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SHARP $L\log^{\alpha}L$ INEQUALITIES FOR CONJUGATE FUNCTIONS

by M. ESSÉN, D.F. SHEA & C.S. STANTON

1. Introduction.

Suppose f is a real-valued harmonic function on the unit disc $\mathbb{D} \subset \mathbb{C}$ such that

$$\|f\|_{h^p} = \sup_{r<1} \left(\int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

Let $\tilde{f}$ be the harmonic conjugate of f, normalized so that $\tilde{f}(0) = 0$, and $F = f + i\tilde{f}$. Then, by a famous theorem of M. Riesz [19], if $1 < p < \infty$,

$$\|\tilde{f}\|_{h^p} \leqslant c_p \, \|f\|_{h^p}, \tag{1.1}$$

and

$$\|F\|_{H^p} \leqslant C_p \, \|f\|_{h^p}. \tag{1.2}$$

S. K. Pichorides [18] determined the sharp constants c_p in (1.1) : $c_p = \tan \frac{\pi}{2p}$ for $1 < p \leqslant 2$ and $c_p = \cot \frac{\pi}{2p}$ for $p > 2$. In the same paper, Pichorides proved that for any $A > 2/\pi$ there exists a B depending only on A such that

$$\|\tilde{f}\|_{h^1} \leqslant A \|f\|_{L\log L} + B, \tag{1.3}$$

giving the best constant in a theorem originally due to Zygmund [22]. The sharp constants in (1.2) are (cf. Essén [6], Verbitsky [21]) $C_p = \sec(\pi/2p)$ for $1 < p \leqslant 2$ and $C_p = \csc(\pi/2p)$ for $p > 2$.

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The class $L\log L$ is defined to be those functions harmonic on the unit disc such that

$$\|f\|_{L\log L} = \sup_{r<1} \int_0^{2\pi} |f| \log(e+|f|) \frac{d\theta}{2\pi} < \infty.$$

We will also work with the classes $L\log^\alpha L$ of harmonic functions on the disc such that

$$\|f\|_{L\log^\alpha L} = \sup_{r<1} \int_0^{2\pi} |f| \left(\log(e+|f|)\right)^\alpha \frac{d\theta}{2\pi} < \infty,$$

and $L\log\log L$ of functions with

$$\|f\|_{L\log\log L} = \sup_{r<1} \int_0^{2\pi} |f| \log\log(e+|f|) \frac{d\theta}{2\pi} < \infty.$$

Remark. — Of course, none of these is actually a norm. We note that for any real α and $b>0$ there exist positive constants $C(\alpha,b)$ such that for any $x>e+b$,

$$\begin{aligned}(1.4)\quad x(\log x)^\alpha - C(b,\alpha)(\log x)^{\alpha-1} &\leqslant x\left(\log(x+b)\right)^\alpha \\ &\leqslant x(\log x)^\alpha + C(b,\alpha)(\log x)^{\alpha-1},\end{aligned}$$

so that our choice of $e+|f|$ in these definitions is largely one of convenience.

About the same time as Pichorides did his work, B. Cole also found the sharp constants in the h^p inequalities, while also providing a general framework for finding the best constant in many such inequalities. His theorem is (see [14]):

COLE'S THEOREM. — *Let $H:\mathbb{C}\to\mathbb{R}$ be a continuous function. Then*

$$(1.5)\qquad \int_0^{2\pi} H\left(f(e^{i\theta}),\tilde f(e^{i\theta})\right) d\theta \geqslant 0$$

for all harmonic polynomials f with normalized conjugates $\tilde f$ if and only if H has a subharmonic minorant h such that h is non-negative on the real axis.

Using Cole's theorem, the problem of finding a sharp inequality becomes one of determining which functions H have subharmonic minorants with the desired positivity on the real axis. However, determining whether or not a given H has such a subharmonic minorant may be difficult. *Our idea in this paper is to start with an appropriate subharmonic function h,* *and then try to determine the optimal (minimal) H having h as a minorant.* The subharmonic functions h we start with are of a very particular form, motivated by the simple choices of h and H used for the special case of Cole's proof of Riesz's L^p-inequality. These functions are harmonic splines formed by joining two harmonic functions along the imaginary axis. We then find a function $H(x,y) = \phi(x) - \psi(y)$ which is minorized by h. This function H is chosen so that $H - h$ vanishes to first order on the curve where $h = 0$. Note that if h is harmonic and H subharmonic on the curve where $h = 0$, then this first order vanishing ensures that $h \leqslant H$ in a neighborhood of $h = 0$.

In most cases the inequality we obtain by this method will involve implicitly defined functions, which we will have to approximate to obtain a more useful explicit inequality. After stating and proving our basic theorem (Theorem 6 below), we give applications. The first of these will show how Pichorides' sharp form of Riesz's inequality (1.1) follows from our methods. Our main applications yield new results for $L\log^{\alpha} L$ inequalities. We note that, unlike our earlier results in [8] and [10], in all cases the right hand sides of our inequalities depend only on $|f|$ and not on $|\tilde{f}|$. We prove

THEOREM 1. — *Suppose that $F = f + i\tilde{f}$ is analytic in the unit disc $\mathbb{D}$ and that $\alpha > 1$. Then there are positive constants C_1 and C_2 depending only on α such that*

$$\begin{aligned} \int_0^{2\pi} |\tilde{f}| \log^{\alpha-1}(e + |\tilde{f}|) \leqslant{} & \frac{2}{\pi\alpha} \int_0^{2\pi} |f| \log^{\alpha}(e + |f|) \\ & + \frac{2}{\pi} \int_0^{2\pi} |f| \log^{\alpha-1}(e + |f|) \log\log(e + |f|) \\ & + C_1 \int_0^{2\pi} |f| \log^{\alpha-1}(e + |f|) + C_2. \end{aligned} \tag{1.6}$$

Moreover, the constants $\frac{2}{\pi\alpha}$ and $\frac{2}{\pi}$ in (1.6) are sharp.

Remark. — We can choose $C_1 = 0$ for $\alpha > \alpha_0$ and $C_1 = \frac{2\alpha}{\pi(\alpha-1)}$ for $1 < \alpha \leqslant \alpha_0$, where $\alpha_0 = 2.916\ldots$ is the solution of $\frac{\alpha}{\alpha-1} = \log(\pi\alpha/2)$.

This theorem extends a result from [10], where the authors showed that $\frac{2}{\pi\alpha}$ was the appropriate leading constant in a version of (1.6) for $1 \leqslant \alpha \leqslant 2$ where the error term depended on $|\tilde{f}|$. Here and below, integrals over $[0, 2\pi]$ are taken with respect to normalized Lebesgue measure $\frac{d\theta}{2\pi}$.

For the $\alpha = 1$ case, we offer

THEOREM 2. — *Suppose that $F = f + i\tilde{f}$ is analytic in the unit disc $\mathbb{D}$. Then there are positive constants C_1 and C_2 such that*

$$\int_0^{2\pi} |\tilde{f}| \leqslant \frac{2}{\pi} \int_0^{2\pi} |f| \log(e + |f|) + \frac{4}{\pi} \int_0^{2\pi} |f| \log\log(e + |f|) + C_1 \int_0^{2\pi} |f| + C_2. \tag{1.7}$$

The constant $\frac{2}{\pi}$ in (1.7) is sharp, and the constant $\frac{4}{\pi}$ cannot be replaced by a constant less than $\frac{2}{\pi}$.

This theorem improves the result (1.3). In [18], Pichorides proves that the constant A in (1.3) cannot be reduced to $\frac{2}{\pi}$ even if an error term using the h^1 norm of f is introduced. Our theorem shows that $\|f\|_{L \log\log L}$ is the appropriate error term in the sharp form of Zygmund's inequality.

Remark added October 19, 2001. Using a different method, we can now prove that the constant $\frac{4}{\pi}$ in the right hand side of (1.7) can be replaced by $\frac{2}{\pi}$ (with a somewhat different error term) and that also this constant is best possible (cf. [13]).

For $0 < \alpha < 1$ we have

THEOREM 3. — *Suppose that $F = f + i\tilde{f}$ is analytic in the unit disc $\mathbb{D}$ and that $0 < \alpha < 1$. Then there are positive constants C_1 and C_2, depending only on α, such that*

$$\int_0^{2\pi} |\tilde{f}| \log^{\alpha-1}(e + |\tilde{f}|) \leqslant \frac{2}{\pi\alpha} \int_0^{2\pi} |f| \log^{\alpha}(e + |f|) + C_1 \int_0^{2\pi} |f| + C_2. \tag{1.8}$$

Moreover, the constant $\frac{2}{\pi\alpha}$ is sharp.

We need the following result for the proofs of Theorem 2 and Theorem 3. This theorem may be regarded as the $\alpha = 0$ case of the theorems above:

THEOREM 4. — *Suppose that $F = f + i\tilde{f}$ is analytic in the unit disc $\mathbb{D}$. Then there is an absolute constant C such that*

$$\int_0^{2\pi} |\tilde{f}| \log^{-1}(e + |\tilde{f}|) \leqslant \frac{2}{\pi} \int_0^{2\pi} |f| \log\log(e + |f|) + \int_0^{2\pi} |f| + C. \tag{1.9}$$

Moreover, the constant $\frac{2}{\pi}$ is sharp.

Remark. — Inequality (1.9) is a candidate for the "true form" of a limiting case of certain inequalities considered by Zygmund (let $\beta \to 0$ in (7.7) below!) (cf. [11]).

We can also prove a result giving inequalities for $|F|$. In the case that $\alpha > 0$, it turns out that we get the same inequalities as for $|\tilde{f}|$:

THEOREM 5. — *Theorems 1, 2 and 3 remain true if $|\tilde{f}|$ is replaced by $|F|$.*

A survey of our results is given in [12]. Orlicz spaces of type $L\log^\alpha L$ arise naturally in many settings in analysis. The reader is urged to compare our results with recent papers on sharp inequalities for Riesz transforms on $\mathbb{R}^n$ by Iwaniec and Martin [16] and by Bañuelos and Wang [2], for instance. Interesting related results are due to Burkholder (cf. [3] and [4]).

2. Functions with a specified subharmonic minorant.

We begin by constructing a subharmonic function h on the complex plane $\mathbb{C}$ of a particular form. This is done by forming a harmonic spline from the real part g of an analytic map $G = g + i\tilde{g}$ on the right half–plane $\Pi^+ = \{z \mid \operatorname{Re} z > 0\}$. We will assume that G extends to be analytic on a neighborhood of the imaginary axis with the origin omitted, and that $G(0) = 0$. For $z = x + iy \in \mathbb{C}$, we define

$$h(x,y) = g(|x| + iy). \tag{2.1}$$

The properties we require G to have are collected in the following list:

HYPOTHESES. — *We assume that the function G is an analytic function on a neighborhood of the set $\overline{\Pi^+} \setminus \{0\}$ and that*

$\mathcal{H}(1)$ *G maps the positive real axis onto itself,*

$\mathcal{H}(2)$ *G' maps the first quadrant to the first quadrant,*

$\mathcal{H}(3)$ *G'' maps the first quadrant to the lower half plane,*

$\mathcal{H}(4)$ *Let γ be the part of the curve defined by $\operatorname{Re} G(x+iy) = 0$ in the first quadrant. For each $x > 0$ there is a unique $\overline{y}(x)$ such that $(x, \overline{y}(x))$ is on γ, and for each $y > 0$ there is a unique $\underline{x}(y)$ such that $(\underline{x}(y), y)$ is on γ.*

$\mathcal{H}(5)$ *If h is defined by (2.1) then h is subharmonic in the complex plane $\mathbb{C}$.*

Remark. — Since G is real on the real axis, $G(\bar{z}) = \overline{G(z)}$. Assumption $\mathcal{H}(2)$ is equivalent to requiring that if $g = \operatorname{Re} G$ then $D_1 g(x,y) > 0$ and $D_2 g(x,y) < 0$ in the first quadrant. (We will systematically use the notation $D_1 g(x,y) = \frac{\partial}{\partial x} g(x,y)$ and $D_2 g(x,y) = \frac{\partial}{\partial y} g(x,y)$.) Assumption $\mathcal{H}(3)$ is equivalent to requiring $D_1 D_2 g(x,y) > 0$ in the first quadrant. By assumption $\mathcal{H}(4)$, the curve γ is non–empty, and by assumption $\mathcal{H}(2)$, γ is the graph of a monotone increasing function. The function h is harmonic (and hence subharmonic) everywhere except on the imaginary axis. The derivative condition in assumption $\mathcal{H}(2)$ ensures that near the imaginary axis h is, except possibly at 0, locally the maximum of two harmonic functions. Hence h is subharmonic on the imaginary axis except possibly at the origin. Thus, $\mathcal{H}(5)$ reduces to a requirement on the integral means of h near 0.

We define, assuming that the derivatives are integrable at 0,

$$\phi(x) = \int_0^x D_1 g\,(t, \overline{y}(t))\, dt, \tag{2.2}$$

and

$$\psi(y) = \int_0^y -D_2 g\,(\underline{x}(t), t)\, dt. \tag{2.3}$$

If γ is defined as in $\mathcal{H}(4)$ we have

LEMMA 1. — *Let $H(x,y) = \phi(x) - \psi(y)$. Then $H = 0$ on γ, and $\nabla(H - h) = 0$ on γ.*

Proof. — It suffices to prove that $\phi(x) = \psi(y)$ on γ. This is true, since $\phi(0) = \psi(0) = 0$, and

$$\begin{aligned} \frac{d}{dx}\left(\phi(x) - \psi(\overline{y}(x)\right) &= D_1 g\,(x, \overline{y}(x)) + D_2 g\,(\underline{x}\,(\overline{y}(x)), \overline{y}(x)\,)\,\overline{y}\,'(x) \\ &= D_1 g\,(x, \overline{y}(x)) + D_2 g\,(x, \overline{y}(x)) \left(\frac{-D_1 g\,(x, \overline{y}(x))}{D_2 g\,(x, \overline{y}(x))} \right) \\ &= 0. \end{aligned}$$

The vanishing of the gradient is immediate from (2.2) and (2.3). □

THEOREM 6. — *Suppose $G = g + i\tilde{g}$ satisfies the hypotheses $\mathcal{H}(1)$–$\mathcal{H}(5)$, and that ϕ and ψ are defined by (2.2) and (2.3) respectively. Then for any analytic polynomial $F = f + i\tilde{f}$ with $\tilde{f}(0) = 0$,*

$$\int_0^{2\pi} \psi(|\tilde{f}(e^{i\theta})|)\, d\theta \leqslant \int_0^{2\pi} \phi(|f(e^{i\theta})|)\, d\theta. \tag{2.4}$$

Remark. — Inequality (2.4) is best-possible for a class of functions G satisfying some additional regularity conditions. This class contains all functions G used in the proofs of Theorems 1-5. See Theorem 8 in Section 9 below.

Proof. — Let $H(x,y) = \phi(x) - \psi(y)$. It suffices, by Cole's theorem, to show that h minorizes H. The functions H and h depend only on $|x|$ and $|y|$, so it suffices to show that $H - h \geqslant 0$ in the first quadrant. We fix $x_0 > 0$, and calculate

$$\begin{aligned}\frac{\partial}{\partial y}\left(H(x_0,y) - h(x_0,y)\right) &= -\psi'(y) - D_2 g(x_0,y)\\ &= D_2 g\left(\underline{x}(y),y\right) - D_2 g(x_0,y)\\ &= D_{1,2} g(\xi,y)\left(\underline{x}(y) - x_0\right)\end{aligned}$$

for some ξ between x_0 and $\underline{x}(y)$. By our hypothesis $\mathcal{H}(3)$, $D_{1,2}g \geqslant 0$ in the first quadrant. Since γ is the graph of an increasing function of x, $\underline{x}(y) - x_0$ has the same sign as $y - \overline{y}(x_0)$. From this it follows that $H(x_0,y) - h(x_0,y)$ has a minimum at $y = \overline{y}(x_0)$, i.e., on the curve γ. By Lemma 1, we conclude that $H - h \geqslant 0$. □

We can also formulate a theorem for comparing the means of the analytic function F to those of its real part f. For other recent work on $|F|$ vs. $|f|$ inequalities, see [2]. We introduce some more notation. Let γ be a curve in the first quadrant such that any positive value of either of x or r determines a unique point $x + iy = re^{i\theta}$ on the curve γ. We write $\overline{y}(x)$, $\overline{r}(x)$, and $\overline{\theta}(x)$ for the values determined by x, and $x^*(r)$, $y^*(r)$, and $\theta^*(r)$ for those determined by r.

If G is an analytic function with a real part g which vanishes on such a curve γ, we wish to define functions $\Phi(x)$ and $\Psi(r)$ such that $\Phi(x) - \Psi(r) - g(x,y)$ vanishes to first order on γ. To do this, we define, assuming that the derivatives are integrable at 0,

$$\Psi(r) = \int_0^r -D_2 g\left(x^*(t), y^*(t)\right) \csc\theta^*(t)\, dt, \tag{2.5}$$

and

$$\Phi(x) = \int_0^x \left(D_1 g\left(t, \overline{y}(t)\right) + \Psi'\left(\overline{r}(t)\right)\cos\overline{\theta}(t)\right) dt. \tag{2.6}$$

LEMMA 2. — *Let $H(x,y) = \Phi(x) - \Psi(r)$ Then $H = 0$ on γ, and $\nabla(H - h) = 0$ on γ.*

Proof. — We calculate the partial $\partial(H-g)/\partial x$ at a point (x,y) on γ:

$$\begin{aligned}\frac{\partial}{\partial x}&(\Phi(x)-\Psi(r)-g(x,y))\\ &=\Phi'(x)-\Psi'(r)\frac{\partial r}{\partial x}-D_1g(x,y)\\ &=D_1g\,(x,\overline{y}(x))+\Psi'\,(\overline{r}(x))\cos\overline{\theta}(x)-\Psi'(r)\cos\theta-D_1g(x,y)\\ &=0,\end{aligned}$$

since $\overline{y}(x)=y$, $\overline{r}(x)=r$ and $\overline{\theta}(x)=\theta$ on γ. Next, we calculate $\partial(H-g)/\partial y$:

$$\begin{aligned}\frac{\partial}{\partial y}(\Phi(x)-\Psi(r)-g(x,y))&=-\Psi'(r)\frac{\partial r}{\partial y}-D_2g(x,y)\\ &=D_2g\,(x^*(r),y^*(r))\csc\theta^*(r)\sin\theta-D_2g(x,y)\\ &=0,\end{aligned}$$

since $x^*(r)=x$, $y^*(r)=y$, and $\theta^*(r)=\theta$ on γ. □

THEOREM 7. — *Let Ψ and Φ be defined by (2.5) and (2.6) respectively. Suppose that in addition to hypotheses $\mathcal{H}(1)$–$\mathcal{H}(5)$, the function $g=\operatorname{Re}G(x+iy)$ satisfies in the first quadrant*

$$\frac{\partial}{\partial\theta}(D_2g(r\cos\theta,r\sin\theta)\csc\theta)<0\qquad\left(0<\theta<\frac{\pi}{2}\right).\tag{2.7}$$

Then for any analytic polynomial $F=f+i\tilde{f}$ with $\tilde{f}(0)=0$,

$$\int_0^{2\pi}\Psi(|F(e^{i\theta})|)\,d\theta\leqslant\int_0^{2\pi}\Phi(|f(e^{i\theta})|)\,d\theta.\tag{2.8}$$

Proof. — We calculate at a point $x+iy=re^{i\theta}$ in the first quadrant:

$$\begin{aligned}\frac{\partial}{\partial y}&(\Phi(x)-\Psi(r)-g(x,y))\\ &=0+D_2g(x^*(r),y^*(r))\csc\theta^*(r)\sin\theta-D_2g(x,y)\\ &=\sin\theta\,(D_2g(x^*(r),y^*(r))\csc\theta^*(r)-D_2g(x,y)\csc\theta)\\ &=\sin\theta\frac{\partial}{\partial\theta}(D_2g(r\cos\theta_1,r\sin\theta_1)\csc\theta_1)\,(\theta^*(r)-\theta),\end{aligned}$$

for some θ_1 between $\theta^*(r)$ and θ. Note that $\theta<\theta^*(r)$ if $y<\overline{y}(x)$, while $\theta>\theta^*(r)$ if $y>\overline{y}(x)$. So, by the additional hypothesis (2.7), the partial

$$\frac{\partial}{\partial y}(\Phi(x)-\Psi(r)-g(x,y))$$

is negative if $y<\overline{y}(x)$, and positive if $y>\overline{y}(x)$. We conclude that for a fixed x that $\Phi(r)-\Psi(y)-g(x,y)$ has a minimum if $y=\overline{y}(x)$, i.e., on γ. Since this difference is zero on γ, it is positive everywhere else. □

3. Pichorides' theorem.

Our first application of Theorem 6 gives an alternative proof (for $1 < p < 2$) of Pichorides' theorem giving the best constants in M. Riesz's theorem. We use the same subharmonic minorant as Pichorides, but avoid having to derive some trigonometric inequalities by using our Theorem 6. We take $G(z) = z^p$. It is easily checked that G satisfies the hypotheses of the theorem. We first calculate that $G'(z) = pz^{p-1}$, and so $\operatorname{Re} G'(re^{i\theta}) = pr^{p-1}\cos((p-1)\theta)$ where $r = |z|$ and $\theta = \arg z$. To calculate $\phi(x)$ in order to apply Theorem 6, we note that on the curve $\gamma = \{z : \operatorname{Re} z^p = 0\}$ in the first quadrant, we have $r = \left(\sec\frac{\pi}{2p}\right)x$ and $\theta = \frac{\pi}{2p}$. Hence,

$$\begin{aligned} D_1 g(x, \overline{y}(x)) &= pr^{p-1}\cos((p-1)\theta) \\ &= p\left(\sec\frac{\pi}{2p}\right)^{p-1} x^{p-1}\cos\left((p-1)\frac{\pi}{2p}\right) \\ &= p\left(\sec\frac{\pi}{2p}\right)^{p-1}\sin\left(\frac{\pi}{2p}\right)x^{p-1}. \end{aligned}$$

Integrating with respect to x gives

$$\phi(x) = \left(\sec\frac{\pi}{2p}\right)^{p-1}\sin\left(\frac{\pi}{2p}\right)x^p,$$

and, similarly,

$$\psi(y) = \left(\csc\frac{\pi}{2p}\right)^{p-1}\cos\left(\frac{\pi}{2p}\right)y^p.$$

We conclude from Theorem 6 that, if f is a harmonic polynomial with conjugate $\tilde{f}$ normalized so that $\tilde{f}(0) = 0$,

$$\begin{aligned} &\left(\csc\frac{\pi}{2p}\right)^{p-1}\cos\left(\frac{\pi}{2p}\right)\int_0^{2\pi}|\tilde{f}(e^{i\theta})|^p\,d\theta \\ &\qquad\leqslant \left(\sec\frac{\pi}{2p}\right)^{p-1}\sin\left(\frac{\pi}{2p}\right)\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta. \end{aligned}$$

This is easily seen to reduce to Pichorides' theorem.

It is not difficult to construct examples to show that $c_p = \tan(\pi/2p)$ is the sharp constant for $1 < p < 2$ (see [18], [14]). Another way to prove that this constant is sharp is via Cole's theorem. For if $c < c_p$ and $H(x,y) = c^p|x|^p - |y|^p$, then H vanishes on the rays $y = \pm c|x|$ in the right half–plane. These rays form a sector with an angle of less than π/p. The maximum principle applies (by a Phragmén-Lindelöf argument) to any subharmonic function in this sector of growth $O(|z|^p)$. Thus, any subharmonic minorant of H must be non-positive in this sector. Since $H \leqslant 0$ in the rest of the half-plane, the minorant must be $\leqslant 0$ in the half-plane. The same argument applies in the left half-plane, and hence the minorant must be constant. However, $H(0,y) \to -\infty$ as $|y| \to \infty$, so this is impossible. We conclude that H has no subharmonic minorant, and hence by Cole's theorem that (1.1) cannot hold for this value of c.

Starting with $G(z) = z^p$ and $1 < p \leqslant 2$, we can also apply Theorem 7. The definitions (2.5) and (2.6) give

$$\Psi(r) = \cot \frac{\pi}{2p}\, r^p,$$

and

$$\Phi(x) = \left(\sec \frac{\pi}{2p}\right)^{p-1} \left\{\sin \frac{\pi}{2p} + \cot \frac{\pi}{2p} \cos \frac{\pi}{2p}\right\} x^p.$$

We apply Theorem 7 and simplify to recover (1.2) in the case $1 < p \leqslant 2$:

$$\|F\|_p \leqslant \sec\left(\frac{\pi}{2p}\right) \|f\|_p.$$

4. Conjugate function inequalities.

Our main application of Theorem 6 deals with conjugate function inequalities for functions in the class $L \log^\alpha L$, $0 < \alpha < \infty$. For this, we study the basic mapping properties of the functions defined on Π^+:

$$G_\alpha(z) = \int_0^z \log^\alpha(\zeta + b)\, d\zeta \quad (b = e^\alpha, \quad 0 < \alpha < \infty), \tag{4.1}$$

as well as

$$G_0(z) = \int_0^z \log\log(\zeta + e)\, d\zeta. \tag{4.2}$$

Remark. — In the proof of Theorem 5, b is chosen differently.

LEMMA 3. — *For any $\alpha \geqslant 0$, G_α satisfies hypotheses $\mathcal{H}(1)$–$\mathcal{H}(5)$.*

Proof. — We will use the following notation throughout this paper:

$$z = x + iy = re^{i\theta}, \qquad z + b = Re^{i\varphi}, \qquad \log(z+b) = te^{is} = \rho + i\varphi. \tag{4.3}$$

Condition $\mathcal{H}(1)$, that G_α map the positive real axis to itself, is obvious, for any $\alpha \geqslant 0$. Now, take $\alpha > 0$. For $\mathcal{H}(2)$, we observe that $\arg(G'_\alpha) = \alpha s$. Using our choice $b = e^\alpha$, we see that for $z \in Q = \{x, y > 0\}$,

$$0 < s < \tan s = \frac{\varphi}{\log R} < \frac{\varphi}{\log b} < \frac{\pi}{2\alpha}, \tag{4.4}$$

as required.

For $\mathcal{H}(3)$, we note that $G''_\alpha(z) = (\alpha/(z+b)) \log^{\alpha-1}(z+b)$, and hence $\arg G''(z) = (\alpha - 1)s - \varphi$. Thus we need to verify

$$0 > (\alpha - 1)s - \varphi > -\pi. \tag{4.5}$$

In fact, the negativity is immediate if $\alpha \leqslant 1$. For $\alpha > 1$, we have $s < \varphi/\rho \leqslant \varphi/\alpha$ and

$$(\alpha - 1)s - \varphi < \varphi\left(\frac{\alpha - 1}{\alpha} - 1\right) < 0.$$

The right-hand inequality in (4.5) follows easily from (4.4) in both cases.

The validity of hypothesis $\mathcal{H}(4)$ is a consequence of the the fact that $\overline{y}(0) = 0$, that $\overline{y}'(x) > 0$ which follows from $\mathcal{H}(2)$, and the estimate $x < \overline{y}(x) < Cx \log x$ for large x which follows from Lemma 6 below.

Finally, $\mathcal{H}(5)$ follows easily from analyticity of $G_\alpha(z)$ at 0, with

$$G_\alpha(z) = (\log^\alpha b)z + O(r^2) \qquad (z \to 0),$$

so that

$$h(x, y) = g(|x| + iy) = \alpha^\alpha r |\cos\theta| + O(r^2) \qquad (z \to 0)$$

must satisfy the mean value inequality for subharmonicity at 0.

To check $\mathcal{H}(2)$–$\mathcal{H}(5)$ for G_0, we use the notations (4.3) but now take $b = e$. Since

$$\operatorname{Re} G'_0(z) = \log|\log(z + e)| = \log t > 0, \quad \operatorname{Im} G'_0(z) = s > 0,$$

$\mathcal{H}(2)$ follows. For $\mathcal{H}(3)$, observe that

$$\operatorname{Im} G''_0(z) = -R^{-1}t^{-1}\sin(\varphi + s) < 0$$

since $\varphi, s \in (0, \frac{\pi}{2})$.

The validity of hypothesis $\mathcal{H}(4)$ is obtained as in the case $\alpha > 0$, this time using Lemma 8.

To check the behavior of $h(x, y) = \operatorname{Re} G_0\,(|x| + iy)$ near 0, we use

$$G_0(z) = a_2 z^2 + a_3 z^3 + O(r^4) \qquad (r \to 0)$$

to see that

$$h(x,y) = a_2 r^2 \cos 2\theta + a_3 r^3 |\cos\theta| \left(\cos^2\theta - 3\sin^2\theta\right) + O(r^4),$$

and thus the required mean–value inequality holds, since

$$0 = \frac{h(0,0)}{r^3} < (-a_3)\frac{1}{2\pi}\int_0^{2\pi} |\cos\theta| \left(4\sin^2\theta - 1\right) d\theta + O(r)$$

where $a_3 = -1/3e^2$ is negative. □

5. The curve $\gamma = \{\operatorname{Re} G_\alpha = 0\}$.

The inequality (2.4) for ϕ–means of f and ψ–means of $\tilde{f}$ given by Theorem 6 is useful only if ϕ and ψ closely approximate interesting functions. For instance, in (5.19) below we establish the approximation

$$\phi(x) \doteq x\log^\alpha x + \alpha x \log^{\alpha-1} x \log\log x \qquad (x \to \infty)$$

when $\alpha > 0$. A similar expansion for $\psi(y)$ is given in (5.21), and the case $\alpha = 0$ is treated in Section 6. These estimates depend on the properties of the level curve $\gamma = \{\operatorname{Re} G_\alpha = 0\}$.

In this section we assume $\alpha > 0$ and study the asymptotic behavior of G_α defined in (4.1). Clearly G_α is univalent on Π^+, and each $G_\alpha(\Pi^+)$ contains $\Pi^+ \setminus \{0\}$. Thus, we can define the analytic curve

(5.1) $$\gamma(v) = G_\alpha^{-1}(iv) = x + iy, \qquad (0 < v < \infty)$$

with x and y positive, smooth functions of v. To get good estimates on γ, it is useful to consider its image σ in the $\varphi - \rho$ plane,

(5.2) $$\sigma(v) = \log\left(G_\alpha^{-1}(iv) + b\right) = \rho + i\varphi$$

with $\sigma(0) = \log b = \alpha$. Then

$$\begin{aligned}\sigma'(v) &= \frac{i}{\left(G_\alpha^{-1}(iv) + b\right) G'_\alpha\left(G_\alpha^{-1}(iv)\right)}\\ &= \frac{i}{\left(G_\alpha^{-1}(iv) + b\right)\log^\alpha\left(G_\alpha^{-1}(iv) + b\right)},\end{aligned}$$

so that by (5.1) and (4.3) we obtain, for $\rho + i\varphi$ on σ,

$$\begin{aligned}\frac{d\rho}{dv} + i\frac{d\varphi}{dv} &= \frac{1}{Re^{i\varphi}}\cdot\frac{1}{t^\alpha e^{i\alpha s}}\, i\\ &= \frac{1}{Rt^\alpha} e^{i(\pi/2 - (\varphi + \alpha s))},\end{aligned}$$

where $\rho + i\varphi = \log(x + iy + b)$, $t = |\rho + i\varphi|$, and $s = \arctan(\varphi/\rho)$ are all smooth functions of v. Taking the real and imaginary parts, we see that

$$\begin{aligned} \frac{d\rho}{dv} &= \frac{\sin(\varphi + \alpha s)}{Rt^\alpha} \\ \frac{d\varphi}{dv} &= \frac{\cos(\varphi + \alpha s)}{Rt^\alpha}. \end{aligned} \tag{5.3}$$

It follows from (4.4) that $0 < \varphi + \alpha s < 2\varphi < \pi$, and so it is clear that we can consider v an increasing function of ρ. In turn, we can regard φ, t, s, x, and y as functions of ρ, and from (5.3) we conclude that φ satisfies the differential equation

$$\frac{d\varphi}{d\rho} = \cot(\varphi + \alpha \arctan(\varphi/\rho)), \qquad \varphi(\alpha) = 0. \tag{5.4}$$

We denote the solution to (5.4) by $\hat{\varphi}$, and $\arctan(\hat{\varphi}(\rho)/\rho)$ by $\hat{s}$.

LEMMA 4. — *Suppose $\alpha > 0$. The function $\hat{\varphi} = \hat{\varphi}(\rho)$ defined in (5.4) satisfies*

$$0 < \hat{\varphi} + \alpha\hat{s} < \pi/2 \qquad (\alpha < \rho < \infty).$$

Thus, by (5.4), $\hat{\varphi}$ is strictly increasing for all $\rho > \alpha$ and $\hat{\varphi}(\rho) \to \pi/2, \rho \to \infty$.

Proof. — We put $k(\varphi, \rho) = \varphi + \alpha \arctan(\varphi/\rho) = \varphi + \alpha s$ and consider the region

$$\mathcal{R} = \left\{(\rho, \varphi) : \rho > \alpha, 0 < \varphi < \frac{\pi}{2}, 0 < k(\varphi, \rho) < \frac{\pi}{2}\right\}.$$

Note that k increases in φ, for each $\rho > \alpha$, and

$$\frac{\pi}{2} < k\left(\frac{\pi}{2}, \rho\right) \leqslant \frac{\pi}{2} + \frac{\pi\alpha}{2\rho} < \pi,$$

so that $\cot k(\frac{\pi}{2}, \rho) < 0$. Further,

$$\left\{(\rho, \varphi) : \quad \rho > \alpha, \quad 0 < \varphi \leqslant \frac{\pi}{4}\right\} \subset \mathcal{R}$$

since $0 < \varphi \leqslant \frac{\pi}{4}$ implies $0 < k(\varphi, \rho) < \frac{\pi}{4} + \frac{\pi\alpha}{4\rho} < \frac{\pi}{2}$. For each $\rho > \alpha$, define $\varphi_1(\rho)$ by $k(\varphi_1(\rho), \rho) = \pi/2$ and notice that $\varphi_1'(\rho) > 0$. A simple comparison argument will show that $\hat{\varphi}(\rho)$ satisfies

$$0 < \hat{\varphi}(\rho) < \varphi_1(\rho) \quad (\alpha < \rho < \infty).$$

In fact, $0 < \hat{\varphi}(\rho) \leqslant \pi/4 < \varphi_1(\rho)$ for $\alpha < \rho \leqslant \rho_0$. Now suppose there is a ρ such that $\hat{\varphi}(\rho) = \varphi_1(\rho)$. Let ρ_1 be the first such ρ. Then $\hat{\varphi}'(\rho_1) = 0$, while $\varphi_1'(\rho_1) > 0$ so that $\varphi_1(\rho) < \hat{\varphi}(\rho)$ for some $\rho < \rho_1$, a contradiction. □

We now proceed to make estimates on the solution $\hat{\varphi}(\rho)$. For $(\varphi,\rho) = (\hat{\varphi}(\rho),\rho)$, we write

$$\cot(\varphi+\alpha s) = \frac{\pi}{2} - \left(\varphi + \alpha\frac{\varphi}{\rho}\right) + h(\rho), \tag{5.5}$$

and note that $h(\rho) > 0$. Thus the nonlinear differential equation (5.4) becomes a less explicit linear differential equation:

$$\frac{d\varphi}{d\rho} = \frac{\pi}{2} - \left(\varphi + \alpha\frac{\varphi}{\rho}\right) + h(\rho). \tag{5.6}$$

This first-order linear differential equation has the solution

$$\begin{aligned}\hat{\varphi}(\rho) &= \exp\left[-\int_\alpha^\rho \left(1+\frac{\alpha}{\tau}\right)\,d\tau\right]\\ &\quad\times\left\{\int_\alpha^\rho \left(\frac{\pi}{2}+h(\tau)\right)\exp\left[\int_\alpha^\tau \left(1+\frac{\alpha}{\sigma}\right)\,d\sigma\right]\,d\tau\right\}\\ &= \frac{\pi}{2}\rho^{-\alpha}e^{-\rho}\int_\alpha^\rho \tau^\alpha e^\tau\,d\tau + \rho^{-\alpha}e^{-\rho}\int_\alpha^\rho h(\tau)\tau^\alpha e^\tau\,d\tau.\end{aligned}$$

The first integral we integrate by parts. Writing $h_1(\rho)$ for the second integral, we obtain

$$\hat{\varphi}(\rho) = \frac{\pi}{2}\left(1-\frac{\alpha}{\rho}+\frac{\alpha(\alpha-1)}{\rho^2}+O\left(\frac{1}{\rho^3}\right)\right)+h_1(\rho).$$

Here, and in the remainder of this section, all O statements for functions of ρ are to be taken as $\rho\to\infty$. Since $h(\rho)>0$, we see that $\pi/2-\varphi = O(1/\rho)$, and hence $h(\rho) = O(1/\rho^3)$. This in turn implies that $h_1(\rho) = O(1/\rho^3)$, and we have the following

LEMMA 5. — *Suppose that $\alpha > 0$ and that the curve σ is defined by (5.2) in the first quadrant. Then for $\rho + i\varphi \in \sigma$ we have, as $\rho\to\infty$,*

$$\frac{\pi}{2} - \varphi = \frac{\pi\alpha}{2\rho} - \frac{\pi\alpha(\alpha-1)}{2\rho^2} + O\left(\frac{1}{\rho^3}\right), \tag{5.7}$$

and since $s = \arctan(\varphi/\rho)$,

$$s = \frac{\pi}{2\rho} - \frac{\pi\alpha}{2\rho^2} + \frac{A(\alpha)}{\rho^3} + O\left(\frac{1}{\rho^4}\right), \tag{5.8}$$

with $A(\alpha) = (\pi/2)\left(\alpha(\alpha-1)-\pi^2/12\right)$.

We can now estimate the values of x and y on the curve γ. We have

$$y = R\sin\varphi = e^\rho\left(1-\frac{\pi^2\alpha^2}{8\rho^2}+O\left(\frac{1}{\rho^3}\right)\right), \tag{5.9}$$

and

$$x = R\cos\varphi - b = e^\rho\left(\frac{\pi\alpha}{2\rho} - \frac{\pi\alpha(\alpha-1)}{2\rho^2} + O\left(\frac{1}{\rho^3}\right)\right). \tag{5.10}$$

From these, we obtain asymptotic estimates for ρ:

LEMMA 6. — *Suppose $\alpha > 0$ and that the curve γ is defined by* $\operatorname{Re} G_\alpha = 0$ *in the first quadrant. Then*

$$\rho = \log x + \log\log x + \log\frac{2}{\pi\alpha} + \frac{\log\log x}{\log x} + O\left(\frac{1}{\log x}\right) \qquad (x \to \infty), \tag{5.11}$$

and

$$\rho = \log y + O\left(\frac{1}{\log^2 y}\right) \qquad (y \to \infty). \tag{5.12}$$

Proof. — From (5.9) we have

$$\log y = \rho + \log\left(1 - \frac{\pi^2\alpha^2}{8\rho^2} + O\left(\frac{1}{\rho^3}\right)\right),$$

and hence

$$\begin{aligned}\rho &= \log y + O\left(\frac{1}{\rho^2}\right)\\ &= \log y + O\left(\frac{1}{\log^2 y}\right),\end{aligned}$$

since $\log y \leqslant \log R$, and thus we have (5.12).

To prove (5.11), we start with (5.10), and obtain

$$\begin{aligned}\log x &= \rho + \log\left(\frac{\pi\alpha}{2\rho} + O\left(\frac{1}{\rho^2}\right)\right)\\ &= \rho + \log\frac{\pi\alpha}{2} - \log\rho + O\left(\frac{1}{\rho}\right),\end{aligned}$$

and hence

$$\rho = \log x + \log\rho - \log\frac{\pi\alpha}{2} + O\left(\frac{1}{\rho}\right). \tag{5.13}$$

To eliminate the $\log\rho$ term from the right side, we use $1/\rho \leqslant 1/\log x$, and see that

$$\rho = \log x\left(1 + \frac{\log\rho}{\log x} + O\left(\frac{1}{\log x}\right)\right),$$

from which it follows that

$$\log\rho = \log\log x + \frac{\log\rho}{\log x} + O\left(\frac{1}{\log x}\right).$$

From this we deduce

$$\log \rho = \log\log x + \frac{\log\log x}{\log x} + O\left(\frac{1}{\log x}\right).$$

Substituting this estimate into (5.13), we have (5.11). □

We now obtain bounds for the functions $\phi(x)$ and $\psi(y)$. It follows from the definition (2.2) of φ that

(5.14) $$\phi'(x) = \operatorname{Re} G'_\alpha(x + i\overline{y}(x)) = t^\alpha \cos \alpha s.$$

We observe that

(5.15) $$t = \left(\rho^2 + \varphi^2\right)^{1/2} = \rho \left(1 + \left(\frac{\varphi}{\rho}\right)^2\right)^{1/2} = \rho\left(1 + O\left(\frac{1}{\rho^2}\right)\right),$$

and that by (5.8), $\cos \alpha s = 1 + O\left(1/\rho^2\right)$, We deduce that

(5.16) $$t^\alpha \cos \alpha s = \rho^\alpha \left(1 + O(1/\rho^2)\right).$$

We now estimate

(5.17) $$\rho^\alpha = \log^\alpha x + \alpha \log^{\alpha-1} x \, \log\log x + \alpha \log \frac{2}{\pi\alpha} \log^{\alpha-1} x + O\left(\log^{\alpha-2} x \, (\log\log x)^2\right).$$

It follows from (5.15), (5.16), and (5.17) that ϕ' satisfies the same estimate as ρ^α:

(5.18) $$\phi'(x) = \log^\alpha x + \alpha \log^{\alpha-1} x \, \log\log x + \alpha \log \frac{2}{\pi\alpha} \log^{\alpha-1} x + O\left(\log^{\alpha-2} x \, (\log\log x)^2\right).$$

It follows that

(5.19) $$\phi(x) = x \log^\alpha x + \alpha x \log^{\alpha-1} x \log\log x - \alpha\left(1 - \log \frac{2}{\pi\alpha}\right) x \log^{\alpha-1} x + O\left(x \log^{\alpha-2} x \, (\log\log x)^2\right).$$

For $\psi(y)$, we have $\psi(y) = \operatorname{Im} G'_\alpha(\underline{x}(y) + iy) = t^\alpha \sin \alpha s$, and deduce that

$$\begin{aligned}\psi'(y) &= t^\alpha \sin \alpha s \\ &= \rho^\alpha \left(1 + O(1/\rho^2)\right) \left(\frac{\pi\alpha}{2\rho} - \frac{\pi\alpha^2}{2\rho^2} + O\left(\frac{1}{\rho^3}\right)\right),\end{aligned}$$

and thus

(5.20) $$\psi'(y) = \frac{\alpha\pi}{2} \log^{\alpha-1} y - \frac{\pi\alpha^2}{2} \log^{\alpha-2} y + O(\log^{\alpha-3} y).$$

It follows by an integration by parts that

(5.21) $$\psi(y) = \frac{\alpha\pi}{2} y \log^{\alpha-1} y - \frac{\pi\alpha(2\alpha-1)}{2} y \log^{\alpha-2} y + O(y \log^{\alpha-3} y).$$

6. The case $\alpha = 0$.

We now consider the case $\alpha = 0$. In analogy to the $\alpha > 0$ case, we parameterize the curve γ by the inverse function $G_0^{-1}(iv)$, with $0 < v < \infty$, and consider the image σ_0 of γ in the ρ–φ plane, defined by the equation

$$\sigma_0(v) = \log(G_0^{-1}(iv) + e) = \rho + i\varphi. \tag{6.1}$$

Then

$$\begin{aligned}\sigma_0'(v) &= \frac{1}{\left(G_0^{-1}(iv)+e\right)}\cdot\frac{1}{G_0'\left(G_0^{-1}(iv)\right)}\,i\\ &= \frac{1}{G_0^{-1}(iv)+e}\cdot\frac{1}{\log\log\left(G_0^{-1}(iv)+e\right)}\,i.\end{aligned}$$

We introduce the notation $\log(te^{is}) = \lambda e^{i\xi}$. Then, substituting $z = G_0^{-1}(iv)$ and using the notation (4.3),

$$\begin{aligned}\sigma_0'(v) &= \frac{1}{Re^{i\varphi}}\cdot\frac{1}{\lambda e^{i\xi}}\,i\\ &= \frac{1}{R\lambda}e^{i(\pi/2-(\varphi+\xi))}.\end{aligned}$$

Taking the ratio of imaginary and real parts, this time we get the differential equation

$$\frac{d\varphi}{d\rho} = \cot(\varphi+\xi) \qquad \varphi(e) = 0. \tag{6.2}$$

Since $\xi = \arctan(s/\log t)$, we may linearize this as

$$\frac{d\varphi}{d\rho} = \frac{\pi}{2} - \left(\varphi + \frac{\varphi}{\rho\log\rho}\right) + h(\rho). \tag{6.3}$$

It follows, as in the case for $\alpha > 0$, that (6.3) has a solution $\hat\varphi$ given by

$$\hat\varphi(\rho) = \frac{\pi}{2}\frac{1}{e^\rho\log\rho}\int_e^\rho \log\tau e^\tau\,d\tau + \frac{1}{e^\rho\log\rho}\int_e^\rho h(\tau)\log\tau e^\tau\,d\tau,$$

from which follows the analog of Lemma 5:

LEMMA 7. — *Suppose σ_0 is defined by (6.1) in the first quadrant. Then for $\rho + i\varphi \in \sigma_0$ we have, as $\rho \to \infty$,*

$$\frac{\pi}{2} - \varphi = \frac{\pi}{2}\frac{1}{\rho\log\rho} + \frac{\pi}{2}\frac{1}{\rho^2\log\rho} + O\left(\frac{1}{\rho^3}\right), \tag{6.4}$$

for $s = \arctan(\sigma/\rho)$

$$s = \frac{\pi}{2}\frac{1}{\rho} - \frac{\pi}{2}\frac{1}{\rho^2\log\rho} + O\left(\frac{1}{\rho^3}\right), \tag{6.5}$$

and for $\xi = \arctan(s/\log\rho)$

$$\xi = \frac{\pi}{2\rho\log\rho} - \frac{\pi}{2\rho^2\log^2\rho} + O\left(\frac{1}{\rho^3}\right). \tag{6.6}$$

We deduce that on the curve γ

$$y = R\sin\varphi = e^\rho\left(1 - \frac{1}{2}\left(\frac{\pi}{2}\frac{1}{\rho\log\rho}\right)^2 + O\left(\frac{1}{\rho^3}\right)\right),$$

and

$$x = R\cos\varphi - e = e^\rho\left(\frac{\pi}{2\rho\log\rho} + \frac{\pi}{2\rho^2\log^2\rho} + O\left(\frac{1}{\rho^3}\right)\right),$$

and hence

$$\log y = \rho + O\left(\frac{1}{(\rho\log\rho)^2}\right), \quad (\rho\to\infty),$$

and

$$\log x = \rho - \log\rho - \log\log\rho + \log\frac{\pi}{2} + O\left(\frac{1}{\rho\log\rho}\right), \quad (\rho\to\infty).$$

We can solve these for ρ as in Lemma 6, and obtain

LEMMA 8. — *Suppose* γ *is defined by* $\operatorname{Re} G_0 = 0$ *in the first quadrant. Then on* γ,

$$\rho = \log y + O\left(\frac{1}{(\log y\log\log y)^2}\right), \quad (y\to\infty), \tag{6.7}$$

and

$$\rho = \log x + \log\log x + \log\log\log x - \log\frac{\pi}{2} + o(1), \quad (x\to\infty). \tag{6.8}$$

We can now obtain bounds on $\phi(x)$ and $\psi(y)$. Recall that $\phi'(x) = \operatorname{Re} G_0'(x + i\overline{y}(x)) = \lambda\cos\xi$, and observe that

$$\begin{aligned}\lambda &= \left(\log^2 t + s^2\right)^{1/2}\\ &= \log\rho + O\left(\frac{1}{\rho^2}\right)\\ &= \log\log x + O\left(\frac{\log\log x}{\log x}\right), \quad (x\to\infty),\end{aligned}$$

and that $\cos\xi = 1 + O\left(1/\rho^2\log^2\rho\right)$ by Lemma 7. Thus, we have

$$\begin{aligned}\phi'(x) &= \left(\log\log x + O\left(\frac{\log\log x}{\log x}\right)\right)\left(1 + O\left(\frac{1}{\rho^2\log^2\rho}\right)\right)\\ &= \log\log x + O\left(\frac{\log\log x}{\log x}\right), \quad (x\to\infty).\end{aligned}$$

From this, we conclude that

$$\phi(x) = x\log\log x + O\left(\frac{x\log\log x}{\log x}\right), \quad (x\to\infty). \tag{6.9}$$

For ψ, we have

$$\begin{aligned}\psi'(y) &= \operatorname{Im} G_0'\left(\underline{x}(y)+iy\right) = \lambda\sin\xi\\ &= \frac{\pi}{2\rho} - \frac{\pi}{2\rho^2\log\rho} + O\left(\frac{1}{\rho^3}\right)\\ &= \frac{\pi}{2\log y} - \frac{\pi}{2\log^2 y\log\log y} + O\left(\frac{1}{\log^3 y}\right), \quad (y\to\infty),\end{aligned}$$

and hence

$$\psi(y) = \frac{\pi y}{2\log y} + \frac{\pi y}{2\log^2 y} - \frac{\pi y}{2\log^2 y\log\log y} + O\left(\frac{y}{\log^3 y}\right), \quad (y\to\infty). \tag{6.10}$$

7. Proofs of conjugate function inequalities.

In this section we prove the inequalities stated in our Theorem 1–Theorem 4. The sharpness assertions are discussed in Sections 9 and 10.

By Theorem 6 and Lemma 3, if ϕ and ψ are defined by (2.2) and (2.3) respectively, with $G = G_\alpha$ defined by either (4.1) or (4.2), then

$$\int_0^{2\pi} \psi(|\tilde f|) \leqslant \int_0^{2\pi} \phi(|f|). \tag{7.1}$$

We now use the estimates derived for ϕ and ψ in Sections 5 and 6 to prove the more explicit theorems stated in Section 1. In writing integrals over the unit circle like (7.1), we shall sometimes delete the limits 0 and 2π.

We first prove Theorem 4. We deduce from (6.9) that there is a constant B_1 such that

$$\phi(|f|) \leqslant |f|\,\log\log(e+|f|) + |f| + B_1. \tag{7.2}$$

It follows from (6.10) that there is a constant B_2 such that

$$\psi(|\tilde f|) \geqslant \frac{\pi}{2}|\tilde f|\log^{-1}(e+|\tilde f|) - B_2, \tag{7.3}$$

and now (7.1) implies

$$\int |\tilde f|\log^{-1}(e+|\tilde f|) \leqslant \frac{2}{\pi}\Big(\int |f|\log\log(e+|f|) + \int |f| + B_1 + B_2\Big),$$

proving (1.9).

Now we consider the case $\alpha > 0$. We deduce from (5.21) that there are non-negative constants c_1 and c_2, depending on α, such that

$$\begin{aligned} \frac{2}{\pi\alpha}\psi(y) \geqslant y\log^{\alpha-1}(e+y) - (2\alpha-1)y\log^{\alpha-2}(e+y) \\ -c_1 y\log^{\alpha-3}(e+y) - c_2 \qquad (0 \leqslant y < \infty). \end{aligned} \tag{7.4}$$

We deduce from (5.19) that

$$\begin{aligned} \phi(x) \leqslant x\log^{\alpha}(e+x) + \alpha x\log^{\alpha-1}(e+x)\log^+\log x \\ -\alpha(1+\log\alpha+\eta)x\log^{\alpha-1}(e+x) + c_3 \qquad (0 \leqslant x < \infty), \end{aligned} \tag{7.5}$$

where η is any constant less than $\log\frac{\pi}{2}$ and c_3 depends on only on α and η. Then (7.1) implies

$$\begin{aligned} \int |\tilde f|\log^{\alpha-1}(e+|\tilde f|) \leqslant \frac{2}{\pi\alpha}\int |f|\log^{\alpha}(e+|f|) \\ +\frac{2}{\pi}\int |f|\log^{\alpha-1}(e+|f|)\log^+\log\ |f| \\ -\frac{2}{\pi}(1+\log\alpha+\eta)\int |f|\log^{\alpha-1}(e+|f|) \\ +(2\alpha-1)\int |\tilde f|\log^{\alpha-2}(e+|\tilde f|) + c_4 \\ +c_1\int |\tilde f|\log^{\alpha-3}(e+|\tilde f|), \end{aligned} \tag{7.6}$$

where $c_4 = c_2 + \frac{2c_3}{\pi\alpha}$.

To prove Theorems 1, 2, and 3 we must estimate the integrals of the conjugate function $\tilde f$ which appear on the right-hand side of (7.6). To do this, we consider the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ separately.

Suppose first that $\alpha > 1$. Then $\beta = \alpha - 1 > 0$ and we can apply the inequality

$$\int |\tilde f|\log^{\beta-1}(e+|\tilde f|) \leqslant A_\beta\int |f|\log^{\beta}(e+|f|) + B_\beta, \tag{7.7}$$

where A_β is any constant larger than $\frac{2}{\pi\beta}$ and B_β depends on A_β. With other (larger) A_β, this is classical (cf. [11]). With $\beta = 1$, this is Pichorides' inequality (1.3).

The proof of (7.7) is a simple application of our estimates. Let $\sigma > 1$ be given and use (7.4),(7.5) to deduce that there are constants c_5, c_6 only depending on β and σ such that

$$\frac{2\sigma}{\pi\beta}\psi(y) > y\log^{\beta-1}(e+y) - c_5,$$

$$\varphi(x) < \sigma x \log^\beta(e+x) + c_6,$$

for all $x > 0$, $y > 0$. Then (7.7) follows from (7.1).

Returning to our estimate (7.6) with $\alpha > 1$, we first absorb the last term in the right hand side into the previous two terms by increasing c_4 and the coefficient $(2\alpha - 1)$, and then use (7.7) to obtain (1.6) with

$$\begin{aligned} C_1 &= -\frac{2}{\pi}(1+\log\alpha+\eta) + (2\alpha-1+\varepsilon)\frac{2+\varepsilon}{\pi(\alpha-1)} \\ &= \frac{2}{\pi}\left(\frac{\alpha}{\alpha-1} - \log\alpha - \log\left(\frac{\pi}{2}\right) + \varepsilon'\right), \end{aligned}$$

where $\varepsilon' \to 0$ when $\varepsilon \to 0$ and $\eta \to \log(\frac{\pi}{2})$.

Let $\alpha_0 = 2.916\ldots$ be the root of

$$\frac{\alpha}{\alpha-1} - \log\left(\frac{\pi\alpha}{2}\right) = 0, \qquad (\alpha > 1);$$

given $\alpha > \alpha_0$, we can find constants $\varepsilon' > 0$ and c_4 so that (1.6) holds with $C_1 \leqslant 0$. For $1 < \alpha \leqslant \alpha_0$, we may as well take $C_1 = \frac{2\alpha}{\pi(\alpha-1)}$.

To prove Theorem 2, we start from (7.6) with $\alpha = 1$, and use Theorem 4 to estimate the next-to-last term. Thus

$$\begin{aligned} \int |\tilde f| \leqslant \frac{2}{\pi}\int |f|\left(\log(e+|f|) + \log^+\log|f| + \log\log(e+|f|)\right) \\ + c_1\int |\tilde f|\log^{-2}(e+|\tilde f|) + c_6. \end{aligned}$$

To complete our proof, we rewrite the last integral above in terms of the distribution function $m(t) = |\{e^{i\theta} : |\tilde f(e^{i\theta})| \geqslant t\}|$, where $|\cdot|$ is normalized Lebesgue measure on the circle ($m(0) = 1$). Thus

$$\begin{aligned} \int |\tilde f|\log^{-2}(e+|\tilde f|) &= -\int_0^\infty t\log^{-2}(e+t)dm(t) \\ &\leqslant \int_0^\infty m(t)\log^{-2}(e+t)dt \qquad (7.8) \\ &\leqslant e + \int_e^\infty m(t)(\log^{-2} t)dt \leqslant e + A||f||_{h^1}, \end{aligned}$$

where we have used Kolmogorov's estimate $tm(t) \leqslant A||f||_{h^1}$ with e.g. $A < 2$ (cf. [14]). This completes the proof of (1.7).

Now suppose that $0 < \alpha < 1$ and rewrite (7.6) as

$$\begin{aligned} \int |\tilde f|\log^{\alpha-1}(e+|\tilde f|) \leqslant \frac{2}{\pi\alpha}\int |f|\log^\alpha(e+|f|) + \frac{2}{\pi}\log\left(\frac{1}{\alpha}\right)\int |f| \qquad (7.9) \\ + c_7 + (2\alpha-1)\int |\tilde f|\log^{\alpha-2}(e+|\tilde f|) + c_1\int |\tilde f|\log^{-2}(e+|\tilde f|), \end{aligned}$$

where c_7 depends only on α. We have already estimated the last term in (7.9) and a similar argument shows that (1.8) holds with

$$C_1 = \frac{2}{\pi}\log\left(\frac{1}{\alpha}\right) + \frac{2}{1-\alpha} + c_8,$$

where c_8 is bounded for $0 < \alpha < 1$.

8. Proof of Theorem 5.

We will now prove Theorem 5 giving inequalities between $|F|$ and $|f|$. We will need the following

LEMMA 9. — *Let $\alpha > 0$. Then G_α satisfies (2.7) if $b = b(\alpha)$ is chosen sufficiently large.*

We will first prove Theorem 5 before giving the proof of the lemma. We note that G_α satisfies $\mathcal{H}(1) - \mathcal{H}(5)$ for any $b > e^\alpha$ and thus Theorem 7 is available.

The essence of the proof that follows is that $\Psi(r)$ and $\Phi(x)$ differ only in lower order terms from $\psi(y)$ and $\phi(x)$, so that the statement of the theorems for $|F|$ vs. $|f|$ are unchanged (cf (8.3) below). By (2.5) and (4.1),

$$\Psi'(r) = t^\alpha \sin\alpha s \csc\theta, \tag{8.1}$$

with t, s, and θ defined by (4.3) evaluated at $z = x^*(r) + iy^*(r)$. On the curve γ, we have by (5.12) that $\rho = \log y + O(1/\log^2 y)$ as $y \to \infty$. Since $\rho = \log R$ and $y < r < R$, this implies that on γ,

$$\rho = \log r + O\left(\frac{1}{\log^2 r}\right). \tag{8.2}$$

We also deduce from Lemma 5 and the obvious fact that $\theta > \varphi$ that

$$\csc\theta^*(r) = 1 + O\left(\frac{1}{\log^2 r}\right). \tag{8.3}$$

It now follows that $\Psi'(r)$ satisfies the same estimate in r that (5.20) gives for $\psi'(y)$ in the variable y, and hence that $\Psi(r)$ satisfies (5.21) with y replaced by r and ψ replaced by Ψ.

We now turn to estimating $\Phi(x)$. Applying the definition (2.6) gives

$$\Phi'(x) = t^\alpha \cos\alpha s + t^\alpha \sin\alpha s \cos\theta, \tag{8.4}$$

with t, s, and θ evaluated at $(x, \overline{y}(x))$. Since $\theta > \varphi$ we have $\cos\theta < \cos\varphi = O(1/\rho)$, and we also have $\sin\alpha s = O(1/\rho)$ by (5.8). Hence,

$$\Phi'(x) = t^{\alpha}\cos\alpha s\left(1 + O\left(\frac{1}{\rho^2}\right)\right), \tag{8.5}$$

and thus $\Phi'(x)$ satisfies the same estimate (5.18) as $\phi'(x)$. Thus, $\Phi(x)$ also satisfies (5.19) with ϕ replaced by Φ. Since $\Phi(x)$ and $\Psi(r)$ satisfy the same estimates as $\phi(x)$ and $\psi(y)$ respectively, we can use the argument in Section 7 to prove the inequalities in Theorem 5 with b replaced by $b(\alpha)$ It is now easy to prove Theorem 5 as stated by using inequality (1.4). Since $|F| \geqslant |\tilde{f}|$, the sharpness in Theorem 1 implies the inequality here must also be sharp.

Proof of Lemma 9. — We need to show that $t^{\alpha}\sin\alpha s\csc\theta$ is increasing in θ for $0 < \theta < \pi/2$. We first note that, using the notation of (4.3),

$$\frac{\partial\varphi}{\partial\theta} = \frac{r(r + b\cos\theta)}{r^2 + 2br\cos\theta + b^2} \geqslant 0, \tag{8.6}$$

$$\frac{\partial\rho}{\partial\theta} = \frac{1}{R}\frac{\partial R}{\partial\theta} = \frac{-br\sin\theta}{r^2 + 2br\cos\theta + b^2}\frac{1}{R} \leqslant 0, \tag{8.7}$$

and thus

$$\frac{\partial s}{\partial\theta} = \frac{\partial\varphi}{\partial\theta}\frac{\rho}{\rho^2 + \varphi^2} - \frac{\partial\rho}{\partial\theta}\frac{\varphi}{\rho^2 + \varphi^2} \geqslant 0. \tag{8.8}$$

It also follows from (4.3) that $t = \rho/\cos s = \varphi/\sin s$, and hence we may write

$$\log\left(t^{\alpha}\sin\alpha s\csc\theta\right) = \log\frac{\varphi\rho^{\alpha-1}}{\sin\theta} + \log\frac{\sin\alpha s}{(\cos s)^{\alpha-1}\sin s}. \tag{8.9}$$

We require one further consequence of (4.3),

$$\cot\theta = \frac{\partial\rho}{\partial\theta} + \cot\varphi\frac{\partial\varphi}{\partial\theta}, \tag{8.10}$$

which is obtained by differentiating the identity $\log r + \log(\sin\theta) = \rho + \log(\sin\varphi)$ with respect to θ. We then calculate

$$\frac{\partial}{\partial\theta}\log\frac{\varphi\rho^{\alpha-1}}{\sin\theta} = \frac{1}{\varphi}\frac{\partial\varphi}{\partial\theta} + (\alpha-1)\rho\frac{\partial\rho}{\partial\theta} - \cot\theta,$$

which by (8.10) gives

$$\frac{\partial}{\partial\theta}\log\frac{\varphi\rho^{\alpha-1}}{\sin\theta} = \left(\frac{1}{\varphi} - \frac{1}{\tan\varphi}\right)\frac{\partial\varphi}{\partial\theta} + \left(-\frac{\partial\rho}{\partial\theta}\right)\left(1 - \frac{(\alpha-1)}{\rho}\right). \tag{8.11}$$

Meanwhile,

$$\frac{\partial}{\partial\theta}\log\frac{\sin\alpha s}{(\cos s)^{\alpha-1}\sin s} = q(s)\frac{\partial s}{\partial\theta} \tag{8.12}$$

where

$$q(s) = (\alpha - 1)\tan s + \alpha \cot \alpha s - \cot s = \frac{(\alpha-1)(2-\alpha)}{3} s + O(s^3) \tag{8.13}$$
$$\geqslant -Cs$$

for some $C > |(\alpha - 1)(2-\alpha)|$ and $0 \leqslant s \leqslant B(C)$. It now follows from (8.8), (8.9), (8.11), (8.12), and (8.13) that

$$\begin{aligned}
\frac{\partial}{\partial\theta}\log\left(t^\alpha \sin\alpha s\,(\sin\theta)^{-1}\right) &\geqslant \left(\frac{1}{\varphi} - \frac{1}{\tan\varphi}\right)\frac{\partial\varphi}{\partial\theta} + \left(-\frac{\partial\rho}{\partial\theta}\right)\left(1 - \frac{\alpha-1}{\rho}\right) \\
&\quad - Cs\left(\frac{\partial\varphi}{\partial\theta}\frac{\rho}{\rho^2+\varphi^2} - \frac{\partial\rho}{\partial\theta}\frac{\varphi}{\rho^2+\varphi^2}\right) \\
&= \frac{\partial\varphi}{\partial\theta}\left[\frac{1}{\varphi} - \frac{1}{\tan\varphi} - \frac{Cs\rho}{\rho^2+\varphi^2}\right] \\
&\quad + \left(-\frac{\partial\rho}{\partial\theta}\right)\left[1 - \frac{\alpha-1}{\rho} - \frac{Cs\varphi}{\rho^2+\varphi^2}\right].
\end{aligned} \tag{8.14}$$

We note that $s\rho \leqslant \varphi$ and hence both bracketed terms are positive if ρ is large enough,which will be the case if b is chosen sufficiently large. Thus, by (8.6) and (8.7), $\frac{\partial}{\partial\theta}\log\left(t^\alpha \sin\alpha s\,(\sin\theta)^{-1}\right) > 0$ if $b = b(\alpha)$ is chosen sufficiently large. This finishes the proof of Lemma 9. □

9. A general result on sharpness.

Let G be an analytic function in $\Pi^+ = \{\operatorname{Re} z > 0\}$ satisfying the hypotheses in Section 2. We note that it follows from our assumptions that $G(z)$ defines a conformal mapping of

$$\Omega = \{z = x + iy : g(x,y) > 0, x > 0\} = \{z = x + iy : |y| < \bar{y}(x)\}$$

onto Π^+.

We can prove that Theorem 6 is best possible in the case when $G'(z)$ "behaves like a logarithm" in the following sense: we assume that

$$G'(z)/G'(|z|) \to 1, \qquad (z \to \infty,\ z \in \Pi^+), \tag{9.1}$$

$$zG'(z)/G(z) \to 1, \qquad (z \to \infty,\ z \in \Pi^+), \tag{9.2}$$

$$\phi(x)/(xD_1 g(x, \bar{y}(x))) \to 1, \qquad (x \to \infty), \tag{9.3}$$

$$D_{11} g(z) \geqslant 0, \qquad (z \in \gamma,\ z \text{ large}), \tag{9.4}$$

$$\underline{x}(y)/y \text{ is decreasing when } y \text{ is large.} \tag{9.5}$$

$$\liminf_{R\to\infty} \phi(R)/g(R) < \infty. \tag{9.6}$$

It is clear that these assumptions hold when $G = G_\alpha$, $\alpha \geqslant 0$.

THEOREM 8. — *Assume that $E = \Pi^+ \setminus \Omega$ is not minimally thin at infinity in Π^+. Then (2.4) is best possible in the sense that for any $\varepsilon > 0$, no inequality of the form*

$$\int \psi(|\tilde{f}|) \leqslant (1-\varepsilon) \int \phi(|f|), \tag{9.7}$$

can hold for all analytic polynomials $f + i\tilde{f}$ with $\tilde{f}(0) = 0$.

In the proof, we shall use Cole's theorem. A simple argument of this type, using a classical Phragmén-Lindelöf theorem, can be found in Section 1 in [12] (cf. also Section 3 in the present paper). To deduce a more general theorem of Phragmén-Lindelöf type, to be used in the proof of Theorem 8, we need some results on minimal thinness.

As a general reference on minimal thinness, we use the lecture notes [1] of Aikawa and Essén. The set E in Theorem 8 will be minimally thin at infinity with respect to Π^+ if and only if

$$\int_0^\infty \underline{x}(y)(1+y^2)^{-1} dy < \infty, \tag{9.8}$$

(cf. Theorem 15.1, p. 82 or Corollary 7.4.6, p. 158 in [1]).

If we write $\Omega = \{z : |\arg z| < \frac{\pi}{2} - \Phi(|z|)\}$, an equivalent condition is given by

$$\int_0^\infty \Phi(t)(1+t)^{-1} dt < \infty. \tag{9.9}$$

Remark. — To explain the equivalence, we note that it follows from our conditions that $dr = dy(1+o(1))$ as $y \to \infty$ along γ (cf. (9.16)).

LEMMA 10. — *If E is not minimally thin at infinity in Π^+ and h is a minimal harmonic function in Ω with pole at infinity, then*

$$\lim_{x\to\infty} h(x)/x = \infty. \tag{9.10}$$

Proof. — We start by observing that since h is positive in Ω with boundary values zero and Ω is symmetric with respect to $\mathbb{R}$, we have

$$\sup_y h(x+iy) = h(x). \tag{9.11}$$

This follows from a $*$-function argument similar to the one used in the proof of Lemma 9.2, p. 88, in [5].

Let $\omega_t^\sharp$ be the harmonic measure of $\{x = t\} \cap \Omega$ in $\Omega \cap \{x < t\}$.

LEMMA A. — *A necessary and sufficient condition for E not to be minimally thin at infinity in Π^+ is that for all $z \in \Omega$, we have*

$$\lim_{t \to \infty} t\omega_t^\sharp(z) = 0. \tag{9.12}$$

For the proof, we refer to Lemma 2 in [7].

If (9.10) is false, there exists a sequence $\{x_n\}$ increasing to infinity and a constant A such that $h(t) \leqslant At$, $t \in \{x_n\}$. Using (9.11) and the maximum principle, we deduce that

$$h(z) \leqslant h(t)\omega_t^\sharp(z), \qquad (z \in \Omega \cap \{x < t\}).$$

It follows from (9.12) that

$$h(x) \leqslant At\omega_t^\sharp(x) \to 0, \qquad (t \in \{x_n\}, \quad n \to \infty).$$

This is impossible and we have proved that (9.10) holds which concludes the proof of Lemma 10. □

Let $\varepsilon > 0$ be given. We shall study the curve

$$\gamma_\varepsilon = \{(x, y) : (1 - \varepsilon)\phi(x) = \psi(y), x > 0, y > 0\}$$

and the domain

$$\Omega_\varepsilon = \{(x, y) : (1 - \varepsilon)\phi(x) > \psi(y)\}.$$

LEMMA 11. — *Assume that E is not minimally thin at infinity in Π^+. Then $\Omega \setminus \Omega_\varepsilon$ is not minimally thin at infinity in Ω.*

Proof. — Since minimal thinness is conformally invariant, it suffices to prove that $G(\Omega \setminus \Omega_\varepsilon)$ is not minimally thin at infinity in Π^+.

We work in the first quadrant. From (9.1), we see that

$$D_1 g(z)/D_1 g(|z|) \to 1, \tag{9.13}$$

$$D_1 \tilde{g}(z)/D_1 g(z) = -D_2 g(z)/D_1 g(z) \to 0, \tag{9.14}$$

in both cases when $z \to \infty$ in Π^+. On γ, we have

$$\frac{dy}{dx} = -\frac{D_1 g(z)}{D_2 g(z)} \to \infty, \qquad (z \to \infty,\ z \in \Pi^+). \tag{9.15}$$

If $z = x + iy = re^{i\theta}$, it is easy to see that

$$dr = dy(1 + o(1)), \qquad (z \to \infty,\ z \in \gamma). \tag{9.16}$$

We claim that $\phi(x)$ is a convex function. Let $z = x + iy \in \gamma$. According to the proof of Theorem 6, the function

$$\phi(t) - \psi(y) - g(t, y), \quad (t > 0),$$

has a minimum at x. It follows that

$$\phi''(x) \geqslant D_{11}g(z) \geqslant 0,$$

(cf. (9.4)) which proves the claim.

Let $z = x + iy$ and $Z = X + iY$ be the points of intersection of a circle of radius r centered at the origin and the curves γ and γ_ε, respectively. We write $X + iY = re^{i(\theta+\Delta\theta)}$, where $\Delta\theta < 0$. We note that $z = x + i\bar{y}(x) = \underline{x}(y) + iy$. In the following discussion, we always let $z \to \infty$ along γ.

Let us first prove that $dY/dX \to \infty$ as $z \to \infty$. Differentiating the expression used to define γ_ε, we see that

$$(1-\varepsilon)D_1g(\underline{x}(Y), Y) \leqslant (1-\varepsilon)D_1g(X, \bar{y}(X)) = -D_2g(\underline{x}(Y), Y)\frac{dY}{dX},$$

where the inequality holds since ϕ' is increasing. Hence (cf. (9.15)

$$\frac{dY}{dX} \geqslant -(1-\varepsilon)\frac{D_1g(\underline{x}(Y), Y)}{D_2g(\underline{x}(Y), Y)} \to \infty, \quad (z \to \infty).$$

To compare $\phi(x)$ and $\phi(X)$, we note that

$$\phi(x) \leqslant \phi(X) = (1-\varepsilon)^{-1}\psi(Y) \leqslant (1-\varepsilon)^{-1}\psi(y) = (1-\varepsilon)^{-1}\phi(x). \tag{9.17}$$

To find an estimate of $\Delta\theta$, we note that since $z \in \gamma$ and $Z \in \gamma_\varepsilon$, we have

$$\varepsilon\phi(X) = \phi(X) - \phi(x) - (\psi(Y) - \psi(y)) = \Delta\theta(\phi'(x_0)(-y_0) - \psi'(y_0)x_0),$$

where $x_0 + iy_0$ is a point on the arc $(re^{i(\theta+\Delta\theta)}, re^{i\theta})$. It is clear that

$$Y/X \leqslant y_0/x_0 \leqslant y/x. \tag{9.18}$$

From (9.14), (9.15), the definitions of ϕ and ψ and the fact that ϕ' is increasing, we deduce that that as $z \to \infty$,

$$0 \geqslant -\frac{\psi'(y_0)}{\phi'(x_0)} = \frac{D_2g(\underline{x}(y_0), y_0)}{D_1g(x_0, \bar{y}(x_0))} = \frac{D_2g(\underline{x}(y_0), y_0)}{D_1g(\underline{x}(y_0), y_0)} \to 0, \tag{9.19}$$

$$\Delta\theta = -\frac{\varepsilon\phi(X)(1+o(1))}{y_0D_1g(x_0, \bar{y}(x_0))} = -\frac{\varepsilon x_0\phi(X)(1+o(1))}{y_0\phi(x_0)}, \tag{9.20}$$

(cf. (9.3)). According to (9.17) and (9.18),

$$|\Delta\theta| \leqslant \varepsilon(1-\varepsilon)^{-1}(X/Y) \to 0, \qquad (z \to \infty).$$

Since $\Delta\theta \to 0$ as $z \to \infty$, we know that

$$X + iY - (x+iy) = iz\Delta\theta(1+o(1)).$$

Let us now estimate $\arg G(ze^{i\Delta\theta})$ and $\rho := |G(ze^{i\Delta\theta})|$. Starting from

$$G(Z) - G(z) = D_1 g(z) iz\Delta\theta(1+o(1)),$$

we have (cf. (9.18), (9.20) and (9.3))

$$\begin{aligned}(9.21) \qquad \mathrm{Re}(G(ze^{i\Delta\theta}) - G(z)) &= -y\Delta\theta D_1 g(z)(1+o(1)) \\ &= \varepsilon\,\frac{yx_0\phi(x)(1+o(1))}{xy_0(1-\varepsilon)} \\ &\geqslant \varepsilon\phi(x)(1+o(1))/(1-\varepsilon),\end{aligned}$$

and

$$(9.22) \qquad \mathrm{Im}(G(ze^{i\Delta\theta}) - G(z)) = x\Delta\theta D_1 g(z)\left(1 + o\left(\frac{y}{x}\right)\right).$$

Using (9.2), it is easy to see that

$$(9.23) \qquad \tilde{g}(\underline{x}(y), y) = yD_1 g(z)(1+o(1)).$$

Combining (9.22) and (9.23), we obtain

$$(9.24) \qquad \mathrm{Im}\, G(ze^{i\Delta\theta}) = yD_1 g(z)(1+o(1)).$$

Since $g(z)$ vanishes for $z \in \gamma$, (9.21) and (9.24) imply that there is a positive constant c such that

$$\Phi_\varepsilon(\rho) := \frac{\pi}{2} - \arg G(ze^{i\Delta\theta}) \geqslant c\,\varepsilon\frac{x}{y}, \quad (z \text{ large}).$$

The image domain $G(\Omega_\varepsilon)$ is of the form

$$\left\{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta > 0, \quad |\arg\zeta| < \frac{\pi}{2} - \Phi_\varepsilon(\rho)\right\}.$$

To prove Lemma 11, it suffices to prove that $\int^\infty \frac{x(y)}{y} d\rho/\rho$ is divergent (cf. (9.9)). Integrating by parts, forgetting constants and using (9.5), we have

$$\begin{aligned}\int^\infty \frac{x(y)}{y}\,d\log\rho = \int^\infty \log\rho\; d(-\underline{x}(y)/y) &\geqslant \int^\infty \log y\; d(-\underline{x}(y)/y) \\ &= \int^\infty \frac{x(y)}{y^2}dy.\end{aligned}$$

Since we have assumed that E is not minimally thin at infinity, the last integral is divergent (cf. (9.8)) which finishes the proof of Lemma 11.□

Proof of Theorem 8. — Let H be a minimal harmonic function in $G(\Omega_\varepsilon)$ with pole at infinity. Since $\Pi^+ \setminus G(\Omega_\varepsilon)$ is not minimally thin at infinity in Π^+, it follows from Lemma 10 that

$$\lim_{x\to\infty} H(x)/x = \infty.$$

It is clear that $h(z) = H \circ G(z)$ is a minimal harmonic function in Ω_ε with pole at infinity and that

$$\lim_{x\to\infty} h(x)/g(x) = \lim_{t\to\infty} H(t)/t = \infty. \tag{9.25}$$

We note that

$$\max_\theta h(re^{i\theta}) = h(r), \qquad (r > 0), \tag{9.26}$$

(cf. the beginning of the proof of Lemma 10 for a similar remark).

LEMMA 12. — *Let $\omega_R(\cdot, \Omega_\varepsilon)$ be the harmonic measure of $\{|z| = R\} \cap \Omega_\varepsilon$ in $\Omega_\varepsilon \cap \{|z| < R\}$ and let*

$$M(R) = M(R, h) = \sup_{|z|=R} h(z) = h(R).$$

For any $z_0 \in \Omega_\varepsilon$, there exists a positive constant $A(z_0)$ such that

$$A(z_0)^{-1} \leqslant \omega_R(z_0, \Omega_\varepsilon) M(R) \leqslant A(z_0),$$

for all large values of R.

Before proving Lemma 12, let us complete the proof of Theorem 8. Assume that there exists $\varepsilon > 0$ such that (9.7) holds for all analytic polynomials $f + i\tilde{f}$ with $\tilde{f}(0) = 0$. By Cole's theorem, we know that $(1 - \varepsilon)\phi(|x|) - \psi(|y|)$ has a subharmonic minorant u in $\mathbb{C}$. We note that

$$u(z) \leqslant 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{9.27}$$

$$\sup_y u(x + iy) \leqslant (1 - \varepsilon)\phi(x), \qquad (x > 0). \tag{9.28}$$

From (9.6), we see that there is a sequence $\{R_n\}$ tending to infinity such that

$$\begin{aligned} u(z) &\leqslant M(R, u)\omega_R(z, \Omega_\varepsilon) \\ &\leqslant (1 - \varepsilon)\phi(R)\omega_R(z, \Omega_\varepsilon) \\ &\leqslant \text{Const.}\, g(R)\omega_R(z, \Omega_\varepsilon)) \\ &= o(h(R)\omega_R(z, \Omega_\varepsilon)) = o(1), \qquad (R \to \infty,\ R \in \{R_n\}); \end{aligned}$$

(cf. also (9.25) and (9.27)). It follows that u is non-positive in Ω_ε and thus that u is non-positive in $\mathbb{C}$. The only subharmonic functions in $\mathbb{C}$ which are bounded above are the constant ones (cf. Theorem 2.14 in [15]). On the other hand, we see that

$$u(iy) \leqslant -\psi(y) \to -\infty, \qquad (y \to \infty),$$

and thus u cannot be constant and $(1-\varepsilon)\phi(|x|) - \psi(|y|)$ does not have a subharmonic minorant. We have proved Theorem 8. □

Remark. — A sufficient condition for (9.6) to hold is

$$\limsup_{R\to\infty} D_1 g(R,0)/D_1 g(R,\bar{y}(R)) > 0. \tag{9.29}$$

In fact, if (9.6) doesn't hold, we have $g(R,0)/\phi(R) \to 0, R \to \infty$, and it follows from (9.2) and (9.3) that

$$R\, D_1 g(R,0) = g(R,0)(1+o(1)) = o(\phi(R)) = o(R\, D_1 g(R,\bar{y}(R)),\ R \to \infty,$$

which contradicts (9.29).

Proof of Lemma 12. — Without loss of generality, we assume that $z_0 = 1$. According to the maximum principle, $h(1) \leqslant M(R)\omega_R(1,\Omega_\varepsilon)$.

To prove an inequality going the other way, we define

$$\Gamma_R = \partial\Omega_\varepsilon \cap \left\{ z : |z| = R,\ |\arg z| < \frac{\pi}{4} \right\},$$

and let $\omega_R(\cdot, \Gamma_R, \Omega_\varepsilon)$ be the harmonic measure of Γ_R in $\Omega_\varepsilon \cap \{|z| < R\}$. To every Brownian curve starting at 1 and ending on $(\Omega_\varepsilon \cap \{|z| = R\}) \setminus \Gamma_R$, there is a Brownian curve starting at 1 and ending on Γ_R: just reflect the last part of the first curve in $\arg z = \frac{\pi}{4}$ or $\arg z = -\frac{\pi}{4}$. Hence $\omega_R(1,\Omega_\varepsilon) \leqslant 2\omega_R(1,\Gamma_R,\Omega_\varepsilon)$. Applying Harnack's inequality and using the fact that $h(R) = M(R)$, we see that there is an absolute constant C such that

$$h(1) \leqslant M(R)\omega_R(1,\Omega_\varepsilon) \leqslant 2M(R)\omega_R(1,\Gamma_R,\Omega_\varepsilon) \leqslant 2C^{-1}h(1). \qquad \square$$

10. Examples for sharpness.

Our Theorem 8 shows that our Theorem 1 through Theorem 4 are sharp with regard to the constants appearing in the principal terms. In this section we give examples showing that the sharpness of these theorems extends to the secondary terms in several cases.

For examples, it is convenient to consider the family of functions

$$G_p(z) = \left(\frac{1+z}{1-z}\right)^p, \quad (0 < p < 1).$$

The functions G_p are in H^q for $q < \frac{1}{p}$, so we may work on the unit circle to estimate integral means in $L(\log L)^\alpha$. We write G_p in terms of its real and imaginary parts:

$$G_p\left(e^{i\theta}\right) = g_p(\theta) + i\tilde{g}_p(\theta).$$

A direct calculation gives

$$|g_p(\theta)| = \cos\left(\frac{\pi p}{2}\right)\left|\cot\left(\frac{\theta}{2}\right)\right|^p, \tag{10.1}$$

and

$$|\tilde{g}_p(\theta)| = \sin\left(\frac{\pi p}{2}\right)\left|\cot\left(\frac{\theta}{2}\right)\right|^p. \tag{10.2}$$

Both $|g_p|$ and $|\tilde{g}_p|$ are even functions of θ, so we work on the interval $(0,\pi)$. We define the distribution function λ_g on $(0,\pi)$ by

$$\lambda_g(t) = \left|\{\theta \in (0,\pi) : \ |g(e^{i\theta})| > t\}\right|.$$

From (10.1) we calculate

$$\lambda_{g_p}(t) = 2\arctan\left[\left(\frac{\cos\frac{\pi p}{2}}{t}\right)^{1/p}\right], \tag{10.3}$$

and from (10.2) it follows that

$$\lambda_{\tilde{g}_p}(t) = 2\arctan\left[\left(\frac{\sin\frac{\pi p}{2}}{t}\right)^{1/p}\right]. \tag{10.4}$$

We wish to calculate $\|g_p\|_{L\log^\alpha L}$ and $\|\tilde{g}_p\|_{L\log^{\alpha-1} L}$ as defined in the introduction. However, it is more direct to calculate expressions of the type $\int |g|\left(\log^+|g|\right)^\beta$. We define

$$\|g\|_{1,\beta} = \frac{1}{2\pi}\int_{\{|g(e^{i\theta})|>1\}} \left|g\left(e^{i\theta}\right)\right| \left(\log\left|g\left(e^{i\theta}\right)\right|\right)^\beta d\theta \qquad (\beta \geqslant 0), \tag{10.5}$$

with the modification

$$\|g\|_{1,\beta} = \frac{1}{2\pi}\int_{\{|g(e^{i\theta})|>e\}} \left|g\left(e^{i\theta}\right)\right| \left(\log\left|g\left(e^{i\theta}\right)\right|\right)^\beta d\theta \qquad (\beta < 0). \tag{10.6}$$

We also define

$$\|g\|_{1,\beta,1} = \frac{1}{2\pi}\int_{\{|g(e^{i\theta})|>e\}} |g(e^{i\theta})|(\log|g(e^{i\theta})|)^\beta \log\log(|g(e^{i\theta})|)\, d\theta. \tag{10.7}$$

These integral means have growth comparable to that of the means defined in the introduction.

LEMMA 13. — *There exist positive constants A_β independent of p such that*

$$(10.8)\qquad \|g_p\|_{1,\beta} - A_\beta \leqslant \|g_p\|_{L\log^\beta L} \leqslant \|g_p\|_{1,\beta} + A_\beta \quad (\beta > 0),$$

$$(10.9)\qquad \|\tilde{g}_p\|_{1,\beta} - A_\beta \leqslant \|\tilde{g}_p\|_{L\log^\beta L} \leqslant \|\tilde{g}_p\|_{1,\beta} + A_\beta \quad (\beta > -1),$$

and

$$(10.10)\qquad \|g_p\|_{1,\beta,1} - A_\beta \leqslant \int_0^{2\pi} |g_p|\,(\log|g_p|)^\beta \log\log(e+|g_p|) \leqslant \|g_p\|_{1,\beta,1} + A_\beta \qquad (\beta > -1).$$

Proof. — For (10.8), it follows from (1.4) that the difference between the expressions $\|\cdot\|_{L\log^\beta L}$ and $\|\cdot\|_{1,\beta}$ is bounded by a constant times $\int(\log|g_p|)^{\beta-1}$. It is easily shown by the methods used in the proof of Lemma 14 below that $\int(\log|g_p|)^{\beta-1} < c_\beta$ for some constant c_β independent of p. The boundedness of the integrands when $|g_p| < e$ completes the proof. The proofs of (10.9) and (10.10) are similar. □

We will prove that

LEMMA 14. — *The following estimates hold as $p \to 1$:*

$$(10.11)\qquad \|g_p\|_{1,\alpha} = \frac{\alpha\Gamma(\alpha)}{(1-p)^\alpha} - \frac{\alpha\Gamma(\alpha)\log\left(\frac{1}{1-p}\right)}{(1-p)^{\alpha-1}} + O\left(\frac{1}{(1-p)^{\alpha-1}}\right) + O(1) \qquad (\alpha > 0),$$

$$(10.12)\qquad \|\tilde{g}_p\|_{1,\alpha-1} = \frac{2}{\pi}\frac{\Gamma(\alpha)}{(1-p)^\alpha} + O\left(\frac{1}{(1-p)^{\alpha-1}}\right) \qquad (\alpha \geqslant 1),$$

$$(10.13)\qquad \|\tilde{g}_p\|_{1,\alpha-1} = \frac{2}{\pi}\frac{\Gamma(\alpha)}{(1-p)^\alpha} + O(1) \qquad (0 < \alpha < 1),$$

$$(10.14)\qquad \|\tilde{g}_p\|_{1,-1} = \frac{2}{\pi}\log\frac{1}{1-p} + O(1),$$

and

$$(10.15)\qquad \|g_p\|_{1,\alpha-1,1} = \frac{\Gamma(\alpha)\log\left(\frac{1}{1-p}\right)}{(1-p)^{\alpha-1}} + O(1) \qquad (\alpha \geqslant 0).$$

With these estimates, we can prove that Theorems 1–4 are sharp. Here and for the remainder of this section, we use "O" and "o" to indicate behavior as $p \to 1$, with constants possibly depending on α. By comparing (10.11), (10.12), and (10.15) and letting $p \to 1$, we see that

$$\|\widetilde{g}_p\|_{1,\alpha-1} = \frac{2}{\pi\alpha}\|g_p\|_{1,\alpha} + \left(\frac{2}{\pi} + o(1)\right)\|g_p\|_{1,\alpha-1,1} \qquad (\alpha \geqslant 1),$$

which establishes the sharpness of both the $2/\pi\alpha$ and $2/\pi$ constants in Theorem 1. The same comparison with $\alpha = 1$ shows the leading $2/\pi$ is sharp in Theorem 2, and that the coefficient of the $L\log\log L$ term must be at least $2/\pi$. A similar comparison as $p \to 1$ of (10.11), (10.14) gives

$$\|\widetilde{g}_p\| = \left(\frac{2}{\pi\alpha} + o(1)\right)\|g_p\| \qquad (0 < \alpha < 1),$$

which establishes the sharpness of $2/\pi\alpha$ in Theorem 3. Finally, by comparing (10.14) and (10.15) with $\alpha = 1$ we see that

$$\|\widetilde{g}_p\|_{1,-1} = \left(\frac{2}{\pi} + o(1)\right)\|g_p\|_{1,0,1},$$

establishing the sharpness of $2/\pi$ in Theorem 4.

Proof of Lemma 14. — We need three basic estimates:

$$\frac{p^\alpha}{(1-p)^\alpha} = \frac{1}{(1-p)^\alpha} - \alpha\frac{1}{(1-p)^{\alpha-1}} + O\left(\frac{1}{(1-p)^{\alpha-2}}\right), \tag{10.16}$$

$$\left(\sin\frac{\pi p}{2}\right)^p = 1 + O\left((1-p)^2\right), \tag{10.17}$$

$$\left(\cos\frac{\pi p}{2}\right)^{1/p} = \frac{\pi}{2}(1-p) + \frac{\pi}{2}(1-p)^2\log\left(\frac{\pi}{2}(1-p)\right) + o\left((1-p)^2\right). \tag{10.18}$$

We will also make use of the fact that for any $\alpha > -1$, the substitution $v = \frac{1-p}{p}\log t$ gives

$$\begin{aligned}\int_1^\infty t^{-1/p}(\log t)^\alpha\,dt &= \frac{p^{\alpha+1}}{(1-p)^{\alpha+1}}\int_0^\infty e^{-v}v^\alpha\,dv \\ &= \frac{p^{\alpha+1}}{(1-p)^{\alpha+1}}\Gamma(\alpha+1).\end{aligned} \tag{10.19}$$

Since $\arctan(x) = x + O(x^3)$ for $0 < x < 1$, it follows from (10.3) that

$$\lambda_{g_p}(t) = 2\left(\frac{\cos\frac{\pi p}{2}}{t}\right)^{1/p} + O\left(t^{-3/p}\right). \tag{10.20}$$

Thus, by (10.5), (10.19), and (10.20), we have for any $\alpha > 0$

$$\begin{aligned}\|g_p\|_{1,\alpha} &= \frac{1}{\pi}\int_1^\infty \lambda_{g_p}(t)\left((\log t)^\alpha + \alpha(\log t)^{\alpha-1}\right)dt \\ &= \frac{2}{\pi}\left(\cos\frac{\pi p}{2}\right)^{1/p}\int_1^\infty t^{-1/p}\left((\log t)^\alpha + \alpha(\log t)^{\alpha-1}\right)dt \\ &\quad + \int_1^\infty O(t^{-3/p})\left((\log t)^\alpha + \alpha(\log t)^{\alpha-1}\right)dt \\ &= \frac{2}{\pi}\left(\cos\frac{\pi p}{2}\right)^{1/p}\left(\frac{p^{\alpha+1}}{(1-p)^{\alpha+1}}\Gamma(\alpha+1) + \frac{p^\alpha}{(1-p)^\alpha}\alpha\Gamma(\alpha)\right) \\ &\quad + O(1).\end{aligned}$$

We now deduce from (10.16) and (10.18) that

$$\begin{aligned}\|g_p\|_{1,\alpha} &= \frac{2}{\pi}\left(\frac{\pi}{2}(1-p) + \frac{\pi}{2}(1-p)^2\log\frac{\pi}{2}(1-p) + o\left((1-p)^2\right)\right) \\ &\qquad \times\left(\frac{p^{\alpha+1}}{(1-p)^{\alpha+1}}\Gamma(\alpha+1) + \frac{p^\alpha}{(1-p)^\alpha}\alpha\Gamma(\alpha)\right) + O(1),\end{aligned}$$

and thus we have proved (10.11):

$$\|g_p\|_{1,\alpha} = \frac{1}{(1-p)^\alpha}\alpha\Gamma(\alpha) - \frac{\log\left(\frac{\pi}{2}\left(\frac{1}{1-p}\right)\right)}{(1-p)^{\alpha-1}}\alpha\Gamma(\alpha) + O\left(\frac{1}{(1-p)^{\alpha-1}}\right) + O(1).$$

Next, we estimate the integral mean of the conjugate function in the case $\alpha > 1$. Thus, by (10.5),

$$\begin{aligned}\|\tilde{g}_p\|_{1,\alpha-1} &= \frac{1}{\pi}\int_1^\infty \lambda_{\tilde{g}_p}\left((\log t)^{\alpha-1} + (\alpha-1)(\log t)^{\alpha-2}\right)dt \\ &= \frac{2}{\pi}\left(\sin\frac{\pi p}{2}\right)^{1/p}\int_1^\infty t^{-1/p}\left((\log t)^{\alpha-1} + (\alpha-1)(\log t)^{\alpha-2}\right)dt \\ &\quad + \int_1^\infty O(t^{-3/p})\left((\log t)^{\alpha-1} + (\alpha-1)(\log t)^{\alpha-2}\right)dt \\ &= \frac{2}{\pi}\left(\sin\frac{\pi p}{2}\right)^{1/p}\left(\frac{p^\alpha}{(1-p)^\alpha}\Gamma(\alpha) + \frac{p^{\alpha-1}}{(1-p)^{\alpha-1}}(\alpha-1)\Gamma(\alpha-1)\right) \\ &\quad + O(1).\end{aligned}$$

Now applying (10.16) and (10.17), we deduce

$$\|\tilde{g}_p\|_{1,\alpha-1} = \frac{2}{\pi}\left(\frac{\Gamma(\alpha)}{(1-p)^\alpha}\right) + O\left(\frac{1}{(1-p)^{\alpha-1}}\right), \quad (\alpha > 1). \tag{10.21}$$

In the case $\alpha = 1$, the $(\log t)^{\alpha-2}$ term is absent, and we find

$$\|\tilde{g}_p\|_{1,0} = \frac{2}{\pi(1-p)} + O(1). \tag{10.22}$$

Using the two equations (10.21) and (10.22) together, we obtain (10.12).

For the case $0 < \alpha < 1$, we first note that as $0 < p < 1$ both

$$\int_e^\infty t^{-1/p}(\log t)^{\alpha-2}\,dt = O(1),$$

and also that

$$\int_1^e t^{-1/p}(\log t)^{\alpha-1}\,dt = O(1),$$

with bounds independent of p. A calculation similar to that which gave (10.21), incorporating these last two estimate, proves (10.14).

To calculate the integral mean $\|g_p\|_{1,\alpha-1,1}$, we note that, using (10.19),

$$\begin{aligned}\int_1^\infty & t^{-1/p}\,(\log t)^{\alpha-1}\log\log t\,dt\\ &= \frac{d}{d\alpha}\int_1^\infty t^{-1/p}\,(\log t)^{\alpha-1}\;dt\\ &= \left(\frac{p}{1-p}\right)^\alpha \log\left(\frac{p}{1-p}\right)\Gamma(\alpha) + \left(\frac{p}{1-p}\right)^\alpha \Gamma'(\alpha), \quad (\alpha>0).\end{aligned}$$

Thus for $\alpha > 0$ we have

$$\begin{aligned}\|g_p\|_{1,\alpha-1,1} = \frac{1}{\pi}\int_e^\infty \lambda_{g_p}(t)\,&\big\{(\log t)^{\alpha-1}\log\log t\\ &+ \;((\alpha-1)\log\log t + 1)\,(\log t)^{\alpha-2}\big\}\;dt.\end{aligned}$$

From this we deduce

$$\begin{aligned}\|g_p\|_{1,\alpha-1,1} &= \frac{2}{\pi}\left(\cos\frac{\pi p}{2}\right)^{1/p}\int_1^\infty t^{-1/p}(\log t)^{\alpha-1}\log\log t\,dt\\ &\quad + o\left(\frac{1}{(1-p)^{\alpha-1}}\right) + O(1)\\ &= \frac{1}{(1-p)^{\alpha-1}}\log\left(\frac{1}{1-p}\right)\Gamma(\alpha) + \frac{1}{(1-p)^{\alpha-1}}\Gamma'(\alpha)\\ &\quad + o\left(\frac{1}{(1-p)^{\alpha-1}}\right) + O(1),\end{aligned}$$

which proves (10.15). □

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