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SPECTRAL SHIFT AND MULTIPLICITY OF THE FIRST EIGENVALUE OF THE MAGNETIC SCHRÖDINGER OPERATOR IN TWO DIMENSIONS

by László ERDŐS (*)

1. Introduction.

Multiplicity of low lying eigenvalues of Laplace-type operators over a compact Riemannian manifold M has been of great interest since its intrinsic relation to topology and geometry. In case of the Laplacian on functions, the lowest eigenvalue ($= 0$) is always simple. The second and higher eigenvalues can have arbitrary degeneracy if $\dim(M) \geq 3$ (see [CdV86]), but in two dimensions the degeneracy is subject to topological constraints [C], [Be], [HON].

Another celebrated example is the square of the Dirac operator (Pauli operator) on a Spin^c-bundle, where the index theorem may give a constraint on the ground state multiplicity. In two dimensions, in particular, the multiplicity of the zero eigenvalue is at least the total curvature of the connection, or, with physics terminology, the total flux of the magnetic field, $\Phi := \frac{1}{2\pi} \int_M B$ (Aharonov-Casher theorem, see [AC], [ES]). Note that Φ is a topological invariant of the Spin^c-bundle: it is the Chern number of the corresponding determinant line bundle.

In this paper we investigate the magnetic Schrödinger operator (magnetic Laplacian) on a line bundle L over M. First we prove that the lowest eigenvalue is bounded from above essentially by the average L^1-norm, $\frac{1}{|M|} \int_M |B|$, of the magnetic field (Theorem 2.1). The constant in

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the estimate depends on the geometry of M and we show, via an example, that this dependence is unavoidable (Section 6.2).

We then show that $\int_M |B|$ also controls the multiplicity of the lowest eigenvalue (Theorem 2.2). This answers to a question posed in [CdVT]. A similar question for graphs [CdV97] raises the conjecture that the constant in this estimate should be universal. Our proof only yields a constant that depends on the geometry of M, since along the proof we use the bound on the lowest eigenvalue whose constant is geometrical.

However, our estimate depends on the curvature of the bundle in an optimal way. Let $M = S^2$ for definiteness, however similar constructions work for other manifolds as well. The Riemann-Roch theorem easily gives that the ground state is $(\Phi + 1)$-fold degenerate for a constant field B on the nontrivial line bundle with Chern number $\Phi = \frac{1}{2\pi}\int_M B$. It has been shown in [CdVT] that if an additive scalar potential is also allowed, then arbitrarily high multiplicity of the ground state is possible even on the *trivial* bundle. A recent example in [BCC] has answered affirmatively to a question of [CdVT] as to whether an arbitrary degeneracy is possible on the trivial bundle *without* scalar potential. In this construction, however, the L^1-norm of the magnetic field was at least an exponential function of the multiplicity.

We present a different construction (Section 6.3) with a magnetic field whose L^1-norm is comparable to the ground state multiplicity. This example shows that the bound $\int_M |B|$ on the multiplicity is optimal, modulo geometric constants. In fact the proof works for any fixed multiplicity pattern of a finite part of the low lying spectrum.

This result is in strong contrast to the case of the Pauli operator on S^2 whose ground state multiplicity is given by the modulus of the Chern number, unless this number is zero [ES]. From the physical side, this is another manifestation that inclusion of the spin substantially changes the spectral properties of the corresponding free kinetic energy operator (Schrödinger vs. Pauli).

We emphasize that we use only the L^1-norm of the magnetic field B in our estimates (see Remark 3. after Theorem 2.2). Similar results involving other L^p-norms ($p > 1$) are much easier to prove (for example an L^2-bound is given in [BCC]), but such bounds have no apparent topological flavor. Note that, for example, if B has a definite sign, then $\frac{1}{2\pi}\int_M |B| = |n|$, where n is the Chern number of the bundle.

2. Statement of the results.

Let (M, g) be an oriented compact connected surface without boundary and with a Riemannian metric g. Let $d(x, y)$ be the distance function on M and let $v_g = \star 1$ be the volume (area) form of the metric g associated with the orientation of M where $\star$ is the Hodge dual. We denote the Riemannian volume of a set $S \subset M$ by $|S| = \int_S v_g$ and the geodesic ball (disk) of radius r about $x \in M$ by $D(x, r)$.

Three numbers, $c_1, c_2, c_3 > 0$, will control the geometry of M. We assume that the Gauss curvature $\kappa(x)$ and the injectivity radius $\iota(x)$ satisfy

$$|\kappa(x)| \leq c_1 \quad \text{and} \quad \iota(x) \geq c_2 \quad \forall x \in M. \tag{2.1}$$

Furthermore we assume a positive lower bound on the isoperimetric constant $I(M)$ of the manifold M,

$$I(M) \geq c_3. \tag{2.2}$$

We recall that

$$I(M) := \inf_N \frac{\ell(N)^2}{\min\{|M_1|, |M_2|\}},$$

where the infimum is over all closed curves N that separate the manifold M into two pieces, M_1 and M_2.

We consider a complex line bundle L over M equipped with a hermitian scalar product $\langle\, , \rangle$ and a compatible smooth connection ∇. The scalar product on sections of L is defined as $(\xi, \eta) = \int_M \langle \xi, \eta \rangle v_g$ and $\|\xi\| = (\xi, \xi)^{1/2}$ is the L^2-norm of ξ. We denote the corresponding Hilbert space of L^2-sections by $\mathcal{H}$. Let

$$R_\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

be the curvature of ∇, then

$$\beta(X, Y) := i R_\nabla(X, Y)$$

is a real 2-form on M, which is also called the *magnetic field*. The intensity of the magnetic field $B \in C^\infty(M)$ is defined by $\beta = B v_g$, i.e., $B = \star\beta$. It is well known that the total flux $\Phi := \frac{1}{2\pi} \int_M \beta$ is an integer, the Chern number of L. For $M = S^2$ this number topologically classifies the complex line bundles over M.

The *magnetic Schrödinger operator (magnetic Laplacian)* $H_\nabla = \nabla^*\nabla$ is defined on L^2-sections of L via the Friedrichs extension of its quadratic form

$$Q(\xi,\eta) := \int_M \sum_{j=1}^{2} \langle \nabla_{e_j}\xi, \nabla_{e_j}\eta\rangle v_g, \quad \xi,\eta \in \Gamma(L), \tag{2.3}$$

for a local orthonormal basis $\{e_1, e_2\}$. The quadratic form is independent of the basis.

The bundle L has a canonical complex structure by defining

$$\partial_{\bar z} := \frac{1}{2}(\nabla_{\partial_x} + i\nabla_{\partial_y}), \quad \partial_z := \frac{1}{2}(\nabla_{\partial_x} - i\nabla_{\partial_y}),$$

in a local conformal coordinate system where $g = \mathrm{e}^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$ with a real function u. The quadratic forms

$$q_-(\xi,\eta) := \int_M \langle \partial_{\bar z}\xi, \partial_{\bar z}\eta\rangle \,\mathrm{d}x\,\mathrm{d}y, \quad q_+(\xi,\eta) := \int_M \langle \partial_z\xi, \partial_z\eta\rangle \,\mathrm{d}x\,\mathrm{d}y,$$

are independent of the coordinates, and their selfadjoint extensions define the positive operators $\partial_{\bar z}^*\partial_{\bar z}$ and $\partial_z^*\partial_z$. We have the basic relations: $Q = 4q_\pm \mp B$, i.e.,

$$H_\nabla = 4\partial_{\bar z}^*\partial_{\bar z} + B, \quad H_\nabla = 4\partial_z^*\partial_z - B. \tag{2.4}$$

On a local chart, $U \subset M$, we choose a normalized smooth section ϕ, $\langle\phi,\phi\rangle \equiv 1$ (in fact, ϕ can be chosen globally on M away from a few points). From $d\langle\phi,\phi\rangle = 0$ we obtain $\nabla\phi = -i\alpha\phi$ with some real 1-form α. Clearly $\mathrm{d}\alpha = \beta$, and α is called the (local) vector potential. Let

$$\widehat{Q}_\alpha(f,f) := \int_M |||\,\mathrm{d}f - i\alpha f|||^2 v_g,$$

where $|||\cdot|||$ denotes the norm on T^*M obtained from the metric g. Then $Q(\xi,\xi) = \widehat{Q}_\alpha(f,f)$ for $\xi = f\phi$, $f \in C_0^\infty(U)$.

Let

$$E := \inf\left\{Q(\xi,\xi) : \xi \in \Gamma(L),\ \|\xi\| = 1\right\}$$

be the bottom of the L^2-spectrum of H_∇. By gauge invariance, E depends only on B in the simply connected case, otherwise it also depends on the fractional part of the fluxes $\Phi_j = \frac{1}{2\pi}\int_{C_j}\alpha$, $j = 1, 2, \ldots, q$. Here C_j's are a basis of nontrivial homology cycles of M.

As a special case, on a flat trivial bundle with $\Phi_j = 0$ we can assume that $\alpha \equiv 0$ globally. Considering a global normalized parallel section ϕ with $\nabla\phi = 0$ we then see that H_∇ is unitarily equivalent to $H_0 := -\Delta$ on functions. We use the physics convention that the Laplacian, Δ, is a negative operator, i.e., $\Delta = -(\delta d + d\delta)$.

Our main results are the following:

THEOREM 2.1 (Ground state energy estimate). — *Assume that* (M, g) *satisfies* (2.1).

(i) *There exists a constant* K^* *depending only on* c_1, c_2 *such that*

$$E \leq K^* + \frac{K^*}{|M|} \int_M |B| v_g. \tag{2.5}$$

The constant K^* *goes to zero as* $c_1 \to 0$ *and* $c_2 \to \infty$.

(ii) *The bound* (2.5) *would not hold in general if* K^* *were not allowed to depend on* c_2.

THEOREM 2.2 (Estimate on the multiplicity). — *Assume that* (M, g) *satisfies* (2.1) *and* (2.2). *There exists a constant* K^{**} *depending only on* c_1, c_2, c_3 *such that*

$$\dim \mathcal{N} \leq K^{**}|M| + K^{**} \int_M |B| v_g, \tag{2.6}$$

where $\mathcal{N} = \{\xi : Q(\xi, \xi) = E\|\xi\|^2\}$. *The constant* K^{**} *goes to zero as* $c_1 \to 0$ *and* $c_2 \to \infty$.

THEOREM 2.3. — *Given a finite set of positive integers,* $\{m_1, \dots, m_k\}$ *and* $\mu > 0$, *there exists a connection* ∇ *on the trivial line bundle* L_0 *over* S^2 *with a magnetic field* $\beta = B v_g$ *such that the multiplicity of the* ℓ*-th eigenvalue of the magnetic Schrödinger operator* $\nabla^*\nabla$ *is exactly* m_ℓ *for* $\ell = 1, 2, \dots, k$ *and*

$$\left| \frac{1}{2\pi} \int_M |B| v_g + 2 - 2\sum_{\ell=1}^{k} m_\ell \right| < \mu. \tag{2.7}$$

In particular, the multiplicity of the ground state can be comparable to the L^1*-norm of* B *on a trivial bundle.*

Remark 1. — A lower bound on E that is comparable with the upper bound (2.5) is false in general. On a small geodesic disk D of radius r_0 one can trivialize L, where r_0 depends only on c_1, c_2. If B is chosen to be zero on D, then a trial function localized on D shows that E is at most Kr_0^{-2}, i.e., it does not increase with the magnetic field. However, if B has a definite sign, then $E \geq \inf_M |B|$ by (2.4).

Remark 2. — A smooth trial function and a vector potential chosen by the Poincaré formula on D show that $E \leq \sup_M |B| + K(c_1, c_2)$. The key point in our theorem is that we do not assume any other control on the magnetic field apart from $B \in L^1(v_g)$.

Remark 3. — A B-independent C^∞ trial function localized on a small geodesic disk D does *not* give the bound (2.5). This would require a (local) vector potential α with $\int_D |||\alpha|||^2 v_g \leq K \int_D |B| v_g$ but such bound is not true even in the flat case. The local L^2-norm of an appropriate vector potential can be controlled only by the L^p norm of B for any $p > 1$ (Young's inequality), but not with $p = 1$. The point is that the trial function must be small where B is large, in particular a field $B \in L^1$ can have such a singularity which forces the trial function go to zero at a point in some sense. This phenomenon is well known for Dirac delta magnetic fields (see [LW] for a Hardy inequality with such magnetic field), but it can occur for much more complicated fields as well where one cannot use the explicit form of the singularity. See [EV] for more details on this delicate issue for the Pauli operator.

Remark 4. — A version of Theorem 2.3 was obtained in [BCC] but the constructed magnetic field did not satisfy any effective bound in terms of the multiplicities. The control (2.7) is important since it shows the optimality of the bound (2.6). We recall that the construction of [BCC] uses the fact that a strong magnetic field acts like a strong effective potential barrier (see also [HH]). In particular, the ground states in different angular momentum sectors can be localized in space and they can be tuned independently by a properly chosen field. The separation of the sectors requires a huge, practically uncontrolled magnetic field. Our construction relies on a different idea (see Section 6.3).

Remark 5. — For simplicity, we assume that we are in the smooth category, the connection and the metric are C^∞. Since the constants K^*, K^{**} are independent of B, only the L^1-norm of B is involved in the estimates. By a limiting argument one can obtain similar results for less regular connections.

2.1. Manifold with boundary.

Since the proofs of our main results rely on local arguments, the theorems can be easily extended to manifolds with boundary. We explain the necessary technical modifications and we add an extra condition to control the irregularities of the boundary. This section and Section 5 with the proofs are independent of the rest of the paper.

The Friedrichs extension of the quadratic form Q in (2.3) defines the Neumann quadratic form; to obtain the Dirichlet form one has to restrict Q to sections vanishing at ∂M. Our main results are valid for both cases and we simply use the notations Q and H_∇ for both boundary conditions. We will indicate when this distinction is necessary (Section 5).

However, the relations (2.4) hold only for Dirichlet boundary conditions. Consequently, the lower bound $E \geq \inf_M |B|$ (see Remark 1 above) holds for the Dirichlet ground state if B has a definite sign, but it does not hold for the Neumann case. In fact, the Neumann ground state can be much smaller than $\inf_M |B|$ even in the flat case, see e.g., [HM].

To state the theorem for manifolds with boundary, we first modify the second part of condition (2.1) to

$$(2.8)\quad \big|\kappa(x)\big| \leq c_1 \ \forall x \in M, \ \text{ and } \iota(x) \geq c_2, \ \forall x \in M, \ \text{ with } d(x, \partial M) \geq c_2,$$

i.e., the bound on the injectivity radius is required only for points away from the boundary.

The additional assumption is a uniform cone condition. We define the cones with center $x \in M$, radius $r > 0$ and angle $\alpha < \pi$ in T_xM as

$$C(x,r,\alpha,\gamma) := \big\{\xi : g(\xi,\xi) \leq r^2, \gamma \leq \arg\xi \leq \gamma+\alpha\big\} \subset T_xM, \quad \gamma \in [0,2\pi].$$

Let $\Gamma(x,r,\alpha)$ be the set of those γ's such that the exponential map $\exp_x : T_xM \to M$ is diffeomorphism on $C(x,r,\alpha,\gamma)$, its image is disjoint from ∂M. We assume that there exists a positive constant $c_4 > 0$ such that

$$(2.9)\qquad \Gamma(x,c_2,c_4) \neq \emptyset, \quad \forall x \in M.$$

THEOREM 2.4. — *Let M be a manifold with boundary and we assume* (2.8), (2.2) *and* (2.9). *Let E be ground state of the Dirichlet or Neumann quadratic form Q. Then the statements of Theorems* 2.1 *and* 2.2 *hold with constants K^* and K^{**} that depend on c_1, c_2, c_3 and c_4.*

Remark. — The analogue of Theorem 2.1 at least in the Dirichlet case does not hold with a constant K^* independent of c_3 and c_4. To see this, we can consider the flat manifold $M \subset \mathbb{R}^2$ which is the union of the disk $D = D(0, c_2)$ and the rectangle $R = [-N, N] \times [\delta, -\delta]$ with $N \gg c_2$, $0 < \delta \ll c_2$. Let the magnetic field be zero on $M \setminus D$ and uniform on D with strength $B_0 > 0$. The average flux, $|M|^{-1} \int_M |B| v_g = \pi c_2^2 B_0 / |M|$, goes to zero as $N \to \infty$ for any B_0 and δ. On the other hand, an easy calculation shows that E is at least of order $\min\{B_0, \delta^{-2}\}$ as $B_0 \to \infty$, $\delta \to 0$, independently of N. Roughly speaking, the ground state energy is of order B_0 on the disk D by (2.4), and it is of order δ^{-2} on $M \setminus D$.

3. Proof of part (i) of Theorem 2.1.

We construct an appropriate trial section to bound the lowest eigenvalue. We will not keep track of the exact dependence of the constant K^* on c_1, c_2, but the fact that $K^* \to 0$ as $c_1 \to 0$, $c_2 \to \infty$ can be easily seen from the proof.

The core of the proof is a local argument (Section 3.1), the trial section will be supported on a small geodesic disc and local geometry does not play much role. A covering argument will complete the proof (Section 3.2).

We first give an intuitive outline of the local argument. For simplicity, here we assume that $B \geq 0$. The first identity of (2.4) indicates that the magnetic energy is large in regions where B is large. The trial section therefore must be small in these regions. On the other hand, the magnetic energy is always bounded by $\|\nabla \xi\|^2$ from below using the diamagnetic inequality (see (4.2) later). Therefore the major difficulty is that if the $\{B \leq \text{const.}\}$ level sets are very complicated, then it may be impossible to localize a section on them without too large H^1 norm.

The key idea is that locally the trial section will be a function f of the form $f = \mathrm{e}^h$, where the real function h is a specific solution to $\Delta h = B$. In the flat noncompact case, $M = \mathbb{R}^2$, the solution would be

$$h(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| B(y) \, \mathrm{d}y. \tag{3.1}$$

Notice that if B is very large near a point x, then with this choice $h(x)$ would be very negative, hence $f(x)$ would almost vanish. Such function therefore properly takes into account the possible local "peaks" of B. The calculation of the energy will show that this intuition is also correct quantitatively.

Several technicalities are necessary to carry this idea through. First, we have to remove the assumption $B \geq 0$ by treating the positive and negative parts separately. Second, we need to localize the problem on a sufficiently small geodesic disk D where the local geometry is almost flat. Third, we have to use the Green's function, $G(x, y)$, of this disk instead of the Green's function, $\frac{1}{2\pi} \log |x - y|$, of the flat plane. We will need only that $G(x, y)$ behaves essentially logarithmically near the diagonal $x = y$ and away from the boundary. Finally, we have to estimate the L^2 norm of $f = \mathrm{e}^h$ from below. We will show that $\int_D \mathrm{e}^{2h}$ is comparable with $\mathrm{e}^{2 \max h}$ using that h is subharmonic. This argument requires that $\max_D h$ be attained away from the boundary of the disk. In order to ensure this, we will modify the magnetic field B to $\widetilde{B} = B\chi - B_0$. Here χ is a cutoff function supported near the center of D and B_0 is a strong, essentially uniform magnetic field within the support of χ. The trial function will eventually be constructed from the solution of $\Delta h = \widetilde{B}$. If B_0 is sufficiently large, then h will be large near the center of the disk and the maximum will be attained away from the boundary. On the other hand, the additional field B_0 is regular with a total flux $O(1)$. For such fields it will be easy to argue that they change the energy only by an $O(1)$ amount.

3.1. Trial function on a small geodesic disk.

The goal of this section is to prove the local version of part (i) of Theorem 2.1:

PROPOSITION 3.1. — *Assume that the manifold (M, g) satisfies (2.1). Let $R_0 := \frac{1}{8} \min\{c_2, c_1^{-1/2}\}$ and let $D = D(m, R) \subset M$ be a geodesic disk of radius $R \leq R_0$ about $m \in M$. Let $\Phi_0 < 1$ be fixed and suppose that*

$$\frac{1}{2\pi} \int_D |B| v_g \leq \Phi_0. \tag{3.2}$$

Let

$$E(D) := \inf \left\{ Q(\xi, \xi) : \xi \in \Gamma(L),\ \|\xi\| = 1,\ \operatorname{supp}(\xi) \subset D \right\}$$

be the bottom of the spectrum of the magnetic operator H_∇ restricted onto D with Dirichlet or Neumann boundary conditions. Then there exists a constant K, depending only on Φ_0, c_1 and c_2 such that

$$E(D) \leq \frac{K}{|D|} \cdot$$

Convention. — In this proof letters $K_1, K_2, \ldots$ denote specific positive constants depending only on Φ_0 and c_1, c_2 from (2.1). The letter K denotes a generic positive constant, depending on Φ_0, c_1, c_2, whose value may change from line to line. Universal positive constants are denoted by (const.).

Proof of Proposition 3.1. — The proof is given in the following subsections. D is simply connected since $R_0 < c_2$. The line bundle L is therefore trivial, $L = D \times \mathbb{C}$, and for any connection ∇ there exists a global vector potential α on D. We know that the spectrum of H_∇ is gauge invariant, i.e., $E(D)$ indeed depends only on the magnetic field. The Neumann ground state is lower than the Dirichlet one, so we will construct a low energy Dirichlet trial section.

Fix a number $0 < \varrho \le \frac{1}{7}R$. We define six concentric geodesic disks $D_j := D(m, j\varrho) \subset D$ about m with radius $j\varrho$, $j = 1, 2, \ldots, 6$. They will play the following roles. The additional magnetic field B_0 will be essentially uniform on D_1 and supported on D_2, smoothly cut-off in between. The maximum of h will be attained on $\overline{D}_2$. The trial function f will be equal to e^h on D_3, supported on D_4 and smoothly cut-off in between. The magnetic field B will be cut-off between D_4 and D_5. Finally the Green's function of D will be approximated by the logarithm of the distance function on D_6.

Let $0 \le \chi_j \le 1$ be C_0^∞ cutoff functions for $j = 3, 4$, with $\chi_j \equiv 1$ on D_j, $\mathrm{supp}(\chi_j) \subset D_{j+1}$ and $|||\,\mathrm{d}\chi_j\,||| \le (\mathrm{const.})\varrho^{-1}$.

We let $s_y(x) = d(x, y)$ be the Riemannian distance function and let $s(x) := s_m(x)$ denote the distance from the center m of D. We also introduce the notation σ_g for the arc-length measure inherited from v_g. For a set $D \subset M$ with regular boundary we use the notation $|\partial D|$ for the σ_g-measure of the boundary ∂D. Finally, let $G(x, y)$ be the Green's function of the Laplacian on D (see, e.g., Theorem 4.17 of [A], but with our sign convention for Δ). In particular it has the following properties: G is smooth on $D \times D$ away from the diagonal, $G(x, y) = G(y, x)$, $G(x, y) = 0$ if $x \in \partial D$, and finally $\Delta_y G(x, y) = \delta_x$; more precisely

$$\varphi(x) = \int_D G(x, y)\Delta\varphi(y) v_g(\mathrm{d}y) + \int_{\partial D} \nu \cdot \nabla_y G(x, y)\varphi(y)\sigma_g(\mathrm{d}y), \tag{3.3}$$

where $\varphi \in C^2(D) \cap C(\overline{D})$ (ν is the outward normal).

We collected a few standard results from Riemannian geometry at the end of the proof in Section 3.1.6. These will be used with proper reference below.

3.1.1. *Modification of the magnetic field.* — Let $B_+ := \frac{1}{2}(|B| + B)$ and $B_- := \frac{1}{2}(|B| - B)$ be the positive and negative parts of B. We choose a smooth, increasing real function $k(t)$ for $t \geq 0$ such that

$$k(t) := \begin{cases} t & \text{for } t \leq \varrho^2, \\ 2\varrho^2 & \text{for } t \geq (2\varrho)^2 \end{cases} \tag{3.4}$$

and $0 \leq k'(t) \leq 1$, $-\varrho^{-2} \leq k''(t) \leq 0$.

We define the following smooth functions:

$$\widetilde{B}_\pm := B_\pm \chi_4 - F_\pm \Delta\big[k(s^2)\big], \quad \widetilde{B} := \widetilde{B}_+ + \widetilde{B}_-,$$

where $F_\pm \geq 0$ is chosen as

$$F_\pm := \gamma \varrho^{-2} \int_{D_5} B_\pm \chi_4 v_g \tag{3.5}$$

with some constant γ to be determined later (see Lemma 3.3). Clearly, $\operatorname{supp}(\widetilde{B}_\pm) \subset D_5$. Moreover,

$$\int_D |\widetilde{B}_\pm| v_g \leq (\text{const.})(\Phi_0 + \gamma) \tag{3.6}$$

using the flux condition (3.2), the volume comparison (3.36) from Section 3.1.6 and the fact that

$$\big|\Delta[k(s^2)]\big| \leq 21. \tag{3.7}$$

The last estimate follows from $\Delta\big[k(s^2)\big] = k''(s^2)|\nabla s^2|^2 + k'(s^2)\Delta s^2$ and from the choice of k, using (3.35) of Section 3.1.6 and $|\nabla s^2|^2 = 4s^2 \leq 16\varrho^2$ on the support of $\Delta\big[k(s^2)\big]$.

3.1.2. *Properties of the solution to* $\Delta h_\pm = \widetilde{B}_\pm$. — We define a solution to $\Delta h_\pm = \widetilde{B}_\pm$ on D as

$$h_\pm(x) := \int_{D_5} \Big[G(x,y) - \frac{1}{2\pi}\log\varrho\Big] \widetilde{B}_\pm(y) v_g(\mathrm{d}y), \quad x \in D, \tag{3.8}$$

and we collect a few properties of $h_\pm$ which will be proved later in Section 3.1.5.

First we need that the functions $\mathrm{e}^{2h_\pm}$ satisfy a reversed Hölder inequality on D_4. On a flat space such property is related to the A_2-weight class used in harmonic analysis (see [St]). In a similar two dimensional magnetic context it has been exploited in [EV].

LEMMA 3.2. — *There exists a constant* $K_1 = K_1(\Phi_0, c_1, c_2)$ *such that*

$$\Big(\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{-2h_\pm} v_g\Big)\Big(\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{2h_\pm} v_g\Big) \le K_1. \tag{3.9}$$

Next we estimate the maxima of $\mathrm{e}^{h_\pm}$. Clearly $h_\pm$ are continuous, let $p_\pm \in \overline{D}_6 \subset D$ be points where the maxima of $h_\pm$ on $\overline{D}_6$ are attained,

$$h_\pm(p_\pm) = h_\pm^{\max} := \max_{\overline{D}_6} h_\pm. \tag{3.10}$$

Lemma 3.4 below asserts that the maxima of $\mathrm{e}^{h_\pm}$ can be estimated by their averages on a disk of radius ϱ about the maximum points $p_\pm$ if ϱ is sufficiently small. Beforehand, we need to ensure that the disks $D(p_\pm, \varrho)$ lie strictly inside D_6 at least for large enough γ because we can control the Green's function and $h_\pm$ only within D_6.

LEMMA 3.3. — *There exists* $\gamma = \gamma(c_1, c_2) > 0$ *such that the maxima* $h_\pm^{\max} = \max_{\overline{D}_6} h_\pm$ *are attained on* $\overline{D}_2$, *i.e.,* $p_\pm \in \overline{D}_2$.

LEMMA 3.4. — *There exist* $K_2 = K_2(c_1, c_2) > 0$ *and* $\varrho_0 = \varrho_0(c_1, c_2) > 0$ *such that for all* $\varrho \le \varrho_0$,

$$\frac{1}{|D(p_\pm, \varrho)|}\int_{D(p_\pm,\varrho)} \mathrm{e}^{h_\pm} v_g \ge K_2\, \mathrm{e}^{h_\pm^{\max}}. \tag{3.11}$$

Remark. — If M has no boundary, then one could alternatively define $h_\pm$ using the global Green's function of M and the analogue of Lemma 3.3 would follow directly from $\Delta h_\pm = \widetilde{B}_\pm \ge 0$ outside of D_2. However, we wish to emphasize that our construction is local and we will use it for the case $\partial M \neq \emptyset$ as well (see Section 5).

For the rest of the proof we fix $\gamma = \gamma(c_1, c_2)$ from Lemma 3.3 and let

$$\varrho := \min\Big\{\frac{1}{7}R, \varrho_0(c_1, c_2)\Big\} \tag{3.12}$$

with ϱ_0 from Lemma 3.4.

3.1.3. *Energy estimate for the trial section.* — We define $h := h_+ + h_-$ and $f := \chi_3\, \mathrm{e}^h$. Let ϕ be any normalized section on D with $\nabla\phi = -i\alpha\phi$ and $\mathrm{d}\alpha = Bv_g = \beta$. Clearly

$$\Delta\big[h_+ - h_- + (F_+ - F_-)k(s^2)\big] = B$$

on D_4, i.e., $\mathrm{d}\alpha = \mathrm{d}\star \mathrm{d}[h_+ - h_- + (F_+ - F_-)k(s^2)]$. Since D_4 is simply connected, $\alpha = \star\mathrm{d}[h_+ - h_- + (F_+ - F_-)k(s^2)] + \mathrm{d}g$ with some function g. The trial section is defined as $\xi := f\,\mathrm{e}^{ig}\phi$ and note that $\mathrm{supp}(\xi) \subset D_4$.

LEMMA 3.5. — *The energy of the trial section $\xi = f\,\mathrm{e}^{ig}\phi$ satisfies*

$$Q(\xi,\xi) \le K_3\,\mathrm{e}^{2(h_-^{\max}+h_+^{\max})}. \tag{3.13}$$

Proof. — Let $F := F_+ + F_-$ and we recall from (3.5) and (3.2) that $F \le 2\pi\gamma\varrho^{-2}$. We have

$$\begin{aligned} Q(\xi,\xi) &= \int_{D_4} \big|\!\big|\!\big| \,\mathrm{d}f - if\star \mathrm{d}[h_+ - h_- + (F_+ - F_-)k(s^2)]\big|\!\big|\!\big|^2 v_g \\ &\le (\mathrm{const.})\Big[\,\mathrm{e}^{2(h_-^{\max}+h_+^{\max})}(\varrho^{-2}+F^2\varrho^2)|D_4| \\ &\qquad + \int_{D_4} \mathrm{e}^{2h}(|\!|\!|\,\mathrm{d}h_+|\!|\!|^2 + |\!|\!|\,\mathrm{d}h_-|\!|\!|^2)v_g\Big] \end{aligned} \tag{3.14}$$

$$\le K\Big[\,\mathrm{e}^{2(h_-^{\max}+h_+^{\max})} + \int_{D_4} \mathrm{e}^{2h}(|\!|\!|\,\mathrm{d}h_+|\!|\!|^2 + |\!|\!|\,\mathrm{d}h_-|\!|\!|^2)v_g\Big], \tag{3.15}$$

using the estimate on $\mathrm{d}\chi_3$, volume comparison (3.36) and that $|\!|\!|\,\mathrm{d}(s^2)|\!|\!|^2 = 4s^2 \le (\mathrm{const.})\varrho^2$. By the coarea formula,

$$\begin{aligned} \int_{D_4} \mathrm{e}^{2h}|\!|\!|\,\mathrm{d}h_+|\!|\!|^2 v_g &\le \mathrm{e}^{2h_-^{\max}}\int_{D_4} \mathrm{e}^{2h_+}|\!|\!|\,\mathrm{d}h_+|\!|\!|^2 v_g \\ &= \mathrm{e}^{2h_-^{\max}}\int_{-\infty}^{h_+^{\max}} \mathrm{e}^{2c}\Big(\int_{\{h_+=c\}\cap D_4} |\!|\!|\,\mathrm{d}h_+|\!|\!|\sigma_g\Big)\,\mathrm{d}c, \end{aligned} \tag{3.16}$$

where the inner integration is over the level set $\{x\in D_4 : h_+(x) = c\}$.

We recall that $h_\pm$ is constant on the boundary of D and $h_\pm \in C^\infty(D)\cap C(\overline{D})$ by elliptic regularity from $\Delta h_\pm = \widetilde{B}_\pm$, $\widetilde{B}_\pm \in C^\infty(D)\cap C(\overline{D})$. In particular the level sets of $h_\pm$ are smooth curves for almost all values of c by Sard's theorem. Since the boundary of D itself belongs to one level set, any other level set within the interior of D is a union of closed smooth curves and these curves separate the sets $\{h>c\}$ and $\{h<c\}$ for a.e. c.

Since $(\star\,\mathrm{d}h_+)(X) = |\!|\!|\star\,\mathrm{d}h_+|\!|\!| = |\!|\!|\,\mathrm{d}h_+|\!|\!|$ for a properly oriented unit vectorfield X tangent to these level curves, we can orient the curves such that

$$\begin{aligned} \int_{\{h_+=c\}} |\!|\!|\,\mathrm{d}h_+|\!|\!|\sigma_g &= \Big|\int_{\{h_+=c\}} \star\,\mathrm{d}h_+\Big| \\ &= \Big|\int_D (\mathrm{d}\star\,\mathrm{d}h_+)\mathbf{1}\{h_+\ge c\}\Big| \le \int_D |\widetilde{B}_+|v_g \le K, \end{aligned} \tag{3.17}$$

for $c > h_+(\partial D)$, where we used Stokes' theorem, (3.6) and that γ depends on c_1, c_2. For $c < h_+(\partial D)$ the set $\mathbf{1}\{h_+\ge c\}$ is replaced by $\mathbf{1}\{h_+\le c\}$ in (3.17).

Therefore we can estimate the integral in (3.16) by (3.17) to obtain

$$\int_{D_4} \mathrm{e}^{2h} ||| \,\mathrm{d}h_+ |||^2 v_g \le K\,\mathrm{e}^{2(h_-^{\max}+h_+^{\max})},$$

and similar estimate is valid for the $\mathrm{d}h_-$ term in (3.15). This completes the proof of (3.13). □

3.1.4. *Lower bound on the L^2 norm of ξ.* — By Schwarz inequality and the support properties of χ_3,

$$(3.18)\qquad \|\xi\|^2 = \int_D \chi_3^2\,\mathrm{e}^{2(h_-+h_+)} v_g \ge \frac{\left(\int_D \chi_3^2\,\mathrm{e}^{h_-} v_g\right)^2}{\int_D \chi_3^2\,\mathrm{e}^{-2h_+} v_g} \ge \frac{\left(\int_{D_3} \mathrm{e}^{h_-} v_g\right)^2}{\int_{D_4} \mathrm{e}^{-2h_+} v_g}.$$

We recall that $D(p_\pm,\varrho)\subset D_3$ by Lemma 3.3. Now we can use Lemma 3.2 and 3.4 together with (3.36) to obtain

$$(3.19)\qquad \begin{aligned} \|\xi\|^2 &\ge \frac{1}{K_1|D_4|^2}\Big(\int_{D(p_-,\varrho)} \mathrm{e}^{h_-} v_g\Big)^2 \int_{D(p_+,\varrho)} \mathrm{e}^{2h_+} v_g \\ &\ge K\varrho^2\,\mathrm{e}^{2(h_-^{\max}+h_+^{\max})}. \end{aligned}$$

Combining (3.13) and (3.19) we obtain that the lowest eigenvalue E of the quadratic form Q is estimated by

$$E \le \frac{Q(\xi,\xi)}{\|\xi\|^2} \le K\varrho^{-2}.$$

Recalling the definition of ϱ (3.12) and the volume comparison (3.36), we completed the proof of Proposition 3.1. □

3.1.5. *Proofs of the properties of $h_\pm$.*

Proof of Lemma 3.2. — We prove (3.9) for h_+, the proof for h_- is identical. Let

$$\widetilde{h}_+(x) := \frac{1}{2\pi}\int_{D_5} (\log s_x) B_+\chi_4 v_g, \quad \Phi := \frac{1}{2\pi}\int_{D_5} B_+\chi_4 v_g.$$

We can assume that $\Phi > 0$, otherwise $h_+\equiv 0$. We use (3.7), (3.2), (3.38) and (3.39) to estimate

$$\begin{aligned} &\big|h_+(x) - \widetilde{h}_+(x) + \Phi\log\varrho\big| \\ &\qquad \le K\int_{D_5} (B_+ v_g + 21F_+) v_g + 21F_+\int_{D_5}\Big|\log\frac{s_x}{\varrho}\Big| v_g \le K_4 \end{aligned}$$

for $x \in D_4$. We have

$$\Big(\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{-2h_+} v_g\Big)\Big(\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{2h_+} v_g\Big) \le \mathrm{e}^{4K_4}\Big(\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{-2\widetilde{h}_+} v_g\Big)\Big(\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{2\widetilde{h}_+} v_g\Big).$$

By Jensen's inequality (applied to the probability measure $(2\pi\Phi)^{-1}B_+\chi_4 v_g$)

$$\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{-2\widetilde{h}_+} v_g \le \frac{1}{|D_4|}\int_{D_4} \frac{1}{2\pi\Phi}\int_{D_5} \big[d(x,y)\big]^{-2\Phi} (B_+\chi_4)(y) v_g((\mathrm{d}y) v_g(\mathrm{d}x) \le K\varrho^{-2\Phi} \tag{3.20}$$

using that $2\Phi \le 2\Phi_0 < 2$ by (3.2). We performed the x-integration first and used that the singularity is integrable uniformly for all $y \in D_5$.

Similarly we obtain

$$\frac{1}{|D_4|}\int_{D_4} \mathrm{e}^{2\widetilde{h}_+} v_g \le K\varrho^{2\Phi},$$

which, together with (3.20), finishes the proof of Lemma 3.2. □

Proof of Lemma 3.3. — We can assume that $F_\pm > 0$, otherwise $h_\pm \equiv 0$. We estimate

$$\int_{D_1} h_\pm v_g \ge -\Big(\sup_{y\in D_5}\int_{D_1}\Big|G(x,y) - \frac{1}{2\pi}\log\varrho\Big| v_g(\mathrm{d}x)\Big)\int_{D_5} B_\pm\chi_4 v_g - F_\pm\int_{D_1}\int_{D_5}\Big[G(x,y) - \frac{1}{2\pi}\log\varrho\Big] \Delta\big[k(s^2)\big](y) v_g(\mathrm{d}y) v_g(\mathrm{d}x). \tag{3.21}$$

The first term is estimated from below by $-KF_\pm\varrho^4\gamma^{-1}$ with $K = K(c_1, c_2)$, using

$$\int_{D_1}\Big|G(x,y) - \frac{1}{2\pi}\log\varrho\Big| v_g(\mathrm{d}x) \le K\varrho^2, \quad y \in D_5, \tag{3.22}$$

which follows from (3.39), (3.36) and (3.38).

For the second term in (3.21) we recall that $k(s^2) = 2\varrho^2$ outside D_2. Hence

$$\begin{aligned}\int_{D_5}\Big[G(x,y) - \frac{1}{2\pi}\log\varrho\Big]\Delta\big[k(s^2)\big](y) v_g(\mathrm{d}y) &= \int_{D_5} G(x,y)\Delta\big[k(s^2) - 2\varrho^2\big](y) v_g(\mathrm{d}y) \\ &= k(s^2(x)) - 2\varrho^2 \le -\varrho^2\end{aligned} \tag{3.23}$$

using (3.3) and that $s(x) \le \varrho$. By (3.36) the second term in (3.21) is estimated from below by $\frac{1}{2}\pi F_\pm \varrho^4$, hence

$$\int_{D_1} h_\pm v_g \ge \frac{1}{4} F_\pm \pi \varrho^4 \tag{3.24}$$

if γ is big enough, depending only on c_1, c_2. In particular using (3.36) again

$$\max_{\overline{D}_1} h_\pm \ge \frac{1}{8} F_\pm \varrho^2. \tag{3.25}$$

On the other hand, if x is on the boundary of D_6, i.e., $s(x) = 6\varrho$, then we have from (3.39)

$$\begin{aligned} h_\pm(x) \le & \int_{D_5} \Big| G(x,y) - \frac{1}{2\pi} \log \varrho \Big| (B_\pm \chi_4)(y) v_g(\mathrm{d}y) \\ & - F_\pm \int_{D_2} \Big[G(x,y) - \frac{1}{2\pi} \log \varrho \Big] \Delta\big[k(s^2)\big](y) v_g(\mathrm{d}y) \\ \le & \Big(K_5 + \frac{1}{2\pi} \log 11 \Big) \int_{D_5} B_\pm \chi_4 v_g \\ & - F_\pm \int_{D_2} G(x,y) \Delta\big[k(s^2) - 2\varrho^2\big](y) v_g(\mathrm{d}y). \end{aligned} \tag{3.26}$$

Here we used (3.39) to estimate $G(x,y)$ by $H(x,y)$ and then the fact that $\varrho \le d(x,y) \le 11\varrho$ for $x \in \partial D_6$, $y \in D_5$. The second term in (3.26) is zero by (3.3) and (3.4), hence from (3.5)

$$h_\pm(x) \le F_\pm K \varrho^2 \gamma^{-1}, \quad x \in \partial D_6. \tag{3.27}$$

By comparing (3.25) and (3.27), we can therefore fix a $\gamma := \gamma(c_1, c_2)$ large enough so that

$$\max_{\overline{D}_1} h_\pm > \max_{\partial D_6} h_\pm.$$

Hence the maximum of $h_\pm$ on $\overline{D}_6$ is attained in the interior of D_6. Let $p_\pm \in D_6$ be (one of) these points. Since $\Delta h_\pm \ge 0$ outside of D_2, we obtain $p_\pm \in \overline{D}_2$. □

Proof of Lemma 3.4. — For simplicity, we drop the $\pm$ indices, the argument below is valid for both choices. We let $G_p(y) := G(p,y) - \frac{1}{2\pi}\log \varrho$, i.e., $\Delta G_p = \delta_p$. We define a measure μ on $D(p,\varrho)$ via a linear functional on continuous functions $\psi \in C(\overline{D}(p,\varrho))$ as follows:

$$(3.28)\qquad (\psi,\mu) = \int_{D(p,\varrho)} \psi \,\mathrm{d}\mu := 2\pi \int_0^{\varrho} \Big(\int_{\partial D(p,u)} \psi(\nu\cdot\nabla G_p)\sigma_g \Big) u \,\mathrm{d}u,$$

where ν is the outer normal of $\partial D(p,u)$. We have

$$\frac{1}{2\pi u} - K_6 \le \nu\cdot\nabla G_p \le \frac{1}{2\pi u} + K_6$$

on $\partial D(p,u)$ by (3.39) and $\int_{\partial D(p,u)} \sigma_g \le 4\pi u$ by (3.36). Thus (3.28) defines a continuous functional, hence a Borel measure μ. For small enough ϱ (depending only on c_1, c_2) this measure is nonnegative. By standard properties of the exponential map and σ_g, the measure μ is absolutely continuous with respect to v_g and its Radon-Nikodym derivative satisfies

$$(3.29)\qquad \frac{1}{2} \le \frac{\mathrm{d}\mu}{v_g} \le 2$$

on $D(p,\varrho)$, $p \in \overline{D}_2$, for small enough ϱ. Using (3.29) and Jensen's inequality

$$(3.30)\qquad \begin{aligned} \frac{1}{|D(p,\varrho)|}\int_{D(p,\varrho)} \mathrm{e}^h v_g &\ge \frac{1}{4\mu(D(p,\varrho))}\int_{D(p,\varrho)} \mathrm{e}^h \,\mathrm{d}\mu \\ &\ge \frac{1}{4}\exp\Big[\frac{1}{\mu(D(p,\varrho))}\int_{D(p,\varrho)} h \,\mathrm{d}\mu\Big]. \end{aligned}$$

For any $0 \le u \le \varrho$ and any smooth function ψ on $D(p,\varrho)$,

$$(3.31)\qquad \begin{aligned} \psi(p) &= \int_{D(p,u)} (\Delta G_p)\psi\, v_g \\ &= \int_{D(p,u)} G_p(\Delta\psi) v_g + \int_{\partial D(p,u)} \psi(\nu\cdot\nabla G_p)\sigma_g \\ &\qquad - \int_{\partial D(p,u)} G_p(\nu\cdot\nabla\psi)\sigma_g \end{aligned}$$

by Stokes' theorem. In particular for $\psi \equiv 1$,

$$1 = \int_{\partial D(p,u)} (\nu\cdot\nabla G_p)\sigma_g$$

for any u. Integrating it with respect to $2\pi u\,\mathrm{d}u$ from 0 to ϱ we obtain

$$(3.32)\quad \pi\varrho^2 = 2\pi\int_0^{\varrho} u\,\mathrm{d}u = 2\pi\int_0^{\varrho}\Big(\int_{\partial D(p,u)} (\nu\cdot\nabla G_p)\sigma_g\Big) u\,\mathrm{d}u = \mu\big(D(p,\varrho)\big)$$

by (3.28).

Now let $\psi = h$, we apply (3.31), integrate as before and use (3.28)

$$h(p)\pi\varrho^2 = 2\pi\int_0^\varrho h(p)u\,\mathrm{d}u$$
$$= \int_{D(p,\varrho)} h\,\mathrm{d}\mu + 2\pi\int_0^\varrho\Big[\int_{D(p,u)} G_p(\Delta h)v_g - \int_{\partial D(p,u)} G_p(\nu\cdot\nabla h)\sigma_g\Big]u\,\mathrm{d}u.$$

Combining this with (3.30) and (3.32) and recalling $h(p) = h^{\max}$, the statement (3.11) would follow from

$$2\pi\int_0^\varrho\Big[\int_{D(p,u)} G_p(\Delta h)v_g - \int_{\partial D(p,u)} G_p(\nu\cdot\nabla h)\sigma_g\Big]u\,\mathrm{d}u \le K\varrho^2. \tag{3.33}$$

To estimate the first term in (3.33), we recall that $\Delta h_\pm = B_\pm - F_\pm\Delta\big[k(s^2)\big]$ on $D(p_\pm, u)\subset D_3$ and $F_\pm \le K\varrho^{-2}$. Using (3.39) we have

$$\int_{D(p,u)} G_p(\Delta h)v_g \le K_5\int_{D(p,u)} |\Delta h|v_g + \frac{F}{2\pi}\int_{D(p,u)}\Big|\log\frac{s_p}{\varrho}\Big|\cdot\big|\Delta\big[k(s^2)\big]\big|\,v_g$$

which is bounded by a constant K using (3.6), (3.7) and (3.38). We also used that

$$\frac{1}{2\pi}\int_{D(p,u)}\Big(\log\frac{s_p}{\varrho}\Big)B_\pm v_g \le 0, \quad p = p_\pm,$$

since $B_\pm \ge 0$ and $s_{p_\pm} \le \varrho$ on $D(p_\pm, u)$ for $u \le \varrho$.

For the second term in (3.33) we use (3.39) again

$$\Big|\int_{\partial D(p,u)} G_p(\nu\cdot\nabla h)\sigma_g\Big| \le \frac{1}{2\pi}\Big|\Big(\log\frac{u}{\varrho}\Big)\int_{\partial D(p,u)}(\nu\cdot\nabla h)\sigma_g\Big| + K_5\int_{\partial D(p,u)}|\nabla h|\sigma_g. \tag{3.34}$$

We consider these two terms separately. In the first term we use Stokes's theorem and (3.6)

$$\Big|\int_{\partial D(p,u)}(\nu\cdot\nabla h)\sigma_g\Big| = \Big|\int_{D(p,u)}\Delta h v_g\Big| \le K$$

which gives a term of order ϱ^2 after the $\mathrm{d}u$-integration in (3.33).

For the second term in (3.34) we use the explicit form of h (3.8) and $|\nabla_x G(x,y)| \le K d(x,y)^{-1}$ for $x \in D(p,\varrho)$, $y \in D_5$ which follows from (3.39). Hence

$$\begin{aligned}
2\pi \int_0^{\varrho} &\Big[K_5 \int_{\partial D(p,u)} |\nabla h| \sigma_g\Big] u \,\mathrm{d}u \\
&\le K \int_0^{\varrho} \int_{\partial D(p,u)} \Big[\int_{D_5} |\nabla_x G(x,y)| \cdot |\widetilde{B}_\pm(y)| v_g(\mathrm{d}y)\Big] \sigma_g(\mathrm{d}x) u \,\mathrm{d}u \\
&\le K \int_{D_5} |\widetilde{B}_\pm(y)| \Big[\int_0^{\varrho} \Big(\int_{\partial D(p,u)} \frac{1}{s_y} \sigma_g\Big) u \,\mathrm{d}u\Big] v_g(\mathrm{d}y) \\
&\le K \int_{D_5} |\widetilde{B}_\pm(y)| \Big[\int_{D(p,\varrho)} \frac{s_p}{s_y} v_g\Big] v_g(\mathrm{d}y) \\
&\le K \varrho^2 \int_{D_5} |\widetilde{B}_\pm| v_g,
\end{aligned}$$

and finally we use (3.6) to complete the proof of (3.33). In the last but one step we used that $s_p = u$ on $\partial D(p,u)$ and that the measure $\int_0^{\varrho} \int_{\partial D(p,u)} [\cdots] \sigma_g \,\mathrm{d}u$ is bounded by $2v_g$ on $D(p,\varrho)$ for small enough ϱ (depending only on c_1, c_2). This follows from standard properties of the exponential map in a small enough neighborhood. □

3.1.6. *Standard estimates from Riemannian geometry.* — In the proof above we used the following information based on standard comparison results in Riemannian geometry. The lemma below guarantees that the geometry is approximately flat on a lengthscale smaller than R_0.

LEMMA 3.6. — *Let (M,g) be a compact two dimensional Riemann surface without boundary satisfying the geometric conditions (2.1) on the curvature and the injectivity radius. Let $R_0 := \frac{1}{8}\min\{c_2, c_1^{-1/2}\}$. Let $D(x,R)$ be the geodesic disk of radius R about $x \in M$ and assume that $R \le R_0$.*

(i) *The distance function $s_y(x) = d(x,y)$ satisfies*

$$3 \le \Delta_x s_y^2(x) \le 5 \quad \text{for } d(x,y) \le R. \tag{3.35}$$

(ii) *The area and perimeter of the geodesic disk $D(x,R)$ satisfies*

$$\frac{1}{2}\pi R^2 \le \big|D(x,R)\big| \le 2\pi R^2, \quad \pi R \le \big|\partial D(x,R)\big| \le 4\pi R. \tag{3.36}$$

(iii) (*Approximate mean value property*). *For any real function* $h \in C^2(D(x,R)) \cap C(\overline{D}(x,R))$,

$$\left|2\pi h(x) - \int_{D(x,R)} (\Delta h) \log \frac{s_x}{R} v_g - \frac{1}{R} \int_{\partial D(x,R)} h\sigma_g\right| \leq 2c_1 \int_{D(x,R)} |h| v_g. \tag{3.37}$$

(iv) (*Integral of the logarithm of the distance*). *For any positive numbers* λ, μ *with* $\lambda + \mu < 1$ *there exist two positive constants* $m_j = m_j(\lambda,\mu)$, $j = 1,2$, *such that for any* $R \leq R_0$, *and any* y, z *with* $d(y,z) \leq \mu R$,

$$-m_1 R^2 \leq \int_{D(y,\lambda R)} \log \frac{s_z}{R} v_g \leq -m_2 R^2. \tag{3.38}$$

(v) (*Estimates on the Green's function*). *Let* $G(x,y)$ *be the Green's function of the Laplacian on* $D = D(m,R)$ (*see* (3.3)) *and let*

$$H(x,y) := \frac{1}{2\pi} \log d(x,y),$$

then for $x, y \in D$,

$$\big|G(x,y) - H(x,y)\big| \leq K_5, \quad \big|\nabla_y [G(x,y) - H(x,y)]\big| \leq K_6, \tag{3.39}$$

with constants K_5, K_6 *depending only on* c_1, c_2.

Proof. — Parts (i), (ii), (iii) are standard (see e.g. [J], Section 4.6 and 4.7). To prove the lower bound in (3.38) we extend the integration to $D(z,(\lambda+\mu)R)$ then we apply part (iii) with $h = s_z^2$ and use (3.35). The upper bound follows from $\log(s_z/R) \leq \log(\mu+\lambda)$ and from (3.36). Finally the estimates on the Green's functions easily follow from the proof of Theorem 4.17 of [A]. □

3.2. Choice of the localization domain.

In this section we complete the proof of part (i) of Theorem 2.1. We need to find a geodesic disk $D(m,R) \subset M$ such that the flux condition (3.2) is satisfied but the area is not too small. This is the content of the following proposition. The estimate (2.5) then obviously follows from Proposition 3.1. □

PROPOSITION 3.7. — *There exist a constant K depending only on c_1, c_2 and a geodesic disk $D = D(m, R)$ around some $m \in M$ with radius $R \leq R_0 = \frac{1}{8}\min\{c_2, c_1^{-1/2}\}$ such that*

$$\frac{1}{2\pi}\int_D |B|v_g \leq \frac{1}{2}, \quad |D| \geq K \min\Big\{1, \frac{|M|}{\int_M |B|v_g}\Big\}. \tag{3.40}$$

Proof. — Around each point $x \in M$ we consider the geodesic disk $D(x, R(x))$ of radius $R(x) \leq R_0$ given by the condition

$$\int_{D(x,R(x))} |B|v_g = \pi, \tag{3.41}$$

or if $\int_{D(x,R_0)} |B|v_g < \pi$, then $R(x) := R_0$. If $R(x) = R_0$ for some x, then we can choose $D := D(x, R_0)$ and K accordingly to satisfy (3.40); henceforth we can assume that (3.41) is valid for all $x \in M$.

We can choose a finite covering $M \subset \bigcup_j \widetilde{D}_j$ with disks $\widetilde{D}_j := D(x_j, 4R(x_j))$ such that the disks $D_j := D(x_j, R(x_j))$ are disjoint. To do that, we first choose a finite covering from $M \subset \bigcup_{x\in M} D(x, 4R(x))$ by compactness, then we choose $\widetilde{D}_j$'s successively. Let $\widetilde{D}_1$ be the disk with the biggest radius. Once $\widetilde{D}_1, \widetilde{D}_2, \dots, \widetilde{D}_j$ are chosen, we choose $\widetilde{D}_{j+1}$ to be the disk of biggest radius among those whose center is not included in $\bigcup_{k=1}^{j} D(x_k, 2R(x_k))$. Let N be the number of disks selected in this way, it is easy to see that $\widetilde{D}_j$'s cover M and D_j's are disjoint ($j = 1, 2, \dots, N$).

We define $D := D_1$. By covering and (3.36) we have

$$|M| \leq \sum_{j=1}^{N} |\widetilde{D}_j| \leq (\text{const.}) \sum_{j=1}^{N} |D_j| \leq (\text{const.}) N\, |D|.$$

Furthermore,

$$\int_M |B|v_g \geq \sum_{j=1}^{N} \int_{D_j} |B|v_g = N\pi.$$

This completes the proof of (3.40). □

4. Proof of Theorem 2.2.

This proof is an easy consequence of part (i) of Theorem 2.1 since the multiplicity of E can be controlled by E itself (the same idea was used in [BCC]). We consider the trace of the heat kernel of H_∇ for $t > 0$:

$$\dim(\mathcal{N})\, \mathrm{e}^{-tE} \leq \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{-tH_\nabla}. \tag{4.1}$$

On the other hand

$$\mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{-tH_\nabla} \le \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t\Delta} = \int_M \mathrm{e}^{t\Delta}(x,x) v_g(\mathrm{d}x) \tag{4.2}$$

by the diamagnetic inequality, where Δ is the Laplacian on functions on M. We recall that the diamagnetic inequality (semigroup domination) on complex line bundles follows directly from the Feynman-Kac formula as in the flat case (see, e.g., Theorem 3.5 in [R]). Another proof can be obtained via Trotter product formula and Kato's inequality following the corresponding proof in the flat case (see [HLMW]).

The estimate on the heat kernel is standard:

$$\mathrm{e}^{t\Delta}(x,x) \le (\mathrm{const.})\Big(\frac{1}{|M|} + \frac{1}{t}\cdot\frac{1}{I^2(M)}\Big), \quad x \in M, \tag{4.3}$$

see, e.g., formulas (2.9) and (1.18) in [CL].

By choosing $t := E^{-1}$ we obtain (2.6) from (4.1), (4.2), (4.3) and (2.5). □

5. Manifolds with boundary.

In this section we indicate the additional ideas that are needed for the proof of Theorem 2.4. The core of the argument (Section 3.1) is local and is valid for the $\partial M \neq \emptyset$ case as well if the center of the disk $D = D(m,R)$ in Proposition 3.1 satisfies $\mathrm{dist}(m, \partial M) \ge R$.

For any $\varrho_0 > 0$ let

$$\widetilde{M} := \Big\{ x \in M : d(x, \partial M) \ge \frac{1}{2}\varrho_0 \Big\}$$

be the set of points that are at distance at least $\frac{1}{2}\varrho_0$ away from the boundary. The threshold ϱ_0 will depend only on c_1, c_2, c_4 (see (2.1) and (2.9)). The following lemma shows that the volumes of $\widetilde{M}$ and M are comparable if ϱ_0 is small.

LEMMA 5.1. — *Assuming* (2.1) *and* (2.9), *there exist positive constants* ϱ_0 *and* K', *depending only on* c_1, c_2 *and* c_4, *such that*

$$|\widetilde{M}| \ge K'|M|. \tag{5.1}$$

Proof. — For sufficiently small ϱ the cone condition (2.9) guarantees that for any $x \in M$ there exists a point $p(x) \in M$, $\mathrm{dist}(x, p(x)) \le c_2$ such that the disk $D(p(x), \varrho_0)$ is disjoint from ∂M hence the disk $D(p(x), \frac{1}{2}\varrho_0)$

is fully contained in $\widetilde{M}$. By a standard covering argument one can select a finite collection of points x_i such that the disks $D(p(x_i), \frac{1}{2}\varrho_0)$ be disjoint and the disks $D(p(x_i), c_2)$ cover. A volume comparison estimate (3.36) gives (5.1). □

For the rest of the proof of Theorem 2.1 (i), we replace M with $\widetilde{M}$ in Proposition 3.7, this results in a disk $D = D(m, R)$ with center in $\widetilde{M}$. We choose the radius R to be smaller than ϱ_0 and apply the argument of Section 3.1. □

For the proof of Theorem 2.2 in the $\partial M \neq \emptyset$ case we repeat the argument in Section 4. The diamagnetic inequality is valid for manifolds with boundary; the Laplacian being replaced with the Dirichlet or Neumann Laplacian, Δ^D or Δ^N, depending on the boundary condition of H_∇ (see [HLMW] for a recent careful treatment of both boundary conditions). The diagonal elements of the Dirichlet heat kernel can be estimated by the Neumann one. The Neumann heat kernel is estimated in (2.9) of [CL] as

$$\mathrm{e}^{t\Delta^N}(x,x) \leq (\mathrm{const.})\Big(\frac{1}{|M|} + \frac{1}{t}\cdot\frac{1}{I^2(M)}\Big).$$

The estimate (2.2) on $I(M)$ completes the proof of Theorem 2.4 on the multiplicity similarly to the proof of Theorem 2.2. □

6. Examples.

We explore the sharpness of our theorems via a few examples and in particular we prove part (ii) of Theorem 2.2 and Theorem 2.3.

6.1. Example 1: spectral shift with potentials.

Theorem 2.1 states that the inclusion of the magnetic field raises the spectrum of the Laplacian by at most the absolute flux per unit volume (apart from geometric constants depending on c_1, c_2, c_3). A similar statement with a potential would claim ($H_0 = -\Delta$),

$$\inf \mathrm{Spec}\big(H_\nabla + V\big) \leq \inf \mathrm{Spec}(H_0 + V) + K + \frac{K}{|M|}\int |B| v_g,$$

but this is *false*. To see this, let $V(x)$ be equal to the constant $-U \ll 0$ on a small ε-neighborhood of a point $m \in M$, $x \in D(m, \varepsilon)$, and zero elsewhere.

We can assume flat geometry around m. An appropriately scaled trial function localized on $D(m,\varepsilon)$ gives $\inf \mathrm{Spec}(H_0 + V) \leq (\mathrm{const.})\varepsilon^{-2} - U$ for any ε. Let $B = U$ on $D(m,\varepsilon)$ and zero elsewhere. Using (2.4) it is easy to see that $H_\nabla + V \geq 0$. Choosing $U \gg \varepsilon^{-2}$, we see that the spectral shift due to the magnetic field is at least of order U, while the average flux can be kept tiny if $|M| \gg U\varepsilon^2$.

6.2. Example 2: proof of part (ii) of Theorem 2.1: bound on the injectivity radius is necessary.

PROPOSITION 6.1. — *For any $\delta > 0$, there exists a sequence of magnetic Schrödinger operators H_n on the trivial bundle over appropriate manifolds M_n, $\partial M_n = \emptyset$, with uniformly bounded Gauss curvature, $|\kappa(M_n)| \leq \delta$, and with a magnetic field B_n such that*

$$\limsup_{n\to\infty} \left[\inf \mathrm{Spec}(H_n) \right] = \infty$$

but $|M_n|^{-1} \int_{M_n} |B_n|$ remains bounded. The injectivity radius of M_n, of course, tends to zero.

Proof. — Let $F : \mathbb{R} \to \mathbb{R}_+$ be a function supported on $[0, L]$, $F \in C^\infty(0, L)$, $F(0) = F(L) = 0$, such that $F(x) \equiv F(L - x)$, $F \equiv \varepsilon$ on $[\ell, L - \ell]$, with $L \gg \ell \gg 1$, $\varepsilon \ll 1$ chosen later. Let $M = M_F \subset \mathbb{R}^3$ be the surface of revolution of F around the x-axis with the metric inherited from $\mathbb{R}^3$. For any $\varepsilon, \delta > 0$ one can choose $\ell = \ell(\varepsilon, \delta)$ large enough so that for any $L \geq 2\ell$ there exists an $F = F_{\varepsilon,\delta,L}$ so that the Gauss curvature of M satisfies $|\kappa| \leq \delta$ on the whole M. Of course, $F'(0 + 0) = \infty$ and $F'(L - 0) = -\infty$ and $\ell(\varepsilon, \delta) \to \infty$ as $\varepsilon, \delta \to 0$. Roughly, M looks like two distant big spheres connected with a narrow tube and with smoothed out joints (see Fig. 1). In fact, the joints have to be long to keep the curvature almost zero and get the necessary narrowing. For such manifold $|\kappa|$ is small, the injectivity radius is also small (of order ε) and the volume can be arbitrarily big as $L \to \infty$.

To define the magnetic field, we partition M into five pieces $M = \bigcup_{j=1}^5 M_j$ such that

$$M_1 := M \cap \{x \leq \ell\}, \qquad M_2 := M \cap \{\ell < x < 2\ell\},$$

$$M_3 := M \cap \{2\ell \leq x \leq L - 2\ell\}, \qquad M_4 := M \cap \{L - 2\ell < x < L - \ell\},$$

$$M_5 := M \cap \{L - \ell \leq x \leq L\}.$$

We assume $L > 4\ell$.

Now we choose B such that $B(x) = U \gg 0$ for $x \in M_1 \cup M_2$, $B(x) = 0$ on M_3 and $B(x) = -U$ for $x \in M_4 \cup M_5$. We also ensure that $\frac{1}{2\pi}\int_{M_1\cup M_2} Bv_g$ be a half integer, i.e., $U|M_1 \cup M_2| = 2\pi(N + \frac{1}{2})$ with some $N \in \mathbb{N}$. Notice that $\int_M Bv_g = 0$ by symmetry, hence one can realize this magnetic field as the curvature of a connection ∇ on the trivial line bundle. The field B is not smooth to simplify our construction, but it can be smoothed out on a very small lengthscale without changing the conclusion.

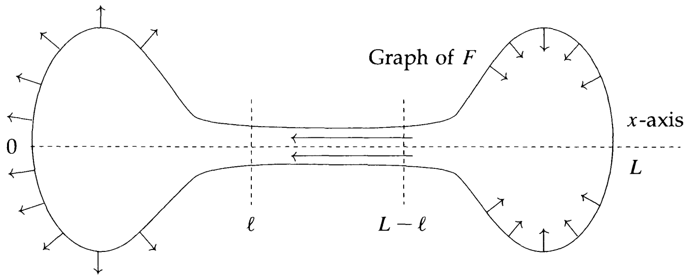

Figure 1. Surface of revolution with magnetic field vectors

We claim that

$$\limsup_{\varepsilon,\delta\to 0} \limsup_{\substack{U\to\infty \\ U|M_1\cup M_2|\in 2\pi(\mathbb{N}+\frac{1}{2})}} \limsup_{L\to\infty} \Big[\inf \operatorname{Spec}(H_\nabla)\Big] = \infty, \tag{6.1}$$

on the other hand, it is clear that

$$\limsup_{\varepsilon,\delta\to 0} \limsup_{\substack{U\to\infty \\ U|M_1\cup M_2|\in 2\pi(\mathbb{N}+\frac{1}{2})}} \limsup_{L\to\infty} \frac{1}{|M|}\int_M |B|v_g = 0.$$

These statements prove Proposition 6.1 and then part (ii) of Theorem 2.1.

To show (6.1), we define a partition of unity $1 \equiv \chi_1^2 + \chi_3^2 + \chi_5^2$ such that $\chi_j \equiv 1$ on M_j, $j = 1, 3, 5$ and $\mathrm{d}\chi_j$ is supported on a compact subset of $M_2 \cup M_4$. We also assume that $|||\,\mathrm{d}\chi_j|||\le (\text{const.})\ell^{-1}$. By Schwarz inequality

$$Q(\xi,\xi) \ge \frac{1}{2}\sum_{j=1,3,5} Q(\xi\chi_j,\xi\chi_j) - (\text{const.})\ell^{-2}\|\xi\|^2. \tag{6.2}$$

By (2.4)

$$Q(\xi\chi_j,\xi\chi_j) \ge U\|\xi\chi_j\|^2 \tag{6.3}$$

for $j = 1$ and $j = 5$. We write

$$Q(\xi\chi_3, \xi\chi_3) = \sum_{j=2}^{4} \int_{M_j} \left\|\left| \mathrm{d}(f\chi_3) - i\alpha f\chi_3 \right\|\right|^2 v_g \tag{6.4}$$

if $\xi = f\phi$ and $\nabla\phi = -i\alpha\phi$, where ϕ is a global normalized section.

We claim that

$$\int_{M_j} \left\|\left| \mathrm{d}(f\chi_3) - i\alpha f\chi_3 \right\|\right|^2 v_g \geq (\mathrm{const.}) U \int_{M_j} |f\chi_3|^2 v_g \quad \text{for } j = 2, 4, \tag{6.5}$$

and

$$\int_{M_3} \left\|\left| \mathrm{d}(f\chi_3) - i\alpha f\chi_3 \right\|\right|^2 v_g \geq (\mathrm{const.}) \varepsilon^{-1} \int_{M_3} |f\chi_3|^2 v_g \tag{6.6}$$

with positive universal constants. In particular, these estimates inserted into (6.4) imply that

$$Q(\xi\chi_3, \xi\chi_3) \geq (\mathrm{const.}) \min\{U, \varepsilon^{-1}\} \|\xi\chi_3\|^2. \tag{6.7}$$

Finally, from (6.2), (6.3) and (6.7) our claim (6.1) follows.

The estimates (6.5) and (6.6) are given in the following two lemmas applied to $f\chi_3$. In both lemmas we let S^1_ε be the circle of radius ε and let $Z = (a, b) \times S^1_\varepsilon$ be an open cylinder equipped with the standard metric $g = \mathrm{d}x^2 + \varepsilon^2 \,\mathrm{d}\theta^2$, $v_g = \varepsilon \,\mathrm{d}x \,\mathrm{d}\theta$. We use $x \in (a, b)$, $\theta \in [0, 2\pi)$ cylindrical coordinates on Z.

LEMMA 6.2. — *Let f be any smooth function on Z that is zero in a neighborhood of the lower boundary $\{a\} \times S^1_\varepsilon$ of Z. Let α be a 1-form with $\mathrm{d}\alpha = \pm U v_g$, $U \in \mathbb{R}_+$, then there exists a universal positive constant such that*

$$\int_Z \||\mathrm{d}f - i\alpha f\||^2 v_g \geq (\mathrm{const.})\, U \int_Z |f|^2 v_g. \tag{6.8}$$

Similar statement is valid if f vanishes around the upper boundary $\{b\} \times S^1_\varepsilon$ by symmetry.

LEMMA 6.3. — *Let α be a closed 1-form on Z with $\frac{1}{2\pi} \int_C \alpha \in \mathbb{N} + \frac{1}{2}$ where C is a closed curve in Z such that its projection onto S^1_ε winds around once (this integral is independent of the choice of C). Then there exists a universal positive constant such that*

$$\int_Z \||\mathrm{d}f - i\alpha f\||^2 v_g \geq (\mathrm{const.})\, \varepsilon^{-1} \int_Z |f|^2 v_g \tag{6.9}$$

for any smooth function f on Z.

Proof of Lemma 6.2. — For definiteness, we consider the case $\mathrm{d}\alpha = Uv_g$, the other case is identical. Since $\mathrm{d}(\varepsilon Ux\,\mathrm{d}\theta) = Uv_g$, it is easy to see that $\alpha = \varepsilon Ux\,\mathrm{d}\theta + c\,\mathrm{d}\theta + \mathrm{d}\varphi$ with some smooth function φ on Z and some constant c. Changing f to $\mathrm{e}^{i\varphi} f$, it is sufficient to estimate

$$\int_Z \left|\left|\left| \mathrm{d}f - i(\varepsilon Ux + c) f\,\mathrm{d}\theta \right|\right|\right|^2 v_g$$
$$= \varepsilon \int_a^b \int_0^{2\pi} \left[|\partial_x f|^2 + |\varepsilon^{-1}\partial_\theta f - i(Ux + \varepsilon^{-1}c) f|^2 \right] \mathrm{d}\theta\,\mathrm{d}x$$

for any function f on Z. We expand f into Fourier series as

$$f(x,\theta) = \sum_{k=-\infty}^{\infty} f_k(x)\,\mathrm{e}^{ik\theta},$$

hence

$$\int_Z \left|\left|\left| \mathrm{d}f - i(\varepsilon Ux + c) f\,\mathrm{d}\theta \right|\right|\right|^2 v_g$$
$$= 2\pi\varepsilon \sum_{k=-\infty}^{\infty} \int_a^b \left[|f_k'(x)|^2 + |\varepsilon^{-1}(k-c) - Ux|^2 |f_k(x)|^2 \right] \mathrm{d}x.$$

For each $m = \varepsilon^{-1}(k-c)$ we can minimize the quadratic form

$$\Psi(g,g) = \int_a^b \left[|g'(x)|^2 + |m - Ux|^2 |g(x)|^2 \right] \mathrm{d}x$$

of the shifted harmonic oscillator under Dirichlet boundary condition at a; $g(a) = 0$, and free (Neumann) boundary condition at b, i.e., $g'(b) = 0$.

We claim that

$$\Psi(g,g) \geq (\text{const.})\, U \int_a^b |g(x)|^2\,\mathrm{d}x \tag{6.10}$$

from which Lemma 6.2 follows. Notice that for g with Dirichlet boundary conditions on both ends we would have $\Psi(g,g) \geq U \int_a^b |g|^2$ using the explicit lowest eigenvalue U of the harmonic oscillator on $\mathbb{R}$.

If $|m - bU| \geq 5\sqrt{U}$, then $|m - Ux|^2 \geq U$ for all $x \in [b - 4/\sqrt{U}, b]$. In this case we can write $g = g\zeta_1 + g\zeta_2$, where $\zeta_1^2 + \zeta_2^2 \equiv 1$, $\zeta_1(x) \equiv 1$ for $x \leq b - 4/\sqrt{U}$, $\zeta_1 \equiv 0$ for $x \geq b$ and $|\zeta_1'|, |\zeta_2'| \leq \sqrt{U}/3$. Therefore, by Schwarz inequality,

$$\Psi(g,g) \geq \frac{1}{2}\left[\Psi(g\zeta_1, g\zeta_1) + \Psi(g\zeta_2, g\zeta_2)\right] - \left(\|\zeta_1'\|_\infty^2 + \|\zeta_2'\|_\infty^2\right) \int_a^b |g|^2 \geq \frac{5U}{18} \int_a^b |g|^2.$$

If $|m - bU| \le 5\sqrt{U}$, then we can symmetrize the potential around b by defining $W(x) := |m - Ux|^2$ for $x \le b$ and $W(x) := |m - U(2b - x)|^2$ for $x \ge b$. Then the lowest eigenvalue of Ψ with Neumann boundary condition at b will be the same as the lowest eigenvalue of $-\partial^2/\partial x^2 + W(x)$ on $[a, 2b - a]$ with Dirichlet boundary conditions on both ends. On the other hand, $W(x) \ge \eta^2 U^2 (x - b)^2 - \frac{1}{2}\eta U$ for a small enough $\eta > 0$, hence $-\partial^2/\partial x^2 + W(x) \ge \frac{1}{2}\eta U$, using the lowest eigenvalue of the harmonic oscillator on $\mathbb{R}$. This completes the proof of (6.10). □

Proof of Lemma 6.3. — We clearly have

$$\int_Z |||\,\mathrm{d}f - i\alpha f|||^2 v_g \ge \varepsilon^{-1} \int_a^b \int_0^{2\pi} \big|\partial_\theta f - i\alpha(\partial_\theta) f\big|^2 \,\mathrm{d}\theta \,\mathrm{d}x, \tag{6.11}$$

and recall that $f(x, 0) = f(x, 2\pi)$. Let

$$X := \big\{x \in (a, b) : \exists\, \theta_x \text{ s.t.} f(x, \theta_x) = 0\big\}.$$

For any fixed $x \in X$ we have

$$\int_0^{2\pi} \big|\partial_\theta f - i\alpha(\partial_\theta) f\big|^2 \,\mathrm{d}\theta \ge \int_0^{2\pi} \big|\partial_\theta |f|\big|^2 \,\mathrm{d}\theta \ge \int_0^{2\pi} |f|^2 \,\mathrm{d}\theta, \tag{6.12}$$

by the lowest eigenvalue of the Dirichlet Laplacian on S^1, viewing $|f| = |f(x, \cdot)|$ as a periodic function in θ.

For $x \notin X$ we let $f = \mathrm{e}^{i\varphi} g$ with $g = |f|$ and $\varphi(0) - \varphi(2\pi) \in 2\pi\mathbb{Z}$, then

$$\int_0^{2\pi} \big|\partial_\theta f - i\alpha(\partial_\theta) f\big|^2 \,\mathrm{d}\theta \ge \int_0^{2\pi} \big[|\partial_\theta g|^2 + A^2 |g|^2\big] \,\mathrm{d}\theta,$$

where $A = \alpha(\partial_\theta) - \partial_\theta \varphi$, hence $\int_C A \,\mathrm{d}\theta = \int_C \alpha + \varphi(0) - \varphi(2\pi) \in 2\pi(\mathbb{N} + \frac{1}{2})$. With a slight abuse of notation, we dropped the variable x since it is fixed, i.e., we assume that A, g, φ are functions on S^1 and ∂_θ is denoted by prime. It is sufficient to show that for some small universal constant $1 > \eta > 0$ (to be chosen later) we have

$$\int_0^{2\pi} \big[|g'|^2 + A^2 |g|^2\big] \,\mathrm{d}\theta \ge \eta \int_0^{2\pi} |g|^2 \,\mathrm{d}\theta, \quad \forall g,\ g(0) = g(2\pi), \tag{6.13}$$

then (6.9) would follow from (6.11), (6.12) and (6.13). We can assume that $\frac{1}{2\pi}\int_0^{2\pi} g^2 = 1$ and for simplicity we use the notation

$$⨍ \quad \text{for} \quad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta.$$

Let $\bar{g} := \fint g$ and $\widetilde{g} = g - \bar{g}$. By the spectral gap of the Laplacian on S^1 we have

$$\fint |g'|^2 = \fint |\widetilde{g}'|^2 \geq \fint \widetilde{g}^2.$$

If $\fint \widetilde{g}^2 \geq \eta \fint g^2 = \eta$, then (6.13) is proven, hence we can assume that $\fint \widetilde{g}^2 < \eta$. Then

$$|1 - \bar{g}^2| = \Big| \fint (g^2 - \bar{g}^2) \Big| = \fint \widetilde{g}^2 < \eta,$$

in particular $\bar{g}^2 > 1 - \eta$. Let $\mu := \sup |\widetilde{g}|$, then

$$\mu^2 \leq (\sup \widetilde{g} - \inf \widetilde{g})^2 \leq (2\pi)^2 \Big(\fint |\widetilde{g}'| \Big)^2 \leq (2\pi)^2 \fint |\widetilde{g}'|^2 = (2\pi)^2 \fint |g'|^2.$$

If $\mu \geq 2\pi\sqrt{\eta}$, then (6.13) is proven, so we can assume that $\mu < 2\pi\sqrt{\eta}$. In this case we use Schwarz inequality $g^2 \geq \frac{1}{2}\bar{g}^2 - \widetilde{g}^2$ to estimate

$$\fint A^2 g^2 \geq \Big(\frac{1}{2}\bar{g}^2 - \mu^2 \Big) \fint A^2 \geq \frac{1 - (1 + 8\pi^2)\eta}{2} \Big(\fint A \Big)^2$$

if $\eta < (1 + 8\pi^2)^{-1}$. But $\big| \fint A \big| \geq \frac{1}{2}$, hence (6.13) is proven with $\eta = (9 + 8\pi^2)^{-1}$. □

6.3. Proof of Theorem 2.3: any finite sequence of multiplicities can be realized on the trivial bundle.

We will use the method of stable perturbations developed in [CdV86], [CdV87], [CdV88] and used in the magnetic example constructed in [CdVT] with a potential. The magnetic Schrödinger operator on S^2 with a nonzero constant field has a degenerate ground state, but the bundle is nontrivial. Changing the bundle involves a nonperturbative change in the curvature form β. If an additive scalar potential is allowed [CdVT], then one can consider a very strong positive potential supported on a small disk as an "almost" Dirichlet boundary condition and one can trivialize the bundle on the complement. Without potential, the corresponding region of trivialization is not forbidden. Nevertheless, we can keep the spectral effect of the trivialization under control by combining L^∞-estimates obtained from elliptic regularity with a special logarithmic cutoff function known from the proof of the density of $C_0(\mathbb{R}^2 \setminus \{0\})$ in H^1. We then supplement this argument by a Hilbert space perturbation technique of quadratic forms. This second part is similar to [CdV86] so we skip some straighforward steps but we keep the presentation self-contained.

Proof of Theorem 2.3. — We let $M := S^2$ with the standard metric, written as $g = (\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\varphi)^2$ in usual spherical coordinates, where θ is the geodesic distance from the North pole N. Let L_n be the line bundle over M with Chern number $n \in \mathbb{Z}$; this is unique up to bundle isomorphism. Let $n+1 = m_1 + m_2 + \cdots + m_k$, we can assume $n \geq 1$. In this proof we use (const.) to denote various constants depending only on n.

There is a connection ∇^c with constant curvature $\beta_c = \frac{1}{2}nv_g$ on L_n. Using (2.4), $H_{\nabla^c} = (\nabla^c)^*\nabla^c \geq \frac{1}{2}n$, and we let $\mathcal{H}_0 := \mathrm{Ker}(H_{\nabla^c} - \frac{1}{2}n)$; this is the space of the holomorphic sections of L_n. By the Riemann-Roch theorem, $\dim \mathcal{H}_0 = n+1$. By resolvent compactness of H_{∇^c}, there is a gap of size $\gamma = \gamma(n) > 0$ above the ground state in the spectrum of H_{∇^c}.

We will construct a connection ∇ on L_0 that has the same constant curvature as ∇^c apart from a small neighborhood of the South pole $S \in S^2$. The curvatures R_∇ and R_{∇^c} will differ drastically around S to accomodate the different Chern numbers. Nevertheless, $H_\nabla = \nabla^*\nabla$ on L_0 will be considered as an appropriate perturbation of H_{∇^c} on L_n.

To compare operators on L_0 and L_n we introduce coordinates. For an appropriate normalized section $\phi_n \in \Gamma(L_{n|M\setminus\{S\}})$, the connection 1-form of ∇^c becomes $\alpha_c = n(\sin\frac{1}{2}\theta)^2\,\mathrm{d}\varphi$ away from S. We define the following quadratic form on $L^2(M, v_g)$:

$$\widehat{Q}(f,f) = \widehat{Q}_{\alpha_c}(f,f) := \int_M |||\,\mathrm{d}f - i\alpha_c f|||^2 v_g$$

on its maximal domain

$$D_{\max}(\widehat{Q}) := \big\{f \in L^2(M, v_g) : \widehat{Q}(f,f) < \infty\big\},$$

then $Q(f\phi_n, f\phi_n) = \widehat{Q}(f,f)$. The minimal form domain, $D_{\min}(\widehat{Q})$, is defined as the closure of the set

$$\mathcal{C}_n := \Big\{f \in C^\infty(M \setminus \{S\}) : f\,\mathrm{e}^{-in\varphi} \text{ extends to } C^\infty(M \setminus \{N\})\Big\}$$

with respect to the norm $\|\cdot\|_+ := (\|\cdot\|^2 + \widehat{Q}(\cdot,\cdot))^{1/2}$, where $\|\cdot\|$ is the $L^2(M, v_g)$ norm. Using a standard density argument (e.g., [Si]) it is easy to see that $D_{\max}(\widehat{Q}) = D_{\min}(\widehat{Q}) =: D(\widehat{Q})$ since $|||\alpha_c|||$ is regular away from S, and $|||\alpha_c - n\,\mathrm{d}\varphi|||$ is regular away from N.

Let the operator $\widehat{H} := \widehat{H}_{\alpha_c}$ denote the corresponding Friedrichs extension of $\widehat{Q} = \widehat{Q}_{\alpha_c}$. Then the operators H_{∇^c} and $\widehat{H}$ are unitarily equivalent. Similar identification is valid for any connection ∇ on any line bundle L_m, replacing α_c with the connection 1-form α of ∇ determined by a fixed section $\phi_m \in \Gamma(L_{m|M\setminus\{S\}})$.

In particular, we fix a global section ϕ_0 on the trivial bundle L_0. For any connection ∇ on L_0 we will identify the operator $\nabla^*\nabla$ with $\widehat{H}_\alpha$, where $\nabla\phi_0 = -i\alpha\phi_0$. Here the connection 1-form α is global and $\widehat{H}_\alpha$ is defined as the Friedrichs extension of $\widehat{Q}_\alpha(f,f) := \int_M |||\,\mathrm{d}f - i\alpha f|||^2 v_g$ with core $\mathcal{C}_0 = C^\infty(M)$. Explicitly,

$$\widehat{H}_\alpha = (\mathrm{d} - i\alpha)^*(\mathrm{d} - i\alpha) = (\delta - i(\star\alpha)\wedge)(\mathrm{d} - i\alpha). \tag{6.14}$$

We also need (2.4) in coordinates. Let α be a global 1-form, $\mathrm{d}\alpha = \beta = Bv_g$, then

$$\widehat{Q}_\alpha(f,f) = \int_0^{2\pi}\int_0^{\pi}\Big[4|\partial_{\bar z} f - i\alpha(\partial_{\bar z})f|^2 + B|f|^2\Big]\frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta} \tag{6.15}$$

with $\partial_{\bar z} := \frac{1}{2}\left[(\sin\theta)\partial_\theta + i\partial_\varphi\right]$. This is valid for any $f \in C^\infty(M)$ either by a direct calculation or by considering $\xi = f\phi_0 \in \Gamma(L_0)$, then extending it to any function $\widehat{Q}_\alpha(f,f) < \infty$ since $D_{\max}(\widehat{Q}_\alpha) = D_{\min}(\widehat{Q}_\alpha)$. Similarly

$$\begin{aligned} \widehat{Q}(f,f) &= \widehat{Q}_{\alpha_c}(f,f) \\ &= \int_0^{2\pi}\int_0^{\pi}\Big[|\partial_\theta f|^2 + \Big|\frac{i}{\sin\theta}\partial_\varphi f + \frac{n}{2}\tan\frac{\theta}{2} f\Big|^2\Big]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \frac{n}{2}\|f\|^2 + 4\int_0^{2\pi}\int_0^{\pi}\big|\partial_{\bar z} f - i\alpha_c(\partial_{\bar z})f\big|^2\frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta}. \end{aligned} \tag{6.16}$$

This relation is valid for any $f \in \mathcal{C}_n$ by a direct calculation (or using (2.4) for $\xi = f\phi_n \in \Gamma(L_n)$), then by $D_{\max}(\widehat{Q}) = D_{\min}(\widehat{Q})$ it extends to any function f with $\widehat{Q}(f,f) < \infty$.

We can choose a normalized basis $\{\xi_0, \xi_1, \dots, \xi_n\}$ in $\mathcal{H}_0$ by specifying the Taylor polynomial of ξ_k to be z^k at the north pole N. The sections ξ_k can be written as $\xi_k = f_k\phi_n$, where the functions $f_k \in \mathcal{C}_n$ are given in the usual spherical coordinates

$$f_k(\theta,\varphi) := c_k\,\mathrm{e}^{ik\varphi}\Big(\cos\frac{\theta}{2}\Big)^{n-k}\Big(\sin\frac{\theta}{2}\Big)^k, \quad k = 0,1,2,\dots,n$$

with an appropriate normalizing constant. The functions f_k are orthonormal in $L^2(M, v_g)$ and let $\widehat{\mathcal{H}}_0$ be their span; this is the ground state eigenspace of $\widehat{H}$ with eigenvalue $\frac{1}{2}n$ and dimension $n+1$. Let P_0 be the projection onto $\widehat{\mathcal{H}}_0$ in $L^2(M, v_g)$. The gap for H_{∇^c} means that

$$\widehat{Q}(h,h) \geq \Big(\frac{n}{2} + \gamma\Big)\|h\|^2, \quad \forall h \perp \widehat{\mathcal{H}}_0. \tag{6.17}$$

PROPOSITION 6.4. — *For any positive integer n and $0 < \varepsilon \ll 1$ there exists a magnetic Schrödinger operator $H_\varepsilon = \nabla_\varepsilon^* \nabla_\varepsilon$ on L_0 so that the corresponding operator $\widehat{H}_\varepsilon$ on $\widehat{\mathcal{H}} = L^2(M, v_g)$ has the following properties :*

(i) *Let $\widehat{\mathcal{H}}_\varepsilon$ be the spectral subspace of $\widehat{H}_\varepsilon$ with eigenvalues smaller than $\frac{n}{2} + \frac{\gamma}{9n}$. Then $\dim \widehat{\mathcal{H}}_\varepsilon = n+1$ and*

$$\lim_{\varepsilon \to 0} \left\| \left(\widehat{H}_\varepsilon - \frac{n}{2} \right)_{|\widehat{\mathcal{H}}_\varepsilon} \right\| = 0. \tag{6.18}$$

(ii) *For any $f \in \widehat{\mathcal{H}}_\varepsilon$ we have*

$$\|f\|_\infty \leq (\text{const.}) \|f\| \tag{6.19}$$

with a constant depending only on n.

(iii) There exists an orthonormal basis $\{f_0^\varepsilon, f_1^\varepsilon, \ldots, f_n^\varepsilon\}$ in $\widehat{\mathcal{H}}_\varepsilon$ such that

$$\limsup_{\varepsilon \to 0} \max_j \|f_j^\varepsilon - f_j\| = 0, \tag{6.20}$$

and for any $u > 0$,

$$\lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi - u} \left| \partial_{\bar{z}} f_j^\varepsilon - i\alpha_c(\partial_{\bar{z}}) f_j^\varepsilon \right|^2 \frac{\mathrm{d}\theta \, \mathrm{d}\varphi}{\sin\theta} = 0. \tag{6.21}$$

Proof. — For any $0 < \varepsilon \ll 1$ we let

$$\chi^\varepsilon(\theta) := \begin{cases} 1 & \text{for } 0 \leq \theta \leq \pi - \varepsilon, \\ 2 - \dfrac{\log(\pi - \theta)}{\log \varepsilon} & \text{for } \pi - \varepsilon \leq \theta \leq \pi - \varepsilon^2, \\ 0 & \text{for } \pi - \varepsilon^2 \leq \theta \leq \pi \end{cases}$$

and we notice that

$$\int_M ||| \mathrm{d}\chi^\varepsilon |||^2 v_g \leq \frac{(\text{const.})}{|\log \varepsilon|}. \tag{6.22}$$

We also choose a monotone C^∞ cutoff function η^ε on $[0, \pi]$, $0 \leq \eta^\varepsilon \leq 1$, such that $\eta^\varepsilon \equiv 1$ on $\operatorname{supp}(\chi^\varepsilon)$ and $\eta^\varepsilon \equiv 0$ on $[\pi - \frac{1}{2}\varepsilon^2, \pi]$. Let $\alpha_\varepsilon := \eta^\varepsilon \alpha_c$, $\widehat{Q}_\varepsilon(f, f) := \int_M ||| \mathrm{d}f - i\alpha_\varepsilon f |||^2 v_g$ with form core $C^\infty(M)$ and $\widehat{H}_\varepsilon$ be its Friedrichs extension. Clearly

$$\widehat{Q}(\chi^\varepsilon f, \chi^\varepsilon f) = \widehat{Q}_\varepsilon(\chi^\varepsilon f, \chi^\varepsilon f). \tag{6.23}$$

For $\delta > 0$ we define $\widehat{\mathcal{H}}_{\varepsilon,\delta} := \operatorname{Ran}(P_{\varepsilon,\delta})$ and $P_{\varepsilon,\delta} := \Pi_\delta(\widehat{H}_\varepsilon)$, where Π_δ is the spectral projection onto $(-\infty, \frac{1}{2}n + \delta\gamma]$.

LEMMA 6.5. — *Let $\delta \leq \frac{1}{9n}$, then*

(i) *one has*

$$\limsup_{\varepsilon\to 0} \widehat{Q}_\varepsilon(\chi^\varepsilon f, \chi^\varepsilon f) \leq \limsup_{\varepsilon\to 0} \widehat{Q}_\varepsilon(f,f) \tag{6.24}$$

uniformly on $\{f \in C^\infty(M) : \|f\|_\infty \leq K\}$ for any K;

(ii) *one has*

$$\limsup_{\varepsilon\to 0} \big\|(I-P_0)_{|\widehat{\mathcal{H}}_{\varepsilon,\delta}}\big\| = 0, \tag{6.25}$$

$$\limsup_{\varepsilon\to 0} \big\|(I-P_{\varepsilon,\delta})_{|\widehat{\mathcal{H}}_0}\big\| = 0, \tag{6.26}$$

$$\limsup_{\varepsilon\to 0} \big\|\big(\widehat{H}_\varepsilon - \tfrac{1}{2}n\big)_{|\widehat{\mathcal{H}}_{\varepsilon,\delta}}\big\| = 0; \tag{6.27}$$

(iii) *there exists an increasing function $\varepsilon(\delta) > 0$ such that for all $\varepsilon \leq \varepsilon(\delta)$*

$$\dim(\widehat{\mathcal{H}}_{\varepsilon,\delta}) = n+1, \tag{6.28}$$

$$\|f\|_\infty \leq (\text{const.})\|f\| \quad \textit{for any } f \in \widehat{\mathcal{H}}_{\varepsilon,\delta}, \tag{6.29}$$

$$\inf\big\{\widehat{Q}_\varepsilon(f,f) : \|f\| = 1, f \perp \widehat{\mathcal{H}}_{\varepsilon,\delta}\big\} \geq \frac{1}{2}n + \frac{\gamma}{9n}. \tag{6.30}$$

The proof of this lemma is postponed until Section 6.3.1.

Now we complete the proof of Proposition 6.4. Fix $\delta = \frac{1}{9n}$ (we then omit δ from the notation), let $\varepsilon \leq \varepsilon(\frac{1}{9n})$ and apply Lemma 6.5. Parts (i), (ii) of Proposition 6.4 follow directly from (6.27), (6.28) and (6.29). Formula (6.26) implies that the projection P_ε converges to an isometry between $\widehat{\mathcal{H}}_0$ and $\widehat{\mathcal{H}}_\varepsilon$, $\|P_\varepsilon - P_0\| \to 0$, as $\varepsilon \to 0$, hence (6.20) follows.

We apply (6.16) to the truncated functions $\chi^\varepsilon f_j^\varepsilon$,

$$\widehat{Q}(\chi^\varepsilon f_j^\varepsilon, \chi^\varepsilon f_j^\varepsilon) = \frac{n}{2}\|\chi^\varepsilon f_j^\varepsilon\|^2 + 4\int_0^{2\pi}\int_0^\pi \big|\partial_{\bar z}(\chi^\varepsilon f_j^\varepsilon) - i\alpha_c(\partial_{\bar z})(\chi^\varepsilon f_j^\varepsilon)\big|^2 \frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta}$$

using that $\alpha^\varepsilon = \alpha_c$ on the support of $\chi^\varepsilon f_j^\varepsilon$.

By (6.27) we have $\limsup_{\varepsilon\to 0}\widehat{Q}_\varepsilon(f_j^\varepsilon, f_j^\varepsilon) \leq \frac{n}{2}$, hence using (6.23), (6.24) and the uniform boundedness of the functions f_j^ε, we obtain that $\limsup_{\varepsilon\to 0}\widehat{Q}(\chi^\varepsilon f_j^\varepsilon, \chi^\varepsilon f_j^\varepsilon) \leq \frac{1}{2}n$. We also have $(\chi^\varepsilon f_j^\varepsilon, \chi^\varepsilon f_{j'}^\varepsilon) \to \delta_{j,j'}$, hence

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^\pi \big|\partial_{\bar z}(\chi^\varepsilon f_j^\varepsilon) - i\alpha_c(\partial_{\bar z})(\chi^\varepsilon f_j^\varepsilon)\big|^2 \frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta} = 0, \tag{6.31}$$

from which (6.21) follows. □

The rest of the proof of Theorem 2.3 is similar to [CdVT] but the role of the potential is played by the magnetic field. For some small ε we will perturb the magnetic field of $\widehat{H}_\varepsilon$ by a small b. A careful choice of b will turn the $(n+1)$ eigenvalues of $\widehat{H}_\varepsilon$ near $\frac{1}{2}n$ into the multiplicity pattern given by $m_1, m_2, \ldots, m_k$. The key observation is that a perturbation b of the curvature appears as an additive perturbation b up to first order, thanks to (2.4), hence it is like a potential perturbation. However, some care is needed since the estimate (6.21) necessary to control higher order terms does not fully extend to the South pole; a small region has to be treated separately.

Let D be a small geodesic disk around S and let

$$\mathcal{F} := \Big\{ b \in L^\infty(D) : b \text{ real}, \int_M b v_g = 0 \Big\}$$

equipped with the L^∞-norm. By the Poincaré formula, for any $b \in \mathcal{F}$ there is a global 1-form α_b with $\mathrm{d}\alpha^b = b v_g$ and

$$|||\alpha^b(p)||| \leq (\text{const.}) \mathrm{d}(p, S) \|b\|_\infty, \quad p \in M. \tag{6.32}$$

Let α_ε be the global connection 1-form of ∇_ε obtained in Proposition 6.4, i.e., $\nabla_\varepsilon \phi_0 = -i\alpha_\varepsilon \phi_0$. We define $\alpha_\varepsilon^b := \alpha^b + \alpha_\varepsilon$, the quadratic form

$$\widehat{Q}_\varepsilon^b(f, f) := \int_M ||| \mathrm{d}f - i\alpha_\varepsilon^b f |||^2 v_g$$

with core $C^\infty(M)$ and the corresponding operator $\widehat{H}_\varepsilon^b$. Let $\mathrm{Herm}(\widehat{\mathcal{H}}_\varepsilon)$ denote the set of hermitian forms $q(\cdot|\cdot)$ on $\widehat{\mathcal{H}}_\varepsilon$ equipped with the natural norm $\|q\| := \max\{|q(f, g)| : \|f\| = \|g\| = 1\}$.

Lemma 6.6. — *There exist $\varepsilon_0 > 0$ and a finite dimensional subspace $\mathcal{F}_0 \subset \mathcal{F}$ such that for all $\varepsilon \leq \varepsilon_0$ the linear map $\Phi_\varepsilon : (\mathcal{F}_0, \|\cdot\|_\infty) \mapsto (\mathrm{Herm}(\widehat{\mathcal{H}}_\varepsilon), \|\cdot\|)$ given by*

$$\Phi_\varepsilon : b \longmapsto q_b(f \mid g) := \int_M b \bar{f} g \, v_g \tag{6.33}$$

is a bijection such that $\|\Phi_\varepsilon\|$ and $\|\Phi_\varepsilon^{-1}\|$ are bounded uniformly in ε.

Proof. — Consider the restriction to D of the functions $\mathrm{Re}\bar{f}_j f_k$, $\mathrm{Im}\bar{f}_j f_k$, for $j, k = 0, 1, 2, \ldots, n$, $j \leq k$, and the function identically 1 on D. Using their explicit formula, it is easy to check that these $(n+1)(n+2)+1$ functions are linearly independent over $\mathbb{R}$. Hence there exist $b_{jk} \in \mathcal{F}$, for $j, k = 0, 1, 2, \ldots, n$, such that $b_{jk} = b_{kj}$, $\int_M b_{jk} v_g = 0$ and $\int_M b_{jk} \bar{f}_{j'} f_{k'} v_g = \delta_{jj'} \delta_{kk'}$. Let $\mathcal{F}_0$ be the span of $\{b_{jk}\}$.

Using the boundedness of b_{jk}'s and (6.20), we see that for all $\varepsilon \leq \varepsilon_0$ small enough

$$\max_{j,j',k,k'} \left| \int_M b_{jk} \bar{f}^{\varepsilon}_{j'} f^{\varepsilon}_{k'} v_g - \delta_{jj'} \delta_{kk'} \right| \leq \frac{1}{2(n+1)^2},$$

where the functions $\{f^{\varepsilon}_j\}$ are from Proposition 6.4 (iii).

Hence the $(n+1)^2 \times (n+1)^2$ matrix with the $((jk), (j'k'))$-th entry equal to $\int_M b_{jk} \bar{f}^{\varepsilon}_{j'} f^{\varepsilon}_{k'} v_g$ is bounded with a bounded inverse. The bounds are uniform in ε. This is exactly the matrix of Φ_ε in the fixed bases $\{b_{jk}\} \subset \mathcal{F}_0$ and $\{f^{\varepsilon}_j\} \subset \widehat{\mathcal{H}}_\varepsilon$. □

We observe from (6.14) and (6.32) that the operator $\widehat{H}^b_\varepsilon$ is a small perturbation of $\widehat{H}_\varepsilon$ in the following sense:

$$\|(\widehat{H}^b_\varepsilon - \widehat{H}_\varepsilon) f\|^2 \leq (\text{const.}) \|b\|^2_\infty \left(\|f\|^2 + (f, \widehat{H}_\varepsilon f) \right) \quad \text{for } b \in \mathcal{F}. \tag{6.34}$$

The following result follows from standard perturbation theory and we omit its proof.

Lemma 6.7. — *Let H and H_ω be nonnegative selfadjoint operators with discrete spectra, defined on a common core $\mathcal{C}$ in a Hilbert space $\mathcal{H}$, satisfying*

$$\|(H_\omega - H) f\|^2 \leq (\text{const.}) \omega^2 \left(\|f\|^2 + (f, Hf) \right), \quad f \in \mathcal{C}.$$

Let $E \geq 0, \nu > 0$ and let $P := \Pi_{(-\infty, E+\nu]}(H)$ be the spectral projection below the energy $E + \nu$. We assume that

$$H \geq E - \nu \quad \text{and} \quad (I - P) H (I - P) \geq E + \gamma_0$$

with some fixed $\gamma_0 > 0$. Then there exist positive constants $a_0, a_1, \nu_0, \omega_0$ depending on γ_0, E and $\dim(P)$, such that for every $\nu \leq \nu_0, \omega \leq \omega_0$, we have

$$H_\omega \geq E - a_0(\omega + \nu) \quad \text{and} \quad (I - P_\omega) H_\omega (I - P_\omega) \geq E + \frac{\gamma_0}{2},$$

where $P_\omega := \Pi_{(-\infty, E + a_0(\nu+\omega)]}(H_\omega)$ is the spectral projection of H_ω. Furthermore $\|P - P_\omega\| \leq a_1 \omega$ and

$$\|P_\omega (H_\omega - E) P_\omega - P(H_\omega - H) P\| \leq a_1(\nu + \omega^2). \tag{6.35}$$

□

Using (6.34), we apply this lemma to our case (see Proposition 6.4) with $H := \widehat{H}_\varepsilon$, $H_\omega := \widehat{H}^b_\varepsilon$, $E := \frac{1}{2}n$, $P := P_\varepsilon$, $\gamma_0 := \frac{\gamma}{9n}$, $\nu = \nu(\varepsilon) := \|P_\varepsilon - P_0\| \to 0$ as $\varepsilon \to 0$, $\omega = \|b\|_\infty$, $b \in \mathcal{F}_0$. For small enough $\|b\|_\infty$ we obtain $\|P^b_\varepsilon - P_\varepsilon\| \le (\text{const.})\|b\|_\infty$ and

$$\limsup_{\varepsilon\to 0} \Big\| P^b_\varepsilon\Big(\widehat{H}^b_\varepsilon - \frac{1}{2}n\Big)P^b_\varepsilon - P_\varepsilon(\widehat{H}^b_\varepsilon - \widehat{H}_\varepsilon)P_\varepsilon\Big\| \le (\text{const.})\|b\|^2_\infty \tag{6.36}$$

where P^b_ε is the spectral projection of $\widehat{H}^b_\varepsilon$ onto energies below $\frac{n}{2} + \frac{\gamma}{18n}$.

Moreover, we claim that

$$\limsup_{\varepsilon\to 0} \big\| P_\varepsilon(\widehat{H}^b_\varepsilon - \widehat{H}_\varepsilon - b)P_\varepsilon\big\| \le (\text{const.})\|b\|^2_\infty. \tag{6.37}$$

To see this, we use (6.15) for any $f \in \widehat{\mathcal{H}}_\varepsilon$, $\|f\| = 1$,

$$\begin{aligned} \big|(f, [\widehat{H}^b_\varepsilon - \widehat{H}_\varepsilon - b]f)\big| &\le 4\int_0^{2\pi}\!\!\int_0^{\pi} \Big[\big|\partial_{\bar z} f - i(\alpha^\varepsilon + \alpha_b)(\partial_{\bar z})f\big|^2 \\ &\qquad - \big|\partial_{\bar z} f - i\alpha^\varepsilon(\partial_{\bar z})f\big|^2\Big]\frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta} \\ &\le 8\int_0^{2\pi}\!\!\int_0^{\pi}\Big[\big|\partial_{\bar z} f - i\alpha^\varepsilon(\partial_{\bar z})f\big|\cdot\big|\alpha_b(\partial_{\bar z})f\big| \\ &\qquad + |\alpha_b(\partial_{\bar z})f|^2\Big]\frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta}. \end{aligned} \tag{6.38}$$

Since $|\alpha_b(\partial_{\bar z})| \le (\text{const.})\|b\|_\infty(\sin\theta)(\pi - \theta)$ by (6.32), the second term in (6.38) is of order $\|b\|^2_\infty$. For the first term we estimate

$$\big|\partial_{\bar z} f - i\alpha^\varepsilon(\partial_{\bar z})f\big| \le (\text{const.})(\sin\theta)|||\,\mathrm{d}f - i\alpha^\varepsilon f|||.$$

Let $0 < u < \frac{1}{2}\pi$, we split the $\mathrm{d}\theta$ integration and use this estimate for θ near π to obtain, using (6.19) and $\|f\| = 1$, that

$$\begin{aligned} \limsup_{\varepsilon\to 0}&\int_0^{2\pi}\!\!\int_0^{\pi}\big|\partial_{\bar z} f - i\alpha^\varepsilon(\partial_{\bar z})f\big|\cdot\big|\alpha_b(\partial_{\bar z})f\big|\frac{\mathrm{d}\theta\,\mathrm{d}\varphi}{\sin\theta} \\ &\le (\text{const.})\|b\|_\infty \limsup_{\varepsilon\to 0}\int_0^{2\pi}\!\!\int_0^{\pi-u}\big|\partial_{\bar z} f - i\alpha^\varepsilon(\partial_{\bar z})f\big|\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &\qquad + (\text{const.})\|b\|_\infty\limsup_{\varepsilon\to 0}\int_0^{2\pi}\!\!\int_{\pi-u}^{\pi}|||\,\mathrm{d}f - i\alpha^\varepsilon f|||(\pi-\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &\le (\text{const.})u\|b\|_\infty\Big[\limsup_{\varepsilon\to 0}\widehat{Q}_\varepsilon(f,f)\Big]^{1/2}. \end{aligned} \tag{6.39}$$

Here we used (6.21) in the first term, and we used the following estimate with $\pi - \theta \approx \sin\theta$ in the second:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \int_0^{2\pi}\int_{\pi-u}^{\pi} &\||\,\mathrm{d}f - i\alpha^\varepsilon f\||(\pi-\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &\le (\text{const.})\Big(\limsup_{\varepsilon\to 0}\int_0^{2\pi}\int_{\pi-u}^{\pi} \||\,\mathrm{d}f - i\alpha^\varepsilon f\||^2(\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\Big)^{1/2} \\ &\qquad\qquad\times\Big(\int_0^{2\pi}\int_{\pi-u}^{\pi}(\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\Big)^{1/2} \\ &\le (\text{const.})u\big[\limsup_{\varepsilon\to 0}\widehat{Q}_\varepsilon(f,f)\big]^{1/2}.\end{aligned}$$

Since (6.39) is true for any $u > 0$, and $f \in \widehat{\mathcal{H}}_\varepsilon$, $\|f\| = 1$, we proved (6.37).

Hence, for small enough $\|b\|_\infty$, we obtain from (6.36) and (6.37) that

$$\limsup_{\varepsilon\to 0}\Big\|P_\varepsilon^b\Big(\widehat{H}_\varepsilon^b - \frac{n}{2}\Big)P_\varepsilon^b - P_\varepsilon b P_\varepsilon\Big\| \le (\text{const.})\|b\|_\infty^2, \tag{6.40}$$

We fix a unitary transformation U_ε^b on $\widehat{\mathcal{H}} = L^2(M, v_g)$ that maps $\widehat{\mathcal{H}}_\varepsilon$ onto $\widehat{\mathcal{H}}_\varepsilon^b$ isometrically and satisfies

$$\big\|P_\varepsilon^b(U_\varepsilon^b - I)\big\| \le (\text{const.})\|b\|_\infty. \tag{6.41}$$

The existence of such U_ε^b follows from $\|P_\varepsilon^b - P_\varepsilon\| \le (\text{const.})\|b\|_\infty$.

We define the map $\Psi_\varepsilon : b \mapsto (U_\varepsilon^b)^* P_\varepsilon^b(\widehat{H}_\varepsilon^b - \frac{1}{2}n)P_\varepsilon^b U_\varepsilon^b$ from $\mathcal{F}_0$ into $\mathrm{Herm}(\widehat{\mathcal{H}}_\varepsilon)$ by identifying a selfadjoint operator with its quadratic form. Using (6.40) and (6.41) we have

$$\limsup_{\varepsilon\to 0}\big\|\Psi_\varepsilon(b) - P_\varepsilon b P_\varepsilon\big\| \le (\text{const.})\|b\|_\infty^2. \tag{6.42}$$

Notice that $\Psi_\varepsilon(0)$ is a quadratic form of vanishing norm as $\varepsilon \to 0$ by (6.18), and the derivative $\Psi_\varepsilon'(0)$ has a uniformly bounded inverse using Lemma 6.6 and $\lim_{\varepsilon\to 0}\|\Psi_\varepsilon'(0) - \Phi_\varepsilon\| = 0$ that follows from (6.42). Using the inverse function theorem, for small enough ε there exists $b \in \mathcal{F}_0$ such that $\Psi_\varepsilon(b)$ has eigenvalues with any given multiplicities $m_1, m_2, \ldots, m_k$. Since we know that the ground state of $\widehat{H}_\varepsilon^b$ lies in the range of P_ε^b, these are exactly the multiplicities of the lowest eigenvalues of $\widehat{H}_\varepsilon^b$.

Finally, if ε is chosen small enough, then b can also be chosen small, in particular $\int |b| v_g < \frac{1}{2}\mu$ can be achieved. The magnetic field of $\widehat{H}_\varepsilon^b$ is $B = \frac{1}{2}n(\eta_\varepsilon(\theta) + \eta_\varepsilon'(\theta)\tan\frac{1}{2}\theta) + b$ and an easy calculation shows that $\big|\frac{1}{2\pi}\int_M |B| v_g - 2n\big| \le \mu$ for small ε. This completes the proof of Theorem 2.3. □

6.3.1. *Proof of Lemma 6.5.*

Step 1. — For any $f \in D(\widehat{Q}) \cap C^\infty(M)$,

$$\widehat{Q}(\chi^\varepsilon f, \chi^\varepsilon f) \le \widehat{Q}(f,f)\Big(1 + \frac{(\mathrm{const.})}{|\log\varepsilon|^{1/2}}\Big) + (\mathrm{const.})\frac{\|f\|_\infty^2}{|\log\varepsilon|^{1/2}} \tag{6.43}$$

which follows from Schwarz inequality

$$\widehat{Q}(\chi^\varepsilon f, \chi^\varepsilon f) \le (1+s)\widehat{Q}(f,f) + (1+s^{-1})\|f\|_\infty^2 \int_M |||\,\mathrm{d}\chi^\varepsilon|||^2 v_g$$

and (6.22) with an optimal choice of s. Similarly

$$\widehat{Q}_\varepsilon(\chi^\varepsilon f, \chi^\varepsilon f) \le \widehat{Q}_\varepsilon(f,f)\Big(1 + \frac{(\mathrm{const.})}{|\log\varepsilon|^{1/2}}\Big) + (\mathrm{const.})\frac{\|f\|_\infty^2}{|\log\varepsilon|^{1/2}}, \tag{6.44}$$

where $f \in C^\infty(M)$, which proves (6.24). Clearly $\chi^\varepsilon f \in D(\widehat{Q})$ for any $f \in C^\infty(M)$, hence by (6.16) and (6.23) we have

$$\widehat{Q}_\varepsilon(\chi^\varepsilon f, \chi^\varepsilon f) \ge \frac{n}{2}\int_M |\chi^\varepsilon f|^2 v_g \ge \frac{n}{2}\|f\|^2 - (\mathrm{const.})\varepsilon^2\|f\|_\infty^2, \tag{6.45}$$

for $f \in C^\infty(M)$.

Step 2. — Let $\widehat{H}f = Ef$ or $\widehat{H}_\varepsilon f = Ef$, then

$$\|f\|_{L^\infty} \le \mathrm{e}^{tE}\big\|\,\mathrm{e}^{-t\widehat{H}}f\big\|_{L^\infty} \le \mathrm{e}^{tE}\big\|\,\mathrm{e}^{t\Delta}|f|\big\|_{L^\infty} \le (\mathrm{const.})t^{-1/2}\,\mathrm{e}^{tE}\|f\|$$

by the diamagnetic inequality and standard $L^2 \to L^\infty$ heat kernel estimates on $M = S^2$. The same bound is true for the eigenfunction of $\widehat{H}_\varepsilon$. Choosing t appropriately, we obtain that

$$\|f\|_\infty \le (\mathrm{const.})E^{1/2}\|f\|, \quad f \in \mathrm{Ker}(\widehat{H} - E) \text{ or } f \in \mathrm{Ker}(\widehat{H}_\varepsilon - E). \tag{6.46}$$

Step 3. — We show that the lowest eigenvalue E_ε of $\widehat{Q}_\varepsilon$ satisfies

$$\liminf_{\varepsilon\to 0} E_\varepsilon \ge \frac{n}{2}\cdot \tag{6.47}$$

Let f^ε be a corresponding eigenfunction. We can assume that $E_\varepsilon \le \frac{1}{2}n$. Then using (6.44), (6.45) and (6.46), we obtain (6.47).

Step 4. — First we prove the following weaker version of (6.25) and (6.26):

$$\left\|(I-P_0)_{|\widehat{\mathcal{H}}_{\varepsilon,\delta}}\right\|^2 < 3\delta, \tag{6.48}$$

$$\left\|(I-P_{\varepsilon,\delta})_{|\widehat{\mathcal{H}}_0}\right\|^2 < 3\delta. \tag{6.49}$$

Let $\psi_0^\varepsilon, \psi_1^\varepsilon, \psi_2^\varepsilon, \ldots, \psi_k^\varepsilon \in \widehat{\mathcal{H}}_{\varepsilon,\delta}$ be an orthonormal eigenbasis with $k = k(\varepsilon, \delta)$. Using (6.46) we have

$$\|\psi_j^\varepsilon\|_\infty \leq (\text{const.}), \tag{6.50}$$

hence for small enough $\varepsilon \leq \varepsilon(\delta)$,

$$\widehat{Q}(\chi^\varepsilon\psi_j^\varepsilon, \chi^\varepsilon\psi_j^\varepsilon) = \widehat{Q}_\varepsilon(\chi^\varepsilon\psi_j^\varepsilon, \chi^\varepsilon\psi_j^\varepsilon) \leq \frac{n}{2} + \frac{3\delta\gamma}{2} \tag{6.51}$$

for all j by (6.23) and (6.44).

Let $g_j := P_0(\chi^\varepsilon\psi_j^\varepsilon)$, $h_j := (I-P_0)(\chi^\varepsilon\psi_j^\varepsilon)$, then $\|g_j\|^2 + \|h_j\|^2 = \|\chi^\varepsilon\psi_j^\varepsilon\|^2 \geq 1 - (\text{const.})\varepsilon^2\|\psi_j^\varepsilon\|_\infty^2$, hence

$$\widehat{Q}(\chi^\varepsilon\psi_j^\varepsilon, \chi^\varepsilon\psi_j^\varepsilon) = \widehat{Q}(g_j, g_j) + \widehat{Q}(h_j, h_j) \geq \frac{n}{2} - (\text{const.})\varepsilon^2 + \gamma\|h_j\|^2$$

by (6.17). Using (6.51), we obtain $\|h_j\|^2 < 2\delta$ for small enough $\varepsilon \leq \varepsilon(\delta)$.

Since $I - P_0$ is a projection, we obtain

$$\|(I-P_0)\psi_j^\varepsilon\| \leq \|h_j\| + \|(I-P_0)[(1-\chi^\varepsilon)\psi_j^\varepsilon]\| \leq \sqrt{2\delta} + \|(1-\chi^\varepsilon)\psi_j^\varepsilon\| < \sqrt{3\delta}$$

for small enough ε, using (6.50). The orthonormality of $\{\psi_j^\varepsilon\}$ then gives (6.48).

Similar argument is valid for $P_{\varepsilon,\delta}$. Recall that $f_0, f_1, f_2, \ldots, f_n$ is an orthonormal basis in $\widehat{\mathcal{H}}_0$ and their L^∞-norm is uniformly bounded by (6.46). For small enough $\varepsilon \leq \varepsilon(\delta)$,

$$\widehat{Q}_\varepsilon(\chi^\varepsilon f_j, \chi^\varepsilon f_j) \leq \frac{n}{2} + \frac{\delta^2\gamma}{2} \quad \text{and} \quad E_\varepsilon \geq \frac{n}{2} - \delta^2\gamma \tag{6.52}$$

by (6.23), (6.43) and (6.47). We define

$$g_j := P_{\varepsilon,\delta}(\chi^\varepsilon f_j), \quad h_j := (I - P_{\varepsilon,\delta})(\chi^\varepsilon f_j)$$

and we again have $\|g_j\|^2 + \|h_j\|^2 \geq 1 - (\text{const.})\varepsilon^2\|f_j\|_\infty^2$. From

$$\widehat{Q}_\varepsilon(\chi^\varepsilon f_j, \chi^\varepsilon f_j) = \widehat{Q}_\varepsilon(g_j, g_j) + \widehat{Q}_\varepsilon(h_j, h_j) \geq \left(\frac{n}{2} - \delta^2\gamma\right)\|g_j\|^2 + \left(\frac{n}{2} + \delta\gamma\right)\|h_j\|^2$$

and (6.52) we obtain $\|h_j\|^2 < 2\delta$ for $\varepsilon \leq \varepsilon(\delta)$, which implies (6.49).

LEMMA 6.8. — *Let U, V be finite dimensional subspaces of a Hilbert space $\mathcal{H}$ and let $P_V : \mathcal{H} \to V$ be the orthogonal projection onto V. If*

$$\left\|(I - P_V)_{|U}\right\|^2 < \frac{1}{\dim(U)},$$

then $P_{V|U} : U \to V$ is injective, hence $\dim(U) \le \dim(V)$.

Proof. — Let $u_1, \dots, u_k$ be an orthonormal basis in U, $k := \dim(U)$. We decompose $u_j = v_j + w_j$ with $v_j := P_V u_j$, $\|w_j\|^2 < \frac{1}{k}$. If $P_{V|U}$ were not injective, then $P_V u = 0$ for some nonzero vector $u = \sum_j c_j u_j$, $\|u\| = 1$, and $u = \sum_j c_j w_j$. Hence $\|u\|^2 = \|\sum_j c_j w_j\|^2 \le \sum_j \|w_j\|^2 < 1$ since $\sum_j c_j^2 = 1$, but this is a contradiction. □

Step 5. — Now we prove Lemma 6.5. Since $\delta \le \frac{1}{9n}$ and $\dim \widehat{\mathcal{H}}_0 = n+1$, from (6.49) and Lemma 6.8 we obtain that $P_{\varepsilon,\delta} : \widehat{\mathcal{H}}_0 \mapsto \widehat{\mathcal{H}}_{\varepsilon,\delta}$ is injective and $\dim \widehat{\mathcal{H}}_0 \le \dim \widehat{\mathcal{H}}_{\varepsilon,\delta}$.

Suppose that $\dim \widehat{\mathcal{H}}_{\varepsilon,\delta} > \dim \widehat{\mathcal{H}}_0 = n + 1$, then choose a subspace $U \subset \widehat{\mathcal{H}}_{\varepsilon,\delta}$ with $\dim(U) = n + 2$. Using (6.48) and $\delta \le \frac{1}{9n}$, we see that $P_{|U} : U \to \widehat{\mathcal{H}}_0$ is injective and $\dim(U) \le \dim(\widehat{\mathcal{H}}_0) = n + 1$, which is a contradiction. This proves (6.28). The proof of (6.29) easily follows from (6.50).

Finally, we fix $\delta \le \frac{1}{9n}$, let $\delta' < \delta$ and $\varepsilon \le \varepsilon(\delta')$. Then $\widehat{\mathcal{H}}_{\varepsilon,\delta} = \widehat{\mathcal{H}}_{\varepsilon,\delta'}$ by (6.28) and the monotonicity of these subspaces in δ. Therefore

$$\left\|(I - P_0)_{|\widehat{\mathcal{H}}_{\varepsilon,\delta}}\right\|^2 = \left\|(I - P_0)_{|\widehat{\mathcal{H}}_{\varepsilon,\delta'}}\right\|^2 < 3\delta'$$

using (6.48) for δ' and finally, letting $\delta' \to 0$ we obtain (6.25). The proof of (6.26) is similar using (6.49) and the monotonicity of $P_{\varepsilon,\delta}$ in δ.

For the proof of (6.27) we need two sided bounds on $\widehat{Q}_{\varepsilon|\widehat{\mathcal{H}}_{\varepsilon,\delta}}$. The lower bound is given in (6.47). For the upper bound, we use that $\widehat{Q}_\varepsilon \le \frac{1}{2}n + \delta'\gamma$ on $\widehat{\mathcal{H}}_{\varepsilon,\delta'}$, but $\widehat{\mathcal{H}}_{\varepsilon,\delta} = \widehat{\mathcal{H}}_{\varepsilon,\delta'}$ for any $\delta' \le \delta$, $\varepsilon \le \varepsilon(\delta')$ and we let $\delta' \to 0$. The proof of (6.30) is similar by noting that $\widehat{Q}_\varepsilon(f, f)$ is at least $\frac{1}{2}n + \frac{\gamma}{9n}$ if $f \perp \widehat{\mathcal{H}}_{\varepsilon,1/9n}$ but $\widehat{\mathcal{H}}_{\varepsilon,1/9n} = \widehat{\mathcal{H}}_{\varepsilon,\delta}$ if $\varepsilon \le \varepsilon(\delta)$. □

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