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EINSTEIN METRICS ON RATIONAL HOMOLOGY 7-SPHERES

by Ch. P. BOYER, K. GALICKI & M. NAKAMAYE

1. Introduction.

Dimension seven appears to be rather special when it comes to examples of compact Einstein manifolds. It is perhaps the prominent rôle such manifolds have played in physics ever since the early days of Kaluza-Klein supergravity that made both theoretical physicists and mathematicians alike particularly interested in them. Discoveries of many different constructions followed as a result of this interest.

Arguably, today a special place among all compact Einstein 7-manifolds is reserved for the so-called Sasakian-Einstein spaces. They are defined to be Riemannian manifolds with the property that the metric cone on them is a Calabi-Yau 4-fold and, in particular, they are always of positive scalar curvature. All regular Sasakian-Einstein manifolds are circle bundles of Fano 3-folds that admit Kähler-Einstein metrics. Non-regular ones fiber over compact Kähler-Einstein Fano 3-folds with orbifold singularities. An interesting sub-family of the family of Sasakian-Einstein 7-manifolds consists of the so-called 3-Sasakian spaces. They are characterized by fact that their metric cone is not only Calabi-Yau, but also hyperkähler and are all orbifold fibrations over compact Kähler-Einstein Fano 3-folds which admit a complex contact structure.

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Regular and non-regular examples of both Sasakian-Einstein and 3-Sasakian manifolds are now plentiful and they were extensively studied by the first two authors [BG1], [BG2]. There is an example of a regular Sasakian-Einstein $(4n + 3)$-manifold which is worthy of some further discussion. It is the homogeneous Stiefel manifold of 2-frames in $(2n+1)$-dimensional Euclidean space, $V_2(\mathbb{R}^{2n+1}) = SO(2n+1)/SO(2n-1)$ which is a circle bundle over the oriented Grassmannian, $\widetilde{G}_2(\mathbb{R}^{2n+1})$. From the point of view of an algebraic geometer it is a classical fact that $\widetilde{G}_2(\mathbb{R}^{2n+1})$ is diffeomorphic to the complex quadric Q_{2n-1} in $\mathbb{CP}^{2n}$ which is well-known to be Fano and to admit a Kähler-Einstein metric. It is perhaps less well-known that the quadric Q_{2n-1} has the same cohomology groups as $\mathbb{CP}^{2n-1}$, but differs in the ring structure. Hence, $V_2(\mathbb{R}^{2n+1})$ is a rational homology sphere with $H_{n-1}(V_2(\mathbb{R}^{2n+1}), \mathbb{Z}) \approx \mathbb{Z}_2$. Now it has been known for quite some time that $V_2(\mathbb{R}^{2n+1})$ carries a Sasakian-Einstein structure [BFGK], [BG1]. Up to date, apart from $S^{2n+1}, V_2(\mathbb{R}^{2n+1})$, and the 3-Sasakian homogeneous 11-manifold $G_2/Sp(1)$, we are not aware of any other examples of simply connected rational homology spheres which are also known to admit Sasakian-Einstein structures. In this paper we shall demonstrate that for $n = 2$, quite to the contrary, there are many examples of such structures, 184 to be precise. These examples are obtained as hypersurfaces in certain weighted projective 4-spaces, but we certainly expect the phenomena to occur in arbitrary dimension.

The key to this construction is a recent paper of Johnson and Kollár [JK2] which is based on the previous work of Demailly and Kollár [DK]. [JK2] gives a list of 4442 quasi-smooth Fano 3-folds $\mathcal{Z}$ that anticanonically embed in weighted projective 4-spaces $\mathbb{P}(\mathbf{w})$. Moreover, they show that 1936 of these 3-folds admit Kähler-Einstein metrics. According to our general theory [BG1] such Fano 3-folds give rise to Sasakian-Einstein metrics on smooth 7-manifolds M^7. Moreover, these 7-manifolds arise as links of isolated hypersurface singularities associated to certain weighted homogeneous polynomials in $\mathbb{C}^5$. As in [JK1] Johnson and Kollár [JK2] only consider the case when the orbifold Fano index is one, and as the authors showed in [BGN1] for log del Pezzo surfaces, there should be many more interesting examples of quasi-smooth Fano 3-folds with higher orbifold Fano index. This is currently under study.

In this note we prove

THEOREM A. — *There are 1936 distinct Sasakian-Einstein structures on certain 2-connected 7-manifolds $M^7_{\mathbf{w},d}$ realized as links of weighted* *homogeneous polynomials in* $\mathbb{C}^5$ *with weight vector* $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ *and degree d. In particular, there are 184 2-connected rational homology spheres which are listed in a table found on the first two authors web pages.*

We have not answered the question as to whether two distinct or non-conjugate Sasakian-Einstein structures on the same link M^7 could belong to the same underlying Riemannian metric g. Indeed, this can happen, but if g is not the standard round metric on S^7 then by a Theorem of Tachibana and Yu [TaYu], the two Sasakian-Einstein structures must belong to a 3-Sasakian structure. But then by a Theorem of Galicki and Salamon [GS], we must have $b_3 = 0$, so M^7 must be a rational homology sphere. However, we do not know whether any of the rational homology 7-spheres discussed here admit 3-Sasakian structures.

2. The Sasakian geometry of links of weighted homogeneous polynomials.

In this section we briefly review the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by $(z_0, \dots, z_n) \mapsto (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that they are ordered $w_0 \leqslant w_1 \leqslant \cdots \leqslant w_n$ and that $\gcd(w_0, \dots, w_n) = 1$. Let f be a quasi-homogeneous polynomial, that is $f \in \mathbb{C}[z_0, \dots, z_n]$ and satisfies

$$f(\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \dots, z_n), \tag{2.1}$$

where $d \in \mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0, \dots, z_n) = 0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

$$L_f = C_f \cap S^{2n+1}, \tag{2.2}$$

where

$$S^{2n+1} = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, \sum_{j=0}^{n} |z_j|^2 = 1 \right\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n - 1$. Furthermore, it is well-known [Mil] that the link L_f is $(n-2)$-connected.

On S^{2n+1} there is a well-known [YK] "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^{n}$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^{n}(x_i dy_i - y_i dx_i)}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^{n} w_i(x_i\partial_{y_i} - y_i\partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}. The embedding $L_f \hookrightarrow S^{2n+1}$ induces a Sasakian structure on L_f [BG3].

Given a sequence $\mathbf{w} = (w_0, \ldots, w_n)$ of ordered positive integers one can form the graded polynomial ring $S(\mathbf{w}) = \mathbb{C}[z_0, \ldots, z_n]$, where z_i has grading or *weight* w_i. The weighted projective space [Dol], [Fle] $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is defined to be the scheme $\mathrm{Proj}(S(\mathbf{w}))$. It is the quotient space $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*(\mathbf{w})$, where $\mathbb{C}^*(\mathbf{w})$ is the weighted action defined in 2.1, or equivalently, $\mathbb{P}(\mathbf{w})$ is the quotient of the weighted Sasakian sphere $S_{\mathbf{w}}^{2n+1} = (S^{2n+1}, \xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ by the weighted circle action $S^1(\mathbf{w})$ generated by $\xi_{\mathbf{w}}$. As such $\mathbb{P}(\mathbf{w})$ is also a compact complex orbifold with an induced Kähler structure. We have from [BG3]

THEOREM 2.3. — *The quadruple $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ gives L_f a quasi-regular Sasakian structure such that there is a commutative diagram*

$$\begin{tikzcd} L_f \arrow[r] \arrow[d, "\pi"] & S_{\mathbf{w}}^{2n+1} \arrow[d] \\ \mathcal{Z}_f \arrow[r] & \mathbb{P}(\mathbf{w}), \end{tikzcd}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions. Moreover, if $\mathcal{Z}_f$ is Fano, L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$ whose first Chern class in $H^2_{\mathrm{orb}}(\mathcal{Z}_f, \mathbb{Z})$ is $c_1(\mathcal{Z}_f)/I$, where I is the index.

We should also mention that $c_1(\mathcal{Z}_f)$ pulls back to the basic first Chern class $c_1^B \in H^2_B(\mathcal{F}_{\xi_{\mathbf{w}}})$ and $\eta_{\mathbf{w}}$ is the connection in this V-bundle whose curvature is $d\eta = \frac{2ni}{I}\pi^*\omega_{\mathbf{w}}$, where $\omega_{\mathbf{w}}$ is the Kähler form on $\mathcal{Z}_f$.

Now conditions on the weights that guarantee that the hypersurface $C_f \subset \mathbb{C}^{n+1}$ have only an isolated singularity at the origin are well-known [Fle], [JK1]. These conditions become more complicated as the dimension increases [Fle], [JK2]; however, in this paper we are only interested in the $n = 4$ case of hypersurfaces in a weighted complex projective 4-space. These conditions, known as *quasi-smoothness* conditions guarantee that

$\mathcal{Z}_f$ is smooth in the orbifold sense, that is, at a vertex $P_i \in \mathbb{P}(\mathbf{w})$ the preimage of $\mathcal{Z}_f$ in the orbifold chart of $\mathbb{P}(\mathbf{w})$ is smooth. It is easy to see that one can formulate all these conditions as follows [Fle], [JK2]:

Quasi-smoothness conditions 2.4.

I. For each $i = 0, \cdots, 4$ there is a j and a monomial $z_i^{m_i} z_j \in \mathcal{O}(d)$. Here $j = i$ is possible.

II. For all distinct i, j either there is a monomial $z_i^{b_i} z_j^{b_j} \in \mathcal{O}(d)$ or there exist monomials $z_i^{n_1} z_j^{m_1} z_k, z_i^{n_2} z_j^{m_2} z_l \in \mathcal{O}(d)$ with $\{k, l\} \neq \{i, j\}$ and $k \neq l$.

III. For every i, j there exists a monomial of degree d that does not involve either z_i or z_j.

There is another condition apart from quasi-smoothness that assures us that the adjunction theory behaves correctly, and that $\mathbb{P}(\mathbf{w})$ does not have any orbifold singularities of codimension 1. It is [Dol], [Fle]

Well-formedness condition 2.5.

IV. For each i we have $\gcd(w_0, \cdots, \hat{w}_i, \cdots, w_4) = 1$. Here the $\hat{}$ means skip that element.

Condition IV guarantees that the canonical V-bundle $K_{\mathcal{Z}}$ is determined in terms of the degree and index by

$$K_{\mathcal{Z}} \simeq \mathcal{O}(-I) = \mathcal{O}(d - |\mathbf{w}|), \tag{2.6}$$

where $|\mathbf{w}| = \sum_i w_i$.

In this note we shall only consider the anticanonically embedded Fano 3-folds of [JK2], that is, we shall assume hereafter that $I = |\mathbf{w}| - d = 1$. The examples we consider are from the list sporadic.txt of Johnson and Kollár [JK2] which is found at `http://www.math.princeton.edu/~jmjohnso`.

3. The topology of the link $M^7_{\mathbf{w},d}$.

The topology of a link L_f of an isolated hypersurface singularity is encoded in the characteristic polynomial $\Delta(t)$ of the monodromy map. $\Delta(t)$ is an important link invariant that generalizes the Alexander polynomial of a knot, and is often called the "Alexander polynomial" of the link [HZ]. Let us recall the well-known construction of Milnor [Mil] concerning isolated hypersurface singularities: There is a fibration of $(S^{2n+1} - L_f) \longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee S^n \cdots \vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0, \ldots, w_n)$ by the formula [MO]

$$\mu = \mu(L_f) = \prod_{i=0}^{n} \Big(\frac{d}{w_i} - 1\Big). \tag{3.1}$$

The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold whose boundary is precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F, \mathbb{Z}) \approx \mathbb{Z}^\mu$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence [Mil]

(3.2)

$$0 \longrightarrow H_n(L_f, \mathbb{Z}) \longrightarrow H_n(F, \mathbb{Z}) \xrightarrow{\mathbb{I}-h_*} H_n(F, \mathbb{Z}) \longrightarrow H_{n-1}(L_f, \mathbb{Z}) \longrightarrow 0,$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f, \mathbb{Z}) = \ker(\mathbb{I} - h_*)$ is a free Abelian group, and $H_{n-1}(L_f, \mathbb{Z}) = \operatorname{coker}(\mathbb{I} - h_*)$ which in general has torsion, but whose free part is isomorphic to $\ker(\mathbb{I} - h_*)$. There is a well-known algorithm due to Milnor and Orlik [MO] for computing the free part of $H_{n-1}(L_f, \mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I} - h_*)$ of the monodromy map. The Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$. Generally, finding the torsion is much more difficult. However, in the case of rational homology spheres, $b_n(L_f) = b_{n-1}(L_f) = 0$, the group $H_{n-1}(M, \mathbb{Z})$ is a torsion group of order $\Delta(1)$.

It is not our purpose in this note to give a systematic study of the Johnson-Kollár list [JK2]. This requires a computer program for computing the Betti numbers which is currently under study. Here we are content with giving an algorithm for finding special cases when rational homology spheres occur. We have written a MAPLE program which allows us to search the [JK2] list, sporadic.txt and pick out certain rational homology spheres. We have found two distinct conditions on the weights that allow us to find rational homology spheres and they are described in the lemmas and corollaries below. These conditions may determine all rational homology spheres on the [JK2] list, but we do not have a proof of this. The first and simplest of the two conditions is that the weights are all relatively prime to the degree.

LEMMA 3.3. — *Let* $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ *be the weights of a quasi-smooth Fano 3-fold,* $\mathcal{Z}_f$ *of degree* d *and index* 1. *Suppose further that* $\gcd(w_i, d) = 1$ *for all* $i = 0, \cdots, 4$. *Then there exists an integer* $N(\mathbf{w})$ *such that the Alexander polynomial* $\Delta(t)$ *of the link* $M^7_{\mathbf{w},d}$ *has the form*

$$\Delta(t) = \frac{(t^d - 1)^{N(\mathbf{w})}}{t-1}.$$

Hence, the Betti number b_3 *of the link* $M^7_{\mathbf{w},d}$ *is given by* $b_3(M^7_{\mathbf{w},d}) = N(\mathbf{w}) - 1$.

Proof. — The Milnor and Orlik [MO] algorithm for computing the characteristic polynomial of the monodromy operator for weighted homogeneous polynomials is as follows: First associate to any monic polynomial F with roots $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^*$ its divisor

$$\operatorname{div} F = \langle \alpha_1 \rangle + \cdots + \langle \alpha_k \rangle$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$ and let $\Lambda_n = \operatorname{div}(t^n - 1)$. The rational weights w_i' used in [MO] are related to our integer weights w_i by $w_i' = \frac{d}{w_i}$, and we write the $w_i' = \frac{u_i}{v_i}$ in irreducible form. The divisor div Δ is given by

$$\operatorname{div} \Delta = \left(\frac{\Lambda_{u_0}}{v_0} - 1\right) \cdots \left(\frac{\Lambda_{u_4}}{v_4} - 1\right) \tag{3.4}$$

which can be reduced to the form

$$\operatorname{div} \Delta(t) = \sum_j a_j \Lambda_j - 1 \tag{3.5}$$

for some integers a_j upon using the relations $\Lambda_a \Lambda_b = \gcd(a,b)\Lambda_{lcm(a,b)}$. The characteristic polynomial $\Delta(t)$ is then determined from its divisor by

$$\Delta(t) = \frac{\prod (t^j - 1)^{a_j}}{t-1}, \tag{3.6}$$

and the third Betti number is given by

$$b_3(M^7_{\mathbf{w},d}) = \sum_j a_j - 1.$$

In our case we have $\gcd(w_i, d) = 1$ so equation (3.5) must take the form

$$\operatorname{div} \Delta(t) = N(\mathbf{w})\Lambda_d - 1 \tag{3.7}$$

where $N(\mathbf{w})$ is an integer. □

For the Maple program we need a convenient formula for the integer $N(\mathbf{w})$ in terms of the weights and degree. We find

$$N(\mathbf{w}) = \frac{d(dr_{01}r_{23} + r_{01} + r_{23})}{w_4} + \frac{1}{w_4} - (dr_{01}r_{23} + r_{01} + r_{23}), \tag{3.8}$$

where

$$d = |\mathbf{w}| - 1, \qquad r_{ij} = \frac{d}{w_i w_j} - \frac{1}{w_i} - \frac{1}{w_j}.$$

We should remark here that although it is far from manifest in equation (3.8), under the hypothesis of Lemma 3.3 the function $N(\mathbf{w})$ is invariant under a permutation of the weights, i.e., if Σ_5 denotes the permutation group on 5 letters, then $N(\sigma(\mathbf{w})) = N(\mathbf{w})$ for any $\sigma \in \Sigma_5$. We have an immediate

COROLLARY 3.9. — *Let $M^7_{\mathbf{w},d}$ be the link of an isolated hypersurface singularity defined by a weighted homogeneous polynomial f with well-formed weights $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ and degree d which satisfy* $\gcd(w_i, d) = 1$ *for all $i = 0, \cdots, 4$. Then $M^7_{\mathbf{w},d}$ is a rational homology sphere if and only if $N(\mathbf{w}) = 1$. Furthermore, in this case the Milnor number $\mu = d - 1$, and the order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$ equals the degree d.*

Proof. — The only part that we need to compute is the order of H_3. Since for a 2-connected rational homology sphere $H_4(M^7_{\mathbf{w},d}, \mathbb{Z}) = 0$, the exact sequence (3.2) shows [Mil] that the order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$ equals $\Delta(1)$. But from (3.6) and (3.7) we see that in our special case the characteristic polynomial takes the form

$$\Delta(t) = \frac{t^d - 1}{t - 1} = t^{d-1} + \cdots + t + 1$$

from which the result follows. $\square$

We now describe the second type of condition.

LEMMA 3.10. — *Let $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ be the weights of a quasi-smooth Fano 3-fold, $\mathcal{Z}_f$ of degree d and index 1. Suppose further that the degree can be written as $d = m_3 m_2$, where m_2 and m_3 are relatively prime, and that the "rational weights" $\frac{d}{w_i}$ take the form $\frac{m_3}{v_i}$ for 3 values of i and $\frac{m_2}{v_i}$ for the other 2 values of i. Then there exist positive integers $l = l(\mathbf{w})$, and $n = n(\mathbf{w})$, depending on the weights $\mathbf{w}$, such that the Alexander polynomial $\Delta(t)$ of the link $M^7_{\mathbf{w},d}$ takes the form*

$$\Delta(t) = \frac{(t^d - 1)^{ln}(t^{m_3} - 1)^l}{(t - 1)(t^{m_2} - 1)^n}.$$

Hence,

$$b_3(M^7_{\mathbf{w},d}) = (n(\mathbf{w})+1)(l(\mathbf{w})-1).$$

Proof. — Computing as in the proof of Lemma 3.3, we see that from the Milnor-Orlik procedure [MO] that the divisor of the Alexander polynomial must take the form

$$\operatorname{div}\Delta(t) = l(\mathbf{w})n(\mathbf{w})\Lambda_d + l(\mathbf{w})\Lambda_{m_3} - n(\mathbf{w})\Lambda_{m_2} - 1 \tag{3.11}$$

for some positive integers $l(\mathbf{w})$ and $n(\mathbf{w})$ depending on the weights. The above form of the Alexander polynomial then follows from equations (3.5) and (3.6). The explicit form of the functions $l(\mathbf{w})$ and $n(\mathbf{w})$ are also easily calculated. Let i_1, i_2, i_3 denote the 3 indices whose rational weights take the form $\frac{m_3}{v_i}$ and similarly let j_1, j_2 denote the indices corresponding to the rational weights $\frac{m_2}{v_j}$. Then one finds

$$l(\mathbf{w}) = \frac{m_3^2}{v_{i_1}v_{i_2}v_{i_3}} - m_3\Big(\frac{1}{v_{i_1}v_{i_2}} + \frac{1}{v_{i_1}v_{i_3}} + \frac{1}{v_{i_2}v_{i_3}}\Big) + \frac{1}{v_{i_1}} + \frac{1}{v_{i_2}} + \frac{1}{v_{i_3}} \tag{3.12}$$

$$n(\mathbf{w}) = \frac{m_2}{v_{j_1}v_{j_2}} - \frac{1}{v_{j_1}} - \frac{1}{v_{j_2}}. \tag{3.13}$$

The expression for b_3 follows directly from the expression for $\Delta(t)$. □

COROLLARY 3.14. — *Let* $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ *be the weights of a quasi-smooth Fano 3-fold,* $\mathcal{Z}_f$ *of degree* d *and index* 1. *Suppose further that the hypothesis of Lemma 3.10 is satisfied, then* $M^7_{\mathbf{w},d}$ *is a rational homology sphere if and only if* $l(\mathbf{w}) = 1$. *Furthermore, in this case the Milnor number* $\mu = (m_3 - 1)(n(\mathbf{w})m_2 + 1)$, *and the order of* $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$ *is* $m_3^{n(\mathbf{w})+1}$.

Proof. — As in the proof of Corollary 3.9 this follows from Lemma 3.10 by cancelling the $t-1$ factors in $\Delta(t)$ and evaluating at $t = 1$. □

Remark 3.15. — One can also write $d = m_4m_1$ or $d = m_{2,1}m_{2,2}m_1$ where in each case the m_s are pairwise relatively prime positive integers. Also in the first case $d/w_i = m_4/v_i$ for 4 values of i and $d/w_j = m_1/v_j$ for the remaining index. In the second case $d/w_i = m_{2,1}/v_i$ and $d/w_j = m_{2,2}/v_j$ for two pairs of index and $d/w_k = m_1/v_k$ for the remaining index. In both cases one finds rational homology spheres without any further conditions; however, one can also easily show that the weights are not well-formed in either case.

4. Examples of $M^7_{\mathbf{w},d}$.

We have constructed a table of 184 rational homology spheres that admit Sasakian-Einstein metrics. This table can be viewed and/or downloaded at the following URL:

`http://www.math.unm.edu/~cboyer/publications/7mantable.ps.`
Alternatively, the same table is part of an earlier preprint version of this article available as math.DG/0108113 at `http://xxx.lanl.gov` or as MPI-2001-69 at `http://www.mpim-bonn.mpg.de`.

In this section we give a discussion of some examples listed there. First we mention that our table lists the weights $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$, degree d, Milnor number μ, and the order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$ of the rational homology 7-sphere $M^7_{\mathbf{w},d}$. It lists $M^7_{\mathbf{w},d}$'s in increasing order of their weights beginning with w_0. The first entry has weights $(17, 34, 75, 125, 175)$ while the last has weights $(357, 388, 2231, 2975, 5593)$. In the first entry the order of H_3 is huge, 17^{12} a number over 500 trillion, while the lowest order of H_3 is $13^2 = 169$. In general the orders of H_3 tend to be quite large.

From a quick perusal of the table, it is easy to notice the existence of *twins*. These are rational homology 7-spheres with the same degree d, Milnor number μ and order of H_3. Twins often occur as adjacent listings with the same w_0, but this is not always the case as with twins $d = |H_3| = 10881, \mu = 10880$ and $w_0 = 101$ and 109, and the twins $d = |H_3| = 7777$ with $w_0 = 141$ and $w_0 = 167$. Twins may also be members of a larger set, such as the *septuplets* with $d = |H_3| = 5761$ and $\mu = 5760$. These have $w_0 = 157, 157, 185, 205, 214, 253, 271$, respectively. Since twins have the same Milnor number, it is tempting to conjecture that twins correspond to homeomorphic or even diffeomorphic links, but we have no proof as of yet. In fact, except for cases where $|H_3|$ contains no primes of order larger than one in its prime decomposition, we don't even know that twins have isomorphic H_3's.

Another interesting fact is that of the 184 rational homology 7-spheres listed only 10 have even degree, while the remaining 174 have odd degree, and the degree is even if and only if the order of H_3 is even. In addition all 174 rational homology 7-spheres with odd degree have $|H_3| \equiv 1 \bmod(8)$. In [BGN4] we constructed positive Sasakian structures on homotopy 9-spheres using the rational homology 7-spheres listed in the Table. There we showed that the exotic Kervaire sphere can only occur when the degree of the rational homology sphere is even (also see [BGN2]).

Also of interest are invariants of the underlying contact, and almost contact structures. The underlying almost contact structures are classified [Sa] by homotopy classes of maps $[M^7, SO(8)/U(4)]$, and Morita [Mo] shows that for Brieskorn spheres this is a function of the Milnor number μ. It seems reasonable that a similar result holds true in our case. There are candidates for this in the table. For example the rational homology 7-spheres with weights $\mathbf{w} = (196, 2337, 7595, 10127, 17917)$ and degree $d = 38171$, and with weights $\mathbf{w} = (147, 207, 230, 245, 299)$ and degree $d = 1127$ both have $|H_3| = 7^4$, so they could be diffeomorphic. But they have very different Milnor numbers, namely, 37440 and 1152, respectively, so they could belong to distinct almost contact structures. Similarly there are 4 rational homology 7-spheres with $|H_3| = 97^2$, two are twins having the same Milnor number, but the other two have different Milnor numbers both different than the Milnor number of the twins. Moreover, twins probably belong to the same underlying almost contact structures, but could possibly belong to distinct contact structures. It appears that nothing is known beyond homotopy spheres [Us1], [Us2] about distinct contact structures within the same underlying almost contact structures.

5. Some comments on regular rational homology spheres.

In this section we discuss some rational homology spheres that are regular, in particular the homogeneous ones. The following result follows easily from previous work [BG1] together with the well-known classification of del Pezzo surfaces:

PROPOSITION 5.1. — *Let $\mathcal{S} = (g, \xi, \eta, \Phi)$ be a regular positive Sasakian structure on a smooth compact 5-manifold M^5. Then M^5 is a rational homology sphere if and only if M^5 is covered by S^5 and $\mathcal{S}$ is homologous to the standard Sasakian structure with the round metric g_0.*

It is well-known that the standard Sasakian structure is a homogeneous Sasakian-Einstein structure. Dimension seven is a bit more interesting:

THEOREM 5.2. — *Let $\mathcal{S} = (g, \xi, \eta, \Phi)$ be a regular positive Sasakian structure on a smooth compact simply connected 7-manifold M^7. Then M^7 is a rational homology sphere if and only if it is one of the following:*

1. *$M^7 = S^7$ and $\mathcal{S}$ is homologous to the standard Sasakian structure with the round metric.*
2. *$M^7 = V_2(\mathbb{R}^5)$ the Stiefel manifold of 2-frames in $\mathbb{R}^5$ and $\mathcal{S}$ is homologous to the standard homogeneous Sasakian-Einstein structure on $V_2(\mathbb{R}^5)$ [BG1], [BG2].*
3. *M^7 is a circle bundle over a smooth variety V_5 of degree 5 in $\mathbb{CP}^6$ with a compatible Sasakian structure $\mathcal{S}$.*
4. *M^7 is a circle bundle over a smooth variety V_{22} of degree 22 in $\mathbb{CP}^{13}$ with a compatible Sasakian structure $\mathcal{S}$.*

Furthermore, M^7 admits a homogeneous Sasakian-Einstein structure if and only if $M^7 = S^7$ or $V_2(\mathbb{R}^5)$.

Proof. — By [BG1] M^7 is a regular rational homology sphere with a Sasakian structure $\mathcal{S}$ if and only if it is the total space of an S^1 bundle over a smooth projective 3-fold $\mathcal{Z}$ with the same rational homology groups as projective space $\mathbb{CP}^3$. Furthermore, $\mathcal{S}$ is positive [BGN3] if and only if $\mathcal{Z}$ is Fano. Thus, $\mathcal{Z}$ must occur on Iskovskikh's list [Isk] (see Remark 5.3 below) of smooth Fano 3-folds of the first kind, and there are precisely four which have the same rational cohomology groups as $\mathbb{CP}^3$. This gives the four cases above. The last statement follows from Corollary 4.1.3 of [BG2]. □

Remarks 5.3. — (1) Case 4 in Theorem 5.2 has an interesting history. The 3-fold V_{22} was missed by Fano in his original classification of smooth 3-folds with an ample anti-canonical line bundle. It was then found by Iskovskikh [Isk] in his study of Fano's work, but a mistake was made and not all were found. Mukai and Umemura [MU] (See also [IsPr]) produced a V_{22} that is an equivariant compactification of $SL(2,\mathbb{C})/\mathbb{I}$ that was missed by Iskovskikh. Here $\mathbb{I}$ is the icosahedral group. Later Prokhorov (see Proposition 4.3.11 of [IsPr]) showed that the Mukai-Umemura V_{22} completes the Fano-Iskovskikh classification of Fano 3-folds. Recently Tian [Ti1], [Ti2] showed that there are deformations P_a of the Mukai-Umemura V_{22} which do not admit a Kähler-Einstein structure, giving a counterexample to the folklore conjecture that every compact Kähler manifold with no holomorphic vector fields admits a compatible Kähler-Einstein metric. Thus, the Sasakian circle bundle over P_a does not admit a compatible Sasakian-Einstein metric. (2) In the four cases of Theorem 5.2, the corresponding Fano 3-folds are precisely those Fano 3-folds that admit an almost homogeneous with respect to the group $SL(2,\mathbb{C})$. (See [IsPr], p. 116).

There is a straightforward procedure for finding all rational homology spheres M^{2n+1} that admit a homogeneous Sasakian-Einstein structure. By Theorem 3.2 of [BG1] M^{2n+1} must fiber over a generalized flag manifold G/P, where G is a complex semi-simple Lie group, and P is a parabolic subgroup. In order that M^{2n+1} be a rational homology sphere, it is necessary that G/P have the rational homology of a projective space. Hence, we may restrict ourselves to the case where G is simple and P is maximal parabolic. The procedure for computing the cohomology ring of G/P is outlined in Baston and Eastwood [BE]. All G/P's with G simple are realized by crossing out nodes in each Dynkin diagram of G. When P is maximal parabolic only one node is crossed out. The rank of the cohomology groups is determined by the Hasse diagram $W^{\mathfrak{p}}$ which is the coset space $W_{\mathfrak{g}}/W_{\mathfrak{p}}$ where $W_{\mathfrak{g}}$ is the Weyl group of the Lie algebra $\mathfrak{g}$ of G, and $W_{\mathfrak{p}}$ is the Weyl group of the Levi factor of the Lie algebra $\mathfrak{p}$ of P. Then the cohomology groups of G/P will have the same rank as $\mathbb{CP}^n$ if and only if $W^{\mathfrak{p}}$ has precisely one element of length l for each $l=1,\cdots,n$. One then needs to check all maximal parabolics for all Dynkin diagrams, and compute the Hasse diagram for each case. There are many cases and repetitions can and do occur. Here we mention the Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ which are circle bundles over the odd quadrics Q_{2n-1} and the homogeneous 3-Sasakian rational homology sphere $G_2/Sp(1)_+$ (cf. [BGP] and Remark 5.6(2) below). A Gysin sequence or spectral sequence argument shows that

$$H^p(V_2(\mathbb{R}^{2n+1}),\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } p=0,4n-1;\\ \mathbb{Z}_2 & \text{if } p=2n;\\ 0 & \text{otherwise.}\end{cases}$$

$$H^p(G_2/Sp(1)_+,\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } p=0,11;\\ \mathbb{Z}_3 & \text{if } p=4,8;\\ 0 & \text{otherwise.}\end{cases}\tag{5.4}$$

We have

PROPOSITION 5.5. — *The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ and $G_2/Sp(1)_+$ are simply connected rational homology spheres which admit homogeneous Sasakian-Einstein structures.*

Remarks 5.6. — (1) Since $V_2(\mathbb{R}^{2n+1})$ can be represented as the link of the quadric hypersurface singularity, 5.3 can be derived from the Milnor-Orlik algorithm described in Section 3. (2) There are two non-conjugate $Sp(1)$ subgroups of the exceptional Lie group G_2, denoted in

[BGP] as $Sp(1)_{\pm}$. The quotient $G_2/Sp(1)_+$ has a homogeneous 3-Sasakian structure, whereas the quotient $G_2/Sp(1)_-$ does not. It does, however, have a homogeneous Sasakian-Einstein structure, and as homogeneous Sasakian-Einstein manifolds $G_2/Sp(1)_- \approx V_2(\mathbb{R}^7)$.

There is an obvious corollary of Theorem 4.2.6 and Proposition 5.4.4 of [BG2], viz.

COROLLARY 5.7. — *Let M^{4n+3} be a rational homology sphere that admits a 3-Sasakian homogeneous structure. Then M^{4n+3} is either $S^{4n+3}, \mathbb{RP}^{4n+3}$, or $G_2/Sp(1)_+$.*

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