of $[2^v]^*\mathcal{G}(\mathcal{L}_{X_{\bar K}})$. Let $w = (\alpha, a', b')$ be the image of w'' via the map $\mathcal{G}([2^v]^*\mathcal{L}_{X_{\bar K}}) \to \mathcal{G}([2^v]^*\mathcal{L}_{B_{\bar K}})$ defined in 5.3. Let t be a section of $[2^v]^*(L_{B_{\bar K}})$ and let $\bar y$ be an element of $(2^{-v}Y)/2^vY$. Define $\vartheta_{t,\bar y}$ as the associated section of $L_{X_{\bar K}}$. Then,

$$w'' \circ \vartheta_{t,\bar y} = \chi(b, \bar y)\, \vartheta_{w \circ t, \bar y - \bar a}. \tag{5.5.1}$$

See 3.13 for the definition of $\chi(\_,\_)$.

In [Ch], Lemma III.2.3.1 it is proven that Fourier-Jacobi expansion of the value of $\vartheta_{t,\bar y}$ at the rigidification of L_X above the identity element of X is

$$\vartheta_{t,\bar y}(0) = \bigoplus_{2^{-v}Y \ni z \equiv \bar y (\mathrm{mod}\ 2^{<} vY)} t(j(z)) q^{z\otimes z}.$$

In particular, we get a natural action of the theta group $\mathcal{G}(\mathcal{L}_{X_K})$ on q-expansions

$$\begin{aligned} w'' \circ \Big(\bigoplus_{z\equiv \bar y} t(j(z)) q^{z\otimes z}\Big) &:= (w'' \circ \vartheta_{t,\bar y})(0) \\ &= \chi(b, \bar y)\, \vartheta_{w\circ t, \bar y - \bar a}(0) \\ &= \chi(b, \bar y) \bigoplus_{z \equiv \bar y - \bar a} (w \circ t)\,(j(z)) q^{z\otimes z}. \end{aligned} \tag{5.5.2}$$

5.6. Relations between theta functions on G and on X via Mumford's construction.

Let β_{B_K} be the unique non-zero section (up to scalar) of L_{B_K}. Let (a'', b'') be a $1/2^v$-theta characteristic of $(X_{\bar K}, L_{X_{\bar K}})$ (3.16). Factor $a'' = (a', a)$ and $b'' = (b', b)$ as in 5.3. Then,

$$\begin{aligned} \vartheta_{X_K}\begin{bmatrix} a'' \\ b'' \end{bmatrix}(0) &= (1, -h(a''), h(b'')) \circ \Big(\vartheta_{X_K}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\Big)(0) \\ &= \phi^{-1}(1, -h(a''), h(b'')) \circ [2^v]^*(\vartheta_{\beta_{B_K},0})(0) \\ &= \phi^{-1}\big(1, -(h(a'), h(a)), (h(b'), h(b))\big) \circ \big(\vartheta_{[2^v]^*(\beta_{B_K}),0}\big)(0) \end{aligned}$$

$$= \phi^{-1}\left(1, -\big(h(a'), h(a)\big), \big(h(b'), h(b)\big)\right)$$
$$\circ \left(\bigoplus_{z\in Y} [2^v]^*\left(\beta_{B_K}\right)\big(j(z)\big)\, q^{z\otimes z}\right)$$
$$= \bigoplus_{Y\ni z\equiv h(a)} \chi\big(h(b), z-h(a)\big)\,\Big(\big(1, -h(a'), h(b')\big)$$

by (5.5.2)
$$\circ\, [2^v]^*\left(\beta_{B_K}\right)\Big)\big(j(z)\big)\, q^{z\otimes z}$$
$$= \bigoplus_{Y\ni z\equiv h(a)} \chi\big(h(b), z-h(a)\big)\, \vartheta_{B_{\bar K}}\begin{bmatrix} a' \\ b' \end{bmatrix}\Big(j(z)\Big)\, q^{z\otimes z}.$$

In conclusion we get the formula

(5.6.1) $$\vartheta_{X_{\bar K}}\begin{bmatrix} a'' \\ b'' \end{bmatrix}(0) = \bigoplus_{Y\ni z\equiv h(a)} \chi\big(h(b), z-h(a)\big)\vartheta_{B_{\bar K}}\begin{bmatrix} a' \\ b' \end{bmatrix}\big(j(z)\big)\, q^{z\otimes z}.$$

5.7. Corollary (cf. [vG], Lemma 2.4). — *Suppose that* $Y = \mathbb{Z}y$. *In particular,* $y \in V(\underline{Y}_{\bar K})$ *is a vanishing cycle in the sense of 5.4. Let* $a'' = (a', 0)$ *and* $b'' = (b', \delta)$ *with* $\delta = 0, 1$ *and* $\chi(a'', b'') = 1$. *Then*

$$\vartheta_{X_{\bar K}}\begin{bmatrix} a' & 0 \\ b' & \delta \end{bmatrix}(0) = \vartheta_{B_{\bar K}}\begin{bmatrix} a' \\ b' \end{bmatrix}(0) + 2\bigoplus_{n=1}^{\infty}(-1)^{\delta n}\,\vartheta_{B_{\bar K}}\begin{bmatrix} a' \\ b' \end{bmatrix}\big(n\, j(y)\big)\, q^{n^2}.$$

5.8. Corollary (cf. [vG], Proof of Lemma 2.5, [Do1], Proof of Lemma 3.1.1). — *From 5.7, using the same notations, we have the following equality* mod q^4*:*

$$\vartheta_{X_{\bar K}}\begin{bmatrix} a' & 0 \\ b' & 1 \end{bmatrix}(0)\,\vartheta_{X_{\bar K}}\begin{bmatrix} a' & 0 \\ b' & 0 \end{bmatrix}(0) = \vartheta_{B_{\bar K}}\begin{bmatrix} a' \\ b' \end{bmatrix}(0) - 4\,\vartheta_{B_{\bar K}}\begin{bmatrix} a' \\ b' \end{bmatrix}\big(j(y)\big)q^2.$$

5.9. Corollary (cf. [Do1], Proof of Corollary 3.1.3). — *Let* $Y = \mathbb{Z}y$. *Let* $a' = (\epsilon, \epsilon')$ *and* $b' = (\tau, \tau')$ *such that* $\epsilon' = 0, 1$ *and* $\tau' = 0, 1$. *Suppose that* $\chi(a'', b'') = 1$. *Then*

$$\vartheta_{X_{\bar K}}\begin{bmatrix} \epsilon & 0 & a \\ \tau & 0 & b \end{bmatrix}(0)\,\vartheta_{X_{\bar K}}\begin{bmatrix} \epsilon & 0 & a \\ \tau & 1 & b \end{bmatrix}(0)$$
$$= \begin{cases} \vartheta_{B_{\bar K}}\begin{bmatrix} \epsilon & 0 \\ \tau & 0 \end{bmatrix}(0)\,\vartheta_{B_{\bar K}}\begin{bmatrix} \epsilon & 0 \\ \tau & 1 \end{bmatrix}(0) + O(q) & \text{if a=0;} \\ 4\,\vartheta_{B_{\bar K}}\begin{bmatrix} \epsilon & 0 \\ \tau & 0 \end{bmatrix}\left(\tfrac{1}{2}\big(j(y)\big)\right)\vartheta_{B_{\bar K}}\begin{bmatrix} \epsilon & 0 \\ \tau & 1 \end{bmatrix}\Big(\tfrac{1}{2}\, j(y)\Big)q^{\frac{1}{2}} + O(q^2) & \text{if a=1;} \end{cases}$$

5.10. Corollary (cf. [Do1] Proof of Corollary 3.1.2). — *Suppose that* $Y = \mathbb{Z}y$. *Let* $a'' = (a', a)$ *and* $b'' = (b', b)$ *with* $\chi(a'', b'') = 1$. *Then*

$$\vartheta_{X_{\bar K}}\begin{bmatrix} a'' \\ b'' \end{bmatrix}^2(0)$$

$$= \begin{cases} \vartheta_{B_{\bar K}} \begin{bmatrix} a' \\ b' \end{bmatrix}^2 (0) + 4\,(-1)^b\, \vartheta_{B_{\bar K}} \begin{bmatrix} a' \\ b' \end{bmatrix} (0)\, \vartheta_{B_{\bar K}} \begin{bmatrix} a' \\ b' \end{bmatrix} \\ \qquad\qquad\qquad\qquad (j(y))\, q^2 + O(q^4) & \text{if } a = 0; \\ 4\,\vartheta_{B_{\bar K}} \begin{bmatrix} a' \\ b' \end{bmatrix}^2 \Big(\frac{1}{2} j(y)\Big)\, q^{\frac12} + O(q^2) & \text{if } a = 1. \end{cases}$$

Corollary 5.10 has the following geometric interpretation.

5.11. DEFINITION. — *Define*

$$\mathcal{A}_g^{(2,4)} \longrightarrow \vartheta_{2,g}\mathbb{P}^{N_g} := \mathbb{P}\Big(\dots, R_{\left[\begin{smallmatrix} a'' \\ b'' \end{smallmatrix}\right]}, \dots\Big)$$

by

$$\theta_{2,g}\big([X]\big) = \Big(\dots, \vartheta_X \begin{bmatrix} a'' \\ b'' \end{bmatrix}^2 (0), \dots\Big),$$

where (a'', b'') *run through the* $1/2$*-theta characteristics such that*

$$\chi(a'', b'') = 0.$$

5.12. *Remark.* — Let S be as in 5.1 with $v = 1$. Let $\partial(S)$ be the closed subscheme of S defined by $q^{\frac12} = 0$. By 5.10 the map $\vartheta_{2,g}$ is defined on the whole of S. The image of $\partial(S)$ is contained in the locus $Z_{(1,b)}$ defined by the elements

$$R_{\left[\begin{smallmatrix} a' & 1 \\ b' & b \end{smallmatrix}\right]} = 0.$$

The restriction of $\vartheta_{2,g}$ to $\partial(S)$ is the composite of $\vartheta_{2,g-1}$ and the embedding

$$\mathbb{P}^{N_{g-1}} \longrightarrow \mathbb{P}^{N_g}$$

defined by

$$\Big(\dots, y_{\left[\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right]}, \dots\Big) \longmapsto \Big(\dots, x_{\left[\begin{smallmatrix} a'' \\ b'' \end{smallmatrix}\right]}, \dots\Big)$$

with

$$x_{\left[\begin{smallmatrix} a'' \\ b'' \end{smallmatrix}\right]} = \begin{cases} 0 & \text{if } a = 1 \\ y_{\left[\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right]} & \text{if } a = 0. \end{cases}$$

Even more can be said.

5.13. LEMMA. — *The following hold:*

i) *the intersection of* $\vartheta_{2,g}\big(\mathcal{A}_g^{(2,4)}\big)$ *with* $Z_{(1,b)}$ *is empty;*

ii) *the intersection of the scheme theoretic closure of* $\vartheta_{2,g}(S)$ *with* $Z_{(1,b)}$ *is the scheme theoretic image of* $\partial(S)$ *via the composite of* $\vartheta_{2,g-1}$ *and the embedding* $\mathbb{P}^{N_{g-1}} \hookrightarrow \mathbb{P}^{N_g}$.

Proof. — Let (X, L) be an abelian variety with (4,8)-level structure. By 3.17 the following equality of theta nulls holds:

$$\vartheta_X \begin{bmatrix} a' & 1 \\ b' & b \end{bmatrix}^2 (0) = \pm \vartheta_X \begin{bmatrix} a' & 0 \\ b' & b \end{bmatrix}^2 (0).$$

Since $\vartheta_{2,g}$ has no base points, we conclude that (i) holds. The inclusion $\vartheta_{2,g}(\partial(S)) \subset Z_{(1,b)}$ is given in 5.12. Consider the equations

$$\left\{ \vartheta_X \begin{bmatrix} a'' \\ b'' \end{bmatrix}^2 (0) = 0 | \forall (a'', b'') \text{ s. t. } a = 1 \right\}.$$

Since $\vartheta_{B_{\bar{K}}} \left[\begin{smallmatrix} a' \\ b' \end{smallmatrix} \right]^2 \left(\frac{1}{2} \jmath(y) \right)$ is invertible in S for some (a', b') and 2 is invertible in S, we conclude from 5.10 that $q^{\frac{1}{2}} = 0$. This proves (ii).

6. The heat equation.

One of the main ingredients in the proof of Van Geemen's theorem is the heat equation. The paper [We2] gives a substitute of the heat equation for abelian varieties over fields of any characteristic different from 2. We show in this section how Welter's approach can be translated into properties of Mumford's theta functions.

6.1. A review of some results of G.E. Welters.

Fix an abelian variety X over an algebraically closed field k and a line bundle L with a global section s on X. Denote by

$$\Sigma_L^{(n)} = \mathrm{Diff}^n(L, L)$$

the differential operators of order n on L [We2], §1.9, p. 178. In [We2], Formula 1.14, the following commutative diagram with exact rows is defined:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma_L^{(1)} & \longrightarrow & \Sigma_L^{(2)} & \longrightarrow & \bar{S}^2 T_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^1 s} & & \downarrow{\scriptstyle d^2 s} & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & L & \longrightarrow & 0 & \xrightarrow{\sim} & 0. \end{array}$$

Taking the hypercohomology of the columns we get a map

$$H^0(\bar{S}^2 T_X) \longrightarrow \mathbb{H}^1(d^1 s);$$

see [We2], Formula 1.13. By [We2], Proposition 1.2, the first group classifies infinitesimal deformations of X as an abelian variety, while the second group classifies infinitesimal deformations of the triple (X, L, s).

6.2. *Remark.* — We work in the category of schemes over k. Let E be an artinian local k-algebra E. Among the deformations of X (resp. of (X, L, s)) to E there always exists a distinguished one i.e., the pullback of X (resp. (X, L, s)) via the morphism $k \to E$. In particular, the deformations to the dual numbers over k, the so-called "infinitesimal deformations", are not simply a principal homogeneous space under $H^0(\bar{S}^2 T_X)$ (resp. under $\mathbb{H}^1(d^1 s)$). They are completely classified by this cohomology group!

6.3. *Remark.* — If we suppose that $d^1 s(0) = 0$, we get as in [We2], Formula 3.4,

$$\begin{array}{ccc} H^0(\bar{S}^2 T_X) & \longrightarrow & \mathbb{H}^1(d^1 s) \\ \downarrow & & \downarrow \\ \bar{S}^2 T_X(0) & \longrightarrow & \mathbb{H}^1(d^1 s(0)) = L(0). \end{array}$$

Remark that $\bar{S}^2 T_X(0) = \bar{S}^2 T_{X,0}$. Choose elements $\partial/\partial x_1, \ldots, \partial/\partial x_g$ spanning it. For any $\partial^2/\partial x_i \partial x_j$ in $H^0(\bar{S}^2 T_X)$ denote by $s_{x_i x_j}$ the corresponding element of $\mathbb{H}^1(d^1 s)$. The infinitesimal deformation $s + \epsilon_{x_i x_j} s_{x_i x_j}$ of the section s associated to $\partial^2/\partial x_i \partial x_j$ has the property that

$$\frac{\partial}{\partial_{\epsilon_{x_i x_j}}}(s + \epsilon_{x_i x_j} s_{x_i x_j})(0) = s_{x_i x_j}(0) = \frac{\partial^2}{\partial x_i \partial x_j}(s)(0).$$

6.4. *Remark.* — A sufficient condition to guarantee that $d^1 s(0) = 0$ is that the divisor defined by s is invariant under $[-1]$ on X i.e., that s is symmetric. Summarizing we get the following:

6.5. PROPOSITION. — *Let r be a positive integer and let X be an abelian variety over the algebraically closed field k with a theta level structure of type $(2^r, \ldots, 2^r, 2^{r+1}, \ldots, 2^{r+1})$ on L. Let the tangent space of X at 0 be spanned by*

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_g}$$

and let

$$\mathcal{X} \longrightarrow k[\![\ldots, \epsilon_{x_i x_j}, \ldots]\!]_{1 \leqslant i \leqslant j \leqslant g}$$

be the universal deformation space of X over k with the theta structure. Then, for all (a, b) theta characteristics of level $1/2^r$ in the sense of 3.16, *we have*

$$\frac{\partial}{\partial \epsilon_{x_i x_j}} \vartheta_{\mathcal{X}} \begin{bmatrix} a \\ b \end{bmatrix}_{\epsilon_{x_i x_j}=0} (0) = \frac{\partial^2}{\partial x_i \partial x_j} \left(\vartheta_X \begin{bmatrix} a \\ b \end{bmatrix} \right) (0).$$

6.6. COROLLARY. — *Let $I_{g,k}$ be the ideal defining the closure of the image of $\vartheta_{2,g}$ in $\mathbb{P}_k^{N_g}$. Let X be an abelian variety over k with line bundle L and a $(2,4)$-structure on it such that $\vartheta_2([X])$ is a smooth point in the image of $\vartheta_{2,g}$. The k-vector space*

$$\left\langle \bigoplus_{(a,b)} \frac{\partial F}{\partial X_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}} \left(\ldots, \vartheta_X \begin{bmatrix} c \\ d \end{bmatrix}^2 (0), \ldots \right) \vartheta_X \begin{bmatrix} a \\ b \end{bmatrix}^2 \mid F \in I_{g,k} \right\rangle$$

coincides with the subspace

$$\Gamma_{00} := \{ s \in \Gamma(X, L) | m_0(s) \geqslant 4 \}.$$

The notation $m_0(s)$ stands for the multiplicity of s at 0.

Proof. — The proof is formally the same as in [vG], Proposition 2.10.

6.7. PROPOSITION. — *Let $X = \mathrm{Pic}^0(\mathcal{C}/k)$ be the Jacobian of a smooth projective curve. Let L be a line bundle on X with $(2,4)$-level structure compatible with the canonical polarization on X. The zero locus of the sections $s \in \Gamma_{00}$ is known to coincide with the surface $\mathcal{C} - \mathcal{C} := \{ x - y | x, y \in \mathcal{C} \}$ in the following cases:*

1. $g = 3$;
2. $k = \mathbb{C}$ *and* $g \geqslant 5$;
3. $k = \mathbb{C}$, $g = 4$ *and* $\mathcal{C}$ *is hyperelliptic.*

Finally,

4. *if* $k = \mathbb{C}$, $g = 4$ *and* $\mathcal{C}$ *is non-hyperelliptic, then* $\cap_{s \in \Gamma_{00}} \{ s = 0 \} = (\mathcal{C} - \mathcal{C}) \cup \{ \pm (g_3^1 - h_3^1) \}$.

Proof. — Claim (1) follows from [Wel], Proposition 4.17. In loc. cit. one assumes $k = \mathbb{C}$, but the proof of the case $g = 3$ is purely cohomological and works also in positive characteristic. Claims (2)–(4) are the contents of [Wel], Corollary 2.5. This result is based on work of M. Teixidor who works over $k = \mathbb{C}$. The author does not know whether the result holds in positive characteristic or not.

7. Van Geemen's theorem.

In this section we extend Van Geemen's theorem to the case of fields of characteristic different from 2. See 7.6. We follow closely the idea of [vG]. The main technique of [vG] is the study of the behavior of the Schottky-Jung relations deduced in 4.6 for degenerating abelian varieties. The degenerations allowed are semiabelian schemes which geometrically have torus rank at most 1. With the techniques developed so far we go deeper into the boundary than [vG]. This allows us to reduce to boundary points for which the results of [We1] hold also in positive characteristics.

7.1. Definition. — *Let g be a positive integer. Let k be a field of characteristic different from 2. Define*

$$\mathfrak{S}_{g,k}^{\mathrm{small}}$$

as the closure of the intersection over all Hurwitz data H of the images via the forgetful map

$$\mathcal{A}_g^{(4,8),H} \otimes k \longrightarrow \mathcal{A}_g \otimes k$$

of the loci

$$\left(\gamma_g^H\right)^{-1}\left(\vartheta_{2,g-1}\left(\mathcal{A}_{g-1}^{(2,4)}\right)\right);$$

see 4.7 for the notation.

7.2. *Remark.* — To justify the presence of "small" see [Do1] or [Do2], where $\mathfrak{S}_{g,\mathbb{C}}^{\mathrm{big}}$ is defined as in our definition replacing the word intersection with the word union.

7.3. Proposition. — *The boundary of the small Schottky locus in the minimal compactification of $\mathcal{A}_g \otimes k$ is contained in the union of the small Schottky loci of lower dimension.*

Proof. — It follows from the following:

7.4. Lemma. — *The notations and assumptions are as in 5.9. Let $g \geqslant 2$. Assume that $X_{\bar{K}}$ satisfies the Schottky-Jung relations (4.6). Then, B satisfies the Schottky-Jung relations.*

Proof. — Assume that $\gamma_g^H\left[X_{\bar{K}}\right]$ is in the image of $\vartheta_{2,g-1}$. By 5.9, putting $q^{\frac{1}{2}} = 0$, we see that γ_g^H degenerates to γ_{g-1}^H. By 5.13, the intersection of the image of $\vartheta_{2,g-1}$ with the hyperplanes $Z_{(1,b)}$ is the image of $\vartheta_{2,g-2}$. Hence, $\gamma^H_{g-1}[B]$ is in the image of $\vartheta_{2,g-2}$ as wanted.

7.5. LEMMA. — *The notations and assumptions are as in 5.8. Assume that*

1. $X_{\bar{K}}$ *satisfies the Schottky-Jung relations (4.6);*
2. $\vartheta_{2,g-1}([B_{\bar{K}}])$ *is in the smooth locus of the image of* $\vartheta_{2,g-1}$.

Then, $j(y)$ *lies in the closed subscheme* Γ_{00} *of* $B_{\bar{K}}$ *defined in 6.6.*

Proof. — Let F be a homogeneous polynomial vanishing on the image of $\vartheta_{2,g-1}$. Then

$$\begin{aligned}0 &= F\left(\dots,\vartheta_{X_{\bar{K}}}\begin{bmatrix} a & 0\\ b & 0\end{bmatrix}(0)\vartheta_{X_{\bar{K}}}\begin{bmatrix} a & 0\\ b & 1\end{bmatrix}(0),\dots\right)\\ &= \Sigma_{\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]}\frac{\partial F}{\partial X_{\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]}}\left(\dots,\vartheta_{B_{\bar{K}}}\begin{bmatrix} c\\ d\end{bmatrix}^2(0),\dots\right)\ \vartheta_{B_{\bar{K}}}\begin{bmatrix} a\\ b\end{bmatrix}^2(j(y)).\end{aligned}$$

We conclude by 6.6.

7.6. THEOREM (See [vG], Theorem 1.6). — *Let g be a positive integer bigger than 1. Let k be a field of characteristic different from 2. Then, the closure in $\mathcal{A}_g \otimes k$ of the locus of Jacobians $\mathfrak{J}_{g,k}$ is an irreducible component of $\mathfrak{S}^{\text{small}}_{g,k}$.*

Proof. — We can suppose that k is algebraically closed. For $g \leqslant 3$ the statement is clear since $\mathfrak{J}_{g,k}$ is equal to $\mathcal{A}_g \otimes k$. We may assume that $g \geqslant 4$. By 4.6 we know that

$$\mathfrak{J}_{g,k} \subset \mathfrak{S}^{\text{small}}_{g,k}.$$

Let $\overline{\mathcal{A}}_g^{(4,8)} \to \overline{\mathcal{A}}_g$ be compatible toroidal compactifications of $\mathcal{A}_g^{(4,8)}$ and $\mathcal{A}_g$; see [FC], Theorem IV.5.7 for the construction. Let

$$\overline{\mathfrak{J}}_{g,k} \qquad \text{and} \qquad \overline{\mathfrak{S}}^{\text{small}}_{g,k}$$

be the closures of $\mathfrak{J}_{g,k}$ and $\mathfrak{S}^{\text{small}}_{g,k}$ in $\overline{\mathcal{A}}_g$. Let x be a k-valued point of $\overline{\mathcal{A}}_g^{(4,8)} \otimes k$ corresponding to a semiabelian scheme with abelian part of dimension 2. Let $S := \operatorname{Spec}(A) \to \overline{\mathcal{A}}_g^{(4,8)} \otimes k$ be a chart at the point x as in 5.1 for $v = 2$.

We *choose* x so that the semiabelian scheme X_x is the Pic^0 of an irreducible curve $\mathcal{C}_0$. We use the results of 5.2 with $r := g-2$. Fix two distinct indices t and s in $\{1,\dots,r\}$. Then,

- the character group of the torus T is identified with $\mathbb{Z}Q_1 \oplus \cdots \oplus \mathbb{Z}Q_r$;
- $q_{s,t}q_{s,s}^{-1}q_{t,t}^{-1}$ is invertible in S and, hence, $\iota_s(Q_t)$ extends to an S-valued point of G_s.

Define the ideal $I_{t,s} := \langle q_{i,i} | \, i \neq s,t\rangle$. Let $S_{t,s} := \mathrm{Spec}(A_{t,s})$ with $A_{t,s} := A/I_{t,s}$. Denote by $K_{t,s}$ the fraction field of $A_{t,s}$. Let $G_{t,s} := \big(G_t \times_B G_s\big) \times_S S_{t,s}$.

By 5.1 we have $\iota_t(e_t) \in G_t\left(S(q_{t,t}^{-1})\right)$. Hence, the period map $\mathbb{Z}Q_t \to G_t(K_{t,s})$ induced by ι is well-defined. By Mumford's construction [FC], Ch. III it is the uniformization datum of a unique 3-dimensional semiabelian scheme

$$X_t \longrightarrow S_{t,s}.$$

By properties of Mumford's construction [FC], Theorem III.10.2,

1. the abelian part of the semiabelian scheme $X \times_S \mathrm{Spec}(K_{s,t})$ extends to a semiabelian scheme $X_{t,s}$ over $S_{t,s}$ of dimension 4. Its uniformizing datum is the datum
$$\mathbb{Z}Q_t \oplus \mathbb{Z}Q_s \longrightarrow G_{t,s}(K_{t,s})$$
induced by the period map ι;
2. the semiabelian scheme $X_{t,s} \times_{S_{t,s}} S_{t,s}\left(q_{t,t}^{-1}\right)$ degenerates for $q_{s,s} = 0$ to an extension of a 3-dimensional abelian scheme and a 1-dimensional torus. The uniformization datum of its completion along $q_{s,s} = 0$ has abelian part equal to the base change of X_t to $S_{t,s}\left(q_{t,t}^{-1}\right)$.

Since the moduli space $\mathfrak{M}_3$ of curves of genus 3 maps dominantly to the moduli space of principally polarized abelian varieties of dimension 3, we conclude that $X_t = \mathrm{Pic}^0(\mathcal{C}_t/S_{t,s})$ for a geometrically irreducible semistable curve $\mathcal{C}_t \to S_{t,s}$ of genus 3. By 5.2 the base change $\mathcal{C}_t \times_{S_{t,s}} \mathrm{Spec}\,(k(x))$ coincides with the desingularization of $\mathcal{C}_0$ at Q_i for $i \neq t$. Since $K_{s,t}$ maps dominantly to $\mathfrak{M}_{3,k}$, we deduce that $\vartheta_{2,3}\left([\mathcal{C}_t]\right)$ is a smooth point of $\vartheta_{2,3}\big(\mathcal{A}_{3,k}^{(4,8)}\big)$.

Let $\big((0,\ldots,0),(\delta_{s,u}),(0,\ldots,0)\big) \in V(\underline{B}_{\bar{K}}) \times V(\underline{T}_{\bar{K}}) \times V(\underline{Y}_{\bar{K}})$ be the 1/2-theta characteristic corresponding to the orthogonal of the vanishing cycle defined by $e_s \in Y \subset V(\underline{Y}_{\bar{K}})$ in the notation of 5.5. Consider the degeneration of $X_{t,s} \times_{S_{t,s}} S_{t,s}\left(q_{t,t}^{-1}\right)$ along the divisor $q_{s,s} = 0$. By 6.6 and 7.5 a necessary condition for a degeneration to be inside $\overline{\mathfrak{S}}_{4,k}^{\mathrm{small}}$ is that $j(Q_s) \in X_t\big(S_{t,s}\big)$ lies on the surface $\mathcal{C}_t - \mathcal{C}_t \subset X_t = \mathrm{Pic}^0\big(X_t/S_{t,s}\big)$.

Let $\partial S := \mathrm{Spec}\left(A/(q_{1,1},\ldots,q_{r,r})\right)$. It is a complete intersection in S of codimension r. It is equal to V' in the notation of 5.2. Varying t with

$t \neq s$, with the notation of 5.2 and in virtue of 7.3, we conclude that a degeneration to x inside $\overline{\mathfrak{S}}_{g,k}^{\mathrm{small}}$ must satisfy

$$\iota(Q_s) \in \mathrm{Pic}_0(\mathbb{D}_s/V') \cap (\overline{\mathbb{D}}_s - \mathbb{D}_s)$$

for every $s = 1, \ldots, r$. We conclude from 5.2 that the intersections of the image of ∂S in $\overline{\mathcal{A}}_g$ with $\overline{\mathfrak{S}}_{g,k}^{\mathrm{small}}$ and with $\overline{\mathfrak{J}}_{g,k} \bigcap \partial S$ coincide. Hence, by codimension considerations, we get that $\overline{\mathfrak{S}}_{g,k}^{\mathrm{small}} \bigcap S = \overline{\mathfrak{J}}_{g,k} \bigcap S$ as claimed.

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