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FINITENESS RESULTS FOR HILBERT’S IRREDUCIBILITY THEOREM

by Peter MÜLLER

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1. Introduction.

Let k be a finitely generated field extension of \mathbb{Q} , and R a finitely generated subring of k . In a typical situation k is a number field, and $R = \mathcal{O}_k$ is the ring of integers. Let $f(t, X) \in k(t)[X]$ be an irreducible polynomial. By the well-known Hilbert irreducibility theorem there are infinitely many specializations $t \mapsto \bar{t} \in R$ such that $f(\bar{t}, X)$ is irreducible over k . Furthermore, easy examples show that nevertheless $f(\bar{t}, X)$ may be reducible for infinitely many $\bar{t} \in R$.

Denote by $\text{Red}_f(R)$ the set of those $\bar{t} \in R$ for which $f(\bar{t}, X)$ is defined and reducible over k .

The purpose of this paper is to give several sufficient conditions which guarantee that $\text{Red}_f(R)$ is a finite set, and to give non-trivial examples for infinite $\text{Red}_f(R)$.

Our sufficient conditions are of various types. Section 4.3 gives criteria on the ramification of the place $t \mapsto \infty$ of $k(t)$ in a root field of $f(t, X)$ which imply finiteness of $\text{Red}_f(R)$. For instance, if this place is not ramified at all and $f(t, X)$ has odd degree in X , then $\text{Red}_f(R)$ is finite. Some of the results in this section are related to previous work by Dèbes, see [Dèb86].

In Section 4.4 these results are applied to polynomials of special forms. For instance, we extend a result of Langmann on Thue polynomials. An example of this sort is the following. Let $H(t, X) \in k[t, X]$ be a homogeneous and separable polynomial of degree > 2 . Then $\text{Red}_{H(t, x)-1}(R)$ is finite. This was shown by Langmann in [Lan00] under the additional assumptions that $k = \mathbb{Q}$, $R = \mathbb{Z}$, and that H has odd degree. Our method allows to easily obtain results about polynomial of the form $P(X) - tQ(X)$, which again extend previous results by Langmann [Lan90], [Lan94] by removing technical conditions he had to impose to make his diophantine approximation techniques work.

Another sufficient condition which yields finite $\text{Red}_f(R)$ is a transitivity assumption on the Galois group of $f(t, X)$ over $k(t)$. Assume that this Galois group permutes doubly transitively the roots of $f(t, X)$. Then $\text{Red}_f(R)$ is finite, unless $f(t, X)$ is absolutely irreducible, and the curve $f(t, X) = 0$ has genus 0. This is shown in Section 4.5, where we base our proof on a genus estimation of function fields which we consider interesting in its own right.

This is the only situation where we also consider specialization in k .

We prove finiteness of $\text{Red}_f(k)$ under the stronger sufficient (and generally necessary) condition that the curve $f(t, X) = 0$ has genus > 1 .

While we obtain quite satisfactory results if the Galois group of $f(t, X)$ is doubly transitive, the situation changes drastically if we impose the weaker assumption that this Galois group is primitive. If a weak condition on the composition factors is satisfied, then $\text{Red}_f(R)$ is finite. We also prove a converse to this criterion.

A very precise result is possible if $f(t, X)$ has prime degree in X , and $k = \mathbb{Q}$, $R = \mathbb{Z}$. If $f(t, X) = h(X) - t$ with $h(X) \in \mathbb{Z}[X]$, then clearly $|\text{Red}_f(\mathbb{Z})| = \infty$. In Section 4.7 we show that this is essentially the only instance for odd prime degree polynomials $f(t, X)$ with $|\text{Red}_f(\mathbb{Z})| = \infty$. It is interesting that there are exceptions in degree 2. This section is related to [Mül99], the precursor to this paper.

Our main tool for all these results is Siegel's theorem about algebraic curves with infinitely many integral points in a number field, or the extension by Lang to points in a finitely generated integral domain of characteristic 0.

A variation of the classical reduction theorem in the proof of Hilbert's irreducibility theorem gives the following: Assume that $|\text{Red}_f(R)| = \infty$. Then the splitting field of $f(t, X)$ over $k(t)$ contains an element z , such that $t = g(z)$ for $g(Z) \in k(Z)$ a rational function which assumes infinitely many values in R on k . Furthermore, $f(t, X)$ becomes reducible over $k(z)$. The property $|R \cap g(k)| = \infty$ is rather strong, results of Siegel-Lang give precise and restrictive information about the ramification of the places of $k(z)$ which lie above $t \mapsto \infty$.

It is clear in this setting that the Galois group of $g(Z) - t$ over $k(t)$ is a homomorphic image of the Galois groups of $f(t, X)$. Thus one can expect further results if one is able to classify these former Galois groups. This has been carried out in [Mül01]. The proof (as well as the result) is long and involved. In addition, it makes heavy and frequent use of the classification of the finite simple groups. The final Section 5 provides two applications. The first is a sufficient condition on the composition factors of the Galois group of $f(t, X)$ if this group is primitive which guarantees that $\text{Red}_f(R)$ is finite.

The second result says the following: Let $f(t, X) \in \mathbb{Q}(t)[X]$ have a simple Galois group over $\mathbb{Q}(t)$ of order ≥ 3 which is not isomorphic to an alternating group. Then this group is preserved for all but finitely many

specializations $t \mapsto \bar{t} \in \mathbb{Z}$. Even though this is a smooth result, we doubt that a proof of that can be achieved without knowing the list of the finite simple groups.

This work was inspired by M. Fried's observation of the applicability of group theoretic methods in the analysis of Hilbert sets, see [Fri74], [Fri80], [Fri85].

I thank the referee for thoroughly reading the paper and making many useful suggestions.

2. Consequences from Siegel's theorem.

2.1. Description of Hilbert sets.

The following proposition gives a convenient description of the set of specializations which preserve irreducibility. The argument is a variation of the classical reduction argument in the proof of Hilbert's irreducibility theorem (see e.g. [Lan83, Chapter 9]), combined with Lang's extension of Siegel's theorem about integral points on algebraic curves [Lan83, Chapter 8]. An alternative argument for a similar result, which also relies on a reduction to Siegel's theorem, has been given by Fried, see [Fri74].

PROPOSITION 2.1. — *Let k be a field which is finitely generated over \mathbb{Q} , and R a subring of k which is finitely generated over \mathbb{Z} . For an irreducible polynomial $f(t, X) \in k(t)[X]$ of degree ≥ 2 set*

$$\text{Red}_f(R) := \{\bar{t} \in R \mid f(\bar{t}, X) \text{ is defined and reducible}\}.$$

Let L be a splitting field of $f(t, X)$ over $k(t)$. Then there are finitely many $z_i \in L$ and rational functions $g_i(Z) \in k(Z)$ with $g_i(z_i) = t$, such that the following holds:

- (a) $\text{Red}_f(R)$ and $\bigcup_i (g_i(k) \cap R)$ differ by a finite set.
- (b) $f(t, X)$ is reducible over $k(z_i)$.
- (c) $|g_i(k) \cap R| = \infty$.

Proof. — Once we have elements z_i and rational functions g_i fulfilling (a) and (b), we may assume that (c) holds as well by removing those for which (c) does not hold.

In order to prove the proposition, we may replace R by an extension which still fulfills the assumption on R . A finitely generated extension of R allows to assume that k is the quotient field of R . By another finitely generated extension we may assume that R is integrally closed in k , see [Lan83, Chapter 2, Prop. 4.1].

Replace X and $f(t, X)$ by multiples with elements in $k(t)$ to assume that $f(t, X) \in R[t, X]$ is monic in X . Let x_1, x_2, \dots, x_n be the roots of $f(t, X)$ in an algebraic closure of $k(t)$. For each $I \subset \{1, 2, \dots, n\}$ with $1 \leq |I| \leq n - 1$ set

$$F_I(X) := \prod_{i \in I} (X - x_i).$$

Let K_I be the field generated by $k(t)$ and the coefficients of F_I . Let β_I be a primitive element of $K_I/k(t)$. We may assume that β_I lies in the ring generated by $R[t]$ and the coefficients of F_I . In particular, β_I is integral over $R[t]$. Let $P_I(t, Y) \in R[t, Y]$ be the minimal polynomial of β_I over $k(t)$.

Now take $\bar{t} \in \text{Red}_f(R)$ such that $f(\bar{t}, X)$ is separable. (This assumption excludes only finitely many elements \bar{t} from consideration.) Write $f(\bar{t}, X) = u(X)v(X)$ with u, v monic polynomials in $R[X]$. As $k[t][x_1, x_2, \dots, x_n]$ is integral over $k[t]$, the specialization map $t \mapsto \bar{t}$ from $k[t]$ to k extends to a k -algebra homomorphism $\omega : k[t][x_1, x_2, \dots, x_n] \rightarrow k[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n]$, where the \bar{x}_i are the roots of $f(\bar{t}, X)$. Label these roots such that $\omega(x_i) = \bar{x}_i$. Let I be the set of those i such that \bar{x}_i is a root of u . Denote by $\omega(F_I)$ the polynomial F_I with ω applied to its coefficients, thus $\omega(F_I) = u$. Clearly $P_I(\bar{t}, \omega(\beta_I)) = 0$. But, by the construction above, β_I is a polynomial over $k[t]$ in the coefficients of F_I , hence $\omega(\beta_I) \in k$, and then $\omega(\beta_I) \in R$ because $\omega(\beta_I)$ fulfills an integral equation over R and R is integrally closed in k . Thus each such \bar{t} gives rise to a point $(\bar{t}, \omega(\beta_I)) \in R^2$ on P_I for some index set I .

Now consider those I which appear infinitely many times. Thus the curve $P_I(T, Y) = 0$ has infinitely many points with coordinates in R . The Siegel-Lang Theorem [Lan83, Chapter 8] implies that this curve is rational over k , so there is $z_I \in L$ such that $k(t, \beta_I) = k(z_I)$. Thus $t = g_I(z_I)$ for a rational function $g_I \in k(Z)$. From this we obtain (a) and (b), because $F_I(X) \in k(z_I)[X]$ is a proper factor of $f(t, X)$ over $k(z_I)$. Note that (b) is equivalent to $f(g_I(Z), X)$ being reducible over $k(Z)$. □

2.2. Poles of Siegel functions.

Let k be a field which is finitely generated over \mathbb{Q} , and R a subring which is finitely generated over \mathbb{Z} . The functions $g = g_i$ in Proposition 2.1 assume infinitely many values in R on k . This gives rise to the following definition.

DEFINITION 2.2. — *Let k be a field which is finitely generated over \mathbb{Q} . We say that a non-constant rational function $g(Z) \in k(Z)$ is a Siegel function over k if there is a finitely generated subring R of k with $|g(k) \cap R| = \infty$. If this holds, $g(Z)$ is called an R -Siegel function over k .*

The condition to be a Siegel function is quite strong, and puts severe restrictions on the form of g . The basic result is due to Siegel [Sie29] in the number field case, and has been extended by Lang [Lan83, Theorem 8.5.1] to the more general fields k .

If $g(Z)$ is a rational function and $\alpha \in \mathbb{P}^1(\bar{k}) = \bar{k} \cup \{\infty\}$, then we denote by $g^{-1}(\alpha)$ the set of elements $\beta \in \mathbb{P}^1(\bar{k})$ with $g(\beta) = \alpha$.

PROPOSITION 2.3. — *Let k be a field which is finitely generated over \mathbb{Q} , and $g(Z) \in k(Z)$ a Siegel function over k . Then $|g^{-1}(\infty)| \leq 2$.*

If $k = \mathbb{Q}$, $|g^{-1}(\infty)| = 2$, and $g(Z)$ is a \mathbb{Z} -Siegel function, then the two elements in $g^{-1}(\infty)$ are real and algebraically conjugate.

Remark. — Motivated by the necessary conditions on \mathbb{Z} -Siegel function, Dèbes and Fried [DF99] study so called Siegel families. These are certain parameterized families of covers of degree n from genus 0 curves to $\mathbb{P}^1(\mathbb{C})$, such that the fiber of ∞ consists of two real conjugate points. The subject is to describe the nature of the subset of the \mathbb{Z} -Siegel functions in this family.

3. Siegel functions.

3.1. Cycle types of inertia generators.

Let $g(Z) \in k(Z)$ be a non-constant rational function over a field k of characteristic 0, and t a transcendental.

The following lemma is well-known (and easy to prove using Puiseux series, for instance).

LEMMA 3.1. — *Let m_1, m_2, \dots, m_r be the multiplicities of the elements of $\mathbb{P}^1(\bar{k})$ in the fiber $g^{-1}(\alpha)$ for $\alpha \in \bar{k} \cup \{\infty\}$. Let L be a splitting field of $g(Z) - t$ over $k(t)$, and I the inertia group of a place of L lying above the place $t \mapsto \alpha$ of $k(t)$. Then I is cyclic, and generated by an element which has cycle lengths m_1, m_2, \dots, m_r in the action on the roots of $g(Z) - t$.*

3.2. Decomposition groups.

It is clear from Proposition 2.1 that in order to understand the dependency of the Hilbert sets $R \setminus \text{Red}_f(R)$ in terms of $A = \text{Gal}(f(t, X)/k(t))$, one has to get control over the possibilities for the Galois group of $g(Z) - t$ over $k(t)$ for a Siegel function g .

Let k be a field of characteristic 0, and $g(Z) \in k(Z)$ be a non-constant rational function with $|g^{-1}(\infty)| \leq 2$. Denote by L a splitting field of $g(Z) - t$ over $k(t)$. Set $A := \text{Gal}(L/k(t))$, considered as a permutation group on the roots of $g(Z) - t$, and let $G \trianglelefteq A$ be the normal subgroup $\text{Gal}(\bar{k}L/\bar{k}(t))$.

The following lemma is a variation of the branch cycle argument, see [Fri77], [MM99, 2.2.3], or [Völ96, Lemma 2.8].

LEMMA 3.2. — *Let $D \leq A$ and $I \trianglelefteq D$ be the decomposition and inertia group of a place of L lying above the place $t \mapsto \infty$ of $k(t)$, respectively. Then I is generated by an element $\sigma \in I$, and the following holds:*

- (1) σ has at most two cycles, with lengths equal the multiplicities of the elements in $g^{-1}(\infty)$.
- (2) $A = GD$ and $I \leq G \cap D$.

Suppose that $k = \mathbb{Q}$, $|g^{-1}(\infty)| = 2$, and the two elements in $g^{-1}(\infty)$ are real and algebraically conjugate. Then g has even degree $2m$, and the following holds:

- (a) σ is a product of two m -cycles.
- (b) σ^r is conjugate in D to σ for all r prime to m .
- (c) D contains an element which switches the two orbits of $I = \langle \sigma \rangle$.

- (d) D contains an element τ of order 1 or 2, such that $\sigma^\tau = \sigma^{-1}$, and τ fixes the orbits of I setwise.
- (e) If $g^{-1}(\infty) \not\subseteq \mathbb{Q}(\zeta_m)$ (with ζ_m a primitive m -th root of unity), then D contains an element which interchanges the two orbits of I and centralizes I .

Proof. — Assertions (1) and (a) follow from Lemma 3.1.

Assertion (2): Let O_L be the valuation ring of the given place of L , and \mathfrak{P} the corresponding valuation ideal. Then $(O_L \cap k(t))/(\mathfrak{P} \cap \mathfrak{k}(t))$ is naturally isomorphic to k . Using this identification, O_L/\mathfrak{P} is a Galois extension of k with group D/I , see [Ser79, Chapter I, §7, Prop. 20]. On the other hand, $L \cap \bar{k}$ embeds into O_L/\mathfrak{P} , so D/I surjects naturally to $A/G = \text{Gal}(L \cap \bar{k}/k)$. Furthermore, if $\phi \in I$, then $u - u^\phi \in \mathfrak{P}$ for all $u \in L \cap \bar{k}$, hence ϕ is trivial on $L \cap \bar{k}$, so $I \leq D \cap G$.

It remains to prove (b) to (e).

Composing g with linear fractional functions over \mathbb{Q} allows to assume that the two elements in the fiber $g^{-1}(\infty)$ are $\pm\sqrt{d}$, where $d > 1$ is a squarefree integer. Thus, without loss, assume that $g(Z) = h(Z)/(Z^2 - d)^m$, where $h(Z) \in \mathbb{Q}[Z]$ with $\deg(h) \leq 2m$, and $h(\pm\sqrt{d}) \neq 0$.

Let y be a transcendental over \mathbb{Q} , such that $y^m = 1/t$. Fix a square root \sqrt{d} of d , and let $\varepsilon \in \{-1, 1\}$. Substituting $y\tilde{Z} + \varepsilon\sqrt{d}$ for Z in the equation $h(Z) - t \cdot (Z^2 - d)^m = 0$ gives

$$h(y\tilde{Z} + \varepsilon\sqrt{d}) - \tilde{Z}^m(y\tilde{Z} + 2\varepsilon\sqrt{d})^m = 0.$$

This latter equation, by Hensel's Lemma, is solvable in the power series ring $\overline{\mathbb{Q}}[[y]]$.

Thus, for $i = 1, 2, \dots, m$ and $\varepsilon \in \{-1, 1\}$, we can represent the $2m$ roots of $g(Z) - t$ in the form

$$z_{i,\varepsilon} = \varepsilon\sqrt{d} + a_{1,\varepsilon}\zeta^i y + a_{2,\varepsilon}\zeta^{2i} y^2 + \dots \in \overline{\mathbb{Q}}[[y]],$$

where ζ is a primitive m -th root of unity.

Thus L can be regarded as a subfield of $\overline{\mathbb{Q}}((y))$. Each automorphism of $\overline{\mathbb{Q}}((y))$ which fixes $y^m = 1/t$ then restricts to an element in $D \leq A$, and if it is the identity on $\overline{\mathbb{Q}}$, then the restriction to L lies in I .

We will now construct suitable automorphisms of $\overline{\mathbb{Q}}((y))$ which, when restricted to L , give the required actions on the roots of $g(Z) - t$.

To (b). Let $\hat{\tau} \in \text{Gal}(\overline{\mathbb{Q}}((y))/\mathbb{Q}((y)))$ with $\zeta^{\hat{\tau}} = \zeta^r$, and $\tau := \hat{\tau}|_L$. Then $\tau^{-1}\sigma\tau$ is the identity on $\overline{\mathbb{Q}}$, but $y^{\tau^{-1}\sigma\tau} = y^{\sigma\tau} = (\zeta y)^\tau = \zeta^r y$, so $\tau^{-1}\sigma\tau = \sigma^r$.

To (c). Choose $\hat{\tau} \in \text{Gal}(\overline{\mathbb{Q}}((y))/\mathbb{Q}((y)))$ such that $\sqrt{d}^{\hat{\tau}} = -\sqrt{d}$.

To (d). Choose $\hat{\tau} \in \text{Gal}(\overline{\mathbb{Q}}((y))/\mathbb{Q}((y)))$, such that the restriction of $\hat{\tau}$ to $\overline{\mathbb{Q}}$ is the complex conjugation for a fixed embedding of $\overline{\mathbb{Q}}$ into \mathbb{C} . Then $r = -1$ in the notation of case (b).

To (e). If $\sqrt{d} \notin \mathbb{Q}(\zeta)$, then there is an element $\hat{\tau} \in \text{Gal}(\overline{\mathbb{Q}}((y))/\mathbb{Q}((y)))$ such that $\hat{\tau}$ moves \sqrt{d} , but is the identity on $\mathbb{Q}(\zeta)$. Set $\tau := \hat{\tau}|_L$. This gives $r = 1$ in case (b). □

3.3. Indecomposability versus absolute indecomposability.

The main results of this paper do not depend on this section, the following is only used in the proof of Theorem 4.25.

In this section k is any field of characteristic 0. We say that a non-constant rational function $g(Z) \in k(Z)$ is *functionally indecomposable* if $g(Z)$ cannot be written as a composition of rational functions in $k(Z)$ of lower degree. A classical result by M. Fried says that functionally indecomposable polynomials $g(Z) \in k[Z]$ are functionally indecomposable over \bar{k} , [FM69]. We extend this result to rational functions with $|g^{-1}(\infty)| \leq 2$.

We remark that there are many examples of functionally indecomposable rational functions over k which decompose over \bar{k} . An infinite series can be constructed as follows: Let p be an odd prime, and E an elliptic curve over \mathbb{Q} whose p -torsion points generate a field with Galois group $\text{GL}_2(p)$ over \mathbb{Q} . Denote by $[p]$ the multiplication by p map, and by τ the canonical involution on E . Then $[p]$ induces a map $g : E/\langle\tau\rangle \rightarrow E/\langle\tau\rangle$. We may interpret g as a rational function, because $E/\langle\tau\rangle$ is a rational curve. Now g is indecomposable over \mathbb{Q} . For if g would be decomposable over \mathbb{Q} , then also $[p]$ would be a composition of two degree p isogenies defined over \mathbb{Q} , so E would have a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ invariant subgroup of order p , contrary to the transitive Galois action of $\text{GL}_2(p)$ on the p -torsion points. On the other hand, let P be a subgroup of $E(\overline{\mathbb{Q}})$ of order p . Then $[p]$ factors as $E \rightarrow E/P \rightarrow E$, where the second isogeny is the dual of the first one. Dividing by the canonical involution gives a decomposition of $g(Z)$ over $\overline{\mathbb{Q}}$.

THEOREM 3.3. — *Let k be a field of characteristic 0, and $g(Z) \in k(Z)$ be functionally indecomposable over k . Suppose that $|g^{-1}(\infty)| \leq 2$. Then $g(Z)$ is functionally indecomposable over \bar{k} .*

Proof. — The proof is by group theory. Let L be a splitting field of $g(Z) - t$ over $k(t)$, and \hat{k} the algebraic closure of k in L . Set $A := \text{Gal}(L/k(t))$, and $G := \text{Gal}(L/\hat{k}(t)) \trianglelefteq A$. Note that $G = \text{Gal}(g(Z) - t/\bar{k}(t))$. The assumption and Lüroth's theorem give that A is primitive on the roots of $g(Z) - t$. So it remains to show that G is primitive as well. Let I and D be the inertia and decomposition group of a place of L lying above $t \mapsto \infty$. Lemma 3.2 gives $A = GD$, and I has at most two orbits. Thus the theorem follows from the following purely group theoretic result. \square

THEOREM 3.4. — *Let Ω be a finite set, and let $A \leq \mathcal{S}(\Omega)$ be a primitive permutation group on Ω . Let $1 < G \trianglelefteq A$ be a normal subgroup, which contains a cyclic subgroup I with the following properties:*

- (a) I has at most two orbits on Ω , and
- (b) $A = GN_A(I)$, where $N_A(I)$ denotes the normalizer of I in A .

Then G acts primitively on Ω as well.

Proof. — Suppose that G acts imprimitively. Then Ω is a disjoint union of $\Delta = \Delta_1, \Delta_2, \dots, \Delta_m$, where $1 < r = |\Delta_i| < |\Omega|$, and G permutes the m sets Δ_i . We assume that among these systems we have chosen one such that $|\Delta|$ is maximal. This implies that G permutes the Δ_i 's primitively.

If $a \in A$, then the sets $\Delta_i^a, i = 1, \dots, m$, again constitute a system of imprimitivity for G , this follows from $G \trianglelefteq A$. We claim that there is an element $a \in A$ such that Δ^a is not contained in an I -orbit. Suppose that is not the case. Then, for each a , each orbit of I is a union of sets Δ_i^a . So the sets Δ_i^a in an orbit of I are the orbits of a subgroup of I . The size of this subgroup in I depends on $|\Delta|$, but not on a . On the other hand, a subgroup in a cyclic group is uniquely given by its order. We obtain that for each $a \in A$ the sets Δ_i^a are a permutation of the sets Δ_i , thus the Δ_i are a system of imprimitivity for A , contrary to the assumption that A is primitive.

Thus I has two orbits, and we may assume that Δ intersects them both non-trivially. So I permutes the Δ_i transitively. Let $K \triangleleft G$ be the kernel of the action of G on the Δ_i , and let I_Δ the setwise stabilizer of Δ

in I . As IK/K permutes the Δ_i regularly, we obtain that I_Δ fixes each Δ_i setwise, so $I_\Delta \leq K$. As A is primitive, and K is intransitive on Ω , we get $\bigcap_{a \in A} K^a = 1$. From $A = GN_A(I)$ and $K \triangleleft G$ we obtain $\bigcap_{a \in N_A(I)} K^a = 1$. But $I_\Delta \leq K$ and $(I^\Delta)^a = I_\Delta$ for all $a \in N_A(I)$, so $I_\Delta = 1$. On the other hand, I_Δ has two orbits on Δ , so this implies $|\Delta| = 2$.

Choose $\delta \in \Delta$, and let A_δ and G_δ be the stabilizers of δ in A and G , respectively. Also, let G_Δ be the setwise stabilizer of Δ . Clearly $[G_\Delta : G_\delta] = 2$, so G_δ is normal in G_Δ . Furthermore, $G_\delta = A_\delta \cap G$ is normal in A_δ , so G_δ is normal in the group $U := \langle A_\delta, G_\Delta \rangle$. But A_δ is a maximal subgroup of A by primitivity of A , so $U = A_\delta$ or $U = A$. The former possibility cannot hold, because G_Δ is transitive on Δ , so $G_\Delta \leq A_\delta$ cannot hold. Thus $U = A$, so G_δ is normal in A , hence $G_\delta = 1$. So G acts regularly on Ω , $G_\Delta = K$, and G is the direct product of G_Δ and I by order reasons. But then the intransitive group I is normal in $A = GN_A(I)$, contrary to primitivity of A . □

Remark 3.5. — If I has only one orbit on Ω , then we got the claim without using assumption (b).

However, in general we cannot remove the assumption (b), there are infinite series of counterexamples. For instance let $m \geq 3$ be an integer, and $A = (\mathcal{S}_m \times \mathcal{S}_m) \rtimes C_2$, where C_2 flips the two components. Let the action be given on the coset space A/A_1 , where $A_1 = (\mathcal{S}_{m-1} \times \mathcal{S}_{m-1}) \rtimes C_2$. This action is easily be seen to be primitive. Let $G = \mathcal{S}_m \times \mathcal{S}_m$, and I be generated by (a, b) , where a is an m -cycle, and b is an $(m - 1)$ -cycle. Then one verifies that I has two orbits. However, G is not primitive anymore, because $\mathcal{S}_{m-1} \times \mathcal{S}_m$ is a group properly between $G_1 = G \cap A$ and G .

3.4. Decomposing Siegel functions.

In general, the composition of Siegel functions is not a Siegel function. Conversely, if we write a Siegel function as a composition of rational functions, then not all these rational functions need to be Siegel functions. The following lemma clarifies this issue.

LEMMA 3.6. — *Let k be a finitely generated field over \mathbb{Q} , and $g(Z) \in k(Z)$ a Siegel function over k of degree > 1 . Then there is a decomposition $g(Z) = a(b(Z))$ with $a, b \in k(Z)$, such that the following holds:*

- (a) $a(Z)$ is a functionally indecomposable Siegel function of degree > 1 .
- (b) There is $0 \neq \delta \in k$, such that $\delta b(Z)$ is a Siegel function, or the followings holds: There are linear fractional functions $\lambda, \mu \in k_1(Z)$ over a quadratic extension k_1 of k , $m \in \mathbb{N}$, with $b(Z) = \lambda(\mu(Z)^m)$. In this case, $\text{Gal}(b(Z) - t/k(t))$ is solvable.

The same holds if we set $k = \mathbb{Q}$ and replace Siegel function by \mathbb{Z} -Siegel function everywhere.

Proof. — Let R be a finitely generated subring of k such that $|g(k) \cap R| = \infty$. (Or $R = \mathbb{Z}$ in the context of \mathbb{Z} -Siegel functions.)

Write $g(Z) = a(b(Z))$ with $a, b \in k(Z)$ and $a(Z)$ being indecomposable over k of degree > 1 . As $b(k) \subseteq k \cup \{\infty\}$, we have $|a(k) \cap R| = \infty$, so (a) clearly holds.

From $|g^{-1}(\infty)| \leq 2$ we obtain $|a^{-1}(\infty)| \leq 2$.

We first analyze the case $|a^{-1}(\infty)| = 1$. Because $\text{Gal}(\bar{k}/k)$ acts on $a^{-1}(\infty)$, this single element in this fiber must be rational or ∞ . By a linear fractional change we may assume that $a^{-1}(\infty) = \{\infty\}$, so $a(Z)$ is a polynomial. Write $a(Z) = a_r Z^r + a_{r-1} Z^{r-1} + \dots + a_1 Z + a_0$ with $a_i \in k$. By assumption, there are infinitely many $\bar{z} \in k$ such that $\beta = b(\bar{z})$ fulfills $a(\beta) = \rho \in R$. So β is integral over $R' = R[\frac{a_{r-1}}{a_r}, \dots, \frac{a_1}{a_n}, \frac{a_0}{a_n}]$. Thus replace R by a finitely generated ring containing the integral closure of R' in k , using [Lan83, Chapter 2, Proposition 4.1]. So $b(Z)$ is a Siegel function with respect to this ring. Next assume the \mathbb{Z} -Siegel function case. Then $a_i \in \mathbb{Q}$. Let w be a common multiple of the denominators of a_i . The previous consideration shows that $a_n w \beta \in \mathbb{Q}$ is integral over \mathbb{Z} , hence contained in \mathbb{Z} . So $a_n w b(Z)$ assumes infinitely many integral values on \mathbb{Q} , and (b) follows again.

Now assume $|a^{-1}(\infty)| = 2$. Write $a^{-1}(\infty) = \{\lambda_1, \lambda_2\}$. Then the λ_i are either in $k \cup \{\infty\}$, or they generate a quadratic extension k_1 of k . Furthermore, we obtain $b^{-1}(\lambda_i) = \{\mu_i\}$, with $\mu_i \in k \cup \{\infty\}$ in the former case, or $\mu_i \in k_1$ in the latter case. At any rate, there are linear fractional functions $\lambda, \mu \in k_1(Z)$ such that, with $\tilde{b}(Z) := \lambda^{-1}(b(\mu^{-1}(Z)))$, the following holds: $\tilde{b}^{-1}(\infty) = \{\infty\}$, $\tilde{b}^{-1}(0) = \{0\}$, and $\tilde{b}(1) = 1$. This implies $\tilde{b}(Z) = Z^m$. The Galois group of $b(Z) - t$ over $k_1(t)$ is the same one as the Galois group of $\tilde{b}(Z) - t$ over $k_1(t)$. This group is contained in $\text{AGL}_1(n)$, hence solvable. The Galois group of $b(Z) - t$ over $k(t)$ is an extension of the former Galois group by at most the index 2, so is solvable

as well. This proves (b). □

COROLLARY 3.7. — *Let k be a finitely generated field extension of \mathbb{Q} , and $g(Z) \in k(Z)$ a Siegel function over k . Let S be a non-abelian composition factor of $\text{Gal}(g(Z) - t/k(t))$. Then there is a functionally indecomposable Siegel function $\tilde{g}(Z)$ over k , such that S is a composition factor of $\text{Gal}(\tilde{g}(Z) - t/k(t))$.*

The same holds for $k = \mathbb{Q}$ and Siegel function replaced by \mathbb{Z} -Siegel function.

Proof. — If $g(Z) = g_1(g_2(\dots g_r(Z)\dots))$ with functionally indecomposable rational functions $g_i(Z) \in k(Z)$, then S is a composition factor of $\text{Gal}(g_i(Z) - t/k(t))$ for some index i . See Glauber's argument in [GT90, Prop. 2.1] for this fact which is less obvious than it might appear at a first glance.

The assertion now follows from Lemma 3.6. □

4. Applications to Hilbert's irreducibility theorem.

4.1. Not absolutely irreducible polynomials.

In this section k may be any field of characteristic 0.

It is a well-known consequence from Bezout's theorem that if $f(X, Y) \in k[X, Y]$ is irreducible, but not absolutely irreducible, then there are only finitely many $(a, b) \in k^2$ with $f(a, b) = 0$. We show that under certain additional assumptions an analogue of this observation holds in the context of Hilbert sets.

LEMMA 4.1. — *Let $f(t, X) \in k(t)[X]$ be an irreducible polynomial over $k(t)$, and let A be the Galois group of $f(t, X)$ over $k(t)$. Assume that $f(\bar{t}, X)$ is reducible over k for infinitely many $\bar{t} \in k$, and that one of the following holds:*

- (a) A is a simple group; or
- (b) A acts primitively on the roots of $f(t, X)$.

Then $f(t, X)$ is absolutely irreducible over k .

Proof. — We may assume that $f(t, X) \in k[t, X]$ is monic in X . Let \bar{k} be an algebraic closure of k , and G the Galois group of $f(t, X)$ over $\bar{k}(t)$. Suppose that $f(t, X)$ is not absolutely irreducible. Then G is an intransitive normal subgroup of A . Hypothesis (a) implies that $G = 1$. We get $G = 1$ also from hypothesis (b), because the orbits of G are a system of imprimitivity for A . Thus $f(t, X)$ decomposes into linear factors $X - a_i(t)$ with monic polynomials $a_i(t) \in \bar{k}(t)$. We obtain the claim either from [Dèb96, Lemma 2.8(b)], or from the following argument. By separability of $f(t, X)$, the elements $a_i(\bar{t})$ are distinct for all but finitely many $\bar{t} \in k$. This proves the claim, because the transitive action of $\text{Gal}(\bar{k}/k)$ on the set of the $a_i(t)$ is equivariant with respect to the specialization $t \mapsto \bar{t}$. \square

Remark 4.2. — The assertion of the corollary becomes false if we relax the assumption on A . For instance, take $f(t, X) = X^4 + 2(1 - t)X^2 + (1 + t)^2$. Then $f(t, X)$ is irreducible over $\mathbb{Q}(t)$, but $f(t, X) = (X^2 + 2iX - 1 - t)(X^2 - 2iX - 1 - t)$, where $i^2 = -1$. Furthermore, from $f(u^2, X) = (X^2 + 2uX + u^2 + 1)(X^2 - 2uX + u^2 + 1)$ we see that $f(\bar{t}, X)$ is reducible over \mathbb{Q} for each square $\bar{t} \in \mathbb{Q}$.

4.2. Consequences from $|\text{Red}_f(R)| = \infty$.

Let k be a field which is finitely generated over \mathbb{Q} , and R a finitely generated subring. The following lemma summarizes how we use the information that $\text{Red}_f(R)$ is an infinite set for an irreducible polynomial $f(t, X) \in k(t)[X]$.

LEMMA 4.3. — *Let $f(t, X)$ be irreducible, and assume that $|\text{Red}_f(R)| = \infty$. Let L be a splitting field of $f(t, X)$ over $k(t)$, and $x \in L$ a root of $f(t, X)$. Set $A := \text{Gal}(L/k(t))$, and let D and I be the decomposition group and inertia group of a place of L lying above $t \mapsto \infty$, respectively.*

Then there is a rational field $k(t) \subseteq k(z) \subseteq L$, such that the following holds, where A_x and A_z are the stabilizers in A of x and z , respectively:

- (a) A_z acts intransitively on the coset space A/A_x .
- (b) I is cyclic, and has at most two orbits on A/A_z . If $k = \mathbb{Q}$ and $R = \mathbb{Z}$, then these orbits have equal lengths.
- (c) If $k = \mathbb{Q}$ and $R = \mathbb{Z}$, then D is transitive on A/A_z .

Proof. — The existence of the field $k(z)$ with (a) follows from Proposition 2.1. Furthermore, $t = g(z)$ with $g(Z) \in k(Z)$ a Siegel function, so (b) and (c) follow from Lemma 3.2, (1), (a), and (c). □

4.3. Conditions on ramification.

Throughout this section k is a finitely generated field extension of \mathbb{Q} , and R is a finitely generated subring of k .

We obtain finiteness results under suitable conditions on the ramification indices of the places of a root field of $f(t, X)$ which lie above the place $t \mapsto \infty$ of $k(t)$.

THEOREM 4.4. — *Let $f(t, X) \in k(t)[X]$ be irreducible, and assume that the place $t \mapsto \infty$ of $k(t)$ is unramified in the field $k(t, x)$, where x is a root of f . Then one of the following holds:*

- (i) $f(\bar{t}, X)$ is irreducible over k for all but finitely many $\bar{t} \in R$, or
- (ii) There is an element $z \in k(t, x)$, such that $t = g(z)$ with $g(Z) \in k(Z)$ of degree 2.

Remark. — In general one cannot avoid the situation of case (ii). For instance set $g(Z) = 1/(Z^2 - d)$, where $d > 1$ is a squarefree integer. Let z be a root of $g(Z) - t$, and let x be algebraic over $k(z)$ such that the places $z \mapsto \pm\sqrt{d}$ of $k(z)$ are unramified in $k(z, x)$. Then the minimal polynomial $f(t, X)$ of z over $k(t)$ fulfills the assumptions of the theorem. However, there are infinitely many $\bar{z} \in \mathbb{Q}$ with $\bar{t} = g(\bar{z}) \in \mathbb{Z}$, and for each such \bar{t} the polynomial $f(\bar{t}, X)$ is reducible.

Assume the situation of the previous theorem, and let \tilde{x} be a primitive element of the normal closure of $k(t, x)/k(t)$. Apply the theorem to the minimal polynomial of \tilde{x} over $k(t)$. (Note that $t \mapsto \infty$ is unramified in this normal closure too.) Then case (ii) can only appear if the Galois group of $k(t, \tilde{x})/k(t)$ has a subgroup of index 2. Thus we obtain the following

COROLLARY 4.5. — *Let $f(t, X) \in k(t)[X]$ be irreducible, and assume that the place $t \mapsto \infty$ of $k(t)$ is unramified in the field $k(t, x)$, where x is a root of f . Suppose that the Galois group A of $f(t, X)$ over $k(t)$ has no subgroup of index 2. Then $A = \text{Gal}(f(\bar{t}, X)/k)$ for all but finitely many $\bar{t} \in R$.*

In general one cannot relax the assumption about the infinite place without introducing severe other conditions in the theorem. However, if the base field is $k = \mathbb{Q}$, then the following holds.

THEOREM 4.6. — *Let $f(t, X) \in \mathbb{Q}(t)[X]$ be irreducible of odd degree, and x a root of f . Assume that the greatest common divisor of the ramification indices of the places of $\mathbb{Q}(t, x)$ which lie above the place $t \mapsto \infty$ of $\mathbb{Q}(t)$ is 1. Then $f(\bar{t}, X)$ is irreducible over \mathbb{Q} for all but finitely many $\bar{t} \in \mathbb{Z}$.*

A theorem of a similar flavor is

THEOREM 4.7. — *Let $f(t, X) \in \mathbb{Q}(t)[X]$ be irreducible, and x a root of $f(t, X)$. Assume that $\mathbb{Q}(t, x)$ has an unramified place above the place $t \mapsto \infty$ of $\mathbb{Q}(t)$. If this place has odd degree, or if $f(t, X)$ has odd degree, then $f(\bar{t}, X)$ is irreducible over \mathbb{Q} for all but finitely many $\bar{t} \in \mathbb{Z}$.*

Remark. — The condition of an unramified place of $\mathbb{Q}(t, x)$ occurs in previous work, see [Spr83] and [Dèb86], where finiteness results like the above are obtained, however under stronger additional conditions. The methods are different and effectively determine the sets $\text{Red}_f(\mathbb{Z})$. Our approach, based on Siegel's theorem for which no effective version is known, cannot be extended to give effective results. The diophantine approach in [Dèb86] and [Dèb96] even allows to give effective versions over fields with a product formula.

Proof of the theorems. — Let us assume that $f(\bar{t}, X)$ is reducible for infinitely many $\bar{t} \in R$. Let L be a splitting field of $f(t, X)$ over $k(t)$, and $x \in L$ a root of $f(t, X)$. We make frequent use of Lemma 4.3 and the notation introduced there.

Let $\sigma \in A$ be a generator of I .

First assume the situation from Theorem 4.4. This means that the inertia group I is trivial, so $\sigma = 1$. On the other hand, σ has at most two cycles on A/A_z . As A_z is a proper subgroup of A (because A_z is intransitive on A/A_x), this implies $[A : A_z] = 2$. Furthermore, A_z is normal in A , so $A_z A_x$ is a proper subgroup of A . This implies $A_x \subseteq A_z$, so $z \in k(t, x)$, and the claim follows.

Next assume the assumptions from Theorem 4.6. Then σ acts on A/A_z as a product of r cycles of length m , with $r = 1$ or 2 . Let a_1, a_2, \dots, a_j be

the cycle lengths of σ on A/A_x . These lengths are the ramification indices of the places of $\mathbb{Q}(t, x)$ above $t \mapsto \infty$, so the greatest common divisor of the a_1, a_2, \dots, a_j is 1.

Recall that A_x acts intransitively on A/A_z . Let $u < rm = [A : A_z]$ be an orbit length of this action. As $\sigma^{a_i}, 1 \leq i \leq j$ has a fixed point on A/A_x , it is conjugate to an element in A_x . On the other hand, σ^{a_i} has cycle lengths $m/\gcd(m, a_i)$ on A/A_z . Therefore $m/\gcd(m, a_i)$ divides u and hence $\gcd(u, m)$ too. Thus $m/\gcd(m, u)$ divides $\gcd(m, a_i)$ for each a_i . But the a_i have the greatest common divisor 1, hence $m = \gcd(m, u)$, so $r = 2$ and $u = m$. In particular, A_x has two orbits of equal length m on A/A_z . The orbit lengths of A_x on A/A_z are proportional to the sizes of the double cosets $A_z a A_x, a \in A$. Therefore A_z has two orbits of equal length on A/A_x , so $[A : A_x] = \deg_X(f(t, X))$ is even, which proves the theorem.

Finally assume the situation from Theorem 4.7. The condition on the unramified place shows that I fixes a point on A/A_x , so we may assume that $I \leq A_x$. The at most two orbits of I on A/A_z are contained in the at least two orbits of A_x on A/A_z , so both these groups have the same two orbits of equal length on A/A_z . As in the previous paragraph, we obtain that $\deg_X(f(t, X)) = [A : A_x]$ is even. It remains to show that the unramified place in question has even degree. This is equivalent to show that the orbit length $[A_x D : A_x] = [D : A_x \cap D]$ of D on A/A_x through the coset A_x is even. Let $B \geq A_x$ be the biggest subgroup of A which stabilizes the two orbits of A_x on A/A_z . As D is transitive on A/A_z , we obtain that $2 = [BD : B] = [D : B \cap D]$. But $[D : B \cap D]$ divides $[D : A_x \cap D]$, and the theorem follows. □

4.4. Polynomials of special forms.

The following theorem is a generalization of [Lan00, Satz 3.5]. Langmann obtains his result under the following three additional assumptions none of which we need in our approach:

- (a) $k = \mathbb{Q}$ and $R = \mathbb{Z}$,
- (b) the degree of H is odd, and
- (c) t does not divide $H(t, X)$.

THEOREM 4.8. — *Let $H(t, X) \in k[t, X]$ be a homogeneous polynomial of total degree > 2 which is separable with respect to X . Then $H(\bar{t}, X) - 1$ is irreducible for all but finitely many $\bar{t} \in R$.*

Remark. — The theorem is false for degree 2, even over the rationals. To see this set $H(t, X) = X^2 - dt^2$ with $d > 1$ a square-free integer. This is indeed an exception, because the Pellian equation $X^2 - dt^2 = 1$ has infinitely many integral solutions.

A different generalization of Langmann's result is obtained simply by removing the separability assumption on $H(t, X)$ and replacing it by the obviously necessary condition that $H(t, X)$ is not a proper power.

THEOREM 4.9. — *Let $H(t, X) \in \mathbb{Q}[t, X]$ be a homogeneous polynomial of odd degree which is not divisible by t . If $H(t, X)$ is not a proper power in $\mathbb{Q}[t, X]$, then $H(\bar{t}, X) - 1$ is irreducible for all but finitely many $\bar{t} \in \mathbb{Z}$.*

Remark. — This theorem is no longer true for number fields. An example is the following: Let k be a number field with an infinite group of units, and R the ring of integers. Set $H(t, X) := X^2(X - t)$. From

$$H\left(\frac{1 - Z^3}{Z}, X\right) - 1 = \left(X - \frac{1}{Z}\right)(X^2 + Z^2X + Z)$$

and the fact that $\bar{t} = (1 - \bar{z}^3)/\bar{z} \in R$ for each unit \bar{z} we obtain reducibility of $H(\bar{t}, X) - 1$ for infinitely many $\bar{t} \in R$.

CONJECTURE 4.10. — *The assumption that $H(t, X)$ has odd degree n in the above theorem can be dropped if we require the following necessary conditions:*

- (a) $n \neq 2, 4$.
- (b) *If 4 divides n , then $-4H(t, x)$ is not a 4-th power in $\mathbb{Q}[t, X]$. (For otherwise $H(t, X) - 1$ is already reducible).*

Remark. — The group theory got quite involved in an attempt to prove this conjecture. While we feel that we got close to a proof, some difficulties could not be settled. The conjecture is true up to degree 25, at least under the slightly stronger condition that $H(t, X)$ is not a power in $\overline{\mathbb{Q}}[t, X]$. From above we know already that we have to assume $n \neq 2$. The following example shows that $n \neq 4$ is also a necessary condition. This is interesting because the associated curve has genus 1, so the polynomial has a linear factor for only finitely many integral specializations:

Let $d > 1$ be a squarefree integer, and set

$$f(t, X) = -4dX^2(dX^2 - t^2) - 1.$$

Note that

$$f\left(\frac{Z^2 + d}{Z^2 - d}, X\right) = -\left(2dX^2 - \frac{4dZ}{Z^2 - d}X - 1\right)\left(2dX^2 + \frac{4dZ}{Z^2 - d}X - 1\right).$$

There are infinitely many integers u, v with $u^2 - dv^2 = 1$. For $\bar{z} = u/v$ we obtain $\bar{t} = \frac{\bar{z}^2 + d}{\bar{z}^2 - d} = u^2 + dv^2 \in \mathbb{Z}$, and by the above factorization $f(\bar{t}, X)$ is reducible.

The proof of Theorem 4.8 is based on

PROPOSITION 4.11. — *Let m be a positive integer, and $h(X) \in k[X]$ a non-constant separable polynomial. Suppose that $\bar{t}^m h(X) - 1$ is reducible for infinitely many $\bar{t} \in R$. Then $m \leq 2$ and $\deg(h)$ is even.*

Proof. — The polynomial $t^m h(X) - 1$ is irreducible over \bar{k} , for instance by the Eisenstein criterion with respect to a linear factor of $h(X)$. Let x be a root of $t^m h(X) - 1$. By the separability of h and Hensel's Lemma, we can write x as a Laurent series in $1/t^m$ over \bar{k} . Thus the place $t \mapsto \infty$ is unramified in $k(t, x)$, and so is the place $t^m \mapsto \infty$ of $k(t^m)$ in $k(x)$. Theorem 4.4 gives $z \in k(t, x)$ such that $k(z)$ is a quadratic extension of $k(t)$, in particular, $h(X)$ has even degree.

Let $k(y)$ be the intersection of $k(z)$ and the normal closure of $k(x)/k(t^m)$. Clearly, $t^m h(X) - 1$ is reducible over $k(y)$, so $k(y)$ is a proper extension of $k(t^m)$.

From now on we consider only the fields between $k(z)$ and $k(t^m)$. To ease language, we extend the coefficients to \bar{k} . The place $t^m \mapsto \infty$ is totally ramified in $\bar{k}(t)$, so there are at most two places of $\bar{k}(y)$ above $t^m \mapsto \infty$. On the other hand, the place $t^m \mapsto \infty$ is unramified in the normal closure of $k(x)/k(t^m)$, so it is unramified in $\bar{k}(y)$ as well.

Thus $[\bar{k}(y) : \bar{k}(t^m)] = 2$, there are two places of $\bar{k}(y)$ above $t^m \mapsto \infty$, and these two places are the only places which are ramified in $\bar{k}(z)$, because there are at most two places of $\bar{k}(z)$ above $t^m \mapsto \infty$. Let \mathfrak{p} be a place of $\bar{k}(y)$ lying above $t^m \mapsto 0$. From what we saw, m places of $\bar{k}(z)$ lie above \mathfrak{p} . Thus at least m places of $\bar{k}(z)$ lie above $t^m \mapsto 0$. On the other hand, $t^m \mapsto 0$ is totally ramified in $\bar{k}(t)$, so at most two places of $\bar{k}(z)$ lie above $t^m \mapsto 0$. Thus $m \leq 2$, and the claim follows. \square

Proof of Theorem 4.8. — Let n be the total degree of $H(t, X)$. Then $H(t, X) = t^n h(X/t)$, where $h(X) \in k[X]$ is a polynomial of degree $\leq n$.

Note that $H(\bar{t}, X) - 1$ is reducible if and only if $\bar{t}^n h(X) - 1$ is reducible, so the claim follows from Proposition 4.11. \square

Proof of Theorem 4.9. — Let n be the total degree of $H(t, X)$. Write $H(t, X) = t^n h(X/t)$ for a polynomial h . We claim that $H(t, X) - 1$ is irreducible over $\mathbb{Q}(t)$. If this is not the case, then (upon replacing t with $1/T$ and XT with Z) $h(Z) - T^n$ is reducible. Considered as a polynomial in T , it is well known that $h(Z)$ has to be a proper power of a polynomial over \mathbb{Q} , a contradiction.

Let e the greatest common divisor of the multiplicities of the linear factors of $H(t, X)$ over $\overline{\mathbb{Q}}$. Then $H(t, X) = c\tilde{H}(t, X)^e$, where $c \in \mathbb{Q}$ and $\tilde{H}(t, X) \in \mathbb{Q}[t, X]$ is homogeneous of degree n/e and monic in X . The greatest common divisor of the multiplicities of the linear factors of $\tilde{H}(t, X)$ is 1.

Let $\gamma \in \overline{\mathbb{Q}}$ with $\gamma^e c = 1$, and let ζ be a primitive e th root of unity. Then $H(t, X) - 1 = c \prod_{i=1}^e (\tilde{H}(t, X) - \gamma\zeta^i)$. An argument as above shows that $\tilde{H}(t, X) - \gamma$ is irreducible over $\overline{\mathbb{Q}}$. Suppose that $H(\bar{t}, X) - 1$ is reducible for infinitely many $\bar{t} \in \mathbb{Z}$. Then there is a \mathbb{Z} -Siegel function $g(Z) \in \mathbb{Q}(Z)$ such that $H(g(Z), X) - 1$ is reducible over $\mathbb{Q}(Z)$. Let $A(Z, X)$ be a non-trivial factor. We claim that $\tilde{H}(g(Z), X) - \gamma$ is reducible over $\overline{\mathbb{Q}}(Z)$. Suppose that is not the case. As $\tilde{H}(t, X) - \gamma$ divides $H(t, X) - 1$, we may assume that $\tilde{H}(g(Z), X) - \gamma$ divides $A(Z, X)$. However, the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixes $A(Z, X)$, while it permutes transitively the factors $\tilde{H}(g(Z), X) - \gamma\zeta^i$ of $H(g(Z), X) - 1$ (note that $U^e - c$ is irreducible over \mathbb{Q} , for otherwise c were a proper power, and so were $H(t, X)$). Thus $H(g(Z), X) - 1$ divides $A(Z, X)$, a contradiction.

So $\tilde{H}(g(Z), X) - \gamma$ is reducible over $\overline{\mathbb{Q}}(Z)$ with $g(Z)$ a \mathbb{Z} -Siegel function over \mathbb{Q} . Set $\tilde{n} = n/e = \text{deg}(\tilde{H})$, and write $\tilde{H}(t, X) = \gamma t^{\tilde{n}} \tilde{h}(X/t)$ with $\tilde{h}(X) \in \overline{\mathbb{Q}}[X]$. Upon replacing X by $Xg(Z)$, we get that $g(Z)^{\tilde{n}} \tilde{h}(X) - 1$ is reducible over $\overline{\mathbb{Q}}(Z)$. Let L be a splitting field of $t\tilde{h}(X) - 1$ over $\overline{\mathbb{Q}}(t)$, and z be a root of $g(Z)^{\tilde{n}} - t$. So $t\tilde{h}(X) - 1$ is reducible over $\overline{\mathbb{Q}}(z)$. Denote by $\overline{\mathbb{Q}}(y)$ the intersection of $\overline{\mathbb{Q}}(z)$ with L . Of course, $t\tilde{h}(X) - 1$ is reducible over $\overline{\mathbb{Q}}(y)$ as well. Write $t = \tilde{g}(y)$ with $\tilde{g}(Y) \in \overline{\mathbb{Q}}(Y)$. As \tilde{g} , composed with another rational function, gives g , we obtain that the fiber $\tilde{g}^{-1}(\infty)$ contains at most two elements, and that the multiplicities of these elements are the same.

By construction $\tilde{h}(X)$ is a polynomial where the multiplicities of the roots have no common divisor > 1 . These multiplicities are exactly

the ramification indices of the places of $\overline{\mathbb{Q}}(t, x)$ which lie above the place $t \mapsto \infty$ of $\overline{\mathbb{Q}}(t)$, where x is a root of $t\tilde{h}(X) - 1$. Thus the assumptions of Theorem 4.6 are fulfilled except that we are not necessarily over the rationals. Nevertheless, the proof of that theorem covers our situation, because we used the assumption that the base field is \mathbb{Q} only to guarantee that, in the present context, the elements in the fiber $\tilde{g}^{-1}(\infty)$ have the same multiplicities. But we have verified this property above, so the claim follows. \square

Another easy consequence of Theorem 4.4 (and its proof) is

THEOREM 4.12. — *Let $P(X) \in k[X]$ be a polynomial which is relatively prime to the separable polynomial $Q(X) \in k[X]$ of degree $\geq \deg(P) - 1$. Then one of the following holds:*

- (a) $P(X) - \bar{t}Q(X)$ is irreducible for all but finitely many $\bar{t} \in R$, or
- (b) $\max(\deg(P), \deg(Q))$ is even, and there is a rational function $g(Z) \in \bar{k}[Z, \frac{1}{Z}]$ of degree 2, such that $P(X) - tQ(X)$ factors over $\bar{k}(Z)$ in two factors of equal degree in X .

Remark. — This result generalizes [Lan90, Folgerung 6], where this is proven under the assumption that $\deg(Q) = \deg(P) - 1$. Also, the rather technical result [Lan94, Folgerung 3.4] is a very special case of Theorem 4.4.

A direct application of Theorem 4.7 to polynomials of the form $P(X) - tQ(X)$ (which are studied in [Lan90] and [Lan00], too) is

THEOREM 4.13. — *Let $P(X), Q(X) \in \mathbb{Q}[X]$ be relatively prime polynomials, and assume that $Q(X)$ has a simple root α . If one of $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ or $\max(\deg(P), \deg(Q))$ is odd, then $P(X) - \bar{t}Q(X)$ is irreducible for all but finitely many $\bar{t} \in \mathbb{Z}$.*

Remark. — Langmann and other authors, in particularly Dèbes (see [Dèb92]) and Fried (see [Fri85]) have studied irreducibility questions when specializing t in certain subsets of the integers. Examples are the sets of prime powers, or powers of a fixed integer. A recent result of this kind with a completely elementary and elegant proof (in particularly not relying on Siegel's Theorem) is the following by Cavachi [Cav00] (his version is slightly more general): Let $P(X), Q(X) \in \mathbb{Q}[X]$ be relatively prime polynomial with $\deg(P) < \deg(Q)$. Then $P(X) - pQ(X)$ is irreducible for all but finitely many prime numbers p .

4.5. Doubly transitive Galois groups.

The main result of this section is

THEOREM 4.14. — *Let $f(t, X) \in k(t)[X]$ be irreducible, and assume that the Galois group of $f(t, X)$ over $k(t)$ acts doubly transitively on the roots of f . If $f(\bar{t}, X)$ is reducible for infinitely many $\bar{t} \in R$, then the following holds, where x is a root of $f(t, X)$:*

- (a) $f(t, X)$ is absolutely irreducible, and
- (b) $k(t, x)$ has genus 0, and
- (c) there are at most two places of $\bar{k}(t, x)$ above $t \mapsto \infty$.

As a preparation we need a bound on the genus of function fields.

4.5.1. Genus comparison. — Let k be a field of characteristic 0, and $L/k(t)$ be a finite Galois extension. Let \hat{k} be the algebraic closure of k in L . Set $A := \text{Gal}(L/k(t))$ and $G := \text{Gal}(L/\hat{k}(t)) \trianglelefteq A$.

The following is well known (see e.g. [Gro71, Exp. XIII, Cor. 2.12]): Let \mathfrak{p}_i , $i = 1, \dots, r$, be the places of $\bar{k}(t)$ which ramify in $\bar{k}L$. Let I_i be the inertia group of a place of $\bar{k}L$ lying above \mathfrak{p}_i . We identify $\text{Gal}(\bar{k}L/\bar{k}(t))$ with G via restriction to L . We can choose elements $\sigma_i \in G$ such that each σ_i is conjugate to a generator of I_i , and the following holds:

- (a) The σ_i , $i = 1, \dots, r$, generate G .
- (b) $\sigma_1 \sigma_2 \cdots \sigma_r = 1$.

The (not uniquely given) tuple $(\sigma_1, \sigma_2, \dots, \sigma_r)$ is called a *branch cycle description* in G .

If E is a field between $k(t)$ and L with $n = [E : k(t)]$, then A acts as a permutation group on the n conjugates of a primitive element of $E/k(t)$. Let π_E be the homomorphism from A to the symmetric group \mathcal{S}_n .

For a permutation σ on n letters let $\text{ind}(\sigma)$ be “ n minus the number of cycles” of σ . Let E be as above, and assume that $\hat{k} \cap E = k$.

The Riemann-Hurwitz genus formula allows to compute the genus $g(E)$ of E :

$$(1) \quad 2(n - 1 + g(E)) = \sum_{i=1}^r \text{ind}(\pi_E(\sigma_i)).$$

Associated to π_E is the permutation character χ_E , where $\chi_E(\sigma)$ is the number of fixed points of $\pi_E(\sigma)$. In the following lemma a character is understood as a character over the complex numbers. Each character is a unique linear combination of irreducible characters with non-negative integral coefficients. See for example [Gor68] or [Isa76] for more basic facts used in the following.

LEMMA 4.15. — *In the setting from above, let F be another field between $k(t)$ and L , such that $\hat{k} \cap F = k$. Suppose that $\pi_F - \pi_E$ is a character (or 0). Then the following holds:*

- (a) $g(E) \leq g(F)$.
- (b) *For each subgroup $U \leq A$, the number of orbits of $\pi_E(U)$ is not bigger than the number of orbits of $\pi_F(U)$.*

Remark. — To my knowledge part (a) has first been observed by R. Guralnick some years ago. His proof in [Gur00] does not rely on the Riemann-Hurwitz formula and the branch cycle description. Instead, he uses Jacobians of function fields and the action of the Galois group on the ℓ -torsion points for a suitable ℓ . This approach proves (a) in positive characteristic as well. Independently I had found this result by using a linear algebra result of Scott (see below). As Guralnick's proof is not yet published, we supply our elementary proof. This proof, however, does not work in positive characteristic due to the lack of branch cycle descriptions.

The following proposition is an immediate consequence of Scott's result [Sco77, Theorem 1] and Maschke's theorem.

PROPOSITION 4.16. — *Let the finite group G act linearly on the n -dimensional complex vector space V . For M an element or subgroup of G , let $d(M)$ be the dimension of the subspace of fixed vectors under M . Let G be generated by $\sigma_1, \sigma_2, \dots, \sigma_r$, and assume that $\sigma_1\sigma_2 \cdots \sigma_r = 1$. Then*

$$2(n - d(G)) \leq \sum_{i=1}^r (n - d(\sigma_i)).$$

Proof of Lemma 4.15. — Let V_E and V_F be the permutation modules corresponding to π_E and π_F . Considering the set which $\pi_E(G)$ acts on as the natural basis of V_E , we may consider π_E as a homomorphism from G to $GL(V_E)$. With respect to this natural basis, we see the following: If $\pi_E(\sigma)$ has a cycle of length m , then the eigenvalues of $\pi_E(\sigma)$ on the space spanned

by these m cyclically moved elements are just the m -th roots of unity. In particular, the eigenvalue 1 appears exactly once on this subspace. Thus the number of cycles of $\pi_E(\sigma)$ equals $d(\pi_E(\sigma))$. From that we obtain

$$(2) \quad 2([E : k(t)] - 1 + g(E)) = \sum_{i=1}^r ([E : k(t)] - d(\pi_E(\sigma_i)))$$

and likewise

$$(3) \quad 2([F : k(t)] - 1 + g(F)) = \sum_{i=1}^r ([F : k(t)] - d(\pi_F(\sigma_i))).$$

The assumption that $\pi_F - \pi_E$ is a character implies that V_F has a G -submodule which is G -isomorphic to V_E . By Maschke's theorem, there is a G -invariant complement W . Let $\pi : G \rightarrow \text{GL}(W)$ be the associated homomorphism. As π_E and π_F are transitive, they both contain the principal character 1_G with multiplicity 1. Therefore $d(\pi(G)) = 0$. Note that $\dim(W) = [F : k(t)] - [E : k(t)]$. The proposition gives

$$(4) \quad 2([F : k(t)] - [E : k(t)]) \leq \sum_{i=1}^r ([F : k(t)] - [E : k(t)] - d(\pi(\sigma_i))).$$

Clearly $d(\pi_F(\sigma)) - d(\pi_E(\sigma)) = d(\pi(\sigma))$, so (a) follows from (2), (3) and (4).

Claim (b) is obvious, because the number of orbits of $\pi_E(U)$ is the multiplicity of the principal character 1_U in the restriction of π_E to U . \square

Proof of Theorem 4.14. — A doubly transitive permutation group is primitive, so (a) follows from Lemma 4.1.

Again, choose $z \in L$, where L is a splitting field of $f(t, X)$ over $k(t)$, such that t is a Siegel function in z , and A_x is intransitive on A/A_x . Let π_x and π_z be the permutation characters of the action of A on A/A_x and A/A_z , respectively. The scalar product (π_x, π_z) of characters is the number of orbits of A on $A/A_x \times A/A_z$ (by [Gor68, 2.7.4]), which is the same as the number of orbits of A_x on A/A_z , so $(\pi_x, \pi_z) \geq 2$. Each of these characters contains the principal character 1_A with multiplicity 1. Furthermore, $\pi_x - 1_A$ is irreducible, because A is doubly transitive on A/A_x (see [Gor68, Chapter 4, Theorem 3.4]). Thus, as the irreducible characters are an orthonormal basis of the class functions on A , we obtain that the nonprincipal part of π_x occurs in π_z , so $\pi_z - \pi_x$ is a character. Thus (b) follows from Lemma 4.15(a), and (c) follows from applying Lemma 4.15(b) to an inertia generator of a place of L above $t \mapsto \infty$. \square

Remark 4.17. — A weakening of doubly transitivity is primitivity. It is easy to see that an analog does not hold for primitive Galois groups.

For instance let $5 \leq m$ and $2 \leq k \leq m - 2$ be integers with $2k \neq m$. Set $g(Z) = Z^m - Z$. Then $A := \text{Gal}(g(Z) - t/\mathbb{Q}(t)) = \mathcal{S}_m$. Let L be a splitting field of $g(Z) - t$ over $\mathbb{Q}(t)$, and $A_x \cong \mathcal{S}_k \times \mathcal{S}_{m-k}$ be a setwise stabilizer of a k -set in $\{1, 2, \dots, m\}$. Let $f(t, X)$ be a minimal polynomial of a primitive element of the fixed field of A_x over $\mathbb{Q}(t)$. From this setting we obtain that $f(g(Z), X)$ is reducible in $\mathbb{Q}[Z, X]$. Therefore $\text{Red}_f(\mathbb{Z})$ is an infinite set. Furthermore, the genus of the curve $f(T, X) = 0$ goes to infinity with increasing m . For instance if m is prime, then this genus is $1 + \binom{m}{k} \frac{mk - k^2 - m - 1}{2m} \geq 1$.

Remark 4.18. — It does not seem to be obvious that we can replace the conclusion (b) in Theorem 4.14, namely that $k(t, x)$ has genus 0, by the stronger conclusion that $k(t, x)$ is rational. An attempt to prove this stronger property leads to an interesting arithmetic question: Suppose that the assumptions of Theorem 4.14 hold, but that $k(t, x)$ is not rational. We use the notation from the proof of Theorem 4.14. There we have seen that $\pi_z - \pi_x$ is a character, so in particular $[k(z) : k(t)] \geq [k(t, x) : k(t)]$. Let \mathfrak{p}_∞ be the rational place $t \mapsto \infty$, and σ an inertia generator of a place of L above \mathfrak{p}_∞ . If \mathfrak{p}_∞ is totally ramified in $k(z)$, then so is this place in $k(t, x)$ by Lemma 4.15(b), so the field $k(t, x)$ has a rational place, hence is rational. Thus there are two places of $\bar{k}(z)$ above \mathfrak{p}_∞ . Let r and s be their ramification indices. By the argument above, there are two places of $\bar{k}(t, x)$ above \mathfrak{p}_∞ . As they are not rational, they are algebraically conjugate, so they have the same ramification index u . The least common multiple of r and s is the order of σ in the action on A/A_z , while u is the order of σ on A/A_x , where this latter action is faithful. Thus r and s divide u . The field degree estimation from above however gives $r + s \geq 2u$. Thus $r = s = u$. We obtain $\pi_x = \pi_z$. Fields with this equality of permutation characters are said to be *arithmetically equivalent*, see [Kli98] for a book devoted to this subject. Thus we are led to the following

QUESTION 4.19. — *Let k be a field, and $L/k(t)$ a finite Galois extension of the rational field $k(t)$. Let $k(t) \leq k(z) \leq L$ be a rational field, and $k(t) \leq E \leq L$ be a field which is arithmetically equivalent to $k(z)$ over $k(t)$. Does this imply that E is a rational field as well?*

4.6. Rational specializations.

An essential tool in our investigation of integral Hilbert sets is

Siegel's theorem about algebraic curves with infinitely many integral points, combined with the ramification behavior above infinity of Siegel functions. If we look at rational specializations, then an analog of Proposition 2.1 holds, where Siegel's theorem is replaced by Falting's theorem that a curve with infinitely many k -rational points has genus at most 1. The proof of the following proposition is similar to the proof of Proposition 2.1, but simpler because we need not worry about integrality.

PROPOSITION 4.20. — *Let k be a field which is finitely generated over \mathbb{Q} . Let $f(t, X) \in k(t)[X]$ be irreducible. Suppose that $f(\bar{t}, X)$ is reducible for infinitely many $\bar{t} \in k$. Then the splitting field L of $f(t, X)$ over $k(t)$ contains a field $E \supset k(t)$ such that*

- (a) $f(t, X)$ is reducible over E .
- (b) E is either a rational field, or the function field of an elliptic curve with positive Mordell-Weil rank.

An application, whose proof is completely analogously to the proof of Theorem 4.14, is the following finiteness statement.

THEOREM 4.21. — *Let k be a field which is finitely generated over \mathbb{Q} . Let $f(t, X) \in k(t)[X]$ be irreducible, with Galois group acting doubly transitively on the roots of $f(t, X)$. If $f(\bar{t}, X)$ is reducible for infinitely many $\bar{t} \in k$, then the following holds, where x is a root of $f(t, X)$:*

- (a) $f(t, X)$ is absolutely irreducible, and
- (b) $k(t, x)$ has genus ≤ 1 .

4.7. The prime degree case over the rationals.

Here we look at the case that the degree of the irreducible polynomial $f(t, X)$ in X is a prime number p . Let A be the Galois group of $f(t, X)$. By a classical result of Burnside (see e.g. [HB82, Theorem XII.10.8], [DM96, Theorem 3.5B]), either $C_p \leq A \leq \text{AGL}_1(p)$ (where C_p is the cyclic group of order p , and $\text{AGL}_1(p)$ the affine general linear group on p points), a case immediately dealt with, or A is doubly transitive. Though we treated the doubly transitive Galois groups in the previous section, there are a few more things we can do in the prime degree case.

The theorem below is an extension of [Mül99, Theorem 1.2], the method is different though.

Note that $h(X') - t$ with $h(X') \in \mathbb{Z}[X']$ has a root in \mathbb{Q} for each $\bar{t} \in h(\mathbb{Z})$. If x' is a root of $h(X') - t$, and $x \in \mathbb{Q}(x')$ with $\mathbb{Q}(t, x) = \mathbb{Q}(x')$, then the minimal polynomial $f(t, X)$ of x over $\mathbb{Q}(t)$ has a root for the same (up to finitely many exceptions) specializations $\bar{t} \in h(\mathbb{Z})$. The following result shows that the converse holds in the odd prime degree case.

THEOREM 4.22. — *Let $f(t, X) \in \mathbb{Q}(t)[X]$ be irreducible of prime degree $p \geq 3$ in X . Suppose that $f(\bar{t}, X)$ is reducible for infinitely many $\bar{t} \in \mathbb{Z}$. Let x be a root of $f(t, X)$. Then there is $x' \in \mathbb{Q}(t, x)$, such that $\mathbb{Q}(t, x) = \mathbb{Q}(x')$ and $t = h(x')$ with $h(X') \in \mathbb{Q}[X']$.*

Proof. — We use Lemma 4.3. An element in A of order p is a transitive p -cycle on A/A_x . But A_z is intransitive on A/A_x , so the order of A_z is not divisible by p . But $p = [A : A_x]$, so p must divide $[A : A_z]$. Let σ be a generator of the inertia group I . So σ has m cycles of equal length on A/A_z , with $m = 1$ or 2 . As p is odd, p divides these cycle lengths, so in particular σ has order divisible by p on A/A_z . Thus σ acts as a p -cycle on A/A_x . This means that the rational place $t \mapsto \infty$ is totally ramified in $\mathbb{Q}(t, x)$. If A is doubly transitive, then $\mathbb{Q}(t, x)$ in addition has genus 0 by Lemma 4.15, and is a rational field because the unique place above $t \mapsto \infty$ must be rational, the claim follows in this case.

Thus suppose that A is not doubly transitive. Then $C_p \leq A \leq \text{AGL}_1(p)$ in its action on A/A_z . An intransitive subgroup of such a group fixes a point, therefore A_x is contained in a conjugate of A_z . So the fixed field $\mathbb{Q}(t, x)$ of A_x has again genus 0, and we complete the argument as above. □

Remark. — The above proof fails for $p = 2$, because we cannot conclude that p divides the cycles lengths of σ on A/A_z . Indeed, the theorem does not hold for $p = 2$. A counterexample is $f(t, X) = X^2 + X - dt^2$ for a squarefree integer $d > 1$.

4.8. Primitive Galois groups.

While we got a reasonably smooth result about Hilbert sets of polynomials with a doubly transitive Galois group, the results are less pleasant if we weaken the assumption on the Galois group to be merely

primitive. This section contains those results which we achieved without using the classification of the finite simple groups.

DEFINITION 4.23. — *Let k be a finitely generated field of characteristic 0. Denote by $\text{CF}(k)$ the set of those non-abelian simple groups which appear as composition factors of $\text{Gal}(g(Z) - t/k(t))$ for Siegel functions $g(Z)$ over k .*

Similarly, let $\text{CF}_{\mathbb{Z}}$ be the non-abelian composition factors of $\text{Gal}(g(Z) - t/\mathbb{Q}(t))$ for \mathbb{Z} -Siegel functions $g(Z)$ over \mathbb{Q} .

In a bigger project [Mül01], the simple groups classification has been used to determine the sets $\text{CF}(k)$. In particular, we obtained that, except for the alternating groups, $\text{CF}(k)$ is finite. We come back to this in Section 5.

THEOREM 4.24. — *Let k be a finitely generated field extension of \mathbb{Q} , and R a finitely generated subring of k . Let $f(t, X) \in k(t)[X]$ be irreducible, and assume that the Galois group A of $f(t, X)$ over $k(t)$ acts primitively on the roots of $f(t, X)$. Suppose furthermore that A has a non-abelian composition factor which is not contained in $\text{CF}(k)$. Then $\text{Red}_f(R)$ is finite.*

The following result is, in terms of composition factors, a converse to the previous theorem.

THEOREM 4.25. — *Let k be a finitely generated field extension of \mathbb{Q} . Let $S \in \text{CF}(k)$ and $a \in \mathbb{N}$ be arbitrary. Then there exist an irreducible polynomial $f(t, X) \in k(t)[X]$ and a finitely generated subring R of k , such that the following holds:*

- (a) $|\text{Red}_f(R)| = \infty$.
- (b) $\text{Gal}(f(t, X)/k(t))$ acts primitively on the roots of $f(t, X)$.
- (c) S is a composition factor of $\text{Gal}(f(t, X)/k(t))$.
- (d) The genus of the curve $f(T, X) = 0$ is $> a$.

Proof of Theorem 4.24. — We use Lemma 4.3. We first show that A acts faithfully on A/A_z . Suppose the action is not faithful. Then there is a non-trivial normal subgroup $N \triangleleft A$ with $N \leq A_z$. By primitive and faithful action of A on A/A_x we get $A = NA_x$. However, Lemma 4.3(a) says that $A_z A_x$ is a proper subset of A . But $A_z A_x = A_z (NA_x) = A_z A = A$, a contradiction.

Thus A acts faithfully on A/A_z , so each composition factor of A is a composition factor of $\text{Gal}(g(Z) - t/k(t))$, so contained in $\text{CF}(k)$. \square

In order to prove Theorem 4.25 we need an easy group theoretic observation.

LEMMA 4.26. — *Let G be a primitive non-regular permutation group on a set Δ . Let p be a prime, and $C_p \leq H \leq \text{AGL}_1(p)$. Let $W = G^p \rtimes H$ be the wreath product, in the natural imprimitive action on the disjoint union of p copies of Δ . Let \widetilde{W} be a group acting on the same points, and suppose that W is a normal subgroup of \widetilde{W} . Then \widetilde{W} acts imprimitively, respecting the given system of imprimitivity of W . Therefore \widetilde{W} acts naturally and transitively on the cartesian product Δ^p . This action of \widetilde{W} is primitive.*

Proof. — $K = G^p$ is the kernel of the action of W on the system of imprimitivity. Suppose there is $a \in \widetilde{W}$ with $K \neq K^a$. We distinguish two cases. First suppose $K \cap K^a \neq 1$. Then, by primitivity of G , $K \cap K^a$ is transitive on at least one and hence on each block Δ . In particular, the orbits of K^a are unions of K -orbits. On the other hand, K and K^a have the same number of orbits, so the blocks Δ are exactly the K^a orbits. Thus $K = K^a$, a contradiction.

Next assume that $K \cap K^a = 1$. Then K^a acts faithfully and transitively as a normal subgroup of $\text{AGL}_1(p)$ on the system of imprimitivity. So p divides the order of $K^a \cong G^p$, but p^2 does not. This contradiction shows that K is normal in \widetilde{W} .

As the blocks Δ are the K -orbits, we obtain that \widetilde{W} respects that system. By [DM96, Lemma 2.7A)], the action of W on Δ^p is primitive, so this is even more true for \widetilde{W} . \square

Proof of Theorem 4.25. — Let $S \in \text{CF}(k)$, so S is a non-abelian composition factor of $\text{Gal}(g(Z) - t/k(t))$ for a Siegel function $g(Z)$ over k . By Corollary 3.7 we may assume that g is functionally indecomposable. The Galois group G of $g(Z) - t$ over $\bar{k}(t)$ acts primitively on the roots of $g(Z) - t$, because $g(Z)$ is functionally indecomposable over \bar{k} by Theorem 3.3. Let R be a finitely generated ring in k with $|g(k) \cap R| = \infty$.

If $h(Z) \in k(Z)$ is a non-constant rational function of degree n , then we call the elements $\lambda \in \mathbb{P}^1(\bar{k})$ with $|h^{-1}(\lambda)| < n$ the *branch points* of $h(Z)$.

Let p be a prime. Choose $\alpha \in R$ such that the following holds: 0 and ∞ are not branch points of $g(Z) - \alpha$, and the p -th powers of the branch points of $g(Z) - \alpha$ are all distinct. These general position assumptions will be used in a genus computation below. Set $\tilde{g}(Z) := (g(Z) - \alpha)^p$. Let ζ be a primitive p -th root of unity. By our choices, the sets of branch points of the splitting fields of the p functions $g(Z) - \alpha - \zeta^i t^{1/p}$, $i = 1, 2, \dots, p$ over $\bar{k}(t^{1/p})$ are pairwise disjoint. As $\bar{k}(t^{1/p})$ does not possess unramified finite extensions, each of these splitting fields is linearly disjoint to the compositum of the remaining $p - 1$ ones. This implies that the Galois group W of $\tilde{g}(Z) - t$ over $\bar{k}(t)$ is the wreath product $G^p \rtimes C_p$.

Let L be a splitting field of $\tilde{g}(Z) - t$ over $k(t)$, and \hat{k} the algebraic closure of k in L . Again $W = \text{Gal}(L/\hat{k}(t))$. Set $\bar{W} = \text{Gal}(L/k(t))$. By Lemma 4.26, \bar{W} has a maximal subgroup V , such that V is intransitive on the roots of $\tilde{g}(Z) - t$. Indeed, one orbit of V has length p .

We have $|\tilde{g}(k) \cap R| = \infty$. Let $f(t, X) \in k(t)[X]$ be a minimal polynomial of a primitive element of the fixed field of V in L over $k(t)$. We have verified (a), (b), and (c) of our theorem.

It remains to compute the genus of $f(t, X) = 0$. Recall that $f(t, X)$ is absolutely irreducible by Lemma 4.1. We work over $\hat{k}(t)$. Let G_1 be the stabilizer of a point in the given action of G on the roots of $g(Z) - t$. Let x be a root of $f(t, X)$. The stabilizer $V \cap W$ in W of x can be identified with $W_1 := G_1^p \rtimes C_p$. Note that if n is the degree of $g(Z)$, then $f(t, X)$ has degree n^p .

We take advantage of the general position assumptions of the branching locus of $\hat{g}(Z) - t$ in order to get an easy genus computation using the Riemann-Hurwitz genus formula. Let σ_1 and σ_2 be inertia generators belonging to $t \mapsto 0$ and $t \mapsto \infty$, and let τ_1, \dots, τ_r be inertia generators coming from the branch points of $g(Z)$. Let ind refer to the action on W/W_1 . Then σ_i has precisely n fixed points, and moves the remaining $n^p - n$ points in p -cycles. Thus $\text{ind}(\sigma_i) = (n^p - n)(1 - 1/p)$. If the inertia generator belonging to τ_i has orbit lengths v_1, v_2, \dots, v_s on the roots of $g(Z) - t$, then τ_i has the same orbit lengths on W/W_1 , but each one occurs n^{p-1} times. As $\hat{k}(Z)/\hat{k}(g(Z))$ is an extension of genus 0 fields, we obtain

$$\sum_{i=1}^r \text{ind}(\tau_i) = n^{p-1}(2(n - 1)).$$

If g_f is the genus of $\hat{k}(t, x)$, then

$$\begin{aligned}
 2(n^p - 1 + g_x) &= \text{ind}(\sigma_1) + \text{ind}(\sigma_2) + \sum_{i=1}^r \text{ind}(\tau_i) \\
 &= 2(n^p - n) \left(1 - \frac{1}{p}\right) + n^{p-1}(2(n - 1)),
 \end{aligned}$$

so

$$g_x = (n^{p-1} - 1) \frac{np - n - p}{p} > 0,$$

and clearly $g_x \rightarrow \infty$ for $p \rightarrow \infty$. □

5. Applying the simple groups classification.

So far we have used only easy arithmetic and geometric properties of the ramification structure of Siegel functions. In particular, the results are not based on the classification of the finite simple groups. In order to obtain more results, it is indispensable to obtain good information on the Galois groups of $g(Z) - t$ for Siegel functions g . This has been carried out in [Mül01]. There we classify the possible Galois groups, and study which cases live over the rationals. We quote three corollaries from this classification.

THEOREM 5.1. — *Let k be a field of characteristic 0, and $g(Z) \in k(Z)$ a Siegel function. Then each non-abelian composition factor of $\text{Gal}(g(Z) - t/k(t))$ is isomorphic to one of the following groups: \mathcal{A}_j ($j \geq 5$), $\text{PSL}_2(7)$, $\text{PSL}_2(8)$, $\text{PSL}_2(11)$, $\text{PSL}_2(13)$, $\text{PSL}_3(3)$, $\text{PSL}_3(4)$, $\text{PSL}_4(3)$, $\text{PSL}_5(2)$, $\text{PSL}_6(2)$, M_{11} , M_{12} , M_{22} , M_{23} , M_{24} .*

THEOREM 5.2. — *Let $g(Z) \in \mathbb{Q}(Z)$ be a \mathbb{Z} -Siegel function over \mathbb{Q} . Then each non-abelian composition factor of $\text{Gal}(g(Z) - t/\mathbb{Q}(t))$ is isomorphic to one of the following groups: \mathcal{A}_j ($j \geq 5$), $\text{PSL}_2(7)$, $\text{PSL}_2(8)$.*

THEOREM 5.3. — *Let $g(Z) \in \mathbb{Q}(Z)$ be a \mathbb{Z} -Siegel function over \mathbb{Q} . Assume that $A = \text{Gal}(g(Z) - t/\mathbb{Q}(t))$ is a simple group. Then A is isomorphic to an alternating group or C_2 .*

An immediate application of the latter theorem and Lemma 4.3 is

COROLLARY 5.4. — *Let $f(t, X) \in \mathbb{Q}(t)[X]$ be irreducible with Galois group A , where A is a simple group not isomorphic to an alternating*

group or C_2 . Then $\text{Gal}(f(\bar{t}, X)/\mathbb{Q}) = G$ for all but finitely many specializations $\bar{t} \in \mathbb{Z}$.

Remark. — This corollary becomes completely wrong if we allow rational specializations $\bar{t} \in \mathbb{Q}$. Indeed, many interesting simple groups are Galois groups of polynomials $A(X) - tB(X)$ with $A, B \in \mathbb{Q}[X]$, see [MM99, Appendix, Table 10]. So for each specialization $\bar{t} = A(\bar{z})/B(\bar{z})$ with $\bar{z} \in \mathbb{Q}$ we obtain a smaller Galois group, because $A(X) - \bar{t}B(X)$ becomes reducible.

Similarly, Theorems 4.24 and 5.2 give

COROLLARY 5.5. — *Let $f(t, X) \in \mathbb{Q}(t)[X]$ be irreducible, and assume that the Galois group of $f(t, X)$ over $\mathbb{Q}(t)$ acts primitively on the roots of $f(t, X)$ and has a non-abelian composition factor which is not alternating and not isomorphic to $\text{PSL}_2(7)$ or $\text{PSL}_2(8)$. Then $\text{Red}_f(\mathbb{Z})$ is finite.*

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