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GEOMETRY OF COMPACTIFICATIONS OF LOCALLY SYMMETRIC SPACES

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Contents.

For a locally symmetric space M, we define a compactification $M \cup M(\infty)$ which we call the *geodesic compactification.* It is constructed by adding limit points in $M(\infty)$ to certain geodesics in M (see 1.1–1.2).

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The geodesic compactification arises in other contexts. Two general constructions of Gromov for an ideal boundary of a Riemannian manifold give $M(\infty)$ for locally symmetric spaces (see 1.3–1.4). Moreover, $M(\infty)$ has a natural group theoretic construction using the Tits building (see 1.5).

The geodesic compactification plays two fundamental roles in the harmonic analysis of the locally symmetric space: 1) it is the minimal Martin compactification for negative eigenvalues of the Laplacian (see 2.1), and 2) it can be used to parameterize the eigenfunctions of the Laplacian in continuous spectrum on L_2 (see 2.2).

1. Introduction: geometry.

The introduction is in the first two sections. This one contains geometric results, and §2 contains applications to harmonic analysis.

1.1. *The geodesic compactification.* — Let M be a complete Riemannian manifold. A *ray* in M is a map $\gamma : [0, +\infty) \to M$ that is locally isometric, i.e. a half geodesic parameterized by distance. We want to compactify M by adding limit points to rays in M.

Euclidean space. — Consider first the case of Euclidean space $M = \mathbb{R}^n$. Choose a base point $p \in M$. We can compactify M by adding a sphere $M(\infty)$ at infinity, which has a point $q \in M(\infty)$ for every ray emanating from $p \in M$. If we choose a different base point $p' \in M$, then we get the same compactification. A ray eminating from p' converges to the same point as a ray eminating from p if and only if the rays are parallel. So $M(\infty)$ can be identified with the set of parallelism classes of rays in M. (The same construction works in a Hadamard manifold, i.e. a simply connected, nonpositively curved, complete manifold [BGS], §§2–3, except that parallel classes of rays are replaced by equivalence classes of rays defined below.)

More general manifolds M. — If M is, for example, a locally symmetric space, we cannot hope to define a compactification $M \cup M(\infty)$ in which every ray converges to a point in $M(\infty)$. The reason is that some rays don't "go to infinity" at all – they wander around forever, reentering a fixed compact subset of M infinitely often. A ray(1) is called *distance minimizing,* or DM, if it is an isometric embedding of $[0, +\infty)$ into M. These are the

(1) In this paper, every geodesic is directed and has unit speed.

rays that "go to infinity", so these are the rays for which we want to add limit points. All rays on Euclidean space, or on a Hadamard manifold, are DM.[2]

We want to define an equivalence relation on DM rays analogous to parallelism in Euclidean space, determining when two will converge to the same point in the compactification. We say that DM rays γ_1, γ_2 in M are *equivalent* if they remain at finite distance from each other as they go to infinity, i.e., if $\lim_{t\to+\infty} \sup d(\gamma_1(t), \gamma_2(t)) < +\infty$, where $d(\,\cdot\,, \cdot)$ is the distance function on M. Let $M(\infty)$ denote the set of equivalence classes of DM rays in M.

For a general complete Riemannian manifold M, we find two assumptions (see 9.11, 9.16) under which the set of equivalence classes of DM rays $M(\infty)$ forms the ideal boundary of a canonical Hausdorff compactification $M \cup M(\infty)$. We call this *the geodesic compactification* of M (see 9.17). The construction of the topology on $M \cup M(\infty)$ is somewhat technical, but the idea is this: $M(\infty)$ is topologized so that two points in $M(\infty)$ corresponding to nearby rays are nearby. So topologized, we call $M(\infty)$ the *geodesic boundary* of M. The geodesic boundary $M(\infty)$ is glued on to M so that each DM ray in M converges to its equivalence class in $M(\infty)$.

1.2. *Locally symmetric spaces.* — Now we restrict attention to a manifold M which is a locally symmetric space. In other words, $M = \Gamma\backslash X$ where X is a Riemannian symmetric space with automorphism group G, and $\Gamma \subset G$ is an arithmetic subgroup of G that acts properly and discontinuously on X. In this case, the geodesic compactification exists because

THEOREM (see 11.7). — *The space $M = \Gamma\backslash X$ satisfies the assumptions* 9.11 *and* 9.16, *and hence $M \cup M(\infty)$ defines a Hausdorff compactification.*

We will now discuss three other constructions that give the same space $M(\infty)$ at infinity. This shows that the geodesic compactification of M is a natural mathematical object. The first two come from geometry and can,

[2] There is a history of work on DM rays. Hadamard studied rays on nonpositively curved surfaces systematically in [HAD]. For locally symmetric spaces, importance of DM rays has been noticed by Siegel [SI], §23, Selberg [SE1], pp. 101–118, Garland & Raghunathan [GR], Thm. 4.6.

like the geodesic compactification, be defined for more general manifolds. They don't always coincide with $M(\infty)$, but for locally symmetric spaces they do. The third uses the structure of the group G, so it can only be defined for locally symmetric spaces.

1.3. *The Gromov compactification.* — In [BGS], §2, Gromov introduced a compactification $\overline{M}^G$ for any complete Riemannian manifold M, by embedding the manifold into a space of continuous functions using the distance function (see 12.5).

THEOREM (see 12.11). — *The Gromov compactification $\overline{M}^G$ of $M = \Gamma\backslash X$ is homeomorphic to the geodesic compactification $M \cup M(\infty)$. In particular, the Gromov boundary $\partial\overline{M}^G$ can be identified with the geodesic boundary $M(\infty)$.*

1.4. *The tangent cone at infinity.* — In [GR2], §7 and [GR1], 3.16, Gromov introduced the tangent cone at infinity of a metric space M, denoted by $T_\infty M$ (see 5.4). This is a metric cone which reflects shape of the space M viewed from infinity (see the introduction of [GR3]).

THEOREM (§5.16). — *For $M = \Gamma\backslash X$, the tangent cone at infinity $T_\infty M$ exists and is a cone over the geodesic boundary $M(\infty)$.*

1.5. *The Tits compactification.* — Let $\Delta_{\mathbb{Q}}(\mathbf{G})$ be the rational Tits building of $\mathbf{G}$ (see 3.4). Then $\Gamma \subset G$ acts simplicially on $\Delta_{\mathbb{Q}}(\mathbf{G})$ and the quotient $\Gamma\backslash\Delta_{\mathbb{Q}}(\mathbf{G})$ is a finite simplicial complex called the *Tits complex* of $\Gamma\backslash X$ and denoted by $\Delta(\Gamma\backslash X)$ (see 3.6).

THEOREM (see §11.3, 11.8). — *The geodesic boundary $M(\infty)$ of $M = \Gamma\backslash X$ is homeomorphic to the Tits complex $\Delta(\Gamma\backslash X)$.*

Therefore the geodesic compactification $M \cup M(\infty)$ may be identified with $\Gamma\backslash X \cup \Delta(\Gamma\backslash X)$. The compactification $\Gamma\backslash X \cup \Delta(\Gamma\backslash X)$ may be constructed directly using reduction theory, without appeal to geodesics (see Theorem 8.8). In this context, we call it the *Tits compactification* of $\Gamma\backslash X$ and denote it by $\overline{\Gamma\backslash X}^T$.(3) For technical convenience, in this

(3) It follows from the last two characterizations of the geodesic boundary that the tangent cone at infinity $T_\infty M$ is the cone over the Tits complex $\Delta(\Gamma\backslash X)$. This result was stated without proof by Gromov in [Gr3], 3.I$_1$. When $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ and Γ is a congruence subgroup of $\mathrm{SL}(n,\mathbb{R})$, this result on $T_\infty\Gamma\backslash X$ is due to Hattori [HA2], Theorem B, [HA1], Theorem B.

paper we will treat the Tits compactification first and use it to study the compactifications mentioned earlier.

1.6. *The metric link.* — For any point $q \in M(\infty)$ in the geodesic boundary of a complete Riemannian manifold M, we define the *metric link* $S(q)$ of q as follows: The point q corresponds to an equivalence class of rays. $S(q)$ is the quotient of this equivalence class by a finer equivalence relation called *N-equivalence*: Two geodesics are considered N-equivalent if the distance between them to zero as they go to infinity, i.e., if $\lim_{t\to+\infty} \sup d(\gamma_1(t), \gamma_2(t)) = 0$ for suitable parametrizations of γ_1, γ_2. The metric link is a metric space, with a metric induced by $\lim_{t\to+\infty} \sup d(\gamma_1(t), \gamma_2)$.

If M is a locally symmetric space, the metric link is itself a Riemannian manifold which is a product of Euclidean space with a complete Riemannian manifold $\widetilde{S}(q)$ of finite volume. We call $\widetilde{S}(q)$ the *reduced metric link* of q (see 14.10 and 14.13). It is a locally symmetric space of smaller dimension. It depends only on which open simplex q is in (where "simplex" refers to $\Delta(\Gamma\backslash X)$ which is identified with $M(\infty)$). These metric links play an important role in parametrizing the generalized eigenfunctions (see 2.2 below).

1.7. *Other compactifications.* — The reduced metric link is a boundary component of the *reductive Borel-Serre compactification* (see [GHM], [BJ]) of a locally symmetric space $\Gamma\backslash X$, which is important for certain applications. In fact, the reductive Borel-Serre compactification itself can be constructed by placing an equivalence relation on the set of DM rays. This is carried out in §14. The Borel-Serre compactification is similarly constructed.

2. Applications to harmonic analysis.

We give two cases where the geodesic boundary of $\Gamma\backslash X$ solved a problem about the harmonic analysis of $\Gamma\backslash X$. In both cases, the problem is to classify certain eigenfunctions of the Laplacian. The first case is positive functions with negative eigenvalues, leading to the Martin compactification. The second case is the continuous spectrum of the Laplacian on L_2 (with positive eigenvalues).

2.1. *The Martin compactification.* — For any complete Riemannian manifold M and any λ less than the bottom of the spectrum of the Laplace operator Δ on M, there is a Martin compactification $M \cup \partial_\lambda M$ (see 15.2). Each point in the Martin boundary $\partial_\lambda M$ corresponds to a positive solution of $\Delta u = \lambda u$ and these functions generate the cone $C_\lambda(M) = \{u \in C^\infty(M) \mid \Delta u = \lambda u, u > 0\}$ (see 15.4). Extremal elements in $C_\lambda(M)$ are called minimal functions and form the minimal Martin boundary $\partial_{\lambda,\min} M \subset \partial_\lambda M$.

PROPOSITION (see 15.13, 15.15). — *If $M = \Gamma\backslash X$ is a locally symmetric space, the geodesic boundary $M(\infty)$ can be canonically embedded into the minimal Martin boundary $\partial_{\lambda,\min}\Gamma\backslash X$ for any $\lambda < 0$.*[4]

This proposition implies that there exists an injective map $\iota : \Gamma\backslash X \cup \Gamma\backslash X(\infty) \to \Gamma\backslash X \cup \partial_\lambda \Gamma\backslash X$ for any $\lambda < 0$ which restricts to the identity on $\Gamma\backslash X$.

CONJECTURE (see 15.14). — *The map $\iota : \Gamma\backslash X \cup \Gamma\backslash X(\infty) \to \Gamma\backslash X \cup \partial_\lambda\Gamma\backslash X$ is a homeomorphism, so $\Gamma\backslash X(\infty)$ is the Martin boundary $\partial_\lambda\Gamma\backslash X$, and every boundary point in $\partial_\lambda\Gamma\backslash X$ is minimal.*

2.2. *Paramatrization of the continuous spectrum.* — It is a known fact that the continuous spectrum as a subset in $\mathbb{R}$ of a noncompact manifold does not change under compact perturbations. A natural question is to understand relation between the continuous spectrum and the geometry at infinity. The geometry at infinity can be interpreted as compactifications of the space and also as structure of geodesics going out to infinity. One version of the above question is to parametrize the generalized eigenspaces for the continuous spectrum in terms of boundaries of compactifications defined in terms of geodesics. We carry this out for locally symmetric spaces.

Consider triples (q, φ, r) where q is a point in the geodesic boundary $M(\infty)$, φ is a basis element of the space of eigenfunctions of the Laplacian on the reduced metric link $\widetilde{S}(q)$ (see 1.6 above), and r is a positive real number. For every such triple, we define a generalized eigenfunction of the Laplacian on M, $E(q, \varphi, r)$. These are all distinct. The eigenvalue

[4] Since $\Gamma\backslash X$ has finite volume, $\lambda_0(\Gamma\backslash X) = 0$. For $\lambda = 0$, the Martin compactification of $\Gamma\backslash X$ is just one point compactification.

of $E(q, \varphi, r)$ is the r plus the eigenvalue of φ and a positive constant depending only on q.

PROPOSITION (see 13.15). — *The functions $E(q, \varphi, r)$ form a basis of the continuous spectrum of the Laplacian on L_2M.*

We can call the point $q \in M(\infty)$ the *source* of $E(q, \varphi, r)$. The idea is this: By separation of variables, $E(q, \varphi, r)$ gives rise to a solution of the Schroedinger equation on $M \times \mathbb{R}^*$. This solution represents a particle coming in to M from q with internal excitation φ and velocity determined by r. The particle scatters in M and leaves at various points $q' \in M(\infty)$. So this theorem may be regarded as a connection between a "classical" picture (DM rays) and a "quantum" picture (scattering).(5) After the first version of this paper was written, stronger results connecting the rays of $\Gamma\backslash X$ and the generalized eigenfunctions $E(q, \varphi, r)$ have been obtained for $\Gamma\backslash X$ of $\mathbb{Q}$-rank 1 in [JZ].

The functions $E(q, \varphi, r)$ are given by Eisenstein series, and this result is a reinterpretation of Langlands' theory of Eisenstein series [LA], [AR2], [OW2].

2.3. *Other spaces.* — We have developed a paradigm here for locally symmetric spaces: The geodesic boundary, which is defined using the intrinsic geometry of M, classifies certain eigenfunctions of the Laplacian. We expect that this paradigm will apply to some other Riemannian manifolds "of finite type".

For example, consider the case of Euclidean space $M = \mathbb{R}^n$. The analogues of all of the results mentioned above hold (except that the Tits compactification doesn't make sense). As already remarked, geodesic boundary $M(\infty)$ is a sphere. This coincides with the Gromov boundary, and the tangent cone at infinity is the cone over $M(\infty)$. The Martin compactification coincides with the geodesic compactification. The metric link is a Euclidean space $\mathbb{R}^{n-1}$ so the reduced metric link is a point. Therefore triples (q, φ, r) are just pairs (q, r). The function $E(q, r)$ is wave with direction q and wave length $2\pi/r^{1/2}$. The fact that these waves form a basis of the continuous spectrum is just the Fourier transform. See 14.22 and §16 for more discussion.

(5) See the survey by Keller [KE1] for connection between rays and scattering on domains in Euclidean spaces.

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3. Algebraic groups, Tits building $\Delta_{\mathbb{Q}}(X)$, and Tits simplicial complex $\Delta(\Gamma\backslash X)$.

3.1. In this section, we recall some basic facts of algebraic groups. The basic references are [BO1], Chap. III, [BO3] and [BT], §5.

In 3.2, we introduce Weyl chambers and their faces, which are used to describe parabolic subgroups in 3.3. In 3.4, we recall the rational Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ and the finite Tits simplicial complex $\Delta(\Gamma\backslash X)$. In 3.5, we recall the Langlands decomposition of parabolic subgroups (see 3.5.1) and the induced horospherical decomposition of X (see 3.5.2). Finally, we use Weyl chambers and their faces to give a geometric realization $\Delta_{\mathbb{Q}}(X)$ of $\Delta_{\mathbb{Q}}(\mathbf{G})$ (see 3.6.2) and hence of $\Delta(\Gamma\backslash X)$ (see 3.6.3).

3.2. Let $\mathbf{G}$ be an affine algebraic group defined over $\mathbb{Q}$. The radical $\mathbf{R_G}$ of $\mathbf{G}$ is the greatest connected normal solvable subgroup of $\mathbf{G}$, and the unipotent radical $\mathbf{N_G}$ is the greatest unipotent normal subgroup of $\mathbf{G}$. The group $\mathbf{G}$ is called semi-simple if $\mathbf{R_G} = \{e\}$, and reductive if $\mathbf{N_G} = \{e\}$. In the following, we assume that $\mathbf{G}$ is a connected semi-simple algebraic group defined over $\mathbb{Q}$. Denote the real locus $\mathbf{G}(\mathbb{R})$ by G, which is a semisimple Lie group with finitely many connected components.

An algebraic group $\mathbf{T}$ over $\mathbb{Q}$ is an algebraic torus if $\mathbf{T}(\mathbb{C})$, the complex locus of $\mathbf{T}$, is isomorphic to products of $\mathrm{GL}_1(\mathbb{C})$. If $\mathbf{T}$ is isomorphic to products of GL_1 over $\mathbb{Q}$, then $\mathbf{T}$ is said to split over $\mathbb{Q}$ or called a $\mathbb{Q}$-split torus. All the maximal $\mathbb{Q}$-split tori in $\mathbf{G}$ are conjugate to each other by elements of $\mathbf{G}(\mathbb{Q})$, and their common dimension is called the $\mathbb{Q}$-rank of $\mathbf{G}$, denoted by $r_{\mathbb{Q}}(\mathbf{G})$

Let $\mathbf{S}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. Define $X(\mathbf{S})_{\mathbb{Q}} = \mathrm{Mor}_{\mathbb{Q}}(\mathbf{S}, \mathrm{GL}_1)$, the group of characters of $\mathbf{S}$ defined over $\mathbb{Q}$, and $Y(\mathbf{S})_{\mathbb{Q}} = \mathrm{Mor}_{\mathbb{Q}}(\mathrm{GL}_1, \mathbf{S})$, the group of one parameter subgroups in $\mathbf{S}$ defined over $\mathbb{Q}$. Then $X(\mathbf{S})_{\mathbb{Q}}$ and $Y(\mathbf{S})_{\mathbb{Q}}$ are torsion-free abelian groups of rank $r_{\mathbb{Q}}(\mathbf{G})$. Furthermore, there is a unimodular $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle$ on $X(\mathbf{S})_{\mathbb{Q}} \times Y(\mathbf{S})_{\mathbb{Q}}$ such that $X(\mathbf{S})_{\mathbb{Q}}$ and $Y(\mathbf{S})_{\mathbb{Q}}$ are dual to each other.

Since $\mathbf{S}$ consists of semi-simple elements, the adjoint action of $\mathbf{S}$ on $\mathbf{G}$ is diagonalizable:

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(s)X = s^\alpha X,\ s \in \mathbf{S}\}$$

and $\alpha \in X(\mathbf{S})_{\mathbb{Q}}$ (here the action of a character is written in the form so that the group operation on $X(\mathbf{S})_{\mathbb{Q}}$ is addition). If $\mathfrak{g}_\alpha \neq \{0\}$, α is called a rational root of $\mathbf{G}$ with respect to $\mathbf{S}$. The set of all such rational roots is denoted by $\Phi(\mathbf{G}, \mathbf{S})$ and forms a root system in $X(\mathbf{S})^{\mathbb{R}} = X(\mathbf{S})_{\mathbb{Q}} \otimes \mathbb{R}$ (see [BO1], Thm. 11.4). Each $\alpha \in \Phi(\mathbf{G}, \mathbf{S})$ defines a hyperplane $\mathbf{H}_\alpha$ in $Y(\mathbf{S})^{\mathbb{R}} = Y(\mathbf{S})_{\mathbb{Q}} \otimes \mathbb{R}$ by

$$\mathbf{H}_\alpha = \{H \in Y(\mathbf{S})^{\mathbb{R}} \mid \langle \alpha, H \rangle = 0\}.$$

The connected components of the complement of these hyperplanes $\mathbf{H}_\alpha$ in $Y(\mathbf{S})^{\mathbb{R}}$ are called Weyl chambers. Let $N(\mathbf{S})$ and $Z(\mathbf{S})$ are the normalizer and the centralizer of $\mathbf{S}$ in $\mathbf{G}$ respectively. Then the Weyl group $W(\mathbf{G}) = N(\mathbf{S})/Z(\mathbf{S})$ of $\mathbf{G}$ with respect to $\mathbf{S}$ acts simply transitively on the set of these Weyl chambers. Every Weyl chamber C determines a linear order on $\Phi(\mathbf{G}, \mathbf{S})$: a root α is positive if and only if $\langle \alpha, Y \rangle > 0$ for all $Y \in C$. Denote the set of positive roots by $\Phi^+(\mathbf{G}, \mathbf{S})$ and the set of simple roots by $\Phi^{++}(\mathbf{G}, \mathbf{S})$. For any subset $I \subset \Phi^{++}(\mathbf{G}, \mathbf{S})$, we can define a Weyl chamber face C_I by

$$C_I = \{H \in Y(\mathbf{S})^{\mathbb{R}} \mid \langle \alpha, H \rangle = 0,\ \alpha \in I,\ \langle \alpha, H \rangle > 0,\ \alpha \in \Phi^{++}(\mathbf{G}, \mathbf{S}) \setminus I\},$$

which is an open simplex of dimension $r_Q(\mathbf{G}) - |I|$.

3.3. The Weyl chambers and chamber faces can be used to describe the rational parabolic subgroups. Recall that a closed subgroup $\mathbf{P}$ of $\mathbf{G}$ is called parabolic if $\mathbf{P}$ contains a maximal connected solvable subgroup, i.e., a Borel subgroup of $\mathbf{G}$. If $\mathbf{P}$ is defined over $\mathbb{Q}$, then $\mathbf{P}$ is called a rational (or $\mathbb{Q}$) parabolic subgroup.

If $\mathbf{P}$ is a minimal rational parabolic subgroup, then there exists a $\mathbb{Q}$-split torus $\mathbf{S}$ contained in $\mathbf{P}$ such that

$$\mathbf{P} = Z(\mathbf{S})\mathbf{N}_{\mathbf{P}},$$

where $\mathbf{N}_{\mathbf{P}}$ is the unipotent radical of $\mathbf{P}$ and $Z(\mathbf{S})$ is the centralizer of $\mathbf{S}$ in $\mathbf{G}$.

Furthermore, there exists a Weyl chamber C such that the Lie algebra $\mathfrak{n}_{\mathbf{P}}$ of $\mathbf{N}_{\mathbf{P}}$ is given by

$$\mathfrak{n}_{\mathbf{P}} = \sum_{\alpha \in \Phi^+(\mathbf{G},\mathbf{S})} \mathfrak{g}_\alpha.$$

This gives rise to an one-to-one correspondence between the minimal rational parabolic subgroups containing $\mathbf{S}$ and the Weyl chambers C in $\mathbf{S}$ [BT], Cor. 5.9. Because of this correspondence, we also denote $\Phi^+(\mathbf{G},\mathbf{S})$ by $\Phi^+(\mathbf{G},\mathbf{P})$, and $\Phi^{++}(\mathbf{G},\mathbf{S})$ by $\Phi^{++}(\mathbf{G},\mathbf{P})$.

Non-minimal rational parabolic subgroups correspond to proper Weyl chamber faces. More precisely, let $\mathbf{P}$ be a minimal rational parabolic subgroup, and $\mathbf{S}$ a maximal $\mathbb{Q}$-split torus in $\mathbf{P}$ as above. For any subset $I \subset \Phi^{++}(\mathbf{G},\mathbf{S})$, denote by $\mathbf{S}_I$ the identity component of $\bigcap_{\alpha\in I} \ker\alpha \subset \mathbf{S}$. Then $\mathbf{S}_I$ is a $\mathbb{Q}$-split torus of dimension $r_{\mathbb{Q}}(\mathbf{G}) - |I|$. Let $\mathbf{P}_I$ be the subgroup generated by $Z(\mathbf{S}_I)$ and $\mathbf{N}_{\mathbf{P}}$. Then $\mathbf{P}_I$ is a rational parabolic subgroup containing $\mathbf{P}$, whose unipotent radical $\mathbf{N}_{\mathbf{P}_I}$ has Lie algebra $\sum' \mathfrak{g}_\alpha$, where the sum is over all the positive $\mathbb{Q}$-roots which are not linear combinations of elements of I, i.e., all the positive roots which do not vanish on $\mathbf{S}_I$. Such a rational parabolic subgroup is called a standard rational parabolic subgroup associated with $\mathbf{P}$, $\mathbf{S}$. Any rational parabolic subgroup containing the minimal rational parabolic subgroup $\mathbf{P}$ is a standard one [BT], Cor. 5.18, and hence corresponds to a unique Weyl chamber face of the Weyl chamber C associated with $\mathbf{P}$.

Under this correspondence, the opposite of the inclusion relation for the rational parabolic subgroups is the same as the face relation of the Weyl chamber faces. This implies that there are only finitely many rational parabolic subgroups containing any minimal rational parabolic subgroup, while each non-minimal rational parabolic subgroup contains infinitely many minimal rational parabolic subgroups.

3.4. We recall the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{G}$ over $\mathbb{Q}$ (see [TI1], [TI2], Thm. 5.2). Simplexes of $\Delta_{\mathbb{Q}}(\mathbf{G})$ correspond bijectively to proper rational parabolic subgroups of $\mathbf{G}$. Each proper maximal rational parabolic subgroup $\mathbf{Q}$ corresponds to a vertex of $\Delta_{\mathbb{Q}}(\mathbf{G})$, denoted by $\mathbf{Q}$. Vertices $\mathbf{Q}_0, \dots, \mathbf{Q}_k$ are the vertices of a k-simplex if and only if $\mathbf{Q}_0 \cap \dots \cap \mathbf{Q}_k$ is a rational parabolic subgroup, and this simplex corresponds to the parabolic subgroup $\mathbf{Q}_0 \cap \dots \cap \mathbf{Q}_k$.

If $\mathbf{G}$ has $\mathbb{Q}$-rank one, $\Delta_{\mathbb{Q}}(\mathbf{G})$ is a countable collection of points. Otherwise, $\Delta_{\mathbb{Q}}(\mathbf{G})$ is a connected infinite simplicial complex. For any maximal $\mathbb{Q}$-split torus $\mathbf{S}$, all the rational parabolic subgroups containing $\mathbf{S}$ form an apartment in this building. This subcomplex gives a simplicial triangulation of the sphere of dimension $r_{\mathbb{Q}}(\mathbf{G}) - 1$. This is the reason why $\Delta_{\mathbb{Q}}(\mathbf{G})$ is called a spherical building.

The rational points $\mathbf{G}(\mathbb{Q})$ of $\mathbf{G}$ act on the set of rational parabolic subgroups by conjugation and hence on $\Delta_{\mathbb{Q}}(\mathbf{G})$: For any $g \in \mathbf{G}(\mathbb{Q})$ and any rational parabolic subgroup $\mathbf{P}$, the simplex of $\mathbf{P}$ is mapped to the simplex of $g\mathbf{P}g^{-1}$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then by reduction theory (see 4.4) and the discussion in 3.3, there are only finitely many Γ-conjugacy classes of rational parabolic subgroups. Therefore, the quotient $\Gamma\backslash\Delta_{\mathbb{Q}}(\mathbf{G})$ is a finite simplicial complex. This simplicial complex is called the Tits complex of $\Gamma\backslash X$ and denoted by $\Delta(\Gamma\backslash X)$.

3.5. We recall the Langlands decomposition of rational parabolic subgroups and the induced horospherical decomposition of the symmetric space.

For a rational parabolic subgroup $\mathbf{P}$, its Levi quotient $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ is defined over $\mathbb{Q}$. Let $\mathbf{S_P}$ denote the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{L_P}$, and $A_\mathbf{P} = \mathbf{S}(\mathbb{R})^0$ the identity component of the real points $\mathbf{S}(\mathbb{R})$ of $\mathbf{S}$. Let $\mathbf{M_P} = \bigcap_{\alpha \in X(\mathbf{L_P})_{\mathbb{Q}}} \ker \alpha^2$. Then the real locus of the Levi quotient splits as a direct product $\mathbf{L_P}(\mathbb{R}) = \mathbf{M_P}(\mathbb{R}) A_\mathbf{P}$.

Let X be the symmetric space of the maximal compact subgroups of $\mathbf{G}(\mathbb{R})$. Then any point $x_0 \in X$ corresponds to a maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R})$. There is a unique algebraic Cartan involution θ of $\mathbf{G}(\mathbb{R})$ whose fixed point set is K and which extends to $\mathbf{G}$, and there is a unique lift $i_0 : \mathbf{L_P} \to \mathbf{P}$ of the Levi quotient such that the image $\mathbf{L_P}(x_0) = i_0(\mathbf{L_P})$ is θ-invariant [BS], Prop. 1.6 and Cor. 1.9. We also obtain lifts $\mathbf{S_P}(x_0)$, $A_\mathbf{P}(x_0)$ and $\mathbf{M_P}(x_0)$ of the subgroups $\mathbf{S_P}$, $A_\mathbf{P}$ and $\mathbf{M_P}$ respectively.

The real locus $\mathbf{P}(\mathbb{R})$ of $\mathbf{P}$ is denoted by P, $\mathbf{N_P}(\mathbb{R})$ by $N_\mathbf{P}$, and the image $i_0(\mathbf{M_P}(\mathbb{R}))$ by $M_\mathbf{P}(x_0)$. We then have the Langlands decomposition

$$P = N_\mathbf{P} A_\mathbf{P}(x_0) M_\mathbf{P}(x_0), \tag{3.5.1}$$

i.e., the map $(u, a, m) \in N_\mathbf{P} \times A_\mathbf{P}(x_0) \times M_\mathbf{P}(x_0) \mapsto uam \in P$ is a diffeomorphism. Note that $A_\mathbf{P}(x_0)$ commutes with $M_\mathbf{P}(x_0)$, and hence the Langlands decomposition is also written as

$$P = N_\mathbf{P} M_\mathbf{P}(x_0) A_\mathbf{P}(x_0).$$

Since P acts transitively on X, any $x \in X$ can be written as

$$x = u(x)a(x)m(x)x_0,$$

where $u(x) \in N_{\mathbf{P}}$, $a(x) \in A_{\mathbf{P}}(x_0)$, $m(x) \in M_{\mathbf{P}}(x_0)$, and $u(x)$, $a(x)$ are uniquely determined by x. Define $X_{\mathbf{P}} = M_{\mathbf{P}}(x_0)/K_{\mathbf{P}}$, where $K_{\mathbf{P}} = K \cap M_{\mathbf{P}}(x_0)$. Then $X_{\mathbf{P}}$ is product of a symmetric space of non-compact type with a possible Euclidean space and hence called the boundary symmetric space associated with $\mathbf{P}$. Then the above decomposition decomposition of X induces a diffeomorphism

$$(3.5.2) \qquad N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}(x_0) \longrightarrow X\colon\ (u, mK_{\mathbf{P}}, a) \longmapsto uamx_0.$$

This decomposition is called the horospherical decomposition of X with respect to $\mathbf{P}$.

If $\mathbf{Q} \supset \mathbf{P}$ is another rational parabolic subgroup, then $\mathbf{N_Q} \subset \mathbf{N_P}$, $\mathbf{S_Q} \subset \mathbf{S_P}$, and $A_{\mathbf{Q}} \subset A_{\mathbf{P}}$. The above lift i_0 also determines inclusions $\mathbf{N_Q}(x_0) \subset \mathbf{N_P}(x_0)$, $\mathbf{S_Q}(x_0) \subset \mathbf{S_P}(x_0)$, and $A_{\mathbf{Q}}(x_0) \subset A_{\mathbf{P}}(x_0)$. In particular, for a minimal rational parabolic subgroup $\mathbf{P}$ and a standard rational parabolic subgroup $\mathbf{P}_I$ defined in 3.3, $A_{\mathbf{P}_I}(x_0) = \{a \in A_{\mathbf{P}}(x_0) \mid a^\alpha = 1,\ \alpha \in I\} \subset A_{\mathbf{P}}(x_0)$, where the action of the character α on $A_{\mathbf{P}}(x_0)$ is the composition of its action on $A_{\mathbf{P}}$ with the map $i_0 : A_{\mathbf{P}} \to A_{\mathbf{P}}(x_0)$.

If $x_1 \in X$ is a different basepoint, then the lifting map i_1 is conjugate to i_0 by some element of P, and thus the lifts of $\mathbf{L_P}(x_1)$, $\mathbf{S_P}(x_1)$, $A_{\mathbf{P}}(x_1)$, $\mathbf{M_P}(x_1)$ are also conjugate to $\mathbf{L_P}(x_0)$, $\mathbf{S_P}(x_0)$, $A_{\mathbf{P}}(x_0)$, $\mathbf{M_P}(x_0)$, respectively. From now on, this basepoint x_0 will be fixed and omitted from $A_{\mathbf{P}}(x_0)$, $\mathbf{M_P}(x_0)$, *etc.*, unless necessary.

3.6. We use Weyl chambers and faces to give a concrete realization of the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$.

For any rational parabolic subgroup $\mathbf{Q}$, let $\mathfrak{a}_{\mathbf{Q}}, \mathfrak{n}_{\mathbf{Q}}$ be the Lie algebra of $A_{\mathbf{Q}}, N_{\mathbf{Q}}$ respectively. Then $\mathfrak{a}_{\mathbf{Q}}$ acts on $\mathfrak{u}_{\mathbf{Q}}$, and the set of roots is denoted by $\Phi^+(Q, A_{\mathbf{Q}})$, and the subset of the simple roots is denoted by $\Phi^{++}(Q, A_{\mathbf{Q}})$. If $\mathbf{P}$ is a minimal rational parabolic subgroup contained in $\mathbf{Q}$ and $\mathbf{Q}$ is of the standard form $\mathbf{Q} = \mathbf{P}_I$, then $\Phi^{++}(Q, A_{\mathbf{Q}})$ is canonically identified with $\Phi^{++}(\mathbf{G}, \mathbf{P}) \setminus I$.

For the rational parabolic subgroup $\mathbf{Q}$, define an open simplex

$$(3.6.1) \quad A^+_{\mathbf{Q}}(\infty) = \big\{H \in \mathfrak{a}_{\mathbf{Q}} \mid \alpha(H) > 0,\ \langle H, H\rangle = 1,\ \alpha \in \Phi^+(Q, A_{\mathbf{Q}})\big\},$$

and a (closed) simplex

$$\overline{A^+_{\mathbf{Q}}}(\infty) = \{H \in \mathfrak{a}_{\mathbf{Q}} \mid \alpha(H) \geq 0,\ \langle H, H\rangle = 1,\ \alpha \in \Phi^+(Q, A_{\mathbf{Q}})\},$$

where $\langle\, .\, ,\, .\rangle$ is the Killing form on the Lie algebra. It is clear that $\overline{A^+_{\mathbf{Q}}}(\infty)$ is a closed simplex of dimension $r_{\mathbb{Q}}(\mathbf{G}) - r_{\mathbb{Q}}(\mathbf{M_Q}) - 1$, and $A^+_{\mathbf{Q}}(\infty)$ is an open simplex. From the correspondence between the rational parabolic subgroups and the Weyl chamber faces in 3.3, we can see that $\overline{A^+_{\mathbf{Q}}}(\infty)$ is isomorphic to the simplex in $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{Q}$ in 3.4. If $\mathbf{Q}'$ is another rational parabolic subgroup containing $\mathbf{Q}$, then $\overline{A^+_{\mathbf{Q}'}}(\infty)$ is a face of the simplex $\overline{A^+_{\mathbf{Q}}}(\infty)$.

Define a complex

$$\Delta_{\mathbb{Q}}(X) = \bigcup_{\mathbf{Q}} \overline{A^+_{\mathbf{Q}}}(\infty)/\sim,$$

where $\mathbf{Q}$ runs over all the proper rational parabolic subgroups of $\mathbf{G}$, and the equivalence relation is given by the inclusion above for any pair of rational parabolic subgroups $\mathbf{Q}' \supset \mathbf{Q}$. This simplicial complex is a realization of the spherical Tits building for $\mathbf{G}$:

$$\Delta_{\mathbb{Q}}(\mathbf{G}) \cong \Delta_{\mathbb{Q}}(X) = \bigcup_{\mathbf{Q}} \overline{A^+_{\mathbf{Q}}}(\infty)/\sim. \tag{3.6.2}$$

As a set $\Delta_{\mathbb{Q}}(X)$ is a disjoint union of the open simplexes

$$\Delta_{\mathbb{Q}}(X) = \coprod_{\mathbf{Q}} A^+_{\mathbf{Q}}(\infty).$$

For an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$, let $\mathbf{P}_1, \ldots, \mathbf{P}_n$ be a set of representatives of Γ-conjugacy classes of proper rational parabolic subgroups. Then the Tits simplicial complex $\Delta(\Gamma\backslash X)$ in 3.4 has the following realization:

$$\Delta(\Gamma\backslash X) = \bigcup_{i=1}^{n} \overline{A^+_{\mathbf{P}_i}}(\infty)/\sim, \tag{3.6.3}$$

where the equivalence relation is defined as follows: $\overline{A^+_{\mathbf{P}_i}}(\infty)$ is identified with a face of $\overline{A^+_{\mathbf{P}_j}}(\infty)$ if and only if a Γ-conjugate of $\mathbf{P}_i$ contains $\mathbf{P}_j$. Then as a set,

$$\Delta(\Gamma\backslash X) = \coprod_{i=1}^{n} A^+_{\mathbf{P}_i}(\infty). \tag{3.6.4}$$

By the reduction theory in Proposition 4.4, there are only finitely many Γ-conjugacy classes of rational parabolic subgroups, i.e., n is finite, and hence $\Delta(\Gamma\backslash X)$ is a finite simplicial complex.

4. Classical and precise reduction theories.

4.1. In this section, we recall the reduction theory by Borel and Harish-Chandra [BH], Borel [BO1] [BO4], and the refinement by Langlands [LA], Arthur [AR1], Osborne and Warner [OW1], and Saper [SA]. The refined reduction theory plays an important role in the study of the tangent cone at infinity $T_\infty\Gamma\backslash X$ in §5 and the geodesic compactification $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$ in §11.

In 4.2, we define (generalized) Siegel sets associated with parabolic subgroups which are not necessarily minimal. The reduction theory of [BO1] is stated in 4.4, and the refined reduction theory is stated in 4.6.

4.2. We first recall the definition of Siegel sets and generalizations. Let $x_0 \in X$ be the basepoint fixed in 3.5. For any rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, let $\alpha_1, \dots, \alpha_r$ be the simple roots in $\Phi^{++}(P, A_\mathbf{P})$ (see 3.6). For r-tuple of positive numbers $t = (t_1, \dots, t_r)$, define

$$A_{\mathbf{P},t} = \{a \in A_\mathbf{P} \mid \alpha_i(\log a) > t_i,\ i = 1, \dots, r\}, \tag{4.2.1}$$

which is a shift of the positive chamber

$$A_\mathbf{P}^+ = \{a \in A_\mathbf{P} \mid \alpha_i(\log a) > 0,\ i = 1, \dots, r\}. \tag{4.2.2}$$

Since $P = N_\mathbf{P} M_\mathbf{P} A_\mathbf{P}$ by the Langlands decomposition and $G = PK$, G can be written as $G = N_\mathbf{P} M_\mathbf{P} A_\mathbf{P} K$. For any bounded set ω in $N_\mathbf{P} M_\mathbf{P}$, the set $\mathcal{S} = \omega A_{\mathbf{P},t} K$ in G is called a (generalized) Siegel set in G, and the subset $\mathcal{S} x_0 \subset X$ a (generalized) Siegel set in X.

4.3. When $\mathbf{P}$ is a minimal rational parabolic subgroup and $t_1 = \cdots = t_r$, $\mathcal{S} x_0$ is the usual Siegel set in the reduction theory. In terms of the Siegel sets for minimal rational parabolic subgroups, the reduction theory in [BO1], [BO4], Thm. 1.10, can be stated as follows:

4.4. Proposition. — *Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and Γ an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. If $\mathbf{P}$ is a minimal rational parabolic subgroup of $\mathbf{G}$, then $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ is finite, i.e., there are only finitely many Γ-conjugacy classes of minimal rational parabolic subgroups. Furthermore, there exists a Siegel set $\mathcal{S} = \omega A_{\mathbf{P},t} x_0$ associated with $\mathbf{P}$ and a finite subset $C \subset \mathbf{G}(\mathbb{Q})$ such that $\Omega = C\mathcal{S}$ is a fundamental set for Γ, i.e., it satisfies the following conditions :*

1) $\Gamma\Omega = X$;

2) *For any $g \in \mathbf{G}(\mathbb{Q})$, the set $\{\gamma \in \Gamma \mid g\Omega \cap \gamma\Omega \neq \emptyset\}$ is finite.*

4.5. Siegel sets for nonminimal parabolic subgroups are needed to get a fundamental domain for Γ, which is a fundamental set without overlap between the translates by Γ. This is the so-called precise reduction theory and achieved by giving a partion of $\Gamma\backslash X$ into disjoint union of generalized Siegel sets. This theory was developed by Langlands [LA], Arthur [AR1], and Osborne and Warner [OW1], §3. We follow the presentation in [OW1], §3 and [SA].

By the discussion in 3.3 and Proposition 4.4, there are only finitely many Γ-conjugacy classes of rational parabolic subgroups. Let $\mathbf{P}_1^{\max}, \dots, \mathbf{P}_m^{\max}$ be representatives of Γ-conjugacy classes of maximal rational parabolic subgroups. For $i = 1, \dots, m$, let $\mathfrak{a}_i^{\max}$ be the Lie algebra of $A_i^{\max}$, the unique split component of $\mathbf{P}_i^{\max}$ determined by the basepoint x_0 in 3.5. Define a vector space

$$\mathfrak{a} = \bigoplus_{i=1}^{m} \mathfrak{a}_i^{\max}. \tag{4.5.1}$$

Then for any rational parabolic subgroup $\mathbf{P}$, there is a canonically defined map

$$I_{\mathbf{P}} : \mathfrak{a} \longrightarrow \mathfrak{a}_{\mathbf{P}}, \tag{4.5.2}$$

where $\mathfrak{a}_{\mathbf{P}}$ is the Lie algebra of $A_{\mathbf{P}}$.

This map $I_{\mathbf{P}}$ plays an important role in this paper and can be defined briefly as follows (see [OW1], p. 330). For any maximal rational parabolic subgroup $\mathbf{Q}$, there exist an element $\gamma \in \Gamma$ and a unique index i_0 such that $\mathbf{Q} = \gamma \mathbf{P}_{i_0}^{\max} \gamma^{-1}$. Taking the $A_{\mathbf{Q}}$ component of γ into consideration, we define a map $\mathfrak{a}_{i_0}^{\max} \to \mathfrak{a}_{\mathbf{Q}}$ which is independent of the choice of γ. Then $I_{\mathbf{P}}$ is the composition of $\bigoplus_{i=1}^{m} \mathfrak{a}_i^{\max} \to \mathfrak{a}_{i_0}^{\max} \to \mathfrak{a}_{\mathbf{P}}$.

For any rational parabolic subgroup $\mathbf{P}$, let $\mathbf{Q}_1, \dots, \mathbf{Q}_r$ be all the maximal rational parabolic subgroups containing $\mathbf{P}$. Then there is a direct sum decomposition

$$\mathfrak{a}_{\mathbf{P}} = \bigoplus_{j=1}^{r} \mathfrak{a}_{\mathbf{Q}_j} \tag{4.5.3}$$

obtained as follows: For any j, $\mathfrak{a}_{\mathbf{Q}_j}$ is a subspace of $\mathfrak{a}_{\mathbf{P}}$, and the projection map from $\mathfrak{a}_{\mathbf{P}}$ to $\mathfrak{a}_{\mathbf{Q}_j}$ is the orthogonal projection. Note that this decomposition is not orthogonal in general.

Using the decomposition (4.5.3), we define the map $I_{\mathbf{P}}$ as follows: For any $H \in \mathfrak{a}$, the image $I_{\mathbf{P}}(H) \in \mathfrak{a}_{\mathbf{P}}$ is the unique point whose projection in $\mathfrak{a}_{\mathbf{Q}_j}$ is equal to $I_{\mathbf{Q}_j}(H)$.

If $\dim \mathfrak{a}_{\mathbf{P}} = 2$, then the decomposition $\mathfrak{a}_{\mathbf{P}} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is illustrated in Figure 4.5.4, where $\mathfrak{a}_1 = \mathfrak{a}_{\mathbf{Q}_1}$, and $\mathfrak{a}_2 = \mathfrak{a}_{\mathbf{Q}_2}$.

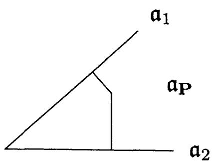

Figure 4.5.4

For any $T \in \mathfrak{a}$, define

$$(4.5.5) \qquad A_{\mathbf{P},T} = \{ e^H \in A_{\mathbf{P}} \mid \alpha(H) > \alpha(I_{\mathbf{P}}(T)),\ \alpha \in \Phi^{++}(P, A_{\mathbf{P}}) \},$$

which is the translate of $A_{\mathbf{P}}^+$ by $I_{\mathbf{P}}(T)$.

Using these canonically chosen (generalized) Siegel sets, we can state the refined, precise reduction theory as follows.

4.6. Proposition (see [OW1], Thm 3.4 and [SA], Thm. 9.6). — *Let $\mathbf{P}_0 = \mathbf{G}$, $\mathbf{P}_1, \ldots, \mathbf{P}_n$ be representatives of Γ conjugacy classes of rational parabolic subgroups of $\mathbf{G}$. Then for any $T \in \mathfrak{a}$, $T \gg 0$ and $i = 0, \ldots, n$, there exist bounded sets $\omega_i \subset N_{\mathbf{P}_i} M_{\mathbf{P}_i}$ such that*

1) *Each Siegel set $\omega_i A_{\mathbf{P}_i,T} x_0$ is mapped injectively into $\Gamma \backslash X$.*

2) *The image of ω_i in $\Gamma \cap P_i \backslash N_{\mathbf{P}_i} M_{\mathbf{P}_i}$ is compact.*

3) *Identify $\omega_i A_{\mathbf{P}_i,T} x_0$ with its image in $\Gamma \backslash X$. Then $\Gamma \backslash X$ can be decomposed into the following disjoint union*

$$\Gamma \backslash X = \coprod_{i=0}^{n} \omega_i A_{\mathbf{P}_i,T} x_0.$$

4.7. *Remark.* — In [SA], Thm. 9.6, Saper obtained a Γ-equivariant tiling of X by manifolds with corners, which also extends canonically to the Borel-Serre completion of X. Projecting this tiling of X to $\Gamma \backslash X$ gives the disjoint decomposition in Proposition 4.6. In fact, his result shows that the image of ω_i in $\Gamma \cap P_i \backslash N_{\mathbf{P}_i} M_{\mathbf{P}_i}$ is a compact manifold with corner, and the image of $\omega_i A_{\mathbf{P}_i,T} x_0$ in $\Gamma \backslash X$ is also a manifold with corner. The corner structure of the image of ω_i does not play any role in this paper, though we use the fact that the image of ω_i is compact as stated in Proposition 4.6, 2).

4.8. Note that the set $\omega_0 A_{\mathbf{P}_0,T} x_0$ corresponding to $\mathbf{G}$ is a compact subset. When $\dim A_{\mathbf{P}} = 2$, the partition of $\Gamma\backslash X$ induces a decomposition of the positive chamber $A_{\mathbf{P}}^+$ in Figure 4.8.1.

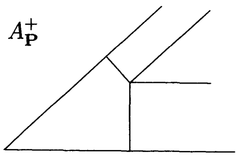

Figure 4.8.1

5. Tangent cone at infinity $T_\infty\Gamma\backslash X$ of $\Gamma\backslash$X.

5.1. In this section, we use the precise reduction theorem (Proposition 4.6) to prove Theorem 1.4 (see 5.16). The basic idea is that Proposition 4.6 reduces the problem to understanding a metric subspace $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_{\text{ind}})$ of $\Gamma\backslash X$ (see 5.7). We compare this subspace with an auxiliary metric space $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_S)$, a polyhedral cone with a simplicial metric (see 5.9).

More precisely, in 5.2, we recall the Hausdorff distance of two metric spaces. Then we define the Gromov-Hausdorff convergence in 5.3. In 5.4, we define the tangent cone at infinity of a metric space. In 5.5, we introduce length spaces, and we show how to associate a length structure to a locally defined distance function in 5.6. In 5.7 and 5.8, we introduce the subspace $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_{\text{ind}})$ and reduce Theorem 1.4 to understanding the metric structure of $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_{\text{ind}})$. In 5.9, we define the auxiliary metric space $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_S)$. Then we use Proposition 4.6 to compare $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_{\text{ind}})$ with $(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_S)$ in 5.11–5.14. We determine $T_\infty(\coprod_{i=0}^n A_{\mathbf{P}_i,T}, d_S)$ in 5.15 and hence $T_\infty(\Gamma\backslash X)$ in 5.16.

The results of this section, in particular, Corollary 5.13 and Lemma 5.14 have been used in [J2], Thm. 7.6, to prove the Siegel conjecture on metric properties of Siegel sets.

5.2. Definition (see [GR1], p. 35). — *If X, Y are two subsets of a metric space (Z, d), then the Hausdorff distance $d_H^Z(X,Y)$ between X, Y in Z is defined as follows:*

$$d_H^Z(X,Y) = \inf\{\varepsilon \mid d(x,Y),\ d(X,y) \le \varepsilon \text{ for any } x \in X,\ y \in Y\}.$$

If X, Y are any two metric spaces, then the Hausdorff distance $d_H(X, Y)$ between them is defined by

$$d_H(X,Y) = \inf_Z d_H^Z(f(X), f(Y)),$$

where Z is a metric space, and $f : X \to Z$, $f : Y \to Z$ are isometric embeddings.

5.3. Definition. — *Let (M_n, d_n, x_n), $n \geq 1$, be a sequence of pointed metric spaces, where d_n is the distance function of M_n and x_n is a basepoint in M_n. Then (M_n, d_n, x_n) is defined to converge to a pointed metric space $(M_\infty, d_\infty, x_\infty)$ in the sense of Gromov-Hausdorff if for all $R > 0$, the Hausdorff distance between the metric ball $B(x_n, R)$ in M_n and the metric ball $B(x_\infty, R)$ in M_∞ goes to zero as $n \to \infty$.*

5.4. Definition. — *Let (M, d) be a metric space. For any $t > 0$, $\frac{1}{t}d$ defines another metric on M. Let $x_0 \in M$ be a basepoint. If the Gromov-Hausdorff limit $\lim_{t\to\infty}(M, \frac{1}{t}d, x_0)$ exists, then it is a metric cone and called the tangent cone at infinity of M, denoted by $T_\infty M$. This limit is clearly independent of the choice of the basepoint x_0.*

5.5. Definition (see [GR1], 1.7). — *A metric space (M, d) is called a length space if the distance between any two points in M is equal to the infimum of the lengths of all curves joining them.*

If (M, g) is a complete Riemannian manifold and d_g is the induced distance function, then (M, d_g) is a length space. If $T_\infty M$ exists, it is also a length space (see [GR1], 3.8).

5.6. Lemma. — *Let M be a topological space with a distance function d defined locally, i.e., when x, y belong to a small neighborhood, $d(x, y)$ is defined. Then there is a canonical length structure ℓ associated with d as in* [GR1], 1.4.

Proof. — It is shown in [GR1], pp. 1–2, that a distance function canonically defines a length structure. Since the dilation is defined locally, the same argument works for a locally defined distance function. □

5.7. Let $\mathbf{P}_0 = \mathbf{G}, \mathbf{P}_1, \ldots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of rational parabolic subgroups of $\mathbf{G}$, and $A_{\mathbf{P}_i,T}$ be the shifted cone as in 4.5.5. Fix a $T \gg 0$ and identify $\coprod_{i=0}^n A_{\mathbf{P}_i,T}$ with the subset $\coprod_{i=0}^n A_{\mathbf{P}_i,T} x_0$

in $\Gamma\backslash X$ as in Proposition 4.6. Then the Riemannian distance function on $\Gamma\backslash X$ induces a distance function on the subspace $\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}$, denoted by d_{ind}. Note that we can not exclude right away the possibility that some points x, y in one $A_{\mathbf{P}_i,T}$ may be connected by a distance minimizing curve not entirely contained in $A_{\mathbf{P}_i,T}$.

5.8. LEMMA. — *If the tangent cone at infinity $T_\infty(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, d_{\mathrm{ind}})$ of the subspace $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, d_{\mathrm{ind}})$ exists, then $T_\infty\Gamma\backslash X$ also exists and is equal to $T_\infty(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, d_{\mathrm{ind}})$.*

Proof. — From the precise reduction reduction theory in Proposition 4.6, it is clear the Hausdorff distance between $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, d_{\mathrm{ind}})$ and $\Gamma\backslash X$ is finite. Then the lemma follows easily. □

5.9. We define another length structure on $\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}$ in order to study this induced distance function d_{ind}.

Identify $A_{\mathbf{P}_i,T}$ with a cone in the Lie algebra $\mathfrak{a}_{\mathbf{P}_i}$ through the exponential map and endow it with the metric defined by the Killing form. Denote this metric by d_S, called the simplicial metric. Then $(A_{\mathbf{P}_i,T}, d_S)$ is a cone over $A^+_{\mathbf{P}_i}(\infty)$, where $A^+_{\mathbf{P}_i}(\infty)$ is the open simplex in the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ associated with $\mathbf{P}_i$ in (3.6.1). In fact, with a suitable simplicial metric on $A^+_{\mathbf{P}_i}(\infty)$, $(A_{\mathbf{P}_i,T}, d_S)$ is a metric cone over $A^+_{\mathbf{P}_i}(\infty)$.[6]

We now glue these metric cones $(A_{\mathbf{P}_i,T}, d_S)$ together to get a local distance function on $\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}$. Since $A_{\mathbf{P}_i,T}$ is a translate of the positive chamber $A^+_{\mathbf{P}_i}$, $(A_{\mathbf{P}_i,T}, d_S)$ is isometric to $(A^+_{\mathbf{P}_i}, d_S)$. Identify $(A_{\mathbf{P}_i,T}, d_S)$ with $(A^+_{\mathbf{P}_i}, d_S)$. Let $(\overline{A^+_{\mathbf{P}_i}}, d_S)$ be the closure of $(A^+_{\mathbf{P}_i}, d_S)$ in $(A_{\mathbf{P}_i}, d_S)$. Any face F of the polyhedral cone $\overline{A^+_{\mathbf{P}_i}}$ is the chamber $\overline{A^+_{\mathbf{P}_F}}$ of a rational parabolic subgroup $\mathbf{P}_F$ containing $\mathbf{P}_i$ (see 3.3 and (4.2.2)). The group $\mathbf{P}_F$ is Γ-conjugate to a unique representative $\mathbf{P}_j$ above. Identify $\overline{A^+_{\mathbf{P}_j}}$ with the face F. Gluing all the spaces $\overline{A^+_{\mathbf{P}_i}}$ together using this face relation gives a topological space $\bigcup_{i=0}^{n} \overline{A^+_{\mathbf{P}_i}}/\sim$. Suppose that $\overline{A^+_{\mathbf{P}_j}}$ is glued onto a face of $\overline{A^+_{\mathbf{P}_i}}$. Since both metrics on $\overline{A^+_{\mathbf{P}_i}}$ and $\overline{A^+_{\mathbf{P}_j}}$ are induced from the Killing form, they coincide on $\overline{A^+_{\mathbf{P}_j}}$. Therefore, all the metric spaces $(\overline{A^+_{\mathbf{P}_i}}, d_S)$ are compatible and can be glued together to give a locally defined distance

(6) The metric on $A^+_{\mathbf{P}_i}(\infty)$ induced from the distance function of $A_{\mathbf{P}_i}$ defined by the Killing form is not a simplicial metric, since, by definition, $A^+_{\mathbf{P}_i}(\infty)$ is a part of the unit sphere in $A_{\mathbf{P}_i}$.

function on $\bigcup_{i=0}^{n}\overline{A_{\mathbf{P}_i}^+}/\sim$. By Lemma 5.6, there is an induced length function on $\bigcup_{i=0}^{n}\overline{A_{\mathbf{P}_i}^+}/\sim$, denoted by ℓ_S.

As a topological space, $\bigcup_{i=0}^{n}\overline{A_{\mathbf{P}_i}^+}/\sim$ is a cone over the Tits complex $\Delta(\Gamma\backslash X)$ in 3.6.3; and as a set, $\bigcup_{i=0}^{n}\overline{A_{\mathbf{P}_i}^+}/\sim$ has a disjoint decomposition $\coprod_{i=0}^{n} A_{\mathbf{P}_i}^+$ that can be identified with $\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}$. Therefore, the length space $(\bigcup_{i=0}^{n}\overline{A_{\mathbf{P}_i}^+}/\sim, \ell_S)$ defines a length structure on $\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}$, denoted by $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, \ell_S)$.

An important property of this length space $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, \ell_S)$ is the following Lemma 5.10. The basic idea of the proof is that since all the minimal rational parabolic subgroups $\mathbf{P}$ (and hence their positive chambers $A_{\mathbf{P}}^+$) are conjugate, there is no shortcut in $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, \ell_S)$ connecting two points in one chamber $\overline{A_{\mathbf{P}}^+}$ by going through other chambers.

5.10. LEMMA. — *For any $i = 0, \dots, n$, $A_{\mathbf{P}_i,T}$ is a convex subspace of the length space $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, \ell_S)$; in other words, for any two points in $A_{\mathbf{P}_i,T}$, any curve that connects x, y and realizes the distance between them is contained in $A_{\mathbf{P}_i,T}$ and is hence a straight line segment contained entirely in $A_{\mathbf{P}_i,T}$. In particular, on each $A_{\mathbf{P}_i,T}$, $\ell_S = d_S$, where d_S is the simplicial distance on $A_{\mathbf{P}_i,T}$ defined by the Killing form.*

Proof. — Suppose $\mathbf{P}_1, \dots, \mathbf{P}_m$ are representatives of minimal rational parabolic subgroups in the list $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_n$. Then $\overline{A_{\mathbf{P}_1,T}}, \dots, \overline{A_{\mathbf{P}_m,T}}$ cover the space $(\coprod_{i=0}^{n} A_{\mathbf{P}_i,T}, \ell_S)$, and hence $\coprod_{i=0}^{n} A_{\mathbf{P}_i,T} = \coprod_{j=0}^{m} \overline{A_{\mathbf{P}_j,T}}$. To prove the lemma, it suffices to prove that that for any $i = 1, \dots, m$ and two points $x, y \in \overline{A_{\mathbf{P}_i,T}}$, any distance minimizing curve connecting x, y is contained in $\overline{A_{\mathbf{P}_i,T}}$. To do this, we notice that any two minimal rational parabolic subgroups are conjugate to each other by an element in $\mathbf{G}(\mathbb{Q})$. This implies that the metric spaces $(\overline{A_{\mathbf{P}_j,T}}, d_S)$, $j = 1, \dots, m$, are isometric to each other. We claim that for every $1 \leq i \leq m$, there is a continuous map $\pi\colon \bigcup_{j=1}^{m}\overline{A_{\mathbf{P}_j,T}} \to \overline{A_{\mathbf{P}_i,T}}$ which restricts to the isometry on each $(\overline{A_{\mathbf{P}_j,T}}, d_S)$. This map is basically given by folding the complex of simplicial cones $\bigcup_{j=1}^{m}\overline{A_{\mathbf{P}_j,T}}$ onto the simplicial cone $\overline{A_{\mathbf{P}_i,T}}$.

To prove the claim, we note that there is a unique isometry from $(\overline{A_{\mathbf{P}_j,T}}, d_S)$ to $(\overline{A_{\mathbf{P}_i,T}}, d_S)$ induced by conjugation of an element in $\mathbf{G}(\mathbb{Q})$. In fact, $\mathbf{P}_j$ is conjugate to $\mathbf{P}_i$, $\gamma\mathbf{P}_j\gamma^{-1} = \mathbf{P}_i$ for some $\gamma \in \mathbf{G}(\mathbb{Q})$. Since A_{P_i} is the unique split component stable under the Cartan involution θ for the basepoint x_0, this proves the existence of the isometry. On the other hand, γ is unique up to left multiplication by an element in $\mathbf{P}_i$, and the uniqueness of the isometry also follows. When the chambers $\overline{A_{\mathbf{P}_j,T}}$ and $\overline{A_{\mathbf{P}_j,T}}$ are contained in a common maximal split torus A, the isometry is given by the action of an element of the Weyl group; in particular, when $\overline{A_{\mathbf{P}_j,T}}$ shares a wall of codimension 1 with $\overline{A_{\mathbf{P}_i,T}}$, the isometry is the folding onto $\overline{A_{\mathbf{P}_i,T}}$ using the reflection associated with the wall. Since $\bigcup_{j=1}^m \overline{A_{\mathbf{P}_j,T}}$ is connected, we can fold any chamber in $\bigcup_{j=1}^m \overline{A_{\mathbf{P}_j,T}}$ onto $\overline{A_{\mathbf{P}_i,T}}$ and get the map π.

Now suppose that there is a distance minimizing curve γ in $(\bigcup_{j=1}^m \overline{A_{\mathbf{P}_j,T}}, \ell_S)$ connecting x, y which is not contained completely in $\overline{A_{\mathbf{P}_i,T}}$. By projecting γ into $\overline{A_{\mathbf{P}_i,T}}$ using the map π, we get a curve $\pi(\gamma)$ in $\overline{A_{\mathbf{P}_i,T}}$ which is not a straight line segment. Since π restricts to an isometry on each $(\overline{A_{\mathbf{P}_j,T}}, d_S)$, $\ell_S(\pi(\gamma)) = \ell_S(\gamma)$. Since γ is distance minimizing and x, y are connected by a line in $\overline{A_{\mathbf{P}_i,T}}$, $\ell_S(\gamma) \leq d_S(x,y)$, and hence $\ell_S(\pi(\gamma)) \leq d_S(x,y)$. But $\pi(\gamma)$ is a not straight line in $\overline{A_{\mathbf{P}_i,T}}$. This is a contradiction. Therefore, γ is contained in $\overline{A_{\mathbf{P}_i,T}}$. This proves the lemma. □

Next we use the precise reduction theory in 4.6 to compare $(\coprod_0^n A_{\mathbf{P}_i,T}, d_{\text{ind}})$, whose metric d_{ind} is induced from the Riemannian distance of $\Gamma\backslash X$, with the length space $(\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S) = (\coprod_0^n \overline{A^+_{\mathbf{P}_i}}, \ell_S)$ defined above. Briefly, the precise reduction theory says that $\coprod_0^n A_{\mathbf{P}_i,T}$ is a skeleton of $\Gamma\backslash X = \coprod_0^n \omega_i A_{\mathbf{P}_i,T} x_0$. The map φ in the following proposition is obtained by shrinking the space to the skeleton. For Riemann surfaces with hyperbolic metric, the map is shown in the following Figure 5.11:

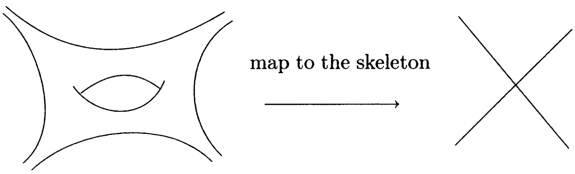

Figure 5.11

5.11. Proposition. — *There is a continuous map $\varphi : \Gamma\backslash X \to (\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S)$ which restricts to the identify map on the subset $\coprod_0^n A_{\mathbf{P}_i,T} = \coprod_0^n A_{\mathbf{P}_i,T} x_0$.*

Proof. — By the precise reduction theory in Proposition 4.6,

$$\Gamma\backslash X = \coprod_0^n \omega_i A_{\mathbf{P}_i,T} x_0,$$

and hence for any $x \in \Gamma\backslash X$, there exist a unique index i and points $z \in \omega_i$, $a \in A_{\mathbf{P}_i,T}$ such that $x = z a x_0$. Define a map $\varphi : \Gamma\backslash X \to (\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S)$

by $\varphi(x) = a$. Clearly, this map φ is well-defined for $T \gg 0$ and depends on the parameter T. We fix the parameter $T \gg 0$ as in 5.7.

It is clear that φ restricts to the identify map on the subset $\coprod_0^n A_{\mathbf{P}_i,T} = \coprod_0^n A_{\mathbf{P}_i,T} x_0$. We need to show that φ is continuous. It is clear that the restriction of φ to each Siegel set $\omega_i A_{\mathbf{P}_i,T} x_0$ is continuous. We need to show that φ is continuous across the boundaries of these Siegel sets. For any $i \in \{0, 1, \ldots, n\}$, suppose x_k is a sequence in $\omega_i A_{\mathbf{P}_i,T} x_0$ converging to $x_\infty \in \Gamma\backslash X$ and $x_\infty \notin \omega_i A_{\mathbf{P}_i,T} x_0$.

Write $x_k = z_k a_k x_0$, where $z_k \in \omega_i$, $a_k \in A_{\mathbf{P}_i,T}$. Since the image of ω_i in $\Gamma \cap \mathbf{P}_i \backslash N_{\mathbf{P}_i} M_{\mathbf{P}_i}$ is compact (4.6.2), the only way for the sequence x_k to leave the set $\omega_i A_{\mathbf{P}_i,T} x_0$ is that the component a_k leaves $A_{\mathbf{P}_i,T}$. Since x_k is bounded, a_k is also bounded. This means that a_k converges to a boundary point of $A_{\mathbf{P}_i,T}$, i.e., for some roots $\alpha \in \Phi^{++}(P_i, A_{\mathbf{P}_i})$,

$$\lim_{k \to \infty} \alpha(\log a_k) = \alpha(I_{\mathbf{P}_i}(T)). \tag{5.11.1}$$

Let I be the subset consisting of all such roots. Then I determines a face $\overline{A^+_{\mathbf{P}_{i,I}}} = \{a \in \overline{A^+_{\mathbf{P}_i}} \mid a^\alpha = 1,\ \alpha \in I\}$ and hence a rational parabolic subgroup $\mathbf{P}_{i,I}$ containing $\mathbf{P}_i$ (see 3.3).

Let $\mathbf{P}_j$ be the unique representative in the list $\mathbf{P}_0, \ldots, \mathbf{P}_n$ (see 5.7) that is Γ-conjugate to $\mathbf{P}_{i,I}$. Then $x_\infty \in \omega_j A_{\mathbf{P}_j,T} x_0$. For simplicity, we can assume that $\mathbf{P}_j = \mathbf{P}_{i,I}$. Since x_k converges to x_∞, the orthogonal projection of a_k on $A_{\mathbf{P}_{i,I}}$ converges to the $A_{\mathbf{P}_{i,I}}$ component of x_∞, which is $\varphi(x_\infty)$. By the gluing procedure of $\coprod_0^n A_{\mathbf{P}_j,T}$ in 5.9 and equation (5.11.1), this implies that $\varphi(x_k)$ converges to $\varphi(x_\infty)$. Therefore φ is continuous. □

5.12. Proposition. — *For any $i = 1, \ldots, n$, the induced metric d_{ind} on $A_{\mathbf{P}_i,T} = A_{\mathbf{P}_i,T} x_0$ is equal to the simplicial metric d_S on $A_{\mathbf{P}_i,T}$.*

Proof. — For any two points $x, y \in A_{\mathbf{P}_i,T} x_0$, let γ be a curve in $\Gamma\backslash X$ connecting them and realizing the distance between them, i.e., the length $|\gamma| = d_{\text{ind}}(x, y)$. Then under the map φ of Proposition 5.11, the image $\varphi(\gamma)$ is a continuous curve in $\coprod_0^n A_{\mathbf{P}_j,T}$ connecting $x = \varphi(x)$ and $y = \varphi(y)$. We claim that

$$d_{\text{ind}}(x, y) \geq |\varphi(\gamma)|,$$

where $|\varphi(\gamma)|$ is the length of the curve $\varphi(\gamma)(t)$, $t \in [0, |\gamma|]$. In fact, let $\gamma : [0, |\gamma|] \to \Gamma\backslash X$ be the unit speed parametrization of γ. Then the claim follows from the following inequality. For each subinterval $[a,b] \subset [0,|\gamma|]$, if $\gamma([a,b]) \subset \omega_i A_{P_i,T} x_0$ for some i, then

$$b - a \geq \left|\varphi(\gamma)_{|[a,\,b]}\right|.$$

Lift both segments $\gamma_{|[a,b]}$ and $\varphi(\gamma)_{|[a,\,b]}$ in $\Gamma_{\mathbf{P}_i}\backslash N_{\mathbf{P}_i} \times X_{\mathbf{P}_i} \times A_{\mathbf{P}_i}$ to $X = N_{\mathbf{P}_i} \times X_{\mathbf{P}_i} \times A_{\mathbf{P}_i}$ with $\tilde{\gamma}(a) = (u_0, m_0 K_{\mathbf{P}_i}, a_0) \in N_{\mathbf{P}_i} \times X_{\mathbf{P}_i} \times A_{\mathbf{P}_i}$ and $\widetilde{\varphi(\gamma)}(a) = (\mathrm{id}, K_{\mathbf{P}_i}, a_0)$. Then $\widetilde{\varphi(\gamma)}_{|[a,\,b]}$ is the projection of $\tilde{\gamma}_{|[a,\,b]}$ into the factor $A_{\mathbf{P}_i}$. By Lemma 10.3.2, this projection is distance decreasing. This implies that $|\widetilde{\varphi(\gamma)}_{|[a,\,b]}| \leq |\tilde{\gamma}_{|[a,\,b]}|$ and hence

$$\left|\varphi(\gamma)_{|[a,\,b]}\right| \leq \left|\gamma_{|[a,\,b]}\right| = b - a.$$

The claim is proved, and hence $d_{\mathrm{ind}}(x,y) \geq |\varphi(\gamma)|$.

Since $\varphi(\gamma)$ is a continuous path in $(\coprod_{i=0}^{n} A_{\mathbf{P},T}, \ell_s)$ connecting x and y, $|\varphi(\gamma)| \geq \ell_S(x,y)$, and hence

$$d_{\mathrm{ind}}(x,y) \geq \ell_S(x,y).$$

According to Lemma 5.10, $\ell_S = d_S$ on each $A_{\mathbf{P}_i,T} x_0$. So

$$d_{\mathrm{ind}}(x,y) \geq d_S(x,y).$$

On the other hand, x and y are connected by a line segment in $A_{\mathbf{P}_i,T} x_0$, and hence

$$d_{\mathrm{ind}}(x,y) \leq d_S(x,y).$$

Therefore, $d_{\mathrm{ind}}(x,y) = d_S(x,y)$. This completes the proof. □

Recall that $\coprod_0^n A_{\mathbf{P}_i,T}$ is identified with the subset $\coprod_0^n A_{\mathbf{P}_i,T} x_0$ in $\Gamma\backslash X$ (see 5.7). Then by the same arguments as above, we get the following.

5.13. Corollary. — *For any two points $x, y \in \coprod_0^n A_{\mathbf{P}_i,T}$,*

$$d_{\mathrm{ind}}(x,y) \geq \ell_S(x,y).$$

5.14. Lemma. — *There exists a finite constant c such that for any $x, y \in \coprod_0^n A_{\mathbf{P}_i,T}$,*

$$d_{\mathrm{ind}}(x,y) \leq \ell_S(x,y) + c.$$

Proof. — Let γ be a curve in $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$ connecting x, y and realizing the distance between them. Since $\ell_S = d_S$ on every $A_{\mathbf{P}_\imath,T}$, γ has at most n linear pieces. Each linear piece in $A_{\mathbf{P}_\imath,T}$ lifts to a line segment in $A_{\mathbf{P}_\imath,T}x_0 \subset \Gamma\backslash X$. So we have a broken curve in $\Gamma\backslash X$. Connecting the ends of this broken curve by distance minimizing curves in $\Gamma\backslash X$, we get a continuous curve $\tilde{\gamma}$ in $\Gamma\backslash X$ connecting x, y. Since ω_i's are bounded and at most $n-1$ curve segments are filled in, there exists a constant c independent of x, y such that $\text{length}(\tilde{\gamma}) \leq \text{length}(\gamma) + c$, and hence $d_{\text{ind}}(x, y) \leq \ell_S(x, y) + c$. □

5.15. PROPOSITION. — *The tangent cone at infinity* $T_\infty(\coprod_0^n A_{\mathbf{P}_\imath,T}, d_{\text{ind}})$ *exists and is equal to* $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$.

Proof. — For any $t > 0$, $(A_{\mathbf{P}_\imath,T}, \frac{1}{t} d_S)$ is isometric to $(A_{\mathbf{P}_\imath,T}, d_S)$. So by Lemma 5.10, $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \frac{1}{t}\ell_S)$ is isometric to $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$; in particular, $T_\infty(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$ exists and is equal to $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$.

By Corollary 5.13 and Lemma 5.14, the Hausdorff distance between $(\coprod_0^n A_{\mathbf{P}_\imath,T}, d_{\text{ind}})$ and $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$ is finite. Therefore, $T_\infty(\coprod_0^n A_{\mathbf{P}_\imath,T}, d_{\text{ind}})$ exists also and is equal to $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$. □

5.16. THEOREM (see 1.4). — *The tangent cone at infinity* $T_\infty(\Gamma\backslash X)$ *exists and is equal to* $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$, *and hence equal to a metric cone over the Tits complex* $\Delta(\Gamma\backslash X)$ *in* (3.6.4).

Proof. — It follows from Lemma 5.8 and the previous proposition that $T_\infty(\Gamma\backslash X)$ exists and is equal to $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$. Since each $(A_{\mathbf{P}_\imath,T}, d_S)$ is a metric cone over $A^+_{\mathbf{P}_\imath,T}(\infty)$, where $A^+_{\mathbf{P}_\imath,T}(\infty)$ is given a suitable simplicial metric (see 5.9), by Lemma 5.10, $(\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$ is a metric cone over the Tits complex $\Delta(\Gamma\backslash X)$. Therefore, $T_\infty(\Gamma\backslash X)$ is a metric cone over $\Delta(\Gamma\backslash X)$. □

6. Define a topology using convergent sequences.

6.1. In this section, we recall a few basic facts concerning how to define a topology using convergent sequences. The reason is that in the following it is easier and more intuitive to describe a topology in terms of convergent sequences than a neighborhood system.

In 6.2, we introduce a closure operator and the induced topology. In 6.4, we define a convergence class of sequences. Finally, in 6.5 and 6.6, we use a convergence class to define a topology.

6.2. A topology on a space can be defined using a closure operator [KU]. More precisely, a closure operator for a space X is a function that assigns to every subset A of X a subset $\overline{A}$ satisfying the following properties:

1) For the empty set $\emptyset$, $\overline{\emptyset} = \emptyset$.

2) For any two subsets $A, B \subset X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

3) For any subset $A \subset X$, $A \subset \overline{A}$.

4) For any subset $A \subset X$, $\overline{\overline{A}} = \overline{A}$.

Once a closure operator is given, then a subset A of X is defined to be closed if and only if $\overline{A} = A$, and a subset B of X is open if and only if its complement is closed. Using the four properties listed above, we can check easily that the open subsets define a topology on X.

6.3. A closure operator on a space can be defined using class of convergent sequences. Let X be a space, and $\mathcal{C}$ be a class of pairs $(\{y_n\}_1^\infty, y_\infty)$ of a sequence $\{y_n\}$ and a point y_∞ in X. If a pair $(\{y_n\}_1^\infty, y_\infty) \in \mathcal{C}$, we say that y_n $\mathcal{C}$-converges to y_∞ and denote it by $y_n \xrightarrow{\mathcal{C}} y_\infty$; otherwise, $y_n \overset{\mathcal{C}}{\not\longrightarrow} y_\infty$.

Motivated by the convergence class of nets in [KE2], Chap. 4, we introduce the following.

6.4. Definition. — *A class $\mathcal{C}$ of pairs $(\{y_n\}_1^\infty, y_\infty)$ is called a convergence class of sequences if the following conditions are satisfied:*

1) *If $\{y_n\}$ is a constant sequence, i.e., there exists a point $y \in X$ such that $y_n = y$ for $n \geq 1$, then $y_n \xrightarrow{\mathcal{C}} y$.*

2) *If $y_n \xrightarrow{\mathcal{C}} y_\infty$, then so does every subsequence of y_n.*

3) *If $y_n \overset{\mathcal{C}}{\not\longrightarrow} y_\infty$, then there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that for any further subsequence $\{y'_{n_i}\}$ of $\{y_{n_i}\}$, $y'_{n_i} \overset{\mathcal{C}}{\not\longrightarrow} y_\infty$.*

4) *Let $\{y_{m,n}\}_{m,n=1}^\infty$ be a double sequence. Suppose that for each fixed m, $y_{m,n} \xrightarrow{\mathcal{C}} y_{m,\infty}$; and the sequence $y_{m,\infty} \xrightarrow{\mathcal{C}} y_{\infty,\infty}$. Then there exists a function $n : \mathbb{N} \to \mathbb{N}$ such that $\lim_{m\to\infty} n(m) = \infty$ and the sequence $y_{m,n(m)} \xrightarrow{\mathcal{C}} y_{\infty,\infty}$.*

6.5. Lemma. — *Suppose $\mathcal{C}$ is a convergence class of sequences in a space X. For any subset A of X, define*

$$\overline{A} = \{y \in X \mid \text{there exists a sequence in } A \text{ such that } y_n \xrightarrow{\mathcal{C}} y\}.$$

Then the operator $A \to \overline{A}$ is a closure operator.

Proof. — The properties 1), 2) and 3) in 6.2 follow directly from the definition. For property 4), we need to show that for any point $y_\infty \in \overline{\overline{A}}$, there exists a sequence y_m in A such that $y_m \xrightarrow{\mathcal{C}} y$. By definition, there exists a sequence $y_{m,\infty}$ in $\overline{A}$ such that $y_{m,\infty} \xrightarrow{\mathcal{C}} y_\infty$. Since $y_{m,\infty} \in \overline{A}$, there exists a sequence $\{y_{m,n}\}_{n=1}^\infty$ in A such that $y_{m,n} \xrightarrow{\mathcal{C}} y_{m,\infty}$. Then (6.4.4) shows that there exists a sequence $y_{m,n(m)}$ in A such that $y_{m,n(m)} \xrightarrow{\mathcal{C}} y_\infty$, and hence $y_\infty \in \overline{A}$. □

6.6. Proposition. — *A convergence class of sequences $\mathcal{C}$ in X defines a unique topology on X such that a sequence $\{y_n\}_1^\infty$ in X converges to a point $y \in X$ with respect to this topology if and only if $(\{y_n\}_1^\infty, y) \in \mathcal{C}$. The topological space X is Hausdorff if and only if every convergent sequence has a unique limit, and X is compact if and only if every sequence in X has a convergent subsequence.*

Proof. — The statement that the convergence class $\mathcal{C}$ defines a unique topology follows from Lemma 6.5 and the discussion in 6.2. For the rest, see [KE2], Chap. 2. □

6.7. *Remark.* — Fréchet [FR] introduced Fréchet L^*-spaces, using class of sequences satisfying only 1), 2) and 3) of 6.4, and hence Fréchet L^*-spaces are not topological spaces in the usual sense. The idea of using convergence class of sequences comes from the Moore-Smith convergence theory of nets (see [KE2], Chap. 2). Since we only deal with metrizable topologies, sequences are sufficient.

7. Borel-Serre compactifications $\overline{\Gamma\backslash X}^{BS}$, $\overline{\Gamma\backslash X}^{RBS}$ of $\Gamma\backslash X$.

7.1. In this section, we recall the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$ [BS] and the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ [ZU1], p. 190, [HZ], 1.3 (b). These compactifications motivate the construction of the Tits compactification $\overline{\Gamma\backslash X}^{T}$ in the next section and are used in the process of classifying DM rays in §10. They also play an important role in parametrizing the continuous spectrum of $\Gamma\backslash X$ in §13.

In 7.2, we give a general procedure of compactifying $\Gamma\backslash X$. In 7.3, we recall $\overline{\Gamma\backslash X}^{BS}$. In 7.5, we define $\overline{\Gamma\backslash X}^{RBS}$. Finally in 7.7, we point out the connection between them.

7.2. Both compactifications $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{RBS}$ are defined using the following procedure:

1) Construct a boundary component for every rational parabolic subgroup of $\mathbf{G}$.

2) Add these boundary components to get a partial compactification of X.

3) Show that the arithmetic subgroup Γ acts continuously on this partial compactification of X with a compact Hausdorff quotient, which is a compactification of $\Gamma\backslash X$.

7.3. The Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$ is defined in [BS] using the geodesic action and the associated corners. We use convergent sequences to describe its topology as in §6. For more detailed discussions of this approach and its equivalence to the approach in [BS], see [BJ].

For any rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, $P = N_{\mathbf{P}} A_{\mathbf{P}} M_{\mathbf{P}}$ is its Langlands decomposition with respect to the basepoint x_0 (3.5.1), and $X_{\mathbf{P}} = M_{\mathbf{P}}/K_{\mathbf{P}}$ is the boundary symmetric space associated with $M_{\mathbf{P}}$ in 3.5.

Define the boundary component $e(\mathbf{P})$ for $\mathbf{P}$ by $e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}$. The Borel-Serre partial compactification $\overline{X}^{BS}$ of X is the set $X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$ with the following topology.

7.3.1. — A unbounded sequence y_n in X is convergent in $\overline{X}^{BS}$ if and only if there exists a rational parabolic subgroup $\mathbf{P}$ such that in terms of the horospherical decomposition, $y_n = (u_n, z_n, \exp(H_n)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ (3.5.2), the components u_n, z_n, H_n satisfy the following conditions:

1) For any $\alpha \in \Phi^{++}(P, A_{\mathbf{P}})$, $\alpha(H_n) \to +\infty$ as $n \to +\infty$.

2) u_n converges to a point $u_\infty \in N_{\mathbf{P}}$, and z_n converges to a point $z_\infty \in X_{\mathbf{P}}$.

Then the limit of y_n in $\overline{X}^{BS}$ is $(u_\infty, z_\infty) \in N_{\mathbf{P}} \times X_{\mathbf{P}} = e(\mathbf{P})$.

7.3.2. — A unbounded sequence $y_n = (u_n, z_n)$ in a boundary component $e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}$ is convergent in $\overline{X}^{BS}$ if and only if the following two conditions hold:

1) u_n converges to some element of $u_\infty \in N_{\mathbf{P}}$.

2) z_n is a unbounded sequence in $X_{\mathbf{P}}$ and converges in the partial compactification $\overline{X_{\mathbf{P}}}^{BS}$ as in 7.3.1 above. Assume that the sequence $z_n \in X_{\mathbf{P}}$ converges to a boundary point $z_\infty = (u'_\infty, z'_\infty) \in e(\mathbf{P}')$, where $\mathbf{P}'$ is some rational parabolic subgroup of $\mathbf{M_P}$. The rational parabolic

subgroup $\mathbf{P}'$ of $\mathbf{M_P}$ corresponds to a unique rational parabolic subgroup $\mathbf{P}''$ of $\mathbf{G}$ contained in $\mathbf{P}$ that satisfies $N_{\mathbf{P}''} = N_{\mathbf{P}} N_{\mathbf{P}'}$, $A_{\mathbf{P}''} = A_{\mathbf{P}} A_{\mathbf{P}'}$ and $M_{\mathbf{P}''} = M_{\mathbf{P}'}$.

Then the limit of y_n in $\overline{X}^{BS}$ is $(u_\infty u'_\infty, z'_\infty) \in N_{\mathbf{P}''} \times X_{\mathbf{P}''} = e(\mathbf{P}'')$.

These are two typical convergent sequences, and general convergent sequences are combinations of them.

7.4. Proposition (see [BS], Prop. 7.6, Thm. 9.3). — *The action of $\mathbf{G}(\mathbb{Q})$ on X extends continuously to $\overline{X}^{BS}$, and every arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ acts properly discontinuously on $\overline{X}^{BS}$. The quotient $\Gamma\backslash\overline{X}^{BS}$ is a Hausdorff compactification of $\Gamma\backslash X$ and denoted by $\overline{\Gamma\backslash X}^{BS}$.*

7.5. The reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ is defined in [ZU1], p. 190, [HZ], 1.3 (b), and plays a crucial role in [GHM], §8. We will explain in §13 that it also plays an important role in parametrizing the continuous spectrum of $\Gamma\backslash X$.

For any rational parabolic subgroup $\mathbf{P}$, its boundary component $\hat{e}(\mathbf{P})$ is defined by $\hat{e}(\mathbf{P}) = X_{\mathbf{P}}$, i.e., the boundary component is obtained from the Borel-Serre boundary component $e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}$ by reducing the nilpotent factor $N_{\mathbf{P}}$ to a point, which is the reason why this compactification is called the reductive Borel-Serre compactification.

The topology of the partial compactification $\overline{X}^{RBS} = X \cup \coprod_{\mathbf{P}} \hat{e}(\mathbf{P})$ is defined as follows:

7.5.1. — A unbounded sequence y_n in X is convergent in $\overline{X}^{RBS}$ if and only if there exists a rational parabolic subgroup $\mathbf{P}$ such that in terms of the horospherical decomposition with respect to $\mathbf{P}$ (3.5.2), $y_n = (u_n, z_n, \exp(H_n)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, the components u_n, z_n, H_n satisfy the following conditions:

1) For any $\alpha \in \Phi^{++}(P, A_{\mathbf{P}})$, $\alpha(H_n) \to +\infty$ as $n \to \infty$.

2) z_n converges to a point $z_\infty \in X_{\mathbf{P}}$.

Then the limit of y_n is $z_\infty \in \hat{e}(\mathbf{P})$.

7.5.2. — A unbounded sequence in a boundary component $\hat{e}(\mathbf{P}) = X_{\mathbf{P}}$ is convergent in $\overline{X}^{RBS}$ if and only if it is convergent in the partial compactification $\overline{X_{\mathbf{P}}}^{RBS}$ as in 7.5.1. Assume the limit y_∞ of y_n in $\overline{X_{\mathbf{P}}}^{RBS}$ belongs to the boundary component $\hat{e}(\mathbf{P}')$ of a rational parabolic subgroup $\mathbf{P}'$ of $\mathbf{M_P}$. Let $\mathbf{P}''$ be the unique rational parabolic subgroup of $\mathbf{G}$ contained in $\mathbf{P}$ corresponding to $\mathbf{P}'$ as in 7.3.2 above. Then the limit of y_n in $\overline{X}^{RBS}$ is $y_\infty \in \hat{e}(\mathbf{P}'') = \hat{e}(\mathbf{P}')$.

These sequences are typical convergent sequences. General convergent sequences are combinations of them. As in 7.4, we have the following [ZU1], Prop. 4.2 (see also [BJ], Prop. 4.4, Thm. 4.6):

7.6. PROPOSITION. — *The action of* $\mathbf{G}(\mathbb{Q})$ *on* X *extends continuously to* $\overline{X}^{RBS}$, *and the quotient* $\Gamma\backslash\overline{X}^{RBS}$ *by an arithmetic subgroup* Γ *is a Hausdorff compactification of* $\Gamma\backslash X$. *This compactification is also denoted by* $\overline{\Gamma\backslash X}^{RBS}$

7.7. From the above descriptions, it is clear that there is a natural surjective continuous map from $\overline{\Gamma\backslash X}^{BS}$ to $\overline{\Gamma\backslash X}^{RBS}$. For each point in the boundary of $\overline{\Gamma\backslash X}^{RBS}$, its inverse image in $\overline{\Gamma\backslash X}^{BS}$ is a nilmanifold. If the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to the $\mathbb{R}$-rank of $\mathbf{G}$, then for any rational parabolic subgroup $\mathbf{P}$, $X_\mathbf{P}$ is a symmetric space of non-compact type, and hence $\overline{\Gamma\backslash X}^{RBS}$ is isomorphic to the maximal Satake compactification $\overline{\Gamma\backslash X}^{S}_{\max}$ in [SA2].

8. Tits compactification $\overline{\Gamma\backslash X}^{T}$.

8.1. In this section, we follow the procedure in 7.2 to define the Tits compactification $\overline{\Gamma\backslash X}^{T}$ (see 2.4, see also 8.12 for a more direct construction). It will be clear from the definition that $\overline{\Gamma\backslash X}^{T}$ is complementary to $\overline{\Gamma\backslash X}^{BS}$, instead of being a quotient of $\overline{\Gamma\backslash X}^{BS}$ (see 8.11). As mentioned in §2, the Tits compactification is the basic compactification in this paper unifying various compactifications of $\Gamma\backslash X$, because of its close relation to the reduction theory.

In 8.2, we define the boundary components of rational parabolic subgroups. In 8.3 and 8.4, we define a topology on the partial compactification $X \cup \Delta_\mathbb{Q}(X)$. In 8.5, we use the conic compactification of X to show that the topology on $X \cup \Delta_\mathbb{Q}(X)$ is Hausdorff. The $\mathbf{G}(\mathbb{Q})$-action on X is extended continuously to $X \cup \Delta_\mathbb{Q}(X)$ in 8.6 and 8.7. Then in 8.8 we prove that the quotient $\Gamma\backslash X \cup \Delta_\mathbb{Q}(X)$ defines the Tits compactification $\overline{\Gamma\backslash X}^{T}$. Finally, we compare it with $\overline{\Gamma\backslash X}^{BS}$ (see 8.11) and point out in 8.12 a more direct construction of $\overline{\Gamma\backslash X}^{T}$ and the advantages of the first approach.

8.2. We first define the boundary components and the boundary of the partial compactification of X. Let $\mathbf{P}$ be any proper rational parabolic subgroup of $\mathbf{G}$. Define its boundary component to be $\overline{A^+_{\mathbf{P}}}(\infty)$ (3.6.1), which is the simplex in the rational Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ corresponding to $\mathbf{P}$. These boundary components $\overline{A^+_{\mathbf{P}}}(\infty)$ are glued together as in (3.6.2) to form the geometric realization $\Delta_{\mathbb{Q}}(X)$ of the rational Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$. We note that the spherical metrics on all the apartments of $\Delta_{\mathbb{Q}}(\mathbf{G})$ are compatible and define a distance function on $\Delta_{\mathbb{Q}}(\mathbf{G})$ (see [TI1], pp. 214–215). The simplicial complex $\Delta_{\mathbb{Q}}(X)$ is the boundary of the partial Tits compactification $X \cup \Delta_{\mathbb{Q}}(X)$ of X, whose topology is defined below.

8.3. We now define the class of convergent sequences of $X \cup \Delta_{\mathbb{Q}}(X)$ and hence the topology.

For any sequence $\{y_n\} \subset X \cup \Delta_{\mathbb{Q}}(X)$, it is defined to be convergent if one of the following alternatives holds:

1) If $\{y_n\}$ belongs to $\Delta_{\mathbb{Q}}(X)$ eventually and converges to a point $y_\infty \in \Delta_{\mathbb{Q}}(X)$ with respect to the simplicial topology, then $\{y_n\}$ is defined to converge to y_∞.

2) If $\{y_n\}$ belongs to X eventually and converges to $y_\infty \in X$ with respect to the topology induced from the invariant Riemannian metric, then $\{y_n\}$ is defined to converge to y_∞.

3) If there exists a rational parabolic subgroup $\mathbf{P}$ so that in the Langlands decomposition with respect to $\mathbf{P}$ (3.5.1),

$$y_n = \ell_n \exp(H_n) x_0,$$

where $\ell_n \in N_{\mathbf{P}} M_{\mathbf{P}}$, $H_n \in \mathfrak{a}_{\mathbf{P}}$, the components ℓ_n and H_n satisfy the following properties:

(a) There exists $H_\infty \in A^+_{\mathbf{P}}(\infty)$ such that as $n \to +\infty$, $H_n/\|H_n\| \to H_\infty$.

(b) Let $d(\,.\,,.)$ be the distance function on X. Then as $n \to +\infty$, $d(\ell_n x_0, x_0)/\|H_n\| \to 0$. Under conditions (a), (b), $\{y_n\}$ is defined to converge to the boundary point $H_\infty \in A^+_P(\infty) \subset \Delta_{\mathbb{Q}}(X)$.

4) If both X and $\Delta_{\mathbb{Q}}(X)$ contain infinitely many terms of $\{y_n\}$, i.e., $\{y_{n_i}\} \subset \Delta_{\mathbb{Q}}(X)$, $\{y_{n'_i}\} \subset X$, $\{y_{n_i}\} \cup \{y_{n'_i}\} = \{y_n\}$, and both sequences $\{y_{n_i}\}$, $\{y_{n'_i}\}$ converge to $y_\infty \in \Delta_{\mathbb{Q}}(X)$ according to 1) and 3) above respectively, then $\{y_n\}$ is defined to converge to y_∞.

8.4. LEMMA. — *The class of convergent sequences defined in* §8.3 *above is a convergence class in the sense of* §6.4 *above and hence defines a topology on* $X \cup \Delta_{\mathbb{Q}}(X)$.

Proof. — We need to check that all the conditions in 6.4 are satisfied. The first three conditions are easily seen to be satisfied. We now show that the condition 4) is also satisfied.

Let $\{y_{m,n}\}_{m,n=1}^{\infty}$ be a double sequence in $X \cup \Delta_{\mathbb{Q}}(X)$ such that for each m, $y_{m,n}$ converges to $y_{m,\infty}$ as $n \to +\infty$, and $y_{m,\infty}$ converges to $y_{\infty,\infty}$ as $m \to +\infty$.

Assume first that $y_{\infty,\infty} \in X$. Then there exists a sequence $\varepsilon_m \to 0$ such that $y_{m,\infty} \in B(y_{\infty,\infty}, \frac{1}{2}\varepsilon_m)$, where $B(y_{\infty,\infty}, \varepsilon)$ is the metric ball of radius ε with center $y_{\infty,\infty}$. For every m, there exists $n(m)$ such that $y_{m,n(m)} \in B(y_{m,\infty}, \frac{1}{2}\varepsilon_m) \subset B(y_{\infty,\infty}, \varepsilon_m)$, and $n(m) \to +\infty$ as $m \to +\infty$. Then the sequence $\{y_{m,n(m)}\}_{m=1}^{\infty}$ converges to $y_{\infty,\infty}$.

Next, we assume that $y_{\infty,\infty} \in \Delta_{\mathbb{Q}}(X)$. Let $\mathbf{P}$ be the unique rational parabolic subgroup such that $y_{\infty,\infty}$ belongs to the open simplex $A_{\mathbf{P}}^{+}(\infty)$, i.e., $y_{\infty,\infty}$ is an interior point of the simplex $\overline{A_{\mathbf{P}}^{+}}(\infty)$.

We assume first that $\mathbf{P}$ is minimal. Then for all $m \gg 0$, either $y_{m,\infty} \in X$ or $y_{m,\infty} \in A_{\mathbf{P}}^{+}(\infty)$. Without loss of generality, we can assume that

1) either for all $m \geq 1$, $y_{m,\infty} \in A_{\mathbf{P}}^{+}(\infty)$;

2) or for all $m \geq 1$, $y_{m,\infty} \in X$.

For case 1), we can assume

(a) either for all $n, m \geq 1$, $y_{m,n} \in A_{\mathbf{P}}^{+}(\infty)$;

(b) or for all $n, m \geq 1$, $y_{m,n} \in X$.

If (a) is true, then it can be shown as above that the condition 4) is satisfied in this case. If (b) is true, then there exists a sequence $\varepsilon_m \to 0$ such that $y_{m,\infty} \in B(y_{\infty,\infty}, \frac{1}{2}\varepsilon_m)$, where $B(y_{\infty,\infty}, \frac{1}{2}\varepsilon_m)$ is the metric ball with respect to the spherical metric on $\Delta_{\mathbb{Q}}(X)$. Since $y_{m,n} \in X$ and $y_{\infty,n} \in A_{\mathbf{P}}^{+}(\infty)$, there exists a sequence $n(m)$ with $n(m) \to +\infty$ as $m \to +\infty$ and an integer m_0 such that

(I) For $m \geq m_0$, $H_{m,n(m)}/\|H_{m,n(m)}\| \in B(y_{m,\infty}, \frac{1}{2}\varepsilon_m) \subset B(y_{\infty,\infty}, \varepsilon_m)$.

(II) For $m \geq m_0$, $d(\ell_{m,n(m)}x_0, x_0)/\|H_{m,n(m)}\| < \varepsilon_m$.

Then the sequence $y_{m,n(m)}$ converges to $y_{\infty,\infty}$ as $m \to +\infty$, and hence condition 4) is also satisfied in this case. The case 2) can be checked similarly.

If $\mathbf{P}$ is not minimal, then either $y_{m,n} \in X$ or $y_{m,n} \in A^+_{\mathbf{P}'}(\infty)$ eventually, where $\mathbf{P}'$ is a rational parabolic subgroup contained in $\mathbf{P}$. Here we have used the fact that $A^+_{\mathbf{P}'}(\infty)$ is a face of $\overline{A^+_{\mathbf{P}}}$ if and only if $\mathbf{P}' \supset \mathbf{P}$. Similarly, we can prove that there exists a subsequence $y_{m,n(m)} \to y_{\infty,\infty}$ and the condition 4) is satisfied.

Therefore, the class defined above forms a convergence class of sequences. By Proposition 6.6, these convergent sequences define a topology on $X \cup \Delta_{\mathbb{Q}}(X)$. □

8.5. Proposition. — *The topology on $X \cup \Delta_{\mathbb{Q}}(X)$ defined in §8.4 is Hausdorff.*

Proof. — We claim that for any unbounded sequence $\{y_n\}$ in X, there is at most one rational parabolic subgroup such that the alternative 3) in 8.3 is satisfied.

To prove this claim, we need to introduce the conic compactification $X \cup X(\infty)$ (see [BGS], §§3–4, and 9.2 below). The boundary $X(\infty)$ is the set of equivalence classes of geodesics in X, where two geodesics γ_1, γ_2 in X are defined to be equivalent if $\lim_{t\to+\infty} \sup d(\gamma_1(t), \gamma_2(t)) < +\infty$. The topology of $X \cup X(\infty)$ is defined as follows: A unbounded sequence y_n in X converges to an equivalence class $[\gamma]$ of geodesics if the geodesic from x_0 to y_n converges to a geodesic in the class $[\gamma]$.

The boundary $X(\infty)$ has a simplicial structure $\Delta(X)$, called the spherical Tits building of X (see [GJT]) and the rational Tits building $\Delta_{\mathbb{Q}}(X)$ is embedded in $\Delta(X)$. Briefly, for each real parabolic subgroup P of G, let A_P be the maximal real split torus in P. Then $A^+_P(\infty)$ can be identified with a subset of $X(\infty)$, and $X(\infty) = \coprod_P A^+_P(\infty)$. For any rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, $P = \mathbf{P}(\mathbb{R})$ is a real parabolic subgroup of G. The maximal real split torus A of P contains the maximal rational split torus $A_{\mathbf{P}}$. Therefore, $A^+_{\mathbf{P}}(\infty) \subset \overline{A^+(\infty)} \subset X(\infty)$.

We can check easily that a sequence satisfying the condition 3) in 8.3 converges in the conic compactification $X \cup X(\infty)$ to the point H_∞ in $A^+_{\mathbf{P}}(\infty) \subset X(\infty)$. Since the compactification $X \cup X(\infty)$ is Hausdorff and $A^+_{\mathbf{P}'}(\infty) \cap A^+_{\mathbf{P}}(\infty) = \emptyset$ for two different rational parabolic subgroups $\mathbf{P}'$ and $\mathbf{P}$, the claim is proved.

Using the claim and the Hausdorff property of X and $\Delta_{\mathbb{Q}}(X)$, we can prove easily that every convergent sequence in $X \cup \Delta_{\mathbb{Q}}(X)$ has a unique limit. Therefore, by Proposition 6.6, the topology of $X \cup \Delta_{\mathbb{Q}}(X)$ is Hausdorff. □

8.6. To show that the $\mathbf{G}(\mathbb{Q})$-action on X extends continuously to $X \cup \Delta_{\mathbb{Q}}(X)$, we first define an action of $\mathbf{G}(\mathbb{Q})$ on the points of $\Delta_{\mathbb{Q}}(X)$. Recall from 3.4 the $\mathbf{G}(\mathbb{Q})$-action on the Tits building $\Delta_{\mathbb{Q}}(X)$. For any $\gamma \in \mathbf{G}(\mathbb{Q})$ and any rational parabolic subgroup $\mathbf{P}$, a maximal parabolic subgroup $\mathbf{Q}$ containing $\mathbf{P}$ is mapped to a maximal parabolic subgroup $\gamma\mathbf{Q}\gamma^{-1}$ containing $\gamma\mathbf{P}\gamma^{-1}$. So the conjugation $\mathbf{P} \mapsto \gamma\mathbf{P}\gamma^{-1}$ induces a bijection from the set of the vertices of the simplex $\overline{A^+_{\mathbf{P}}(\infty)}$ to the set of vertices of the simplex $\overline{A^+_{\gamma\mathbf{P}\gamma^{-1}}(\infty)}$.

Then the $\mathbf{G}(\mathbb{Q})$-action on the underlying topological space of the Tits building $\Delta_{\mathbb{Q}}(X)$ is defined as follows: Any $y \in A^+_{\mathbf{P}}(\infty)$ is mapped to the point in $A^+_{\gamma\mathbf{P}\gamma^{-1}}(\infty)$ with the same barycentric coordinates with respect to the corresponding vertices. Combined with the isometric action on X, this gives a $\mathbf{G}(\mathbb{Q})$-action on $X \cup \Delta_{\mathbb{Q}}(X)$.

8.7. LEMMA. — *The* $\mathbf{G}(\mathbb{Q})$ *action on* $X \cup \Delta_{\mathbb{Q}}(X)$ *defined above is continuous.*

Proof. — First, we express the $N_{\gamma\mathbf{P}\gamma^{-1}}M_{\gamma\mathbf{P}\gamma^{-1}}$ and $A_{\gamma\mathbf{P}\gamma^{-1}}$ components of γx in the decomposition (3.5.1) associated with the parabolic subgroup $\gamma\mathbf{P}\gamma^{-1}$ in terms of the components of x with respect to $\mathbf{P}$. Since $G = KP$, where K is the maximal compact subgroup corresponding to the basepoint x_0, write $\gamma = \gamma_0\gamma_{\mathbf{P}}$, where $\gamma_0 \in K$, $\gamma_{\mathbf{P}} \in P$. Write $\gamma_{\mathbf{P}} = \ell(\gamma)a(\gamma)$, where $\ell(\gamma) \in N_{\mathbf{P}}M_{\mathbf{P}}$, $a(\gamma) \in A_{\mathbf{P}}$. For any $x \in X$, let $x = \ell(x)a(x)x_0$, $\ell(x) \in N_{\mathbf{P}}M_{\mathbf{P}}, a(x) \in A_{\mathbf{P}}$. Then

$$\begin{aligned}\gamma x = \gamma_0\gamma_{\mathbf{P}}x &= \gamma_0\ell(\gamma)a(\gamma)\ell(x)a(x)x_0 \\ &= \gamma_0\ell(\gamma)\big[a(\gamma)\ell(x)a(\gamma)^{-1}\big]a(\gamma)a(x)x_0 \\ &= \gamma_0\big(\ell(\gamma)a(\gamma)\ell(x)a(\gamma)^{-1}\big)\gamma_0^{-1}\gamma_0a(\gamma)a(x)\gamma_0^{-1}x_0.\end{aligned}$$

Since $\gamma_0\mathbf{P}\gamma_0^{-1} = \gamma\mathbf{P}\gamma^{-1}$ and $N_{\mathbf{P}}M_{\mathbf{P}}$ is normalized in $A_{\mathbf{P}}$, it follows that $\gamma_0(\ell(\gamma)a(\gamma)\,\ell(x)a(\gamma)^{-1})\,\gamma_0^{-1} \in N_{\gamma\mathbf{P}\gamma^{-1}}M_{\gamma\mathbf{P}\gamma^{-1}}$ and $\gamma_0a(\gamma)a(x)\gamma_0^{-1} \in A_{\gamma\mathbf{P}\gamma^{-1}}$, and hence

$$\ell_{\gamma\mathbf{P}\gamma^{-1}}(\gamma x) = \gamma_0\big(\ell(\gamma)a(\gamma)\,\ell(x)a(\gamma)^{-1}\big)\gamma_0^{-1}, \quad a_{\gamma\mathbf{P}\gamma^{-1}}(\gamma x) = \gamma_0a(\gamma)a(x)\gamma_0^{-1}.$$

It is then clear that for any $\gamma \in \mathbf{G}(\mathbb{Q})$, if $\{y_n\}$ is a convergent sequence in $X \cup \Delta_{\mathbb{Q}}(X)$ with limit y_∞ in $A^+_{\mathbf{P}}(\infty)$, then the sequence $\{\gamma y_n\}$ converges to the point $\gamma y_\infty = \gamma_0 y_\infty$ in $A^+_{\gamma\mathbf{P}\gamma^{-1}}(\infty) = A^+_{\gamma_0\mathbf{P}\gamma_0^{-1}}(\infty)$. Therefore, $\mathbf{G}(\mathbb{Q})$ acts continuously on $X \cup \Delta_{\mathbb{Q}}(X)$. □

The main result of this section is the following:

8.8. THEOREM (see 1.5). — *The quotient $\Gamma\backslash X \cup \Delta_{\mathbb{Q}}(X)$ is a compact Hausdorff space and hence a compactification of $\Gamma\backslash X$. This compactification is independent of the choice of the basepoint x_0 in 3.5, and its boundary is equal to the Tits simplicial complex $\Delta(\Gamma\backslash X)$ of $\Gamma\backslash X$ defined in 3.4. This compactification is called the Tits compactification and denoted by $\overline{\Gamma\backslash X}^T$.*

A general method to show that the quotient is compact is the following criterion:

8.9. PROPOSITION (see [SA2], Thm. 2.1′). — *Let Γ be a group acting continuously on a Hausdorff space $\overline{X}$. Suppose that there exists a subset $\overline{\Omega}$ of $\overline{X}$ satisfying the conditions:*

1) $\overline{X} = \Gamma\overline{\Omega}$;

2) $\overline{\Omega}$ *is compact*;

3) *there exist finitely many elements γ_i in Γ such that if $\gamma \in \Gamma$ and $\gamma\overline{\Omega} \cap \overline{\Omega} \neq \emptyset$, then $\gamma_{|\overline{\Omega}\cap\gamma^{-1}\overline{\Omega}} = \gamma_{i|\overline{\Omega}\cap\gamma^{-1}\overline{\Omega}}$ for some γ_i.*

Then the quotient $\Gamma\backslash\overline{X}$ is a compact Hausdorff space.

8.10. *Proof of Theorem 8.8.* — We need to construct a subset $\overline{\Omega}$ of $X \cup \Delta_{\mathbb{Q}}(X)$ satisfying the conditions in Proposition 8.9.

Let $\mathbf{P}$ be a minimal rational parabolic subgroup of $\mathbf{G}$, and Ω be the finite union $C\mathcal{S}$ of the Siegel sets in Proposition 4.4, where $\mathcal{S} = \omega A_{\mathbf{P},t}x_0$, and ω is compact. Let $\overline{\Omega}$ be the closure of Ω in the partial compactification $X \cup \Delta_{\mathbb{Q}}(X)$. We claim that $\overline{\Omega}$ satisfies all the conditions in Proposition 8.9.

Let $\bar{\mathcal{S}}$ be the closure of $\mathcal{S}$ in $X \cup \Delta_{\mathbb{Q}}(X)$. Then $\bar{\mathcal{S}} \supset \overline{A^+_{\mathbf{P}}}(\infty)$, because the sequence $\exp(nH), n \geq 1$, converges to $H \in A^+_{\mathbf{P}}(\infty)$. For any element $g \in \mathbf{G}(\mathbb{Q})$, by the proof of Lemma 8.7, there exists a Siegel domain $\mathcal{S}'$ associated with the minimal rational parabolic subgroup $g\mathbf{P}g^{-1}$ such that $g\mathcal{S} \supset \mathcal{S}'$, and hence $g\bar{\mathcal{S}} \supset \overline{A^+}_{g\mathbf{P}g^{-1}}(\infty)$. Since any minimal rational parabolic subgroup is Γ conjugate to one of the groups $g\mathbf{P}g^{-1}$, $g \in C$, it follows that $\Gamma\overline{\Omega} = X \cup \Delta_{\mathbb{Q}}(X)$, and hence condition 1) is satisfied.

To show that $\overline{\Omega}$ is compact, it suffices to show that $\bar{\mathcal{S}}$ is compact.

From 3) of the definition of the topology in 8.3, it follows that $\bar{S} = \omega\overline{A_{\mathbf{P},t}}x_0 \cup \overline{A_{\mathbf{P}}^+}(\infty)$, where $\overline{A_{\mathbf{P},t}}$ is the closure of $A_{\mathbf{P},t}$ in $A_{\mathbf{P}}$. Since ω is compact, $\bar{S}$ is compact.

To check the condition 3), we note that if for some $\gamma \in \Gamma$, $\gamma\overline{\Omega} \cap \overline{\Omega} \neq \emptyset$, then for some $g_1, g_2 \in C$, $\gamma g_1\bar{S} \cap g_2\bar{S} \neq \emptyset$. Thus it suffices to show that there exist finitely many $\gamma_i \in \Gamma$ such that if $\gamma \in \Gamma$ and $\gamma g_1\bar{S} \cap g_2\bar{S} \neq \emptyset$, then $\gamma_{|g_1\bar{S}\cap\gamma^{-1}g_2\bar{S}} = \gamma_{i|g_1\bar{S}\cap\gamma^{-1}g_2\bar{S}}$ for some γ_i. Assume $\gamma g_1\bar{S} \cap g_2\bar{S} \neq \emptyset$. If $\gamma g_1 S \cap g_2 S \neq \emptyset$, then by Proposition 4.4.2, there are only finitely many such γ in Γ. Otherwise, by the previous paragraph, $\gamma g_1\bar{S} \cap g_2\bar{S} = \overline{A^+}_{\mathbf{P}'_I}(\infty)$ for some rational parabolic subgroup $\mathbf{P}'_I$ containing the minimal rational parabolic subgroup $\mathbf{P}' = g_2\mathbf{P}g_2^{-1}$. Since $\gamma g_1 g_2^{-1}\overline{A^+}_{\mathbf{P}'}(\infty) \cap \overline{A^+}_{\mathbf{P}'}(\infty) = (\gamma g_1 g_2^{-1})g_2\bar{S} \cap g_2\bar{S} \cap \Delta_{\mathbb{Q}}(X)$, and $(\gamma g_1 g_2^{-1})g_2\bar{S} \cap g_2\bar{S} = \overline{A^+}_{\mathbf{P}'_I}(\infty)$, it follows that $\gamma g_1 g_2^{-1}\overline{A^+}_{\mathbf{P}'}(\infty) \cap \overline{A^+}_{\mathbf{P}'}(\infty) = \overline{A^+}_{\mathbf{P}'_I}(\infty)$, and hence $\gamma g_1 g_2^{-1}$ leaves $\overline{A^+}_{\mathbf{P}'_I}(\infty)$ invariant, which in turn implies that $\gamma g_1 g_2^{-1} \in \mathbf{P}'_I$. By the definition of the $\mathbf{G}(\mathbb{Q})$-action on $\Delta_{\mathbb{Q}}(X)$ in 8.6, $\gamma g_1 g_2^{-1}$ acts as identity on $A^+_{\mathbf{P}'_I}(\infty)$, and hence γ acts as $g_2 g_1^{-1}$ on $\gamma^{-1}A^+_{\mathbf{P}'_I}(\infty) = g_1\bar{S} \cap \gamma^{-1}g_2\bar{S}$. Therefore the condition 3) is satisfied, and hence $\Gamma\backslash X \cup \Delta_{\mathbb{Q}}(X)$ is compact and Hausdorff.

From the definition, it is clear that $\Gamma\backslash X \cup \Delta_{\mathbb{Q}}(X)$ contains $\Gamma\backslash X$ as a dense open subset and hence is a compactification of $\Gamma\backslash X$.

To show that the compactification $\Gamma\backslash X \cup \Delta_{\mathbb{Q}}(X)$ is independent of the choice of the basepoint x_0, we notice that for any parabolic subgroup $\mathbf{P}$ and the Langlands decomposition with respect to it, choosing a different basepoint is equivalent to conjugating the Langlands decomposition by an element of P (see 3.5). This implies that the convergence of a sequence and its limit point in $X \cup \Delta_{\mathbb{Q}}(X)$ are independent of the basepoint. Therefore, the partial compactification $X \cup \Delta_{\mathbb{Q}}(X)$ and hence the compactification $\Gamma\backslash X \cup \Delta_{\mathbb{Q}}(X)$ are independent of the choice of the basepoint x_0.

The boundary of the compactification is $\Gamma\backslash\Delta_{\mathbb{Q}}(X)$, which is, by definition, the Tits simplicial complex $\Delta(\Gamma\backslash X)$ in (3.6.3).

8.11. *Remark.* — As shown by Zucker in [ZU2], the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$ dominates all Satake compactifications of $\Gamma\backslash X$, in particular, the Baily-Borel compactification for Hermitian locally symmetric spaces. It is natural to ask whether the Tits compactification $\overline{\Gamma\backslash X}^{T}$ is also dominated by the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$. In fact, $\overline{\Gamma\backslash X}^{T}$ is complementary to $\overline{\Gamma\backslash X}^{BS}$, in particular, not dominated by $\overline{\Gamma\backslash X}^{BS}$.

The fact that $\overline{\Gamma\backslash X}^T$ is complementary to $\overline{\Gamma\backslash X}^{BS}$ can be seen from the partial compactifications $\overline{X}^{BS}$ and $X \cup \Delta_{\mathbb{Q}}(X)$. In $\overline{X}^{BS}$, we add at infinity the $N_{\mathbf{P}}M_{\mathbf{P}}$ part of a rational parabolic subgroup $\mathbf{P}$, while in $X \cup \Delta_{\mathbb{Q}}(X)$, we use the spherical section $A_{\mathbf{P}}^{+}(\infty)$ of the cone $A_{\mathbf{P}}^{+}$. Using this description, we can prove easily that the greatest common quotient of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{BS}$ is the one point compactification of $\Gamma\backslash X$ if the $\mathbb{Q}$-rank of $\mathbf{G}$ is greater than 1 and the end compactification if the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 1.

In §13, we show that the pair $\overline{\Gamma\backslash X}^T$ and $\overline{\Gamma\backslash X}^{RBS}$ are needed to parametrize the continuous spectrum of $\Gamma\backslash X$. This gives another evidence that they are complementary to each other.

8.12. *Remarks.* — Using the precise reduction theory in 4.6 instead of the classical reduction theory in 4.4, the compactification $\overline{\Gamma\backslash X}^T$ can also be defined as follows. Let

$$\Gamma\backslash X = \coprod_{i=0}^{n} \omega_i A_{\mathbf{P}_i,T} x_0$$

be the disjoint decomposition in Proposition 4.6. Each cone $A_{\mathbf{P}_i,T}$ can be compactified at infinity by adding $\overline{A_{\mathbf{P}_i}^{+}}(\infty)$, the directions at infinity. This induces a compactification of $\omega_i A_{\mathbf{P}_i,T} x_0$, where the convergence of sequences of points to boundary points does not depend on the component in ω_i. Gluing these compactifications together as in 3.6, we get $\overline{\Gamma\backslash X}^T$. It can be shown that this compactification does not depend on the height parameter T in $\omega_i A_{\mathbf{P}_i,T} x_0$.

But the approach using the rational Tits building $\Delta_{\mathbb{Q}}(X)$ has the following advantages:

1) The boundary $\Gamma\backslash\Delta_{\mathbb{Q}}(X)$ arises as a natural quotient of the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$, and hence construction in this section provides strong support for the philosophy of Tits [TI1], p. 217, concerning compactifications of a Lie group G and its symmetric space X: "The 'most natural' choice for the 'space at infinity' of G or X is 'often' closely related to the spherical Tits building of G".

2) It fits into the general pattern of compactification in 7.2.

3) The relation of $\overline{\Gamma\backslash X}^T$ to $\overline{\Gamma\backslash X}^{BS}$ is easily described in 8.11.

4) It is analogous to the construction of compactifications of symmetric spaces X using the spherical Tits building of X in [GJT].

9. Geodesic compactification $M \cup M(\infty)$ of a Riemannian manifold M.

9.1. In this section we introduce a general method to compactify a noncompact complete Riemannian manifold M in terms of distance minimizing (DM) rays (see 9.17). When M is a Hadamard manifold, i.e., nonpositively curved and simply connected, this compactification is the well-known conic compactification whose boundary is the set of equivalence classes of geodesics (see [BGS], §§3–4, and 9.2 below).

The ideal boundary $M(\infty)$ of the expected compactification is the set of equivalence classes of DM rays or eventually distance minimizing (EDM) geodesics in M, where two rays $\gamma_1(t), \gamma_2(t)$ are defined equivalent if $\lim_{t\to\infty} \sup d(\gamma_1(t), \gamma_2(t)) < +\infty$ as in 1.1. Under suitable conditions on M (see 9.11, 9.16), we can put a topology on $M \cup M(\infty)$ such that it is a Hausdorff compactification of M (see 9.17). Because of its connection with geodesics, this compactification $M \cup M(\infty)$ is called geodesic compactification.

In 9.2, we recall the conic compactification of a Hadamard manifold to motivate the construction of the geodesic compactification. In 9.3 and 9.4, for every compact base subset ω, we define an auxiliary space $\mathcal{R}_\omega$ of pointed rays from ω in order to define a topology on $M \cup M(\infty)$. By the evaluation map, $\mathcal{R}_\omega$ projects to a subspace $M_\omega \cup M_\omega(\infty)$ of $M \cup M(\infty)$ (see 9.10). To get a topology on $M_\omega \cup M_\omega(\infty)$, we introduce Assumption 9.11. In 9.12, we prove that $M_\omega \cup M_\omega(\infty)$ is a compact Hausdorff space under Assumption 9.11. We construct an example in 9.15 such that M_ω is always a proper subset of M for any compact subset ω. Then we introduce Assumption 9.16 to define the geodesic compactification $M \cup M(\infty)$ in 9.17.

9.2. Though we are mainly interested in nonsimply connected manifolds, we recall the conic compactification[(7)] of a Hadamard manifold to motivate the construction in this section.

Let M be a Hadamard manifold, and $M(\infty)$ the set of equivalence classes of rays in M. Note that since M simply connected and nonpositively curved, every ray in M is DM.

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(7) This compactification $M \cup M(\infty)$ of a Hadamard manifold is called the conic compactification in [GJT] because open neighborhoods near the boundary $M(\infty)$ are given by truncated cones and called the conic topology. It seems that it is better to call it the geodesic compactification.

Fix a basepoint $x_0 \in M$. Then the topology on the conic compactification $M \cup M(\infty)$ is defined as follows: A unbounded sequence y_n in M converges to an equivalence class $[\gamma]$ if the ray from x_0 and passing through y_n converges to a ray in the class $[\gamma]$.

To show that the topology does not depend on the choice of the basepoint x_0, we use the following fact: Let x_1 be another basepoint and y be any other point. Denote the ray from x_i to y by γ_i, $i = 0, 1$. Assume that $\gamma_i(t_i) = y$ and $\gamma_i(0) = x_i$. Then for any $0 \leq t \leq \min(t_0, t_1)$,

$$d(\gamma_0(t), \gamma_1(t)) \leq d(x_0, x_1). \tag{9.2.1}$$

This fact can be proved by comparison with the Euclidean space.

For a non-Hadamard manifold, there are several problems with the above definition of topology on $M \cup M(\infty)$. For many points y, there do not exist DM rays passing through x_0 and y (see the example in 9.15). Even if there exist such connecting DM rays, Inequality (9.2.1) does not necessarily hold either.

The basic idea is to replace the basepoint x_0 by a base compact subset and to consider all DM rays issuing from this base compact subset. Then we have to show the independence on the base compact subset.

9.3. We begin our construction of the topology on $M \cup M(\infty)$ by defining an auxiliary space of pointed rays.

Recall from 1.1 that a DM ray is an isometric embedding $\gamma : [0, \infty) \to M$, i.e., for any $t_1, t_2 \geq 0$, $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$.

Let $\omega \subset M$ be a compact subspace. Define

$$\widetilde{\mathcal{R}}_\omega = \{(\gamma, t) \mid \gamma \text{ is a DM ray in } M,\ \gamma(0) \in \omega,\ t \in [0, \infty]\}. \tag{9.3.1}$$

Two pointed rays $(\gamma_1, t_1), (\gamma_2, t_2)$ are defined to be equivalent if and only if

$$t_1 = t_2 = \infty, \quad \text{and} \quad \lim_{t \to \infty} d(\gamma_1(t), \gamma_2(t)) < +\infty. \tag{9.3.2}$$

An equivalence class of pointed rays is denoted by $[\gamma, t]$. The set of equivalence class $[\gamma, t]$ is denoted by $\mathcal{R}_\omega$.

A topology on $\mathcal{R}_\omega$ can be defined as follows. A sequence $[\gamma_n, t_n]$ in $\mathcal{R}_\omega$ is defined to converge to $[\gamma_0, t_0]$ if and only if one of the following alternatives holds:

1) If $t_0 < \infty$, then $t_n \to t_0$ and $\gamma_n(t) \to \gamma_0(t)$ uniformly for t in compact subsets as $n \to \infty$.

2) If $t_0 = \infty$, then for any representative (γ_n, t_n) of $[\gamma_n, t_n]$ and any subsequence $(\gamma_{n'}, t_{n'})$, there exists a further subsequence $(\gamma_{n''}, t_{n''})$ such that $t_{n''} \to \infty$, and $\gamma_{n''}$ converges uniformly for t in compact subsets to a ray equivalent to γ_0.

See Remark 9.5 below for reasons of the restriction to a compact base and the passage to sub-subsequences.

9.4. LEMMA. — *The convergent sequences in* 9.3 *form a convergence class and hence define a topology on* $\mathcal{R}_\omega$.

Proof. — We can see easily that the first three conditions in 6.4 are satisfied. To check the condition 4), we note that a family of rays γ_n converges to γ_0 uniformly for t in compact subsets if and only if $d(\gamma_n, \gamma_0 : [0,1]) \to 0$, where $d(\gamma_n, \gamma_0 : [0,1]) = \sup\{d(\gamma_n(t), \gamma_0(t)) \mid t \in [0,1]\}$.

Now let $[\gamma_{m,n}, t_{m,n}]$ be a double sequence such that $[\gamma_{m,n}, t_{m,n}] \to [\gamma_{m,\infty}, t_{m,\infty}]$ as $n \to \infty$, and $[\gamma_{m,\infty}, t_{m,\infty}] \to [\gamma_{\infty,\infty}, t_{\infty,\infty}]$ as $m \to \infty$. If $t_{\infty,\infty} < \infty$, the condition 4) is clearly satisfied. We assume $t_{\infty,\infty} = \infty$.

Since ω is compact and $\gamma_{m,n}(0) \in \omega$, there exists a subsequence n' such that $\gamma_{m,n'}(0)$, $\mathrm{d}/\mathrm{d}t\, \gamma_{m,n'}(0)$, and $t_{m,n'}$ are convergent. This implies that $\gamma_{m,n'}$ converges to a ray $\gamma'_{m,\infty}$ with $\gamma'_{m,\infty}(0) \in \omega$. Denote the limit $\lim_{n' \to \infty} t_{m,n'}$ by $t'_{m,\infty}$. Then by assumption, $[\gamma'_{m,\infty}, t'_{m,\infty}] = [\gamma_{m,\infty}, t_{m,\infty}]$.

Since $d(\gamma_{m,n'}, \gamma'_{m,\infty} : [0,1]) \to 0$ as $n' \to \infty$, we can choose $n' = n(m)$ such that $d(\gamma_{m,n'}, \gamma_{m,\infty} : [0,1]) < 1/m$; and $t_{m,n'} > t'_{m,\infty} - 1/m$ if $t'_{m,\infty} < \infty$, and $t_{m,n'} > m$ if $t'_{m,\infty} = \infty$. Then we claim that $[\gamma_{m,n(m)}, t_{m,n(m)}] \to [\gamma_{\infty,\infty}]$.

By assumption, $[\gamma'_{m,\infty}, t'_{m,\infty}] = [\gamma_{m,\infty}, t_{m,\infty}] \to [\gamma_{\infty,\infty}, \infty]$. Then for any subsequence m', there exists a subsequence m'' such that $t'_{m'',\infty} \to \infty$, and $\gamma'_{m'',\infty}(t)$ converges uniformly for t in compact subsets to a ray $\gamma''_{\infty,\infty}$ equivalent to $\gamma_{\infty,\infty}$, or equivalently, $d(\gamma'_{m'',\infty}, \gamma''_{\infty,\infty} : [0,1]) \to 0$. By the above choice of $n(m)$, it is clear that $d(\gamma_{m'',n(m'')}, \gamma''_{\infty,\infty} : [0,1]) \to 0$ and $t_{m'',n(m'')} \to \infty$, and hence $[\gamma_{m'',n(m'')}, t_{m'',n(m'')}] \to [\gamma''_{\infty,\infty}, \infty]$ as $m'' \to \infty$. Therefore the condition 4) is satisfied, and the above convergent sequences define a topology on $\mathcal{R}_\omega$. □

9.5. *Remark.* — The complicated condition of passing to a sub-subsequence in 2) above is required by the equivalence relation on $\widetilde{\mathcal{R}}_\omega$ (9.3.2).

For example, take two disjoint equivalent rays γ_1, γ_2. Then $[\gamma_1, n], [\gamma_2, n]$ both converge to the same point $[\gamma_1, \infty] = [\gamma_2, \infty] \in \mathcal{R}_\omega$. But γ_1 is disjoint from γ_2, and hence the sequence of rays γ_n defined by $\gamma_{2n} = \gamma_1$, $\gamma_{2n-1} = \gamma_2$, $n \geq 1$, does not converge to any geodesic uniformly for t in compact subsets as required by the analogue of condition 2) above.

We can see from the above proof of Lemma 9.4 the restriction that the rays start from a fixed base compact is important for the condition 4) in 6.4. In fact, it is necessary. For example, let M be an Euclidean space, and γ_m be disjoint rays parallel to a ray γ with the initial points $\gamma_m(0)$ going to infinity and $t_m = \infty$. Then $[\gamma_m, t_m] = [\gamma, \infty]$. Define $\gamma_{m,n} = \gamma_m$ and $t_{m,n} = n$. Then for every m, $[\gamma_{m,n}, t_{m,n}] \to [\gamma_m, t_m]$, but for any choice of $n = n(m)$, $m \geq 1$, there is no subsequence of $\gamma_{m,m(n)}$ which converges to any ray.

9.6. Lemma. — *The topological space $\mathcal{R}_\omega$ is compact and Hausdorff.*

Proof. — A convergent sequence of type 1) clearly has a unique limit. Let $[\gamma_n, t_n]$ be a convergent sequence of type 2) with limit $[\gamma_0, \infty]$. Let $\gamma_{n'}$ be a subsequence such that $\gamma_{n'}$ converges uniformly over compact subsets to a ray equivalent to γ_0. Assume $[\gamma_0', \infty]$ is another limit of $[\gamma_n, t_n]$. Then $[\gamma_{n'}, t_{n'}] \to [\gamma_0', \infty]$. By definition, there is a further subsequence $\gamma_{n''}$ of $\gamma_{n'}$ such that $\gamma_{n''}$ converges uniformly over compact subsets to a ray equivalent to γ_0', which has to be equivalent to γ_0 since $\lim_{n' \to \infty} \gamma_{n'}$ exists and is equivalent to γ_0. Therefore, $[\gamma_0', \infty] = [\gamma_0, \infty]$, and hence $\mathcal{R}_\omega$ is Hausdorff.

Next we prove the compactness. Let $[\gamma_n, t_n]$ be any sequence in $\mathcal{R}_\omega$. Choose representatives $(\gamma_n, t_n) \in \widetilde{\mathcal{R}}_\omega$. Since $\gamma(0) \in \omega$ and ω is compact, there exists a subsequence $\gamma_{n'}$ such that both $\gamma_{n'}(0)$ and $\mathrm{d}/\mathrm{d}t\gamma_{n'}(0)$ converge as $n' \to \infty$. This implies that $\gamma_{n'}$ converges to a DM ray γ_0 with $\gamma_0(0) \in \omega$. Depending on whether $t_{n'}$ is bounded or not, we can clearly get a further subsequence which is convergent either of type 1) or type 2). This proves that $\mathcal{R}_\omega$ is compact. □

9.7. Lemma. — *If $\omega_1 \subset \omega_2$ are two compact subsets of M, then $\mathcal{R}_{\omega_1}$ is a closed subset of $\mathcal{R}_{\omega_2}$.*

Proof. — Since $\omega_1 \subset \omega_2$, there is clearly an inclusion $\mathcal{R}_{\omega_1} \subset \mathcal{R}_{\omega_2}$. It is clear from the definition that the topology of $\mathcal{R}_{\omega_2}$ restricts to the topology of $\mathcal{R}_{\omega_1}$. Therefore, $\mathcal{R}_{\omega_1}$ is a closed subset of $\mathcal{R}_{\omega_2}$. □

9.8. Recall that $M(\infty)$ is the set of equivalence classes of DM rays in M. Then a map $\pi : \mathcal{R}_\omega \to M \cup M(\infty)$ can be defined as follows:

1) If $t < \infty$, $\pi([\gamma, t]) = \gamma(t) \in M$.

2) If $t = \infty$, $\pi([\gamma, t]) \in M(\infty)$ is the equivalence class containing γ.

This map π may not be surjective (see the example in 9.15). Let $M_\omega = \pi(\mathcal{R}_\omega) \cap M$, then M_ω is the union of DM rays issuing from a point in K.

9.9. Lemma. — *M_ω is a closed subset of M.*

Proof. — Let y be a point in the closure of M_ω in M. Then there exists a sequence of DM rays γ_n with $\gamma_n(0) \in K$, and $t_n \geq 0$ such that $\gamma_n(t_n) \to y$ in M. Since ω is compact and y is at finite distance from ω, by passing to a subsequence if necessary, we can assume that γ_n converges to a ray γ_0 and $t_n \to t_0$. Then clearly $\gamma_0(0) \in \omega$ and $\gamma_0(t_0) = y$. Since γ_n is DM, γ_0 is also DM. Therefore $[\gamma_0, t_0] \in \mathcal{R}_\omega$, and $y = \gamma_0(t_0) \in M_\omega$, and hence M_ω is closed. □

9.10. Let $M_\omega(\infty) = \pi(\mathcal{R}_\omega) \cap M(\infty)$, the set of equivalence classes of rays contained in M_ω. Then π defines a surjective map $\mathcal{R}_\omega \to M_\omega \cup M_\omega(\infty)$. Define a topology on $M_\omega \cup M_\omega(\infty)$ as follows:

1) The topology on M_ω is the induced subset topology from M.

2) Let $[\gamma] \in M_\omega$ be an equivalence class of rays. Then a unbounded sequence y_n in M_ω converges to $[\gamma]$ in $M_\omega \cup M_\omega(\infty)$ if and only if there exist pointed DM rays $(\gamma_n, t_n) \in \widetilde{M}_\omega$ such that $\gamma_n(t_n) = y_n$ and the class $[\gamma_n, t_n] \to [\gamma, \infty]$ in $\mathcal{R}_\omega$.

3) A sequence $[\gamma_n]$ in M_ω converges to a point $[\gamma]$ if and only if $[\gamma_n, \infty] \to [\gamma, \infty]$ in $\mathcal{R}_\omega$.

4) If a sequence y_n in $M_\omega \cup M_\omega(\infty)$ is combination of a sequence $y_{n'}$ in M_ω and a sequence $y_{n''}$ in $M_\omega(\infty)$, then y_n is convergent if and only if both sequences $y_{n'}$ and $y_{n''}$ converge to the same limit in the sense of 2) and 3) above respectively.

To show that these convergent sequences form a convergence class, we need the following

9.11. Assumption. — *Let ω be any compact subset of M and y_n, y_n' be any two sequences in M_ω going to infinity with $d(y_n, y_n')$ bounded.*

Suppose there exist two sequences of pointed rays $(\gamma_n, t_n), (\gamma'_n, t'_n) \in \widetilde{\mathcal{R}_\omega}$ *with* $\gamma_n(t_n) = y_n$ *and* $\gamma'_n(t'_n) = y'_n$. *If* $\gamma_n(t) \to \gamma(t)$, $\gamma'_n(t) \to \gamma'(t)$ *uniformly for* t *in compact subsets as* $n \to \infty$, *then* γ *is equivalent to* γ'.

This assumption says roughly that if two DM rays from ω connect the same point at infinity, then they are equivalent. It is important even for the special case that $y_n = y'_n$, since there could be many DM rays connecting ω and y_n unless M is a Hadamard manifold.

9.12. LEMMA. — *Under Assumption* 9.11, *the convergent sequences in* 9.10 *define a convergence class and hence a topology on* $M_\omega \cup M_\omega(\infty)$.

Proof. — Clearly, the first three conditions of 6.4 are satisfied. We need to check the condition 4). Let $y_{m,n}$ be a double sequence in $M_\omega \cup M_\omega(\infty)$ with $y_{m,n} \to y_{m,\infty}$ and $y_{m,\infty} \to y_{\infty,\infty}$. If $y_{\infty,\infty} \in M_\omega$, then the condition 4) is clearly satisfied. Assume that $y_{\infty,\infty} \in M_\omega(\infty)$.

For simplicity, we assume one of the following two cases holds:

1) For all m, $y_{m,\infty} \in M_\omega(\infty)$.

2) For all m, $y_{m,\infty} \in M_\omega$.

Case 1) is a special case of Lemma 9.4. In Case 2), choose $n = n(m)$ such that $d(y_{m,n(m)}, y_{m,\infty}) \leq 1$. Since $y_{m,n}, y_{m,\infty} \in M_\omega$, there exist pointed rays $(\gamma_{m,n}, t_{m,n})$, $(\gamma_{m,\infty}, t_{m,\infty}) \in \widetilde{\mathcal{R}}_\omega$ such that $\gamma_{m,n}(t_{m,n}) = y_{m,n}$, $\gamma_{m,\infty}(t_{m,\infty}) = y_{m,\infty}$. Let $\gamma_{\infty,\infty}$ be a representative in the class $y_{\infty,\infty}$. We claim that $[\gamma_{m,n(m)}, t_{m,n(m)}] \to [\gamma_{\infty,\infty}, \infty]$, and hence $y_{m,n(m)} \to y_{\infty,\infty}$.

Since $[\gamma_{m,\infty}, t_{m,\infty}] \to [\gamma_{\infty,\infty}, \infty]$, for any subsequence m', there is a further subsequence m'' such that $t_{m'',\infty} \to \infty$, and $\gamma_{m'',\infty} \to \gamma''_{\infty,\infty}$, which is equivalent to $\gamma_{\infty,\infty}$. Since ω is compact, by the same argument as in Lemma 9.6, there is a further subsequence m^* of m'' such that $\gamma_{m^*,n(m^*)}$ converges to a DM ray γ^*_∞. By the choice of $n(m)$, $d(y_{m,n(m)}, y_{m,\infty}) \leq 1$. Then by Assumption 9.11, γ^*_∞ is equivalent to $\gamma''_{\infty,\infty}$ and hence equivalent to $\gamma_{\infty,\infty}$. Therefore, by definition, $[\gamma_{m,n(m)}, t_{m,n(m)}] \to [\gamma_{\infty,\infty}, \infty]$, and the claim is proved. □

9.13. PROPOSITION. — *Under Assumption* 9.11, *the topological space* $M_\omega \cup M_\omega(\infty)$ *is a Hausdorff compactification of* M_ω.

Proof. — Since any ray class $[\gamma]$ can be approximated by points $\gamma(n)$ along the ray γ, it clear that M_ω is dense in $M_\omega \cup M_\omega(\infty)$. The compactness of $M_\omega \cup M_\omega(\infty)$ follows easily from Lemma 9.6.

To prove the Hausdorff property, we need to show that every convergent sequence y_n in $M_\omega \cup M_\omega(\infty)$ has a unique limit. By Lemma 9.6 again, it suffices to consider the case that y_n is a sequence in M_ω going to infinity. The nonuniqueness of the limit of y_n could only come from nonunique choices of pointed rays (γ_n, t_n) with $\gamma(t_n) = y_n$. But by Assumption 9.11, different choices lead to the same ray class in $M_\omega(\infty)$. Therefore, the sequence y_n has a unique limit. □

9.14. Lemma. — *Let $\omega_1 \subset \omega_2$ be two compact subsets of M. Under Assumption 9.11, the inclusion $i : M_{\omega_1} \cup M_{\omega_1}(\infty) \to M_{\omega_2} \cup M_{\omega_2}(\infty)$ is an embedding.*

Proof. — It is clear that i is injective and continuous. To show i is a homeomorphism, it suffices to prove the image $i(M_{\omega_1} \cup M_{\omega_1}(\infty))$ is closed. Let y_∞ be a point in the closure of $i(M_{\omega_1} \cup M_{\omega_1}(\infty))$ in $M_{\omega_2} \cup M_{\omega_2}(\infty)$. If $y_\infty \in M_{\omega_2}$, then $y_\infty \in M_{\omega_1}$ since M_{ω_1} is closed in M by Lemma 9.9. So we assume $y_\infty \in M_{\omega_2}(\infty)$. Let y_n be a sequence in M_{ω_1} that converges to y_∞ with respect to the topology of $M_{\omega_2} \cup M_{\omega_2}(\infty)$. This means that there are pointed rays (γ_n, t_n) in $\widetilde{\mathcal{R}}_{\omega_2}$ with $\gamma_n(t_n) = y_n$ such that $[\gamma_n, t_n]$ is convergent in $\mathcal{R}_{\omega_2}$. On the other hand, $y_n \in M_{\omega_1}$ and is hence connected by pointed rays $(\gamma'_n, t'_n) \in \widetilde{\mathcal{R}}_{\omega_1}$, $\gamma'_n(t'_n) = y_n$. By passing to a subsequence if necessary, we can assume that $[\gamma'_n, t'_n]$ is convergent. By Assumption 9.11, $\lim_{n\to\infty}[\gamma_n, t_n] = \lim_{n\to\infty}[\gamma'_n, t'_n]$. Therefore, $y_\infty \in M_{\omega_1}(\infty)$. □

9.15. *Example.* — If M is a Hadamard manifold, then Assumption 9.11 is clearly satisfied, and for any compact subset ω, in particular a point, $M_\omega = M$, and the compactification $M_\omega \cup M_\omega(\infty)$ defined in 9.13 is the conic compactification recalled in 9.2. In general, M_ω could be a proper subset of M. Such an example can be constructed as follows. Remove small discs whose centers have integral coordinates from the plane $\mathbb{R}^2$ and glue back a very tall hump to every removed disc as in Figure 9.15.1.

Then for any compact subset ω, $M_\omega \neq M$. The reason is that a DM ray does not go through the top of a hump unless it starts from the top since it is quicker to go around it.

To get a compactification of M, we need another assumption.

9.16. Assumption. — *There exists a compact subset $\omega_0 \subset M$ such that $M_{\omega_0} = M$, i.e., any point in M can be reached by a DM ray starting from ω_0.*

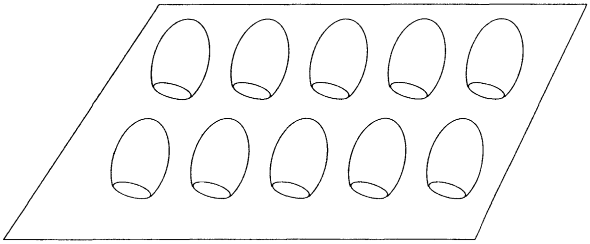

Figure 9.15.1

9.17. THEOREM. — *Under Assumptions* 9.11 *and* 9.16, *for any compact subset* $\omega \supset \omega_0$, $M_\omega \cup M_\omega(\infty)$ *is equal to* $M \cup M(\infty)$ *and defines a compactification of* M. *This compactification does not depend on the choice of the base compact subset* ω *and is called the geodesic compactification.*

Proof. — For any such $\omega \supset \omega_0$, $M_\omega = M$ and $M_\omega(\infty) = M(\infty)$. Then the first statement follows from Proposition 9.13, and the second statement from Lemma 9.14. □

9.18. *Remark.* — Assumptions 9.11, 9.16 should hold for all "geometrically finite" manifolds.(8) The precise meaning of this geometric finiteness is not clear. On the other hand, these conditions seem to be necessary in order to define a compact topology on $M \cup M(\infty)$ in terms of DM rays.

Any compact perturbation of a Hadamard manifold satisfies these assumptions, and locally symmetric spaces of finite volume $\Gamma\backslash X$ also satisfy them (see §11). The latter is the main example we have in mind.

10. DM Rays in $\Gamma\backslash X$.

10.1. In this section, we classify all DM rays or eventually distance minimizing (EDM) geodesics on $\Gamma\backslash X$ (see 10.18) using the Dirichlet domain for Γ (see §10.16). The results of this section play a crucial role in identifying all DM rays in $\Gamma\backslash X$ and hence the geodesic compactification $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$.

(8) The example in §9.15 is not of geometrically finite because of the infinite appearances of the humps.

Recall that a geodesic $\gamma:\mathbb{R}\to\Gamma\backslash X$ is called EDM if there exists a number $t_0 \gg 0$ such that for any $t_1, t_2 \geq t_0$, $d(\gamma(t_1),\gamma(t_2)) = |t_2 - t_1|$, where $d(\,.\,,.)$ is the distance function of $\Gamma\backslash X$. In other words, for $t_0 \gg 0$, the ray $\gamma:[t_0,\infty)\to\Gamma\backslash X$ is DM. For convenience, we will use DM rays and EDM geodesics interchangeably .

In 10.2, we recall the metric behavior of X with respect to the horospherical decomposition (3.5.2). In 10.5, we use the precise reduction theory to construct EDM geodesics in $\Gamma\backslash X$. The rest of the section is to prove that these geodesics in 10.5 exhaust all the EDM ones. Specifically, we prove the crucial separation property of Siegel sets in 10.8 and use it to study the Dirichlet fundamental domain of Γ in 10.16. Using the information on the Dirichlet fundamental domain, we prove in 10.18 that every EDM geodesics in $\Gamma\backslash X$ is one of those constructed in 10.5.

10.2. To list and classify all DM rays or EDM geodesics in $\Gamma\backslash X$, we need to understand the Riemannian metric of X and $\Gamma\backslash X$ near infinity. For any rational parabolic subgroup $\mathbf{Q}$, recall the Langlands decomposition $Q = N_{\mathbf{Q}}A_{\mathbf{Q}}M_{\mathbf{Q}}$ (3.5.1) and the induced horospherical decomposition of X: $X = N_{\mathbf{Q}}\times X_{\mathbf{Q}}\times A_{\mathbf{Q}}$ (3.5.2). Using this, we identify $(u,z,a)\in N_{\mathbf{Q}}\times X_{\mathbf{Q}}\times A_{\mathbf{Q}}$ with the corresponding point $x = uaz$ in X.

10.3. Lemma (see [BO2], Prop. 1.6, Cor. 1.7).

1) *Let dx^2, da^2 and dz^2 be the invariant metrics on X, $A_{\mathbf{Q}}$ and $X_{\mathbf{Q}}$ respectively induced from the Killing form. Then at $(u,z,a)\in X$,*

$$dx^2 = dz^2 + da^2 + \sum_{\alpha\in\Phi^+(Q,A_{\mathbf{Q}})} a^{-2\alpha}\frac{1}{2}h_\alpha(z),$$

where $h_\alpha(z)$ is a metric on the root space $\mathfrak{g}_\alpha$ which depends smoothly on $z\in X_{\mathbf{Q}}$.

2) *For any two points $(u_1,z_1,a_1),(u_2,z_2,a_2)\in X$,*

$$d_X\big((u_1,z_1,a_1),(u_2,z_2,a_2)\big)\geq d_A(a_1,a_2),$$

where $d_X(\cdot,\cdot)$ and $d_A(\cdot,\cdot)$ are the distance functions on X and $A_{\mathbf{Q}}$ respectively.

For any $H\in A^+_{\mathbf{Q}}(\infty)$, $u\in N_{\mathbf{Q}}$ and $z\in X_{\mathbf{Q}}$, define a curve in X: $\tilde\gamma(t) = (u,z,\exp(tH))$, $t\in\mathbb{R}$. From the above description of the metric of X, the following corollary is clear.

10.4. Corollary. — *The curve $\tilde\gamma$ is a geodesic in X.*

10.5. Proposition. — *Let γ be the projection in $\Gamma\backslash X$ of the above geodesic $\tilde{\gamma}$ in X. Then γ is an EDM geodesic in $\Gamma\backslash X$.*

Proof. — Recall from Proposition 4.6 the disjoint decomposition of $\Gamma\backslash X$:

$$\Gamma\backslash X = \coprod_{0}^{n} \omega_i A_{\mathbf{P}_i,T} x_0.$$

Clearly, $\mathbf{Q}$ is Γ-conjugate to a unique $\mathbf{P}_i$. Since we are concerned with the image of $\tilde{\gamma}$ in $\Gamma\backslash X$, we can assume that $\mathbf{Q} = \mathbf{P}_i$. Note that when $T \to +\infty$, the subset ω_i in $A_{\mathbf{P}_i,T}x_0$ increases and converges to $\Gamma_{\mathbf{P}_i}\backslash N_{\mathbf{P}_i} \times X_{\mathbf{P}_i}$. Therefore, when $T \gg 0$, $w \in \omega_i$. Then for $t_0 \gg 0$, the ray $\gamma(t)$, $t \in [t_0, \infty)$, is contained in $wA_{\mathbf{P}_i,T}x_0 \subset \omega_i A_{\mathbf{P}_i,T}x_0$. Then the simplicial metric d_S on $A_{\mathbf{P}_i,T}x_0$ in 5.9 defines a metric d_S on $wA_{\mathbf{P}_i,T}x_0$. The ray $\gamma(t)$, $t \in [t_0, \infty)$, is clearly DM with respect to this metric d_S. By the same proof of Proposition 5.12, we can show that the Riemannian distance of $\Gamma\backslash X$ restricts to d_S on $wA_{\mathbf{P}_i,T}x_0$. Therefore, the ray $\gamma(t)$, $t \in [t_0, \infty)$, is DM, and hence the geodesic γ is EDM in $\Gamma\backslash X$. □

10.6. *Remark.* — Proposition 10.5 is due to Hattori [HA1], [HA2], Thm. A, when $\mathbf{G} = \mathrm{SL}(n)$ and Γ is a congruence subgroup, and due to Leuzinger [LE], Cor. in §4, for general $\mathbf{G}$. Our approach here is intrinsic, dealing with $\Gamma\backslash X$ directly instead of doing computation in the universal covering space X. For another proof of this proposition, see 10.17 below.

10.7. The purpose of this section is to show any EDM geodesic in $\Gamma\backslash X$ is one of the geodesics in Proposition 10.5. For any $t > 0$, define

$$A_{\mathbf{Q},t} = \{a \in A_{\mathbf{Q}} \mid \alpha(\log a) > t,\ \alpha \in \Phi^{++}(Q, A_{\mathbf{Q}})\},$$

which is a special case of (4.2.1) when $t_i = t$, and

$$\overline{A_{\mathbf{Q}}^{+}} = \{a \in A_{\mathbf{Q}} \mid \alpha(\log a) \geq 0,\ \alpha \in \Phi^{++}(Q, A_{\mathbf{Q}})\}.$$

Let $\omega \subset N_{\mathbf{Q}} \times M_{\mathbf{Q}}$ be any compact subset. Then the subset $\omega A_{\mathbf{Q},t}x_0 \subset X$ is a Siegel subset associated with $\mathbf{Q}$ in 4.2.

A technical result of this section is the following separation property of $\omega A_{\mathbf{Q},t}x_0$ under translations by elements in Γ.

10.8. PROPOSITION. — *For any compact subset ω as above, there exists a positive number $t_0 = t_0(\Gamma, \omega)$ such that for any $a_0 \in A_{\mathbf{Q},t_0}$, $a_1 \in \overline{A^+_{\mathbf{Q}}} a_0$, $q_1, q_2 \in \omega$, $\gamma \in \Gamma - \Gamma_Q$, where $\Gamma_Q = \Gamma \cap Q$, the following inequality holds:*

$$d(q_1 a_1 x_0, q_2 a_0 x_0) < d(q_1 a_1 x_0, \gamma q_2 a_0 x_0).$$

10.9. Before proving this theorem, we recall the fundamental representations of $\mathbf{G}$ defined over $\mathbb{Q}$ [BO1], §14, [BT], §12, which will be used to compute the $A_{\mathbf{Q}}$-component of the Langlands decomposition. Let $\mathbf{P}$ be a minimal rational parabolic subgroup of $\mathbf{G}$. For any $\alpha \in \Phi^{++}(\mathbf{G}, \mathbf{P}) \cong \Phi^{++}(P, A_{\mathbf{P}})$ (see 3.2), there is a strongly rational representation $(\pi_\alpha, \mathbf{V}_\alpha)$ of $\mathbf{G}$ whose highest weight λ_α is orthogonal to $\Phi^{++}(\mathbf{P}, A_{\mathbf{P}}) - \{\alpha\}$, and $\langle \lambda_\alpha, \alpha \rangle > 0$. Then the weight space of λ_α is invariant under the maximal parabolic subgroup $\mathbf{P}_{\Phi^{++}-\{\alpha\}}$ [BT], §12.2. Fix an inner product $\|\cdot\|$ on $\mathbf{V}_\alpha(\mathbb{R})$ which is invariant under K and the Weyl group $W(\mathbf{G})$, and with respect to which $A_{\mathbf{P}}$ is represented by self-adjoint operators. Let e_0 be a unit vector in the weight space of λ_α. Let $P_{\Phi^{++}-\{\alpha\}} = N_{\Phi^{++}-\{\alpha\}} M_{\Phi^{++}-\{\alpha\}} A_{\Phi^{++}-\{\alpha\}}$ be the Langlands decomposition of $P_{\Phi^{++}-\{\alpha\}}$ (3.5.1). Then for any $p \in N_{\Phi^{++}-\{\alpha\}} M_{\Phi^{++}-\{\alpha\}}$,

$$\pi_\alpha(p) e_0 = \pm e_0, \tag{10.9.1}$$

and for any $g \in G$,

$$\|\pi_\alpha(g) e_0\| = a(g)^{\lambda_\alpha},$$

where $a(g)$ is the $A_{\Phi^{++}-\{\alpha\}}$ component of g in the Langlands decomposition of $P_{\Phi^{++}-\{\alpha\}}$.

10.10. LEMMA. — *Let $I \subset \Phi^{++}(P, A_{\mathbf{P}})$ be any subset. For $a \in \overline{A^+_{\mathbf{P}_I}}$ and $b \in A_{\mathbf{P}_I}$, if there exist positive constants c_α such that $b^{\lambda_\alpha} \geq a^{\lambda_\alpha} e^{c_\alpha}$ for any $\alpha \in \Phi^{++}(P, A_{\mathbf{P}}) - I$, then*

$$\|\log b\|^2 \geq \|\log a\|^2 + \varepsilon \sum_{\alpha \in \Phi^{++} - I} \alpha(\log a) c_\alpha,$$

where ε is a positive constant independent of a, b, and $\|\cdot\|$ is the norm on $\mathfrak{a}_I$ induced from the Killing form.

Proof. — For any $\alpha \in \Phi^{++}(P, A_{\mathbf{P}})$, let $H_{\lambda_\alpha} \in \mathfrak{a}$ be the unique vector such that $\langle H_{\lambda_\alpha}, H \rangle = \lambda_\alpha(H)$ for all $H \in \mathfrak{a}$, and let $H_\alpha \in \mathfrak{a}$

be the vector dual to the root α, i.e., $\langle H_\alpha, H\rangle = \alpha(H)$ for all $H \in \mathfrak{a}$. Then $\langle H_{\lambda_\alpha}, H_\beta\rangle = \langle \lambda_\alpha, \beta\rangle = d_\alpha \delta_{\alpha\beta}$ for some positive constant d_α. Let $X = \log a$, $Y = \log b$. Then for $\alpha \in \Phi^{++} - I$, $\langle H_{\lambda_\alpha}, Y - X\rangle \geq c_\alpha$. Hence $Y - X = \sum_{\alpha\in\Phi^{++}} b_\alpha H_\alpha$ with $b_\alpha \geq c_\alpha/d_\alpha$ for $\alpha \in \Phi^{++} - I$. Since $X \in \mathfrak{a}_{\mathbf{P}_I}$, we can write $X = \sum_{\alpha\in\Phi^{++}-I} a_\alpha H_{\lambda_\alpha}$. By assumption, $a \in \overline{A^+_{\mathbf{P}_I}}$, $\alpha(X) \geq 0$, and hence $a_\alpha = \alpha(X)/d_\alpha \geq 0$. This implies that

$$\begin{aligned}\langle Y - X, X\rangle &= \sum_{\alpha,\beta\in\Phi^{++}-I} a_\alpha b_\beta \langle H_\alpha, H_{\lambda_\beta}\rangle = \sum_{\alpha\in\Phi^{++}-I} a_\alpha b_\alpha d_\alpha \\ &\geq \sum_{\alpha\in\Phi^{++}-I} \frac{\alpha(X)}{d_\alpha} c_\alpha, \\ \langle Y, X\rangle &= \langle X, X\rangle + \langle Y - X, X\rangle \geq \|X\|^2 + \sum_{\alpha\in\Phi^{++}-I} \frac{1}{d_\alpha}\alpha(X)c_\alpha.\end{aligned}$$

Write $Y = \langle Y, X\rangle X/\|X\|^2 + X^\perp$, $X^\perp \perp X$. Then

$$\begin{aligned}\|Y\|^2 &= \langle Y, X\rangle^2 \frac{1}{\|X\|^2} + \|X^\perp\|^2 \geq \langle Y, X\rangle^2 \frac{1}{\|X\|^2} \\ &\geq \Big(\|X\|^2 + \sum_{\alpha\in\Phi^{++}-I} \frac{1}{d_\alpha}\alpha(X)c_\alpha\Big)^2 \frac{1}{\|X\|^2} \\ &\geq \|X\|^2 + 2\sum_{\alpha\in\Phi^{++}-I} \frac{1}{d_\alpha}\alpha(X)c_\alpha.\end{aligned}$$

Taking $\varepsilon = \min\{2/d_\alpha \mid \alpha \in \Phi^{++} - I\}$, we get

$$\|Y\|^2 \geq \|X\|^2 + \varepsilon \sum_{\alpha\in\Phi^{++}-I} \alpha(X)c_\alpha. \qquad \square$$

10.11. Lemma (see [BO1], Thm. 11.4 and §11.6.4). — *Let $W(\mathbf{G}) = \mathbf{N}(\mathbf{S})/\mathbf{Z}(\mathbf{S})$ be the Weyl group of the maximal $\mathbb{Q}$ split torus $\mathbf{S}$, and $\{w\}$ a set of representatives in $\mathbf{N}(\mathbf{S})(\mathbb{Q})$. Then we have the following Bruhat decomposition :*

$$\mathbf{G}(\mathbb{Q}) = \coprod_{w\in W(\mathbf{G})} \mathbf{N}_P(\mathbb{Q}) w \mathbf{P}(\mathbb{Q}) = \coprod_{w\in W(\mathbf{G})} N'_w w \mathbf{P}(\mathbb{Q}),$$

where $N'_w = (w\mathbf{N}^- w^{-1})(\mathbb{Q}) \cap \mathbf{N}_P(\mathbb{Q})$, $\mathbf{N}^-$ being the unipotent radical of the minimum rational parabolic subgroup $\mathbf{P}^-$ whose Weyl chamber is opposite to the chamber of $\mathbf{P}$.

10.12. *Proof of Proposition 10.8.* — From Lemma 10.3, we get that

$$d^2(q_1a_1x_0, q_2a_0x_0) = d^2(a_1x_0, a_0x_0) + O(1) = \| \log a_0^{-1}a_1\|^2 + O(1),$$

where $O(1)$ only depends on ω. Assume that $\mathbf{Q}$ is a standard rational parabolic subgroup $\mathbf{P}_I$ containing $\mathbf{P}$. Let $g = a_0^{-1}q_2^{-1}\gamma^{-1}q_1a_1$, and $a(g)$ be the A_I component of g in the Langlands decomposition of P_I. Then by Lemma 10.3.2,

$$d(q_1a_1x_0, \gamma q_2a_0x_0) = d(a_0^{-1}q_2^{-1}\gamma^{-1}q_1a_1x_0, x_0) = d(gx_0, x_0) \geq \big\|\log a(g)\big\|.$$

We use the fundamental representations π_α to get a lower bound for $\| \log a(g)\|$. For any $\alpha \in \Phi^{++} - I$, there are two cases to consider:

1) $\gamma \in \mathbf{P}_{\Phi^{++}-\{\alpha\}}$,

2) $\gamma \notin \mathbf{P}_{\Phi^{++}-\{\alpha\}}$.

In case 1), by [BS], Prop. 1.2, $\gamma \in N_{\mathbf{P}_J}M_{\mathbf{P}_J}$, $J = \Phi^{++} - \{\alpha\}$, then $a_1^{-1}(q_2^{-1}\gamma^{-1}q_1)a_1 \in N_{\mathbf{P}_J}M_{\mathbf{P}_J}$, and hence

$$\begin{aligned}(10.12.1)\quad a(g)^{\lambda_\alpha} &= \big\|\pi_\alpha(g)e_0\big\| = \big\|\pi_\alpha(a_0^{-1}a_1a_1^{-1}(q_2^{-1}\gamma^{-1}q_1)a_1)e_0\big\| \\ &= \big\|\pi_\alpha(a_0^{-1}a_1)e_0\big\| = (a_0^{-1}a_1)^{\lambda_\alpha}.\end{aligned}$$

In case 2), using Lemma 10.11, write $\gamma^{-1} = uwtmv$, where $u \in N'_w, t \in A_{\mathbf{P}}, m \in M_{\mathbf{P}}, v \in N_{\mathbf{P}}$. Then we get

$$\begin{aligned} w^{-1}g &= w^{-1}a_0^{-1}q_2^{-1}\gamma^{-1}q_1a_1 \\ &= (w^{-1}a_0^{-1}q_2^{-1}a_0w)w^{-1}a_0^{-1}uwtmvq_1a_1 \\ &= (w^{-1}a_0^{-1}q_2^{-1}a_0w)(w^{-1}a_0^{-1}ua_0w)(w^{-1}a_0^{-1}w)ta_1a_1^{-1}(mvq_1)a_1.\end{aligned}$$

Since $a_0^{-1}q_2^{-1}a_0$ and hence $w^{-1}a_0^{-1}q_2^{-1}a_0w$ belong to compact subsets, $w^{-1}a_0^{-1}ua_0w \in \mathbf{U}^-(\mathbb{Q})$, and $a_1^{-1}(mvq_1)a_1 \in N_IM_I$, it follows that there exists a positive constant $\delta_0 = \delta_0(\omega, \pi_\alpha)$ such that

$$\big\|\pi_\alpha(w^{-1}a_0^{-1}q_2^{-1}a_0w)v\big\| \geq \delta_0\|v\|, \quad \big\|\pi_\alpha(w^{-1}a_0^{-1}ua_0w)e_0\big\| \geq 1,$$

$$\begin{aligned} a(g)^{\lambda_\alpha} &= \big\|\pi_\alpha(g)e_0\big\| = \big\|\pi_\alpha(w^{-1}g)e_0\big\| \\ &= \big\|\pi_\alpha((w^{-1}a_0^{-1}q_2^{-1}a_0w)(w^{-1}a_0^{-1}ua_0w)(w^{-1}a_0w)ta_1)e_0\big\| \\ &\geq \delta_0\big\|\pi_\alpha((w^{-1}a_0^{-1}ua_0w)(w^{-1}a_0w)ta_1)e_0\big\| \\ &= \delta_0\big\|\pi_\alpha(w^{-1}a_0^{-1}ua_0w)e_0\big\|(w^{-1}a_0^{-1}w)^{\lambda_\alpha}t^{\lambda_\alpha}a_1^{\lambda_\alpha} \\ &\geq \delta_0(w^{-1}a_0^{-1}w)^{\lambda_\alpha}a_1^{\lambda_\alpha}t^{\lambda_\alpha},\end{aligned}$$

where in the third equality, we have used the fact that $\pi_\alpha(p)e_0 = \pm e_0$ for $p \in N_I M_I$ (Eq. 10.9.1).

By the proof of [BO1], Cor. 15.3, Eq. 3, there exists a positive number $\delta_1 = \delta_1(\Gamma, \pi_\alpha)$ such that for all $\gamma \in \Gamma$ and any $\alpha \in \Phi^{++}$, the component t of γ^{-1} satisfies that $t^{\lambda_\alpha} \geq \delta_1$. So we get

$$a(g)^{\lambda_\alpha} \geq (a_0^{-1})^{w\circ\lambda_\alpha} a_1^{\lambda_\alpha} \delta_0 \delta_1 = (a_0^{-1})^{w\circ\lambda_\alpha} a_1^{\lambda_\alpha} \delta,$$

where $\delta = \delta_0 \delta_1$.

By assumption, $\gamma \notin \mathbf{P}_{\Phi^{++}-\{\alpha\}}$. If $w \circ \lambda_\alpha = \lambda_\alpha$, then w would fix the Weyl chamber face associated with the subset $\Phi^{++} - \{\alpha\}$ in 3.2, and $w \in \mathbf{P}_{\Phi^{++}-\{\alpha\}}$, and hence $\gamma \in \mathbf{P}_{\Phi^{++}-\{\alpha\}}$. This is impossible, and thus

$$w \circ \lambda_\alpha = \lambda_\alpha - \sum_{\beta \in \Phi^{++}} c_\beta \beta,$$

where c_β are non-negative integers with at least one being non-zero. By [BT], Prop. 12.16, $c_\alpha \geq 1$. Then

$$(a_0^{-1})^{\omega\circ\lambda_\alpha} = (a_0^{-1})^{\lambda_\alpha} \prod_{\beta\in\Phi^{++}} a_0^{c_\beta \beta} \geq (a_0^{-1})^{\lambda_\alpha} a_0^{\alpha} \geq (a_0^{-1})^{\lambda_\alpha} t_0,$$

by the assumption that $a_0 \in A_{\mathbf{Q},t_0}$, and hence

$$a(g)^{\lambda_\alpha} \geq (a_0^{-1})^{\lambda_\alpha} e^{t_0} a_1^{\lambda_\alpha} \delta = (a_0^{-1} a_1)^{\lambda_\alpha} t_0 \delta. \tag{10.12.2}$$

This is the estimate we need for the case 2).

To finish the proof, we apply Lemma 10.10. Since $\gamma \notin \mathbf{P}_I$, there exists at least one $\alpha \in \Phi^{++} - I$ such that $\gamma \notin \mathbf{P}_{\Phi^{++}-\{\alpha\}}$. By assumption, $a_0^{-1} a_1 \in \overline{A_{\mathbf{P}_I}^+}$. Apply Lemma 10.10 to $a = a_0^{-1} a_1$ and $b = a(g)$. Then (10.12.1) and (10.12.2) show that when $t_0 \gg 0$,

$$\left\|\log a(g)\right\|^2 \geq \|\log a_0^{-1} a_1\|^2 + \varepsilon \log(t_0 \delta) > d^2(q_1 a_1 x_0, q_2 a_0 x_0).$$

This completes the proof of Proposition 10.8. □

10.13. We now recall some properties of Dirichlet fundamental domains. For any subgroup $\Gamma \subset \mathrm{Isom}(X) = G$ which acts properly discontinuously and freely on X and any $x_0 \in X$, the domain

$$D = \left\{x \in X \mid d(x, x_0) \leq d(x, \gamma x_0),\ \gamma \in \Gamma\right\}$$

is called the Dirichlet domain for Γ with center x_0. Except for spaces of constant curvature, D is not bounded by totally geodesic hyperplanes, and its shape is complicated. To study global metric property of $\Gamma\backslash X$, it is important to understand D.

10.14. LEMMA (see [SI], §21). — *The Dirichlet domain D is a fundamental domain for the Γ-action and is star shaped, i.e., for any $x \in D$, the geodesic from x_0 to x belongs to D, and no interior point of D is identified with other point of D under Γ.*

10.15. LEMMA. — *Let $x'_0 \in \Gamma\backslash X$ be the projection of $x_0 \in X$. A geodesic ray $\gamma(t)$, $t \geq 0$, in $\Gamma\backslash X$ with $\gamma(0) = x'_0$ is DM if and only if its lift $\tilde{\gamma}(t)$ to X with $\tilde{\gamma}(0) = x_0$ belongs to the Dirichlet domain D with center x_0.*

Proof. — For any $\tilde{x}, \tilde{y} \in X$, denote their projections on $\Gamma\backslash X$ by x, y. Then $d_{\Gamma\backslash X}(x, y) = \inf_{\gamma\in\Gamma} d_X(\tilde{x}, \gamma\tilde{y})$, and the lemma follows from the definition of D. □

10.16. PROPOSITION. — *Assume that Γ is a neat arithmetic subgroup. For any $(u_0, z_0) \in N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ and any sequence $y_n = (u_n, z_n, a_n) \in N_{\mathbf{Q}} \times X_{\mathbf{Q}} \times A_{\mathbf{Q}}$ with $u_n \to u_0$, $u_n \to z_0$, and $\alpha(\log a_n) \to +\infty$ for all $\alpha \in \Phi^+(Q, A_{\mathbf{Q}})$, there exist a compact neighborhood ω of (u_0, z_0) and $n_0 \geq 1$ such that when $n \geq n_0$, the Dirichlet domain D_n for Γ with center (u_n, z_n, a_n) contains $\omega \times \{a_j\}$ when $j \gg n$.*

Proof. — Fix any compact neighborhood ω' of (u_0, z_0). For any n, when $j \gg n$, $a_j \in \overline{A^+_{\mathbf{Q}}} a_n$. By Proposition 10.8, there exists $n_1 \geq 1$ such that for any $n \geq n_1$, $j \gg n$, $q \in \omega'$, $q_n = (u_n, z_n)$, and $\gamma \in \Gamma - \Gamma_Q$,

$$d\big((q, a_j), (q_n, a_n)\big) < d\big((q, a_j), \gamma(q_n, a_n)\big).$$

We next adjust this compact neighborhood ω' of (u_0, z_0) such that the above inequality also holds for $\gamma \in \Gamma_Q$, $\gamma \neq id$. Consider the Dirichlet domain $D_{Q,n}$ for the Γ_Q action on X with center (u_n, z_n, a_n). For any $j \in \mathbb{N}$, let $D_{Q,n,j} = D_{Q,n} \cap (N_{\mathbf{Q}} \times X_{\mathbf{Q}} \times \{a_j\})$. Then $D_{Q,n,j}$ is a cross section of $D_{Q,n}$ at the height a_j. Consider $D_{Q,n,j}$ as a subset of $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$. We claim that there exist a neighborhood D_∞ of (u_0, z_0) and $n_0 \geq n_1$ such that for each $n \geq n_0$, when $j \gg n$, $D_{Q,n,j} \supset D_\infty$.

Assume that this claim is true first. Choose a compact neighborhood ω of (u_0, z_0) contained in both ω' and D_∞. Then when $n \geq n_0$ and $j \gg n$, $D_{Q,n,j} \supset \omega$, and hence for any $\gamma \in \Gamma$, $\gamma \neq$ id, and $q \in \omega$,

$$d\big((q, a_j), (q_n, a_n)\big) < d\big((q, a_j), \gamma(q_n, a_n)\big),$$

and hence $\omega \times \{a_j\} \subset D_n$.

We now prove this claim. By assumption, Γ is neat, and hence Γ_Q is torsion free (see [BO1], §17). By [BS], Prop. 1.2, Γ_Q is contained in $N_{\mathbf{Q}}M_{\mathbf{Q}}$, leaves the $A_{\mathbf{Q}}$ component fixed and hence acts discretely and freely $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$. The subgroup $\Gamma_N = \Gamma_Q \cap N_{\mathbf{Q}}$ acts freely and discretely on $N_{\mathbf{Q}}$, and $\Gamma_X = \Gamma_N \backslash \Gamma_Q$ acts freely and discretely on $X_{\mathbf{Q}}$.

For any $\gamma \in \Gamma_Q$, $\gamma \neq \mathrm{id}$, let

$$D_{n,\gamma} = \big\{(u,z,a) \in X \mid d((u,z,a),(u_n,z_n,a_n)) \\ \leq d((u,z,a),\gamma(u_n,z_n,a_n))\big\},$$

and

$$D_{n,\gamma,j} = D_\gamma \cap \big(N_{\mathbf{Q}} \times X_{\mathbf{Q}} \times \{a_j\}\big),$$

a cross section of $D_{n,\gamma}$ at the height a_j. Consider $D_{n,\gamma,j}$ as a subset of $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$. We need to understand the asymptotic behavior of $D_{n,\gamma,j}$ as $j \to +\infty$. The idea is to use the different shrinking rates of various parts of $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ as $a_j \to \infty$.

Write $\gamma(u_n,z_n,a_n) = (\gamma u_n, \gamma z_n, a_n)$. If $\gamma z_n \neq z_n$, let

$$D_{n,\gamma,X_{\mathbf{Q}}} = \big\{z \in X_{\mathbf{Q}} \mid d(z,z_n) \leq d(z,\gamma z_n)\big\}.$$

From the expression for the metric dx^2 in 10.3,

$$d^2((u,z,a_j),(u_n,z_n,a_n)) = d^2(a_j,a_n) + d^2(z,z_n) + o(1)$$

as $j \to +\infty$, and hence

$$D_{n,\gamma,j} \longrightarrow N_{\mathbf{Q}} \times D_{n,\gamma,X_{\mathbf{Q}}}$$

uniformly over compact subsets as $j \to +\infty$. Since $z_n \to z_0$ as $n \to \infty$, and Γ_X is torsion free, then $\gamma z_0 \neq z_0$ if and only if $\gamma z_n \neq z_n$ for $n \gg 1$. Let

$$D_{0,\gamma,X_{\mathbf{Q}}} = \big\{z \in X_{\mathbf{Q}} \mid d(z,z_0) \leq d(z,\gamma z_0)\big\}.$$

Then

$$\lim_{n\to\infty} D_{n,\gamma,X} = D_{0,\gamma,X_{\mathbf{Q}}}$$

and hence

$$\lim_{n\to\infty}\lim_{j\to\infty} D_{n,\gamma,j} = N_{\mathbf{Q}} \times D_{0,\gamma,X_{\mathbf{Q}}}$$

uniformly over compact subsets.

On the other hand, if $\gamma z_n = z_n$, then $\gamma \in \Gamma_N$ and $\gamma u_n \neq u_n$. Since $\gamma z_n = z_n$,

$$d\big((u,z,a_j),(u_n,z_n,a_n)\big) < d\big((u,z,a_j),(\gamma u_n,\gamma z_n,a_n)\big)$$

if and only if

$$d_{z_n}\big((u,a_j),(u_n,a_n)\big) < d_{z_n}\big((u,a_j),(\gamma u_n,a_n)\big),$$

where $d_{z_n}(\,\cdot\,,\cdot)$ is the distance function on $N_{\mathbf{Q}}\times\{z_n\}\times A_{\mathbf{Q}} \subset N_{\mathbf{Q}}\times X_{\mathbf{Q}}\times A_{\mathbf{Q}}$. For simplicity, denote $d_{z_n}(\,\cdot\,,\cdot)$ by $d(\,\cdot\,,\cdot)$. It thus suffices to compare $d((u,a_j),(u_n,a_n))$ and $d((u,a_j),(\gamma u_n,a_n))$ for $\gamma\in\Gamma_N$, and to determine the asymptotics of

$$D_{n,\gamma,NA,j} = D_{n,\gamma,NA}\cap\big(N_{\mathbf{Q}}\times\{a_j\}\big),$$

a cross section of the set

$$D_{n,\gamma,NA} = \big\{(u,a)\in N_{\mathbf{Q}}\times A_{\mathbf{Q}} \mid d((u,a),(u_n,a_n)) \le d((u,a),(\gamma u_n,a_n))\big\}.$$

For simplicity, we assume that $\dim A_{\mathbf{Q}} = 1$; otherwise, we need to compare rates of $\alpha(\log a_j)$ going to infinity for various $\alpha\in\Phi^+(Q,A_{\mathbf{Q}})$ and decompose the root spaces correspondingly. If there is only one element in $\Phi^+(Q,A_{\mathbf{Q}})$, then

$$D_{n,\gamma,NA,j} = \big\{u\in N_{\mathbf{Q}} \mid d(u,u_n)\le d(u,\gamma u_n)\big\},$$

and hence

$$\begin{aligned} D_{n,\gamma,j} &= D_{n,\gamma,NA,j}\times X_{\mathbf{Q}} \\ &= \big\{u\in N_{\mathbf{Q}} \mid d(u,u_n)\le d(u,\gamma u_n)\big\}\times X_{\mathbf{Q}}, \end{aligned}$$

and

$$\lim_{n\to\infty}\lim_{j\to\infty} D_{n,\gamma,j} = \big\{u\in N_{\mathbf{Q}} \mid d(u,u_0)\le d(u,\gamma u_0)\big\}\times X_{\mathbf{Q}}$$

uniformly over compact subsets. If $\Phi^+(Q,A_{\mathbf{Q}})$ contains more than one elements, say two elements α_1 and α_2, $\alpha_1<\alpha_2$, for simplicity. Let $\mathfrak{n}_{\mathbf{Q}} = \mathfrak{n}_1+\mathfrak{n}_2$ be the decomposition of the Lie algebra of $N_{\mathbf{Q}}$ according to the roots α_1,α_2. Let N_1 and N_2 be the images in $N_{\mathbf{Q}}$ of $\mathfrak{n}_1$ and $\mathfrak{n}_2$ under the exponential map from $\mathfrak{n}_{\mathbf{Q}}$ to $N_{\mathbf{Q}}$. Then $N_{\mathbf{Q}}$ is a bundle over N_2 with fiber N_1. Trivializing the bundle, we write $N_{\mathbf{Q}} = N_1\times N_2$. Correspondingly, write $u_n = (u_{n,1},u_{n,2})$, $\gamma u_n = (\gamma u_{n,1},\gamma u_{n,2})$.

If $\gamma u_{n,1}\ne u_{n,1}$, let

$$D_{n,\gamma,N_1} = \big\{u\in N_1 \mid d(u,u_{n,1})\le d(u,\gamma u_{n,1})\big\}.$$

Then by the metric expression in 10.3, as $j \to \infty$,

$$D_{n,\gamma,NA,j} \longrightarrow D_{n,\gamma,N_1} \times N_2,$$

and hence

$$D_{n,\gamma,j} \longrightarrow D_{n,\gamma,N_1} \times N_2 \times X_{\mathbf{Q}}$$

uniformly over compact subsets.

If $\gamma u_{n,1} = u_{n,1}$, then $\gamma u_{n,2} \neq u_{n,2}$. Let

$$D_{n,\gamma,N_2} = \{u \in N_2 \mid d(u, u_{n,2}) \leq d(u, \gamma u_{n,2})\}.$$

Similarly, as $j \to \infty$,

$$D_{n,\gamma,NA,j} \longrightarrow N_1 \times D_{n,\gamma,N_2}$$

and

$$D_{n,\gamma,j} \longrightarrow N_1 \times D_{n,\gamma,N_2} \times X_{\mathbf{Q}}$$

uniformly over compact subsets.

Write $u_0 = (u_{0,1}, u_{0,2}) \in N_1 \times N_2$. Then $u_{n,1} \to u_{0,1}$ and $u_{n,2} \to u_{0,2}$ as $n \to \infty$. Let $D_{0,\gamma,N_i} = \{u \in N_i \mid d(u, u_{n,i}) \leq d(u, \gamma u_{n,i})\}$, $i = 1, 2$. Then the above discussions show that either

$$\lim_{n\to\infty} \lim_{j\to\infty} D_{n,\gamma,j} = D_{0,\gamma,N_1} \times N_2 \times X_{\mathbf{Q}}$$

or

$$\lim_{n\to\infty} \lim_{j\to\infty} D_{n,\gamma,j} = N_1 \times D_{0,\gamma,N_2} \times X_{\mathbf{Q}}$$

uniformly over compact subsets.

In summary, for any $\gamma \in \Gamma_Q$, $\gamma \neq id$, the double limit

$$\lim_{n\to\infty} \lim_{j\to+\infty} D_{n,\gamma,j}$$

exists and is equal to a hypersurface $D_{0,\gamma}$ in $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ which does not contain (u_0, z_0), i.e., $d_0((u_0, z_0), D_{0,\gamma}) > 0$, where the distance function $d_0(\,\cdot\,,\cdot)$ on $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ is the induced distance by identifying $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ with $N_{\mathbf{Q}} \times X_{\mathbf{Q}} \times \{\mathrm{id}\} \subset X$; and for any positive number c, there are only finitely many such hypersurfaces $D_{0,\gamma}$ such that $d_0((u_0, z_0), D_{0,\gamma}) \leq c$. Then the claim follows easily. The proof of Proposition 10.16 is now complete. □

As a corollary of Proposition 10.16, we get another proof of Proposition 10.5 that is independent of the precise reduction theory in Proposition 4.6 and the results in §5.

10.17. Corollary. — *Assume that Γ is a neat arithmetic subgroup. Then for any $(u,z) \in N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ and $H \in A^+_{\mathbf{Q}}(\infty)$, the projection of the geodesic $\tilde{\gamma}(t) = (u, z, \exp(tH)) \in X$ in $\Gamma\backslash X$ is EDM.*

Proof. — Consider the curve $\tilde{\gamma}(t) = (u, z, \exp(tH))$ instead of a sequence $y_n = (u_n, z_n, a_n)$ in Proposition 10.16, the same proof shows that for $t_0 \gg 0$, the Dirichlet domain for Γ with center $(u, z, \exp(t_0 H))$ contains $\tilde{\gamma}(t)$, $t \gg t_0$. By Lemma 10.14, the Dirichlet domain is star shaped and hence contains the geodesic ray $\tilde{\gamma}(t)$, $t \geq t_0$. Then the projection of this ray in $\Gamma\backslash X$ is DM by Lemma 10.15. Therefore the geodesic $\gamma(t)$, $t \in \mathbb{R}$, is EDM. □

Another application of Proposition 10.16 is the following main result of this section.

10.18. Theorem. — *Any EDM geodesic $\gamma(t)$ on $\Gamma\backslash X$ is the projection of a geodesic in X of the form $\tilde{\gamma}(t) = (u, z, a\exp(tH)) = N_{\mathbf{Q}} \times X_{\mathbf{Q}} \times A_{\mathbf{Q}}$, where $u \in N_{\mathbf{Q}}$, $z \in X_{\mathbf{Q}}$, $a \in A_{\mathbf{Q}}$, and $H \in A^+_{\mathbf{Q}}(\infty)$ for some rational parabolic subgroup $\mathbf{Q}$.*

Proof. — We assume first that Γ is neat. Let $\gamma(t)$, $t \geq 0$, be a DM ray in $\Gamma\backslash X$. Then $\{\gamma(t) \mid t \geq 0\}$ is clearly not contained in any compact subset of $\Gamma\backslash X$. By the compactness of $\overline{\Gamma\backslash X}^{BS}$ (see 7.4), there exist a boundary point $\xi \in \partial(\overline{\Gamma\backslash X}^{BS})$ and a sequence $t_n \to +\infty$ such that $\gamma(t_n) \to \xi$. Let $\mathbf{Q}$ be a rational parabolic subgroup whose boundary component $e(\mathbf{Q}) = N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ contains an inverse image $\tilde{\xi}$ of ξ.

Let $B(\tilde{\xi})$ be a neighborhood of $\tilde{\xi} \in e(\mathbf{Q})$. Then for any $t_0 > 0$, $B(\tilde{\xi})A_{\mathbf{Q},t_0}$ is the intersection of X with a neighborhood of $\tilde{\xi}$ in $\overline{X}^{BS}$. Thus for $t_n \gg 0$, $\gamma(t_n)$ has an inverse image in X of the form $q_n a_n \in B(\tilde{\xi})A_{\mathbf{Q},t_0}$, where $q_n \in B(\tilde{\xi})$, and $a_n \in A_{\mathbf{Q},t_0}$ and for all $\alpha \in \Phi^{++}(Q, A_{\mathbf{Q}})$, $\alpha(\log a_n) \to +\infty$ as $n \to \infty$.

By Proposition 10.16, if $B(\tilde{\xi})$ is a small compact neighborhood and t_0 is large enough, then the interior of the Dirichlet domain D for Γ with center (q_n, a_n), $n \gg 1$, contains $B(\tilde{\xi}) \times \{a_j\}$ when $j \gg n$; in particular, (q_j, a_j) is an interior point of D. On the other hand, $\gamma(t)$, $t \geq 0$, is distance minimizing; hence by Lemma 10.15, the lift $\tilde{\gamma}(t)$ of $\gamma(t)$ with $\tilde{\gamma}(t_n) = (q_n, a_n)$ also belongs to D for $t \geq t_n$. Since (q_j, a_j) and $\tilde{\gamma}(t_j)$ are projected to the same point $\gamma(t_j)$, it follows that $\tilde{\gamma}(t_j) = (q_j, a_j)$. By taking a subsequence, if necessary, we can assume that $\log a_j / \|\log a_j\|$ converges to some $H \in \overline{A^+_{\mathbf{Q}}}(\infty)$. It then follows that

$$\lim_{j\to+\infty} d\big(\tilde\gamma(t_j), \exp(t_j H)\big)/t_j = 0.$$

From the fact that the distance between any two geodesics in X grows at least linearly if they do not converge to the same point in $X(\infty)$, where $X(\infty)$ is the sphere at infinity and classifies the equivalence classes of geodesics in X [BGS], §§3–4, (see also 9.2), it follows that $\tilde\gamma(t)$ converges to $H \in X(\infty)$ as $t \to +\infty$. Since $\tilde\gamma(t_n)$ converges to $\xi \in e(\mathbf{Q})$, $H \in A^+_{\mathbf{Q}}(\infty)$. By the classification of geodesics according to their limits in $X(\infty)$ given in [KA], §7, $\tilde\gamma(t)$ has the form $\tilde\gamma(t) = (u, z, a\exp(tH))$ up to reparametrization, for some $u \in N_{\mathbf{Q}}$, $z \in X_{\mathbf{Q}}$, $a \in A_{\mathbf{Q}}$.

Assume now that Γ is not necessarily neat. By [BO1], Prop. 17.4, Γ admits a neat subgroup Γ' of finite index. If γ is a DM ray in $\Gamma\backslash X$, then any of its lifts in $\Gamma'\backslash X$ is also a DM ray. The previous discussions applied to $\Gamma'\backslash X$ show that these lifts of $\gamma(t)$ and hence $\gamma(t)$ itself is of the form in the proposition. This completes the proof of the proposition. □

To classify EDM geodesics in $\Gamma\backslash X$, we need the following:

10.19. Proposition (see [BS], Prop. 10.3). — *Let $\widetilde\omega$ be a Γ_Q invariant subset of $N_{\mathbf{Q}} \times X_{\mathbf{Q}}$ with compact quotient. Then for any $t_0 \gg 0$, two points in $\widetilde\omega \times A_{\mathbf{Q},t_0}$ are Γ-equivalent if and only if they are Γ_Q-equivalent.*

10.20. Corollary. — *Two geodesics $\tilde\gamma_i(t) = (u_i, z_i, a_i\exp(tH_i)) \in N_{\mathbf{Q}} \times X_{\mathbf{Q}} \times A_{\mathbf{Q}}$, $i = 1, 2$, project to the same EDM geodesic in $\Gamma\backslash X$ up to reparametrization if and only if $H_1 = H_2$, $\log a_1 - \log a_2$ is a multiple of H_1, and $(u_1, z_1) = g(u_2, z_2)$ for some $g \in \Gamma_Q$.*

Proof. — By Theorem 10.18, the projections $\gamma_i(t)$ in $\Gamma\backslash X$ of $\tilde\gamma_i(t)$ are EDM geodesics. By Proposition 10.19, when $t \gg 0$, $\gamma_1(t) = \gamma_2(t+s)$ for some constant s if and only if the conditions in the proposition are satisfied. By the uniqueness of geodesics, $\gamma_1(t) = \gamma_2(t+s)$ is true for all t if and only if it is true for all $t \gg 0$. Then the proposition is clear. □

11. Existence of $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$ and $\Gamma\backslash X \cup \Gamma\backslash X(\infty) \cong \overline{\Gamma\backslash X}^T$.

11.1. In this section, we use the classification of EDM geodesics (or DM rays) in §10 to show the geodesic compactification $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$ exists (see 11.7) and is homeomorphic to the Tits compactification $\overline{\Gamma\backslash X}^T$ (see 11.8).

In 11.2, we show that Assumption 9.16 is satisfied by $\Gamma\backslash X$. We use the precise reduction theory in 4.6 and the results in §5 to study the growth behavior of the distance between two DM rays in $\Gamma\backslash X$ in 11.4 and 11.5 and use it to show that Assumption 9.11 is also satisfied. Then the main results of this section follow easily. In 11.19 we show how to recover the Tits metric on $\Delta(\Gamma\backslash X)$ defined in [TI1], §4, intrinsically from $\Gamma\backslash X$.

11.2. Lemma. — *Assumption* 9.16 *is satisfied for* $\Gamma\backslash X$. *That is, there exists a compact subset* ω *such that every point in* $\Gamma\backslash X$ *is contained in a DM ray starting from* ω.

Proof. — By Proposition 4.6, $\Gamma\backslash X = \coprod_{i=0}^{N} \omega_i A_{\mathbf{P}_i,T} x_0$. Note that $\mathbf{P}_0 = \mathbf{G}$, $A_{\mathbf{P}_0,T}$ consists of the identity element, and $\omega_0 x_0$ is a compact subset of $\Gamma\backslash X$, which can be thought of as the core of $\Gamma\backslash X$. We claim that the compact subset $\omega_0 x_0$ satisfies the property in the lemma.

For any $x \in \Gamma\backslash X - \omega_0 x_0$, there exists a unique $i \geq 1$ and $w \in \omega_i$ and $H \in A_{\mathbf{P}_i}^{+}$ such that $x = w \exp(H + I_{\mathbf{P}_i}(T))x_0$. Define a ray $\gamma(t) = w \exp(tH/|H| + I_{\mathbf{P}_i}(T))x_0$, $t \geq 0$. Then $\gamma(0) = w \exp I_{\mathbf{P}_i}(T)x_0 \in \omega_0$ and $\gamma(|H|) = x$. By Proposition 10.5, γ is a DM ray.

We also need to show that points in ω_0 are also contained in DM rays. For any x in ω_0, take a sequence of points y_n in $\Gamma\backslash X$ diverging to infinity. Connecting x to y_n by a DM geodesic γ_n. Then there exists a subsequence n' such that $\gamma_{n'}$ converges uniformly to a ray γ with $\gamma(0) = x$. Since γ_n is DM, the limit γ is also a DM ray. □

11.3. Proposition (see 1.5). — *Every DM ray* $\gamma(t)$ *in* $\Gamma\backslash X$ *converges to a boundary point in* $\overline{\Gamma\backslash X}^T$ *as* $t \to +\infty$. *Two DM rays in* $\Gamma\backslash X$ *converge to the same boundary point if and only if they are equivalent. And any boundary point of* $\overline{\Gamma\backslash X}^T$ *is the limit of a DM ray in* $\Gamma\backslash X$. *Therefore the set* $\Gamma\backslash X(\infty)$ *of equivalence classes of DM rays in* $\Gamma\backslash X$ *can be identified with the Tits simplicial complex* $\Delta(\Gamma\backslash X)$.[9]

Proof. — By Theorem 10.18, any DM ray γ in $\Gamma\backslash X$ is the projection of a geodesic in X of the form $\tilde{\gamma}(t) = (u, z, a\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ for some parabolic subgroup $\mathbf{P}$, where $u \in N_{\mathbf{P}}, z \in X_{\mathbf{P}}, H \in A_{\mathbf{P}}^{+}(\infty)$, $a \in A_{\mathbf{P}}$. This ray $\tilde{\gamma}$ clearly converges to H in the partial compactification

(9) The last statement that $\Gamma\backslash X(\infty) = \Delta(\Gamma\backslash X)$ has been conjectured by Hattori [HA1] [HA2] and Leuzinger [LE].

$X \cup \Delta_{\mathbb{Q}}(X)$ in 8.5. Therefore, $\gamma(t)$ is convergent in the compactification $\overline{\Gamma\backslash X}^T$ as $t \to +\infty$, and $\lim_{t\to+\infty} \gamma(t) = H \in A^+_{\mathbf{P}}(\infty)$, where we have identify $A^+_{\mathbf{P}}(\infty)$ with its image in $\Delta(\Gamma\backslash X) = \Gamma\backslash\Delta_{\mathbb{Q}}(X)$ (3.6.4).

We now show that two DM rays converge to the same boundary point in $\overline{\Gamma\backslash X}^T$ if and only if they are equivalent.

First we prove that any two DM rays with the same limit are equivalent. For $H \in \Delta(\Gamma\backslash X)$ and the ray $\gamma(t)$ above, suppose that $\gamma_1(t)$ is another DM ray that also converges to H as $t \to +\infty$. We claim that there is a lift in X of $\gamma_1(t)$ that is of the form $(u_1, z_1, a_1 \exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $\mathbf{P}$ is the same rational parabolic subgroup as above.

Let $(u_1, z_1, a_1 \exp(tH_1))$ be a lift of γ_1 in X, where $u_1 \in N_{\mathbf{P}_1}$, $z_1 \in X_{\mathbf{P}_1}$, $H_1 \in A^+_{\mathbf{P}_1}(\infty)$, and $\mathbf{P}_1$ is a rational parabolic subgroup. (By Theorem 10.18, every lift in X of γ_1 is of such a form.) By the above discussions, $\gamma_1(t)$ converges to H_1 in $\overline{\Gamma\backslash X}^T$, in particular, $H_1 = H$ in $\Delta(\Gamma\backslash X)$. This implies that $\mathbf{P}_1$ is Γ-conjugate $\mathbf{P}$, and H_1 is mapped to H under such a conjugation. Therefore, there exists another lift of γ_1 of the form $(u_1, z_1, a_1 \exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ in the claim.

From the claim and Lemma 10.3, it is clear that γ and γ_1 are equivalent.

Next we show that if two DM rays $\gamma_1(t)$ and $\gamma_2(t)$ converge to different limits H_1 and H_2 in $\overline{\Gamma\backslash X}^T$, then they are not equivalent. Let $(u_i, z_i, a_i \exp(tH_i)) \in N_{\mathbf{P}_1} \times X_{\mathbf{P}_i} \times A_{\mathbf{P}_i}$ be the lifts of γ_i in X as above, $i = 1, 2$. By the above discussion, the equivalence classes of the rays do not depend on u_i, z_i, a_i. For simplicity, we assume that $u_i = \mathrm{id}, z_i = x_0$, and $a_i = \exp I_{\mathbf{P}_i}(T)$, where $I_{\mathbf{P}_i} : \mathfrak{a} \to \mathfrak{a}_{\mathbf{P}_i}$ is the map in (4.5.2), and $T \in \mathfrak{a}$ is chosen as in Proposition 4.6. Since $H_1 \neq H_2$ in $\Delta_{\mathbb{Q}}(\Gamma\backslash X)$, either $\mathbf{P}_1$ is Γ-conjugate to $\mathbf{P}_2$ but H_1 is not equal to H_2 under the conjugation, or $\mathbf{P}_1$ is not conjugate to $\mathbf{P}_2$. Then under the map $\varphi : \Gamma\backslash X \to (\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S)$ in Proposition 5.11, $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ are two different rays from the vertex of the polyhedral cone $(\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S)$. By Lemma 5.10, these two rays are clearly not equivalent in $(\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S)$. Then Corollary 5.13 and Lemma 5.14 show that γ_1 and γ_2 are not equivalent either in $\Gamma\backslash X$.

The last statement that any point in $\Delta(\Gamma\backslash X)$ is the limit of a DM ray follows from Proposition 10.5 and the above discussion. □

11.4. Lemma. — *If two DM rays γ_1 and γ_2 in $\Gamma\backslash X$ are not equivalent, then $\lim_{t\to+\infty} \frac{1}{t} d(\gamma_1(t), \gamma_2(t))$ exists and is positive.*

Proof. — As in the proof of Proposition 11.3, the existence and value of

$$\lim_{t\to+\infty}\frac{1}{t}d\big(\gamma_1(t),\gamma_2(t)\big)$$

depend only on the equivalence classes of γ_1,γ_2. So we can assume as above that $\varphi(\gamma_1),\varphi(\gamma_2)$ are two rays in $(\coprod_0^n A_{\mathbf{P}_\imath,T},\ell_S)$ with the vertex as their initial point, where $\varphi:\Gamma\backslash X\to(\coprod_0^n A_{\mathbf{P}_\imath,T},\ell_S)$ is the map in Proposition 5.11. By Lemma 5.10, the limit

$$\lim_{t\to+\infty}\frac{1}{t}d\big(\varphi(\gamma_1(t)),\varphi(\gamma_2(t))\big)$$

exists and is positive. Then Lemma 11.4 follows from Corollary 5.13 and Lemma 5.14. □

11.5. LEMMA. — *If a sequence of DM rays $\gamma_n(t)$ in $\Gamma\backslash X$ converges to a ray $\gamma_0(t)$ uniformly for t in compact subsets, then $\lim_{t\to+\infty}\frac{1}{t}d(\gamma_n(t),\gamma_0(t))\to 0$ as $k\to+\infty$. More generally, for any $t_n\to+\infty$, $\frac{1}{t_n}d(\gamma_n(t_n),\gamma_0(t_n))\to 0$.*

Proof. — From Lemma 5.10, it is clear that the analogous result holds for the polyhedral cone $(\coprod_0^n A_{\mathbf{P}_\imath},\ell_S)$. Then by the same argument as in the previous lemma, the result also holds in $\Gamma\backslash X$. □

11.6. PROPOSITION. — *Assumption* 9.11 *is satisfied by $\Gamma\backslash X$. That is, for any two sequences of DM rays γ_n,γ_n' converging to rays γ_0 and γ_0' respectively, if there exist two sequences of numbers $t_n,t_n'\to+\infty$ such that $d(\gamma_n(t_n),\gamma_n'(t_n'))$ is bounded, then γ_0 is equivalent to γ_o'.*

Proof. — Since the DM rays γ_n and γ_n' start from a compact subset of $\Gamma\backslash X$ and $d(\gamma_n(t_n),\gamma_n'(t_n'))$ is bounded, it is clear that t_n-t_n' is bounded.

Suppose that γ_0 is not equivalent to γ_0'. Then by the same argument as in the proof of Lemma 11.4, there exists a positive constant c such that for any $n\geq 1$,

$$\frac{1}{t_n}d\big(\gamma_0(t_n),\gamma_0'(t_n')\big)>c. \tag{11.6.1}$$

On the other hand, by Lemma 11.5,

$$\frac{1}{t_n}d\big(\gamma_0(t_n),\gamma_n(t_n)\big)\longrightarrow 0,\quad \frac{1}{t_n'}d\big(\gamma_0(t_n'),\gamma_n'(t_n')\big)\longrightarrow 0$$

as $n\to+\infty$. Since $t_n\to+\infty$, t_n-t_n' and $d(\gamma_n(t_n),\gamma_n'(t_n'))$ are bounded, this implies that as $n \to +\infty$,

$$\frac{1}{t_n} d(\gamma_0(t_n), \gamma_0'(t_n')) \longrightarrow 0.$$

But this contradicts Inequality (11.6.1) above. Therefore γ_0 is equivalent to γ_0'. □

11.7. THEOREM (see 1.2). — *The geodesic compactification* $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$ *exists and is Hausdorff.*

Proof. — Lemma 11.2 and Proposition 11.6 show that Assumptions 9.11 and 9.16 are satisfied by $\Gamma\backslash X$. Then Theorem 11.7 follows from Theorem 9.17. □

11.8. THEOREM (see 1.5). — *The Tits compactification* $\overline{\Gamma\backslash X}^T$ *is homeomorphic to* $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$.

Proof. — By Proposition 11.3, the identity map on X extends to a map $\pi : \overline{\Gamma\backslash X}^T \to \Gamma\backslash X \cup \Gamma\backslash X(\infty)$.

We first prove that i is continuous. Assume that y_n is a sequence in $\Gamma\backslash X$ which converges to a boundary point $H_\infty \in \Delta(\Gamma\backslash X)$. Let $\Gamma\backslash X = \coprod_{i=0}^{n} \omega_i A_{\mathbf{P}_i,T} x_0$ be the decomposition of $\Gamma\backslash X$ in Proposition 4.6. Write $y_n = w_n \exp(H_n + I_{\mathbf{P}_i}(T)) x_0 \in \omega_i A_{\mathbf{P}_i,T} x_0$, where $w_n \in \omega_n$, $\exp(H_n) \in A_{\mathbf{P}_i}^+$, and i depends on n. Then $|H_n| \to +\infty$ and $H_n/|H_n| \in \Delta(\Gamma\backslash X)$ converges H_∞.

Define a ray $\gamma_n(t) = w_n \exp(tH_n/|H_n| + I_{\mathbf{P}_i}(T))$, $t \geq 0$. By Proposition 10.5, γ_n is a DM ray with initial point $w_n \in \omega_0$ and passing through y_n. Since w_n belongs to a compact subset and $H_n/|H_n|$ converges to H_∞, it is clear that the pointed ray class $[\gamma_n, t_n]$ is a convergence sequence of type 2) in 9.3 with limit $[\gamma_\infty, \infty]$, where $\gamma_\infty(t) = \exp(tH_\infty + I_{\mathbf{P}_i}(T))x_0$, where $\mathbf{P}_i$ is the subgroup such that $H_\infty \in A_{\mathbf{P}_i}^+(\infty)$. Since $\pi(H_\infty) = [\gamma_\infty, \infty]$, this proves that π is continuous.

Since both $\overline{\Gamma\backslash X}^T$ and $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$ are compact and Hausdorff, the map π is a homeomorphism. □

11.9. *Remark.* — For any two points $[\gamma_1], [\gamma_2] \in \Gamma\backslash X(\infty)$, define

$$d_T([\gamma_1], [\gamma_2]) = \lim_{t\to+\infty} \frac{1}{t} d(\gamma_1(t), \gamma_2(t)).$$

By Proposition 11.3 and Lemma 11.4, this limit is well-defined and gives a metric on $\Delta(\Gamma\backslash X) = \Gamma\backslash X(\infty)$. (Note that this metric is not a simplicial metric.) Using the geometric realization of $\Delta(\Gamma\backslash X)$ as a subspace

$\coprod_1^n A^+_{\mathbf{P}_i}(\infty)$ in $(\coprod_0^n A_{\mathbf{P}_i,T}, \ell_S)$ (3.6.4), where $A_{\mathbf{P}_i,T}$ has been identified with $A^+_{\mathbf{P}_i}$, we get from the proof of Lemma 11.4 that this metric d_T is the restriction of the metric ℓ_S.

By Lemma 5.6, the metric space $(\Delta(\Gamma\backslash X), d_T)$ induces a length structure $(\Delta(\Gamma\backslash X), \ell_T)$. Then the metric ℓ_T is the spherical metric defined in [TI1], §4. More precisely, Tits defined a spherical metric on $\Delta_{\mathbb{Q}}(X)$, and the induced metric on the quotient $\Gamma\backslash\Delta_{\mathbb{Q}}(X) = \Delta(\Gamma\backslash X)$ is equal to ℓ_T.

12. Gromov compactification $\overline{\Gamma\backslash X}^G$ and $\overline{\Gamma\backslash X}^G \cong \overline{\Gamma\backslash X}^T$.

12.1. In this section, we define the Gromov compactification of a complete Riemannian manifold (see 12.5) and prove the Gromov compactification $\overline{\Gamma\backslash X}^G$ is homeomorphic to the Tits compactification $\overline{\Gamma\backslash X}^T$ (see 12.11).

12.2. Let M be a noncompact complete metric space, in particular a Riemannian manifold, its Gromov compactification $\overline{M}^G$ is introduced in [BGS], p. 21, and several key properties are given as exercises.

Let $C(M)$ be the space of continuous functions with the topology of uniform convergence on compact subsets of M, and $C_*(M)$ be the quotient of $C(M)$ by the subspace of constant functions. Denote the distance function on M by $d(\,.\,,.)$. For any $y \in M$, $d_y : x \in M \mapsto d(y,x) \in \mathbb{R}$ defines a continuous function on M. Denote the image of d_y in $C_*(M)$ by $\bar{d}_y$. Then we get a map $i : M \to C_*(M)$ defined by $i(y) = \bar{d}_y$.

12.3. Lemma. — *The map $i : M \to C_*(M)$ is an embedding.*

Proof. — First we show that i is injective. For any two points $y_1, y_2 \in M$, if $i(y_1) = i(y_2)$, then $d_{y_1} - d_{y_2}$ is a constant function, i.e., there exists a constant c such that for any $x \in M$, $d_{y_1}(x) - d_{y_2}(x) = c$. Setting $x = y_1$, we get $-d(y_2, y_1) = c$. Setting $x = y_2$, we get $d(y_1, y_2) = c$. Therefore, $d(y_1, y_2) = 0$ and $y_1 = y_2$. This proves that i is injective.

Since d_y depends continuously on $y \in M$, the map i is continuous. To finish the proof, we need to show that for any sequence $y_n \in M$, if $\bar{d}_{y_n}$ converges to $\bar{d}_{y_0}$ for some $y_0 \in M$, then $y_n \to y_0$.

By definition, there exists a sequence of constants c_n such that

$$d_{y_n}(x) + c_n \longrightarrow d_{y_0}(x) \tag{12.3.1}$$

uniformly for x in compact subsets of M.

Suppose that $y_n \not\to y_0$, i.e., $d(y_0, y_n) \not\to 0$ as $n \to \infty$. For simplicity, we assume that there exists a positive constant d_0 such that $d(y_0, y_n) \geq d_0$ when $n \geq 1$ and the distance minimizing geodesic segment $\gamma_n(t)$ from y_0 to y_n converge to a segment $\gamma_0(t)$ with $\gamma_0(0) = y_0$. Fix a positive t_0 such that $t_0 < d_0$.

Setting $x = y_0$ in (12.3.1), we get

$$d_{y_n}(y_0) + c_n \longrightarrow d_{y_0}(y_0) = 0. \tag{12.3.2}$$

Since $\gamma_n(t_0) \to \gamma_0(t_0)$, we obtain that as $n \to +\infty$,

$$\begin{aligned} \left| d_{y_n}(\gamma_0(t_0)) - d_{y_n}(\gamma_n(t_0)) \right| &\leq d\big(\gamma_0(t_0), \gamma_n(t_0)\big) \longrightarrow 0, \\ d_{y_n}\big(\gamma_0(t_0)\big) + c_n &= d_{y_n}\big(\gamma_n(t_0)\big) + c_n + o(1) \\ &= d_{y_n}\big(\gamma_n(0)\big) + c_n - t_0 + o(1) \\ &= d_{y_n}(y_0) + c_n - t_0 + o(1) \longrightarrow -t_0. \end{aligned}$$

In the second equality, we have used the assumption $t_0 < d_0 < d(y_n, y_0)$, and in the last inequality, we have used (12.3.2).

On the other hand, by (12.3.1), $d_{y_n}(\gamma_0(t_0)) + c_n \to d_{y_0}(\gamma_0(t_0)) = t_0$. This implies that $-t_0 = t_0$, and hence $t_0 = 0$. This contradicts the positivity of t_0. Therefore $y_n \to y_0$. □

12.4. Lemma (see [BGS], p. 21, Exercises 1 and 2). — *The closure of $i(M)$ in $C_*(M)$ is compact.*

Proof. — For any $y \in M$, d_y satisfies the following inequality:

$$\left| d_y(y_1) - d_y(y_2) \right| \leq d(y_1, y_2).$$

For any representative of the class $\bar{d}_y$, the same inequality holds. Fix a basepoint $x_0 \in M$. For any function $\bar{d}_y$, choose a representative $\hat{d}_y$ such that $\hat{d}_y(x_0) = 0$. Then the family $\{\hat{d}_y \mid y \in M\}$ is equicontinuous on any compact subsets on M. It follows that for any sequence y_n in M, there exists a subsequence $y_{n'}$ such that $\hat{d}_{y_{n'}}$ converges uniformly over compact subsets to a continuous function on M. That is, any sequence in $i(M)$ has a subsequence converging to a point on the closure of $i(M)$. Therefore, the closure of $i(M)$ is $C_*(M)$ is compact. □

12.5. Definition. — *The closure of $i(M)$ in $C_*(M)$ is called the Gromov compactification of M and denoted by $\overline{M}^G$.*

12.6. PROPOSITION (see [BGS], §3). — *If M is a Hadamard manifold, i.e., nonpositively curved and simply connected, then $\overline{M}^G$ is the same as the conic (or geodesic) compactification $M \cup M(\infty)$. In particular, for the symmetric space X, $\overline{X}^G = X \cup X(\infty)$.*

This is the only example whose Gromov compactification has been identified before.

12.7. LEMMA. — *If $M = \mathbb{R}^n$, then for any representatives d_∞, d'_∞ of two different boundary points in $\overline{M}^G$, their difference $d_\infty - d'_\infty$ is not bounded.*

Proof. — It follows from the explicit computation in 16.1 that for any $v \in \mathbb{R}^n(\infty)$, i.e, $v \in \mathbb{R}^n$, $|v| = 1$, the associated boundary function is $x \in \mathbb{R}^n \to \langle v, x\rangle$, up to a constant. Then Lemma 12.7 is clear. □

12.8. LEMMA. — *Let $M = (\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$ be the polyhedral cone with the simplicial metric in 5.9. Then $\partial \overline{M}^G = \coprod_1^n A^+_{\mathbf{P}_\imath}(\infty)$, and for any representatives $\ell_\infty, \ell'_\infty$ of two different boundary points in $\overline{M}^G$, their difference $\ell_\infty - \ell'_\infty$ is not bounded.*

Proof. — For any two points $x, y \in (\coprod_0^n A_{\mathbf{P}_\imath,T}, \ell_S)$, we claim that there exist closed conic domains $C_x \ni x$, $C_y \ni y$ and an Euclidean space $\mathbb{R}^r$ such that $(C_x \cup C_y, \ell_S)$ can be isometrically embedded into $\mathbb{R}^r$.

If y is not on the cut locus of x, then there exist conic domains $C_x \ni x$, $C_y \ni y$ such that for any $u \in C_x$, $v \in C_y$, v is not on the cut locus of u and hence connected to u by a unique DM geodesic $\gamma_{u,v}$. By Lemma 5.10, the DM geodesic $\gamma_{u,v}$ is piecewise linear and contained in the same collection of cones $A_{\mathbf{P}_{\imath'},T}$. The union of $\gamma_{u,v}$, where $u \in C_x, v \in C_y$, forms a subset of $\bigcup A_{\mathbf{P}_{\imath'},T}$ that can be straightened out and embedded isometrically into some Euclidean space $\mathbb{R}^r$.

Suppose now that y is on the cut locus of x. The cut locus of x is a cone of codimension 1. Similarly, the cut locus of y is a cone of codimension 1. Let $C_x \ni x$ be a small closed conic domain on one side of the cut locus of y and $C_y \ni y$ be a small closed conic domain on the side of the cut locus of x which is closer to C_x. Then any points $u \in C_x$, $v \in C_y$ are connected by a unique DM geodesic $\gamma_{u,v}$. The same argument as above shows that $(C_x \cup C_y, \ell_S)$ can be isometrically embedded into some Euclidean space $\mathbb{R}^r$. This proves the claim.

Then Lemma 12.8 follows from the claim and Lemma 12.7. □

12.9. Lemma. — *Let* $\mathbf{P}$ *be a rational parabolic subgroup of* $\mathbf{G}$. *For any* $x \in X$, *let* $\exp H_{\mathbf{P}}(x)$ *be the* $A_{\mathbf{P}}$ *component of* x *in the horospherical decomposition* $X = N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ (3.5.2). *Then for any* $H \in \overline{A_{\mathbf{P}}^{+}}(\infty)$,

$$\sup\big\{\langle H, H_{\mathbf{P}}(\gamma x)\rangle \mid \gamma \in \Gamma\big\} < +\infty,$$

and the supremum can be achieved.

Proof. — By [HC], Lemma 21, if H is the vector H_α satisfying $\beta(H_\alpha) = \delta_{\alpha\beta}$, where $\alpha, \beta \in \Phi^{++}(P, A_{\mathbf{P}})$, the above supremum is finite. Since any H is a positive linear combination of such vectors H_α, the supremum is also finite. Since Γ acts properly continuously on the partial Borel-Serre compactification $\overline{X}^{BS}$ (see §7.4) and $H_{\mathbf{P}}(\gamma x) = H_{\mathbf{P}}(x)$ for $\gamma \in \Gamma_P = \Gamma \cap P$, the supremum can be achieved by some element in Γ. □

12.10. Proposition. — *Let* y_n *be a sequence in* $\Gamma\backslash X$ *going to infinity. If* y_n *is convergent in the Tits compactification* $\overline{\Gamma\backslash X}^{T}$, *then* y_n *is also convergent in the Gromov compactification* $\overline{\Gamma\backslash X}^{G}$.

Proof. — Assume that y_n converges to a boundary point $H_\infty \in A_{\mathbf{P}}^{+}(\infty) \subset \Delta(\Gamma\backslash X)$. Then there exists a lift $\tilde{y}_n$ of y_n such that $\tilde{y}_n$ converges to $H_\infty \in A_{\mathbf{P}}^{+}(\infty) \subset \Delta_{\mathbb{Q}}(X)$ in the partial compactification $X \cup \Delta_{\mathbb{Q}}(X)$. In the decomposition $X = N_{\mathbf{P}} M_{\mathbf{P}} A_{\mathbf{P}} x_0$ (3.5.1), $\tilde{y}_n = \ell_n \exp(H_n) x_0$, where $\ell_n \in N_{\mathbf{P}} M_{\mathbf{P}}$ and $H_n \in \mathfrak{a}_{\mathbf{P}}$ satisfy the following conditions in 8.3:

1) $H_n/|H_n| \to H_\infty$,

2) $d_X(\ell_n x_0, x_0)/|H_n| \to 0$ as $n \to \infty$.

We claim that $d_{\Gamma\backslash X}(y_n, x) - |H_n|$ converges to a continuous function on $\Gamma\backslash X$ uniformly for x in compact subsets of $\Gamma\backslash X$. Then Proposition 12.10 follows from the claim.

Let $\tilde{x} \in X$ be a lift of x. Then

$$d_{\Gamma\backslash X}(y_n, x) = \inf\{d_X(\tilde{y}_n, \gamma\tilde{x}) \mid \gamma \in \Gamma\}.$$

For any $\gamma \in \Gamma$, it follows from Lemma 10.3 and the conditions on $\tilde{y}_n$ above that

$$d_X(\tilde{y}_n, \gamma\tilde{x}) = d_X(e^{H_{\mathbf{P}}(\gamma\tilde{x})}, e^{H_n}) + o(1) = |H_n| - \langle H_{\mathbf{P}}(\gamma\tilde{x}), \frac{H_n}{|H_n|}\rangle + o(1)$$

where $o(1)$ stands for functions going to zero uniformly for x in compact subsets as $n \to \infty$. By Lemma 12.9, for every x, there exists $\gamma_0 \in \Gamma$ such that

$$\sup\{\langle H_{\mathbf{P}}(\gamma x), H_\infty\rangle \mid \gamma \in \Gamma\} = \langle H_{\mathbf{P}}(\gamma_0 x), H_\infty\rangle.$$

Since $H_n/|H_n| \to H_\infty$, we get that as $n \to +\infty$,

$$d_X(\tilde{y}_n, \gamma\tilde{x}) = d_X(e^{H_{\mathbf{P}}(\gamma\tilde{x})}, e^{H_n}) + o(1) = |H_n| - \langle H_{\mathbf{P}}(\gamma\tilde{x}), H_\infty\rangle + o(1),$$

and hence as $n \to +\infty$,

$$d_{\Gamma\backslash X}(y_n, x) - |H_n| \longrightarrow \langle H_{\mathbf{P}}(\gamma_0\tilde{x}), H_\infty\rangle$$

uniformly for x in compact subsets. Then the claim, and hence the proposition, follows easily. □

12.11. THEOREM (see 1.3). — *The identity map on X extends to a homeomorphism from $\overline{\Gamma\backslash X}^T$ to $\overline{\Gamma\backslash X}^G$.*

Proof. — From Proposition 12.10, it is clear that the identity map extends to a continuous map from $\overline{\Gamma\backslash X}^T$ to $\overline{\Gamma\backslash X}^G$. Since both $\overline{\Gamma\backslash X}^T$, $\overline{\Gamma\backslash X}^G$ are compact and Hausdorff, to show that the extended map is a homeomorphism, it suffices to prove that it is injective.

Let y_n, y_n' be two sequences in $\Gamma\backslash X$ converging to different limits $H_\infty, H_\infty' \in \Delta(\Gamma\backslash X)$. Denote the limit of y_n, y_n' in $\overline{\Gamma\backslash X}^G$ by $\bar{d}_\infty, \bar{d}_\infty'$ respectively. Let d_∞, d_∞' be representatives of $\bar{d}_\infty, \bar{d}_\infty'$. We claim that $d_\infty - d_\infty'$ is not bounded on $\Gamma\backslash X$, in particular, $\bar{d}_\infty \neq \bar{d}_\infty'$.

Recall the disjoint decomposition of X in 4.6: $\Gamma\backslash X = \coprod_0^n \omega_i A_{\mathbf{P}_i,T} x_0$. Suppose that $H_\infty \in A^+_{\mathbf{P}_i}(\infty)$ and $H_\infty' \in A^+_{\mathbf{P}_i'}(\infty)$. By Proposition 12.10, we can assume that $y_n = \exp(H_n)x_0 \in A_{\mathbf{P}_i,T}x_0$, $y_n' = \exp(H_n')x_0 \in A_{\mathbf{P}_i',T}x_0$. Identify y_n and y_n' with two sequences in the polyhedral cone $(\coprod_0^n A_{\mathbf{P}_i,T}x_0, \ell_S)$ in 5.9.

Recall the map $\varphi : \Gamma\backslash X \to (\coprod_0^n A_{\mathbf{P}_i,T}x_0, \ell_S)$ in 5.11. By Corollary 5.13 and Lemma 5.14, there exists a constant $c > 0$ such that for any $x \in \Gamma\backslash X$,

$$\big|d_{\Gamma\backslash X}(y_n, x) - \ell_S(y_n, \varphi(x))\big| \leq c,$$
$$\big|d_{\Gamma\backslash X}(y_n', x) - \ell_S(y_n', \varphi(x))\big| \leq c.$$

Since $H_\infty \neq H_\infty'$, the claim follows from Lemma 12.8. Therefore, the extended map from $\overline{\Gamma\backslash X}^T$ to $\overline{\Gamma\backslash X}^G$ is injective. □

13. Continuous spectrum and compactifications of $\Gamma\backslash X$.

13.1. In this section, we study relations between the continuous spectrum of $\Gamma\backslash X$ and compactifications of $\Gamma\backslash X$. As mentioned in 2.2, a natural question is to parametrize the generalized eigenspaces of the continuous spectrum using boundaries of geometric compactifications of $\Gamma\backslash X$. The main contribution of this section is to interpret the spectral decomposition of $L^2(\Gamma\backslash X)$ due to Langlands [LA] as a parametrization of the generalized eigenspaces in terms of the pair of compactifications $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$, $\overline{\Gamma\backslash X}^{RBS}$ (see 13.14–13.15).

13.2. As mentioned in 2.2, the continuous spectrum of Δ as a subset in $\mathbb{R}$ of a complete noncompact Riemannian manifold does not change under compact perturbations. This follows from the so-called decomposition principle in [DL], Prop. 2.1. This implies that the continuous spectrum as a set only depends on the geometry of the end of the manifold. On the other hand, behaviors of generalized eigenfunctions of the continuous spectrum under compact perturbations are more complicated. Therefore, a more interesting question is to understand relations between the generalized eigenfunctions and the geometry near infinity.

Since the first version of this paper was written, stronger results on relations between the generalized eigenfunctions and the geodesics which are EDM in both directions have been obtained in [JZ]. The classification of EDM geodesics in this paper palys an important role in [JZ]. See the comments in 13.18 for more details.

13.3. The generalized eigenfunctions of $\Gamma\backslash X$ are given by Eisenstein series. We first recall several basic facts about Eisenstein series and the spectral decomposition of $L^2(\Gamma\backslash X)$ following [LA], [AR2], [MW] and [OW2].

Let $\mathbf{P}$ be a proper rational parabolic subgroup of $\mathbf{G}$, and $\Gamma_{M_\mathbf{P}}$ be the image of Γ_P in $M_\mathbf{P}$ under the projection $A_\mathbf{P} N_\mathbf{P} M_\mathbf{P} \to M_\mathbf{P}$. Then $\Gamma_{M_\mathbf{P}}$ is a cofinite discrete subgroup acting on $X_\mathbf{P} = M_\mathbf{P}/K \cap M_\mathbf{P}$. Let $\mathfrak{a}_\mathbf{P}^*$ be the dual of $\mathfrak{a}_\mathbf{P}$. For any L^2-eigenfunction φ of the Laplace operator Δ on the boundary locally symmetric space $\Gamma_{M_\mathbf{P}}\backslash X_\mathbf{P}$, we define an Eisenstein series $E(\mathbf{P}|\varphi, \Lambda)$, $\Lambda \in \mathfrak{a}_\mathbf{P}^* \otimes \mathbb{C}$, as follows:

$$E(\mathbf{P}|\varphi, \Lambda : x) = \sum_{\gamma \in \Gamma_P\backslash\Gamma} e^{(\rho_\mathbf{P}+\Lambda)(H_\mathbf{P}(\gamma x))}\varphi\big(z_\mathbf{P}(\gamma x)\big),$$

where $\rho_\mathbf{P}$ is the half sum of the positive roots in $\Phi^+(P, A_\mathbf{P})$ with multiplicity equal to the dimension of the root spaces, and

$$x = \big(u_{\mathbf{P}}(x), z_{\mathbf{P}}(x), \exp(H_{\mathbf{P}}(x))\big) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$$

as in (3.5.2). When $\mathrm{Re}(\Lambda) \in \rho_{\mathbf{P}} + \mathfrak{a}_{\mathbf{P}}^{*+}$, the above series converges uniformly for x in compact subsets of X (see [LA], Lemma 4.1). It is a theorem of Langlands [LA], Chap. 7, [MW], Chap. 4, [OW2], Chap. 6, that $E(\mathbf{P}|\varphi, \Lambda)$ can be meromorphically continued as a function of Λ to the whole complex space $\mathfrak{a}_{\mathbf{P}}^* \otimes \mathbb{C}$, and $E(\mathbf{P}|\varphi, \Lambda)$ is regular when $\mathrm{Re}(\Lambda) = 0$.

The Eisenstein series $E(\mathbf{P}|\varphi, \Lambda)$ are clearly Γ-invariant and hence define functions on $\Gamma\backslash X$.

13.4. Lemma. — *If φ has eigenvalue ν, i.e., $\Delta\varphi = \nu\varphi$, then for any $\Lambda \in \sqrt{-1}\,\mathfrak{a}_{\mathbf{P}}^*$,*

$$\Delta E(\mathbf{P}|\varphi, \Lambda) = \big(\nu + |\rho_{\mathbf{P}}|^2 + |\Lambda|^2\big) E(\mathbf{P}|\varphi, \Lambda).$$

Proof. — From the horospherical expression of Δ [HC], p. 19, [MU], Eq. (1.2), p. 479, [KA], Thm. 15.4.1, it is clear that the term $\exp((\rho_{\mathbf{P}} + \Lambda)(H_{\mathbf{P}}(x)))\varphi(z_{\mathbf{P}}(x))$ in the Eisenstein series satisfies the above equation. Since Δ is invariant under G, other terms also satisfy the equation, and hence the Eisenstein series $E(\mathbf{P}|\varphi, \Lambda)$ satisfies the equation in the region of absolute convergence. Since the equation is preserved under the meromorphic continuation, the lemma is proved. □

For any $f \in L^2(\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}}^*)$, define a function $\hat{f}$ on $\Gamma\backslash X$ by

$$\hat{f}(x) = \int_{\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}}^*} f(\Lambda) E(\mathbf{P}|\varphi, \Lambda : x)\, \mathrm{d}\Lambda.$$

It is known that $\hat{f} \in L^2(\Gamma\backslash X)$ (see [OW2], pp. 328–329 for example). For every such pair of $\mathbf{P}$ and φ, denote the span in $L^2(\Gamma\backslash X)$ of all such functions $\hat{f}$ above by $L^2_{\mathbf{P},\varphi}(\Gamma\backslash X)$.

These subspaces induce a decomposition of $L^2(\Gamma\backslash X)$. Denote the subspace of $L^2(\Gamma\backslash X)$ spanned by L^2-eigenfunctions of the Laplace operator Δ by $L^2_{\mathrm{dis}}(\Gamma\backslash X)$, called the discrete subspace, and the orthogonal complement of $L^2_{\mathrm{dis}}(\Gamma\backslash X)$ in $L^2(\Gamma\backslash X)$ by $L^2_{\mathrm{con}}(\Gamma\backslash X)$, called the continuous subspace. Then

$$L^2(\Gamma\backslash X) = L^2_{\mathrm{dis}}(\Gamma\backslash X) \oplus L^2_{\mathrm{con}}(\Gamma\backslash X).$$

The subspace $L^2_{\mathrm{con}}(\Gamma\backslash X)$ can be decomposed into the subspaces $L^2_{\mathbf{P},\varphi}(\Gamma\backslash X)$ [LA], Chap. 7, [OW2], Thm. 7.5).

13.5. PROPOSITION. — *With the notation as above,*

$$L^2_{\text{con}}(\Gamma\backslash X) = \sum_{\mathbf{P}} \sum_{\varphi} L^2_{\mathbf{P},\varphi}(\Gamma\backslash X),$$

where $\mathbf{P}$ *sums over all proper rational parabolic subgroups of* $\mathbf{G}$, *and* φ *is over eigenfunctions of* $\Gamma_{M_\mathbf{P}}\backslash X_\mathbf{P}$ *which form a basis of* $L^2_{\text{dis}}(\Gamma_{M_\mathbf{P}}\backslash X_\mathbf{P})$.

13.6. COROLLARY. — *For any* $\lambda > 0$, *the generalized eigenspace of* $\Gamma\backslash X$ *with eigenvalue* λ, *in the sense defined in* 13.16 *below, is spanned by* $E(\mathbf{P}|\varphi, \Lambda)$, *where* $\mathbf{P}$ *is a parabolic subgroup,* φ *an* L^2 *eigenfunction of* $\Gamma_{M_\mathbf{P}}\backslash X_\mathbf{P}$ *with eigenvalue* ν *satisfying* $\nu \leq \lambda - |\rho_\mathbf{P}|^2$, *and* $\Lambda \in \sqrt{-1}\,\mathfrak{a}^*_\mathbf{P}$ *satisfying* $|\Lambda|^2 = \lambda - |\rho_\mathbf{P}|^2 - \nu$.

Proof. — It follows from Lemma 13.4 and Proposition 13.5. □

The sum of the subspaces in Proposition 13.5 is not direct, and the Eisenstein series $E(\mathbf{P}|\varphi, \Lambda)$ for different $\mathbf{P}$ are not linearly independent. In order to find a basis of the generalized eigenspace, we need to study relation between various Eisenstein series and subspaces $L^2_{\mathbf{P},\varphi}(\Gamma\backslash X)$ spanned by them.

13.7. DEFINITION. — *Two rational parabolic subgroups* $\mathbf{P}_1, \mathbf{P}_2$ *of* $\mathbf{G}$ *are called associated if there exists* $g \in G$ *such that* $\text{Ad}(g)\mathfrak{a}_{\mathbf{P}_1} = \mathfrak{a}_{\mathbf{P}_2}$.

In the above definition, the split components $A_{\mathbf{P}_1}$, $A_{\mathbf{P}_2}$ are the lifts with respect to the fixed basepoint in 3.5, and hence the element $g \in G$ above does not necessarily belong to $\mathbf{G}(\mathbb{Q})$.

Clearly, $\mathbf{P}_1$ and $\mathbf{P}_2$ are associated if they are conjugate. But the converse is not true. Let $\mathcal{C}$ be an association class of rational parabolic subgroups of $\mathbf{G}$, and $\mathcal{C}_1, \cdots, \mathcal{C}_r$ be the G-conjugacy classes in $\mathcal{C}$. Let $\mathbf{P}_i \in \mathcal{C}_i$, $i = 1, \cdots, r$, be representatives of G-conjugacy classes in $\mathcal{C}$.

For any two associate parabolic subgroups $\mathbf{P}_1, \mathbf{P}_2$, denote by $W(\mathfrak{a}_{\mathbf{P}_1}, \mathfrak{a}_{\mathbf{P}_2})$ the set of bijections $w : \mathfrak{a}_{\mathbf{P}_1} \to \mathfrak{a}_{\mathbf{P}_2}$ induced by conjugation under some elements in G.

Denote $\mathfrak{a}_{\mathbf{P}_i}$ by $\mathfrak{a}_i$. Then we have the following decomposition of the set of regular elements of $\mathfrak{a}_i$ [LA], Lemma 2.13, [OW2], p. 67.

13.8. LEMMA. — *For any* $i \in \{1, \cdots, r\}$, *the following union:*

$$\coprod_{j=1}^{r} \coprod_{w_{ji} \in W(\mathfrak{a}_j, \mathfrak{a}_i)} w_{ji}(\mathfrak{a}_j^+)$$

is disjoint and is equal to the set of regular elements in $\mathfrak{a}_i$, *where an element* $H \in \mathfrak{a}_i$ *is regular if for all* $\alpha \in \Phi(P_i, A_{\mathbf{P}_i})$, $\alpha(H) \neq 0$.

When $\mathcal{C}$ is an association class of minimal rational parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$, $r = 1$ and the above lemma is equivalent to the fact that the Weyl group $W(\mathfrak{g}, \mathfrak{a}_i)$ acts simply transitively on the set of chambers in $\mathfrak{a}_i$. On the other hand, when the parabolic subgroups in $\mathcal{C}$ are not minimal, $W(\mathfrak{g}, \mathfrak{a}_i)$ does not act simply transitively on the set of chambers, and the number of orbits is equal to r in the lemma.

For an association class $\mathcal{C}$ of parabolic subgroups, define a subspace $L^2_{\mathcal{C}}(\Gamma\backslash X)$ of $L^2(\Gamma\backslash X)$ by

$$L^2_{\mathcal{C}}(\Gamma\backslash X) = \sum_{\mathbf{P}\in\mathcal{C}} \sum_{\varphi} L^2_{\mathbf{P},\varphi}(\Gamma\backslash X),$$

where φ runs over eigenfunctions of $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$ which form a basis of $L^2_{\text{dis}}(\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}})$.

Then we have the following direct sum decomposition of $L^2_{\text{con}}(\Gamma\backslash X)$ [LA], Lemma 7.1, [OW2], Thm. 7.5.

13.9. Proposition. — *The continuous subspace $L^2_{\text{con}}(\Gamma\backslash X)$ is the direct sum of $L^2_{\mathcal{C}}(\Gamma\backslash X)$ over all association classes $\mathcal{C}$ of proper parabolic subgroups.*

This result implies that Eisenstein series coming from non-associated parabolic subgroups are linearly independent.

Let $\mathcal{C}$ be an association class of parabolic subgroups, and $\mathbf{P}_i$ be representatives of the G-conjugacy classes $\mathcal{C}_i$ in $\mathcal{C}$ as above. For every $i \in \{1, \dots, r\}$, let $\mathbf{P}_{i\mu}$, $1 \le \mu \le r_i$, be representatives of Γ-conjugacy classes in the G-conjugacy class $\mathcal{C}_i$. Then $\mathbf{P}_{i\mu}$, $1 \le i \le r$, $1 \le \mu \le r_i$, are representatives of Γ-conjugacy classes in the association class $\mathcal{C}$.

For a positive number ν, let $\mathcal{E}_{i\mu}(\nu)$ be the L^2 eigenspace with eigenvalue ν of $\Gamma_{M_{\mathbf{P}_{i\mu}}}\backslash X_{\mathbf{P}_{i\mu}}$. The spaces $\mathcal{E}_{i\mu}(\nu)$ are not trivial only when ν is an eigenvalue of $\Gamma_{M_{\mathbf{P}_{i\mu}}}\backslash X_{\mathbf{P}_{i\mu}}$. In the following, we always assume that ν belongs to $\bigcup_{i,\mu} \operatorname{Spec}_d(\Gamma_{M_{\mathbf{P}_{i\mu}}}\backslash X_{\mathbf{P}_{i\mu}})$, the union of the discrete spectrum of $\Gamma_{M_{\mathbf{P}_{i\mu}}}\backslash X_{\mathbf{P}_{i\mu}}$. Denote this union by $\operatorname{Spec}_d(\mathcal{C})$.

For every $i = 1, \dots, r$, define

$$\mathcal{E}_i(\nu) = \bigoplus_{\mu=1}^{r_i} \mathcal{E}_{i\mu}(\nu). \tag{13.9.1}$$

For convenience, identify $\mathfrak{a}_{\mathbf{P}_{i\mu}}$ with $\mathfrak{a}_i = \mathfrak{a}_{\mathbf{P}_i}$ under conjugation of some elements in K, $\mu = 1, \ldots, r_i$.

For any $\varphi_i = (\varphi_{i1}, \cdots, \varphi_{ir_i}) \in \mathcal{E}_i(\nu)$ and $\Lambda_i \in \mathfrak{a}_i \otimes \mathbb{C}$, define

$$E(\mathbf{P}_i|\varphi_i, \Lambda_i) = \sum_{\mu=1}^{r_i} E(\mathbf{P}_{i\mu}|\varphi_{i\mu}, \Lambda_i).$$

13.10. PROPOSITION (see [OW2], Thm. 6.4). — *For any $i, j \in \{1, \ldots, r\}$, $\Lambda_i \in \mathfrak{a}_i^* \otimes \mathbb{C}$, and $w_{ji} \in W(\mathfrak{a}_j, \mathfrak{a}_i)$, there exists a meromorphic family of intertwining operator*

$$c_{ji}(w_{ji}, \Lambda_i) : \mathcal{E}_i(\nu) \to \mathcal{E}_j(\nu)$$

satisfying the following conditions :

1) *For any $k \in \{1, \ldots, r\}$, $w_{ki} \in W(\mathfrak{a}_k, \mathfrak{a}_i)$, $w_{kj} \in W(\mathfrak{a}_k, \mathfrak{a}_j)$ with $w_{ki} = w_{ji} w_{kj}$,*

$$c_{ki}(w_{ki}, \Lambda_i) = c_{kj}(w_{kj}, w_{ji}\Lambda_i) c_{ji}(w_{ji}, \Lambda_i).$$

2) $c_{ji}(w_{ji}, \Lambda_i) c_{ij}(w_{ji}^{-1}, w_{ji}\Lambda_i) = \mathrm{Id}$; *in particular, when* $\mathrm{Re}(\Lambda_i) = 0$, $c_{ji}(w_{ji}, \Lambda_i)$ *is bijective and unitary.*

13.11. PROPOSITION (see [OW2], Thm. 6.2). — *With the notation as above, the Eisenstein series satisfy the following functional equation : For any $i, j \in \{1, \ldots, r\}$,*

$$E(\mathbf{P}_i|\varphi_i, \Lambda_i) = E(\mathbf{P}_j|c_{ji}(w_{ji}, \Lambda_i)\varphi_i, w_{ji}\Lambda_i).$$

13.12. Proposition 13.10.2 shows that $\dim \mathcal{E}_i(\nu) = \dim \mathcal{E}_j(\nu)$ for any $i, j \in \{1, \ldots, r\}$. Denote this common dimension by d. Choose a basis of $\varphi_i^1, \cdots, \varphi_i^d$ of $\mathcal{E}_i(\nu)$ such that each φ_i^k belongs to one subspace $\mathcal{E}_{i\mu}(\nu)$.

For any $f_i = (f_i^1, \cdots, f_i^d)$, where $f_i^k \in L^2(\sqrt{-1}\,\mathfrak{a}_i^*)$, define

$$\hat{f}_i^k(x) = \int_{\sqrt{-1}\,\mathfrak{a}_i^*} f_i^k(\Lambda_i) E(\mathbf{P}_{i\mu}|\varphi_i^k, \Lambda_i) d\Lambda_i,$$

where μ is uniquely determined by the inclusion $\varphi_i^k \in \mathcal{E}_{i\mu}(\nu)$, and

$$\hat{f}_i(x) = \sum_{k=1}^{d} \hat{f}_i^k(x). \tag{13.12.1}$$

Denote by $L^2_{\mathcal{C},\nu}(\sqrt{-1}\,\mathfrak{a}^*)$ the subspace of $\prod_{i=1}^r \prod_{k=1}^d L^2(\sqrt{-1}\,\mathfrak{a}_i^*)$ consisting of functions $f = (f_1, \ldots, f_r) = (f_1^1, \ldots, f_1^d; \cdots; f_r^1, \ldots, f_r^d)$ that satisfy the following condition: For any $i, j \in \{1, \ldots, r\}$, and $w_{ji} \in W(\mathfrak{a}_j, \mathfrak{a}_i)$,

$$c_{ji}(w_{ji}, \Lambda_i) \sum_{k=1}^d f_i^k(\Lambda_i)\varphi_i^k = \sum_{k=1}^d f_j^k(w_{ji}\Lambda_i)\varphi_j^k. \tag{13.12.2}$$

For any $f \in L^2_{\mathcal{C},\nu}(\sqrt{-1}\,\mathfrak{a}^*)$, define a norm

$$\|f\|^2 = \frac{1}{(2\pi)^\ell c} \sum_{i=1}^r \sum_{k=1}^d \int_{\sqrt{-1}\,\mathfrak{a}_i^*} \left|f_i^k(\Lambda_i)\right|^2 \mathrm{d}\Lambda_i,$$

where $\ell = \dim \mathfrak{a}_i$ and c is the number of chambers in $\mathfrak{a}_i$, both of which do not depend on i, and define

$$\hat{f}(x) = \sum_{i=1}^r \hat{f}_i(x). \tag{13.12.3}$$

13.13. Proposition (see [AR2], Main Theorem, p. 256, [OW2], Prop. 7.4). — *The map $f \in L^2_{\mathcal{C},\nu}(\sqrt{-1}\,\mathfrak{a}^*) \mapsto \hat{f} \in L^2(\Gamma\backslash X)$ is an isometric embedding onto $\sum_{i=1}^r \sum_{\mu=1}^{r_i} \sum_{\varphi\in\mathcal{E}_{i\mu}(\nu)} L^2_{\mathbf{P}_{i\mu},\varphi}(\Gamma\backslash X)$. Denote this image by $L^2_{\mathcal{C},\nu}(\Gamma\backslash X)$. Then*

$$L^2_{\mathcal{C}}(\Gamma\backslash X) = \sum_{\nu\in\mathrm{Spec}_d(\mathcal{C})} \bigoplus L^2_{\mathcal{C},\nu}(\Gamma\backslash X) = \sum_{\nu\in\mathrm{Spec}_d(\mathcal{C})} \bigoplus L^2_{\mathcal{C},\nu}(\sqrt{-1}\,\mathfrak{a}^*).$$

This proposition shows that the functional equations for Eisenstein series in 13.11 are the only relations between them. We can reformulate Proposition 13.13 as follows.

13.14. Proposition. — *For any rational parabolic subgroup $\mathbf{P}$ and an L^2-eigenfunction φ on $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$, identify $L^2(\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}}^{*+})$ with a subspace of $L^2(\Gamma\backslash X)$ through the map*

$$f \to \hat{f} = \int_{\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}}^{*+}} f(\Lambda) E(\mathbf{P}|\varphi, \Lambda)\,\mathrm{d}\Lambda.$$

Then the continuous subspace $L^2_{\mathrm{con}}(\Gamma\backslash X)$ is equal to the following direct sum:

$$\sum_{\mathbf{P}} \sum_{\varphi} \bigoplus L^2(\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}}^{*+}),$$

where $\mathbf{P}$ is over all Γ-conjugacy classes of proper rational parabolic subgroups, and φ is over an orthonormal basis of eigenfunctions of $L^2_{\mathrm{dis}}(\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}})$.

Proof. — For any function $f = (f_1, \dots, f_r) \in L^2_{C,\nu}(\sqrt{-1}\,\mathfrak{a}^*)$ above, it follows from the definition and Proposition 13.11 that for any $i, j \in \{1, \dots, r\}$, $w_{ij} \in W(\mathfrak{a}_i, \mathfrak{a}_j)$,

$$\sum_{k=1}^{d} \int_{\sqrt{-1}\, w_{ij}(\mathfrak{a}_j^{*+})} f_i^k(\Lambda_i) E(\mathbf{P}_i | \varphi_i, \Lambda_i)\, \mathrm{d}\Lambda_i = \sum_{k=1}^{d} \int_{\sqrt{-1}\, \mathfrak{a}_j^{*+}} f_j^k(\Lambda_j) E(\mathbf{P}_j | \varphi_j, \Lambda_j)\, \mathrm{d}\Lambda_j.$$

By Lemma 13.8, for each fixed i, and $1 \leq j \leq r$, $w_{ij}(\mathfrak{a}_j^+)$ are disjoint and their union is a dense subset of $\mathfrak{a}_i$ of full measure, i.e., its complement is of measure zero. This implies that

$$\hat{f}(x) = c \sum_{i=1}^{r} \sum_{k=1}^{d} \int_{\sqrt{-1}\, \mathfrak{a}_i^{*+}} f_i^k(\Lambda_i) E\big(P_i | \varphi_i, \Lambda_i\big)\, \mathrm{d}\Lambda_i.$$

By Proposition 13.10.2, $c_{ji}(w_{ji}, \Lambda_i)$ is unitary for $\Lambda_i \in \sqrt{-1}\,\mathfrak{a}_i$, and hence

$$\|f\|^2 = \frac{1}{(2\pi)^\ell} \sum_{i=1}^{r} \sum_{k=1}^{d} \int_{\sqrt{-1}\, \mathfrak{a}_i^{*+}} \big|f_i^k(\Lambda_i)\big|^2\, \mathrm{d}\Lambda_i.$$

The above equations together with Proposition 3.13 implies that

$$L^2_{C,\nu}(\Gamma\backslash X) = \sum_{i=1}^{r} \sum_{k=1}^{d} \bigoplus L^2\big(\sqrt{-1}\,\mathfrak{a}_{ik}^{*+}\big).$$

Since $L^2_C(\Gamma\backslash X) = \bigoplus_\nu L^2_{C,\nu}(\Gamma\backslash X)$, Proposition 13.9 implies that

$$L^2_{\mathrm{con}}(\Gamma\backslash X) = \sum_{C} \sum_{i=1}^{r} \sum_{k=1}^{d} \sum_{\varphi \in \mathcal{E}_{i\mu}} \bigoplus L^2(\sqrt{-1}\,\mathfrak{a}_{ik}^{*+}),$$

which is equivalent to the decomposition in the proposition. □

The decomposition of the continuous subspace $L^2_{\mathrm{con}}(\Gamma\backslash X)$ in 13.14 can be interpreted as a parametrization of the generalized eigenfunction in terms of the pair of compactifications $\overline{\Gamma\backslash X}^T$ and $\overline{\Gamma\backslash X}^{RBS}$.

First we describe a correspondence between the boundary components of $\overline{\Gamma\backslash X}^T$ and $\overline{\Gamma\backslash X}^{RBS}$. Let $\mathbf{P}_1, \cdots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of proper rational parabolic subgroups of $\mathbf{G}$. Then

$$\overline{\Gamma\backslash X}^T = \Gamma\backslash X \cup \coprod_{i=1}^{n} A_{\mathbf{P}_i}^+(\infty),$$

$$\overline{\Gamma\backslash X}^{RBS} = \Gamma\backslash X \cup \coprod_{i=1}^{n} \Gamma_{M_{\mathbf{P}_i}}\backslash X_{\mathbf{P}_i},$$

where $A^+_{\mathbf{P}_i}(\infty)$, $\Gamma_{M_{\mathbf{P}_i}}\backslash X_{\mathbf{P}_i}$ are called the boundary components of $\overline{\Gamma\backslash X}^T$, $\overline{\Gamma\backslash X}^{RBS}$ respectively. These boundary components correspond to each other through convergence of DM rays as follows. For any point $q \in \Gamma\backslash X(\infty)$, let $H \in A^+_{\mathbf{P}_i}(\infty) \subset \Delta(\Gamma\backslash X)$ be the unique element corresponding to q under the identification $\Gamma\backslash X(\infty) = \Delta(\Gamma\backslash X)$ in 11.3. The geodesic $\exp(tH)x_0$, $t \in \mathbb{R}$, in X projects to an EDM ray $\gamma(t)$ in $\Gamma\backslash X$ converging to H in the Tits compactification $\overline{\Gamma\backslash X}^T$. This ray $\gamma(t)$ also converges in $\overline{\Gamma\backslash X}^{RBS}$ to a unique boundary point in $\Gamma_{M_{\mathbf{P}_i}}\backslash X_{\mathbf{P}_i}$. This correspondence between the boundary components is inclusion reversing, i.e., $A^+_{\mathbf{P}}(\infty)$ is a face of $A^+_{\mathbf{P}'}(\infty)$ if and only if $\Gamma_{M_{P'}}\backslash X_{\mathbf{P}'}$ is a boundary component of $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$, i.e., $\Gamma_{M_{P'}}\backslash X_{\mathbf{P}'}$ is contained in the closure of $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$. Because of this correspondence, $\overline{\Gamma\backslash X}^T$ and $\overline{\Gamma\backslash X}^{RBS}$ form a pair of dual compactifications.

For every $q \in \Gamma\backslash X(\infty)$, identify it with $H \in A^+_{\mathbf{P}}(\infty) \subset \overline{\Gamma\backslash X}^T$ as above, and identify H with a point in $\mathfrak{a}^*_{\mathbf{P}}$ using the Killing form. For each L^2-eigenfunction φ on the boundary locally symmetric space $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$ and a positive number r, define

$$E(q,\varphi,r) = E\big(\mathbf{P} \mid \varphi, \sqrt{-1}\,\sqrt{r}H\big).$$

Let $\varphi_1,\dots,\varphi_k,\cdots$ be eigenfunctions on $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$, $\Delta\varphi_k = \nu_k\varphi_k$, which form a basis of $L^2_d(\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}})$. Then we have the following.

13.15. Proposition (see 2.2). — *For $q \in \Gamma\backslash X(\infty)$, $r > 0$ and $n \geq 1$, the set of functions $E(q,\varphi_k,r)$ form a basis of the generalized eigenspace of eigenvalue $\lambda = |\rho|^2 + r^2 + \nu_k$ in the continuous subspace $L^2_{\text{con}}(\Gamma\backslash X)$.*

Proof. — Let $\mathbf{P}_1,\dots,\mathbf{P}_n$ be representatives of all the Γ-conjugacy classes of proper rational parabolic subgroups in $\mathbf{G}$ as above. For every $\mathbf{P}_i$, $1 \leq i \leq n$, let $\varphi_{i,1},\cdots,\varphi_{i,k},\dots,$ be eigenfunctions on $\Gamma_{M_{\mathbf{P}_i}}\backslash X_{\mathbf{P}_i}$ of eigenvalues $\nu_{i,k}$ which form an orthonormal basis of $L^2_{\text{dis}}(\Gamma_{M_i}\backslash X_{\mathbf{P}_i})$. Then by Proposition 13.14, we get

$$L^2_{\text{con}}(\Gamma\backslash X) = \sum_{i=1}^{n}\sum_{k\geq 1} \oplus L^2\big(\sqrt{-1}\,\mathfrak{a}^{*+}_{\mathbf{P}_i}\big),$$

where $L^2(\sqrt{-1}\,\mathfrak{a}^{*+}_{\mathbf{P}_i})$ is identified with a subspace of $L^2_{\text{con}}(\Gamma\backslash X)$ through the map

$$f(\Lambda) \longmapsto \hat{f}(x) = \int_{\sqrt{-1}\,\mathfrak{a}^{*+}_{\mathbf{P}_i}} f(\Lambda)E\big(\mathbf{P}_i \mid \varphi_{i,k},\Lambda\big)\,\mathrm{d}\Lambda.$$

Combined with Corollary 13.6, this implies that for any λ in the continuous spectrum of $\Gamma\backslash X$, the generalized eigenspace of $\Gamma\backslash X$ with eigenvalue λ has a basis of $E(\mathbf{P}_i|\varphi_{i,k},\Lambda)$, where $\nu_{i,k} < \lambda - |\rho_{\mathbf{P}_i}|^2$, and $\Lambda \in \sqrt{-1}\,\mathfrak{a}_{\mathbf{P}_i}^{*+}$, $|\Lambda|^2 = \lambda - |\rho_{\mathbf{P}_i}|^2 - \nu_{i,k}$. Since $\Gamma\backslash X(\infty) = \coprod_1^n A_{\mathbf{P}_i}^{+}(\infty)$, this implies that a basis of the generalized eigenspace of λ is given by $E(q,\varphi,r)$, where $q \in \Gamma\backslash X(\infty)$, the eigenvalue ν of φ satisfies $\nu < \lambda - |\rho_{\mathbf{P}}|^2$, and $r = \sqrt{\lambda - |\rho_{\mathbf{P}}|^2 - \nu}$. This completes the proof of the proposition. □

13.16. *Remark.* — The generalized eigenspaces for the continuous spectrum are defined as follows. When a spectral decomposition of the Laplace operator Δ is obtained by a Fourier type transformation, the set of functions appearing in the transformation satisfy the eigenfunction equation $\Delta E = \lambda E$. If E is square integrable, E is an eigenfunction, otherwise, E is called a generalized eigenfunction. Then for any λ in the continuous spectrum, the set of generalized eigenfunctions of eigenvalue λ span the generalized eigenspace of λ by superposition (or direct integral). In the example of $\mathbb{R}^n$, the generalized eigenfunctions are the exponential functions $e^{\sqrt{-1}\langle x,\Lambda\rangle}$, and for $\Gamma\backslash X$, the generalized eigenfunctions are Eisenstein series. The point of the above proposition is to get a basis of the generalized eigenspace.

There is some ambiguity in the definition of a basis of generalized eigenspaces. As a Hilbert space, $L^2(\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}_i}^{*+}) = L^2(\sqrt{-1}\,\overline{\mathfrak{a}_{\mathbf{P}_i}^{*+}})$. This implies that we can only parametrize a basis of a generalized eigenspace up to a set of measure zero. In order to get a concise statement in Proposition 13.15, we used $L^2(\sqrt{-1}\,\mathfrak{a}_{\mathbf{P}_i}^{*+})$ instead of $L^2(\sqrt{-1}\,\overline{\mathfrak{a}_{\mathbf{P}_i}^{*+}})$.

As mentioned in 2.2, Proposition 13.15 can be interpreted as follows. The wave function $\exp(\rho_{\mathbf{P}} + \sqrt{-1}\sqrt{r}H, H_{\mathbf{P}}(x))\varphi(z_{\mathbf{P}}(x))$ coming from infinity in the direction of H and propagates into $\Gamma\backslash X$ along the geodesic $\gamma(t)$. After scattering in $\Gamma\backslash X$, this wave function produces a wave function on $\Gamma\backslash X$, which is exactly the Eisenstein series $E(\mathbf{P}|\varphi,\Lambda)$, where $\Lambda = \sqrt{-1}\sqrt{r}H$. Proposition 13.14 shows that every wave function on $\Gamma\backslash X$ is obtained this way, and different triples (H,φ,r) produce different wave functions.

In scattering theory of Schrodinger operators on $\mathbb{R}^n$ or its domains, there is a close connection between rays in geometric optics and the quantum scattering (see [KE1] and [ME]). Proposition 13.15 gives the first connection between rays and scattering on $\Gamma\backslash X$.

Since the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ plays a very important role in describing the wave functions (or scattering states) as above, a natural question is the following.

13.17. QUESTION. — *Can* $\overline{\Gamma\backslash X}^{RBS}$ *be described intrinsically in terms of DM rays?*

The answer is positive and worked out in §14. In fact, the boundary components can be identified with the metric links of DM rays as described in 1.6.

The scattering of the wave from infinity

$$\exp\big(\rho_{\mathbf{P}} + \sqrt{-1}\,H, H_{\mathbf{P}}(x)\big)\varphi\big(z_{\mathbf{P}}(x)\big)$$

along directions in $A^+_{\mathbf{P}'}(\infty)$ of another parabolic subgroup $\mathbf{P}'$ is given by the intertwining operators introduced in 13.10. For a function in $L^2(\Gamma\backslash X)$, whether it belongs to the discrete subspace $L^2_{\mathrm{dis}}(\Gamma\backslash X)$ or the continuous subspace $L^2_{\mathrm{con}}(\Gamma\backslash X)$ is determined by its constant terms along rational parabolic subgroups of $\mathbf{G}$. If $\mathbf{P}'$ is associate to $\mathbf{P}$ and φ is a cuspidal eigenfunction, then the constant term $E_{\mathbf{P}'}(\mathbf{P}|\varphi,\Lambda)$ of $E(\mathbf{P}|\varphi,\Lambda)$ along $\mathbf{P}'$ is given by [HC], Thm. 5, p. 44,

$$E_{\mathbf{P}'}(\mathbf{P}\mid\varphi,\Lambda)(x) = \sum_{w\in W(\mathfrak{a}',\mathfrak{a})} \mathrm{e}^{(\rho_{\mathbf{P}'}+w\Lambda)(H_{\mathbf{P}'}(x))}c(w:\Lambda)\varphi(x).$$

Because of this equation, the intertwining operators $c(w:\Lambda)$ are also called scattering matrices. Then a natural question arises:

13.18. QUESTION. — *How to understand the scattering matrices in terms of DM rays?*

If $\Gamma\backslash X$ is a Riemann surface, Guillemin proved in [GU] a beautiful relation between the scattering matrix and the sojourn times of scattering geodesics, where scattering geodesics are those geodesics which are EDM in both directions and the sojourn time is a suitably normalized length of the segment of scattering geodesics contained in the compact core of $\Gamma\backslash X$. This gives a much stronger relation between the geometry at infinity and the scattering theory than Proposition 13.15.

In [JZ], Guillemin's result has been generalized to all $\Gamma\backslash X$ of $\mathbb{Q}$-rank 1, i.e., when the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 1. Instead of an exact expression for the scattering matrices in terms of sojourn times of the scattering geodesics, it is shown in [JZ] that the frequencies of oscillation of the scattering matrices $c(w:\Lambda)$ as $\Lambda \to \infty$ are exactly the sojourn times of the scattering geodesics. The method of [JZ] only works for the $\mathbb{Q}$-rank 1 case. One difficulty in the higher $\mathbb{Q}$-rank case is that instead of 1-dimensional scattering geodesics, we should also consider higher dimensional scattering flats, and the sojourn times need to be generalized.

14. $\overline{\Gamma\backslash X}^{BS}$, $\overline{\Gamma\backslash X}^{RBS}$, and DM rays.

14.1. In this section, we recover the boundary components of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{RBS}$ from DM rays converging to them (see 14.12–14.13), in particular, we identify the boundary components of $\overline{\Gamma\backslash X}^{RBS}$ with reduced metric links of DM rays as mentioned in 1.6, thus answering Question 13.17 positively. Combining this description with the results in §13, we propose a description of the continuous spectrum of a "geometric finite" Riemannian manifold in (14.20). We also identify the boundaries $\partial(\overline{\Gamma\backslash X}^{BS})$ and $\partial(\overline{\Gamma\backslash X}^{RBS})$ with certain equivalence classes of DM rays (see 14.16, 14.19). This section is closely related to the study of geodesics in the symmetric space X in [KA].

In 14.2–14.4, we study the $\mathbb{Q}$-rank 1 case and introduce the N-relation. In 14.5–14.6, we introduce congruence bundles of DM rays and use it to define the rank of a DM ray in 14.7. In 14.8–14.9, we define the L-relation on DM rays. In 14.10, we define the mobility degree of a DM ray. Then in 14.12–14.13, we recover the boundary components of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{RBS}$ in terms of DM rays converging to them. In 14.14, we introduce the R-relation on DM rays. We prove in 14.16 that the set of RL-equivalence classes can be identified with $\partial\overline{\Gamma\backslash X}^{BS}$ and in 14.19 that the set of NRL-equivalence classes can be identified with $\partial\overline{\Gamma\backslash X}^{RBS}$.

14.2. Proposition. — *If the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 1, then the set of EDM geodesics in $\Gamma\backslash X$ corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{BS})$ through the map $\gamma(t) \mapsto \lim_{t\to+\infty}\gamma(t)$.*

Proof. — First, we show that every EDM geodesic is convergent in $\overline{\Gamma\backslash X}^{BS}$. By Theorem 10.18, every EDM geodesic $\gamma(t)$ in $\Gamma\backslash X$ is the projection of $\tilde\gamma(t) = (u, z, \exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ for some rational parabolic subgroup $\mathbf{P}$. From the description of the topology of $\overline{X}^{BS}$ in 7.3, it is clear that $\tilde\gamma(t)$ converges to the boundary point $(u, z) \in e(\mathbf{P})$ in $\overline{X}^{BS}$, and hence $\gamma(t)$ converges to the image of (u, z) in $\partial(\overline{\Gamma\backslash X}^{BS})$ as $t \to +\infty$.

Therefore, the map $\gamma(t) \mapsto \lim_{t\to\infty} \gamma(t)$ is well-defined and is easily seen to be surjective.

By assumption, $\dim A_{\mathbf{P}} = 1$. Then by Corollary 10.20, $\gamma(t)$ is uniquely determined by the image of $(u, z) \in e(\mathbf{P})$ in $\partial(\overline{\Gamma\backslash X}^{BS})$. This implies that this map is also injective. □

14.3. Definition. — *Two EDM geodesics $\gamma_1(t)$, $\gamma_2(t)$ in $\Gamma\backslash X$ are N-related (nil), denoted by $\gamma_1(t) \overset{N}{\sim} \gamma_2(t)$, if $\lim_{t\to+\infty} d(\gamma_1(t), \gamma_2(t)) = 0$ for suitable parametrizations of γ_1, γ_2.*

14.4. Lemma. — *For every rational parabolic subgroup $\mathbf{P}$, and two geodesics $\tilde{\gamma}_i(t) = (u_i, z_i, a_i \exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ with $H \in \mathfrak{a}_{\mathbf{P}}^+$, $\log a_i \perp H$, $i = 1, 2$, when $t \gg 0$,*

$$d_{\Gamma\backslash X}(\gamma_1(t), \gamma_2(t)) \geq d_{\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}\times A_{\mathbf{P}}}((z_1, a_1), (z_2, a_2)) \geq d_{\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}}(z_1, z_2),$$

where (z_i, a_i) also denotes the image of (z_i, a_i) in $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}} \times A_{\mathbf{P}}$, and z_i denotes the image of z_i in $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$.

Proof. — Since $d_{\Gamma\backslash X}(\gamma_1(t), \gamma_2(t)) = \inf_{g\in\Gamma} d_X(\tilde{\gamma}_1(t), g\tilde{\gamma}_2(t))$, it suffices to prove that for every $g \in \Gamma$,

$$d_X(\tilde{\gamma}_1(t), g\tilde{\gamma}_2(t)) \geq d_{\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}\times A_{\mathbf{P}}}((z_1, a_1), (z_2, a_2)).$$

When $g \in \Gamma_P$, $g\tilde{\gamma}_2(t) = (g_N u_2, g_M z_2, a_2 \exp tH)$, where $g_M \in \Gamma_{M_{\mathbf{P}}}$. By Lemma 10.3,

$$\begin{aligned} d_X\big((u_1, z_1, a_1 \exp tH),&(g_N u_2, g_M z_2, a_2 \exp tH)\big) \\ &\geq d_{X_{\mathbf{P}}\times A_{\mathbf{P}}}\big((z_1, a_1), (g_M z_2, a_2)\big) \\ &\geq d_{\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}\times A_{\mathbf{P}}}\big((z_1, a_1), (z_2, a_2)\big). \end{aligned}$$

On the other hand, when $g \notin \Gamma_P = \Gamma \cap P$, Proposition 10.8 implies that for $t \gg 0$,

$$d_X\big(\tilde{\gamma}_1(t), g\tilde{\gamma}_2(t)\big) > d_X\big(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)\big) \geq d_{\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}\times A_{\mathbf{P}}}\big((z_1, a_1), (z_2, a_2)\big).$$

This completes the proof. □

14.5. Proposition. — *Assume $r_{\mathbb{Q}}(\mathbf{G}) = 1$. The N-relation is an equivalence relation, and all the EDM geodesics in one equivalence class converge to the same boundary point in $\partial(\overline{\Gamma\backslash X}^{RBS})$. Furthermore, the set of the N-equivalence classes corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{RBS})$ through the map $\gamma \mapsto \lim_{t\to+\infty} \gamma(t)$.*

Proof. — Since $\overline{\Gamma\backslash X}^{BS}$ dominates $\overline{\Gamma\backslash X}^{RBS}$, it follows from Proposition 14.2 that every EDM geodesic converges to a boundary point in $\overline{\Gamma\backslash X}^{RBS}$, and every boundary point is the limit of some EDM geodesic. We need to show that two EDM geodesics converge to the same boundary point if and only if they are N-related.

If two EDM geodesics $\gamma_i(t)$, $i = 1, 2$, are projections of $\tilde{\gamma}_i(t) = (u_i, z_i, a_i \exp tH_i) \in N_{\mathbf{P}_i} \times X_{\mathbf{P}_i} \times A_{\mathbf{P}_i}$ as in Theorem 10.18, and $\mathbf{P}_1, \mathbf{P}_2$ are not Γ-conjugate, then Proposition 11.3 implies that γ_1, γ_2 are not equivalent and hence not N-related. On the other hand, when $\mathbf{P}_1$, $\mathbf{P}_2$ are Γ-conjugate, we can assume that $\mathbf{P}_1 = \mathbf{P}_2$. By Lemma 11.4, when γ_1 is N-related to γ_2, z_1, z_2 project to the same point in $\Gamma_{M_{\mathbf{P}_1}}\backslash X_{\mathbf{P}_1}$, and hence $\gamma_1(t), \gamma_2(t)$ converge to the same boundary point in $\overline{\Gamma\backslash X}^{RBS}$. On the other hand, when $\gamma_1(t), \gamma_2(t)$ converge to the same boundary point in $\overline{\Gamma\backslash X}^{RBS}$, we can choose lifts $\tilde{\gamma}_1(t)$, $\tilde{\gamma}_2(t)$ such that $z_1 = z_2$. Since $\dim A_{\mathbf{P}_i} = 1$, we can choose $a_1 = a_2 = \mathrm{id}$. Then Lemma 10.3 implies that $\lim_{t\to+\infty} d_X(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) = 0$ and hence γ_1, γ_2 are N-related. □

14.6. To study the higher $\mathbb{Q}$-rank case, we need to introduce further relations between EDM geodesics. For any EDM geodesic γ in $\Gamma\backslash X$, define the congruence bundle(10) of EDM geodesics containing γ as follows:

$$\text{C-Bundle}(\gamma) = \big\{\gamma' \mid d(\gamma(t), \gamma'(t)) = c \text{ a constant, for } t \gg 0\big\},$$

where γ' is given a suitable parametrization. Define $\delta(\gamma, \gamma') = c$, which is also equal to $\lim_{t\to+\infty} d(\gamma(t), \gamma')$, where $d(\gamma(t), \gamma') = \inf\{d(\gamma(t), \gamma'(s)) \mid s \in \mathbb{R}\}$. Similarly, for any two EDM rays $\gamma_1, \gamma_2 \in$ C-Bundle(γ), we can define $\delta(\gamma_1, \gamma_2)$.

14.7. Lemma. — *For any EDM geodesic γ in $\Gamma\backslash X$, $\delta(\,\cdot\,,\cdot)$ defines a metric on* C-Bundle(γ), *and the metric space* (C-Bundle$(\gamma), \delta$) *is complete.*

Proof. — By Theorem 10.18, there exists a parabolic subgroup $\mathbf{P}$ such that a lift in X of $\gamma(t)$ is of the form $\tilde{\gamma}(t) = (u, z, a\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $u \in N_{\mathbf{P}}, z \in X_{\mathbf{P}}$, $a \in A_{\mathbf{P}}$, and $H \in A^+_{\mathbf{P}}(\infty)$, $\log a \perp H$. By Proposition 11.3, any γ' in C-Bundle(γ) has a lift in X of the form $\widetilde{\gamma'}(t) = (u', z', a' \exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $\log a' \perp H$. Since $d_{\Gamma\backslash X}(\gamma(t), \gamma'(t)) = c$ when $t \gg 0$, where c is a constant, there exists a lift $\widetilde{\gamma'}$

(10) The name of a bundle of EDM geodesics is suggested by the bundles of geodesics in [KA].

such that $d_X(\tilde{\gamma}(t), \widetilde{\gamma'}(t)) = c$ when $t \gg 0$. By Lemma 10.3, this implies that $u = u'$. Conversely, when $u = u'$, any two such geodesics $\tilde{\gamma}$, $\widetilde{\gamma'}$ in X project to two congruent geodesics in $\Gamma\backslash X$. These two geodesics $\tilde{\gamma}$, $\widetilde{\gamma'}$ project to the same geodesic in $\Gamma\backslash X$ if and only if (z, a), (z', a) project to the same point in $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}} \times A_{\mathbf{P}}$. Therefore, we get the following identification:

$$\text{C-Bundle}(\gamma) \cong \Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}} \times \text{ Span}(H)^{\perp},$$

where $\text{Span}(H)^{\perp}$ is the orthogonal complement in $\mathfrak{a}_{\mathbf{P}}$ of the line $\text{Span}(H)$ containing H, and $\Gamma_{M_{\mathbf{P}}}$ is the discrete subgroup of $M_{\mathbf{P}}$ projected from Γ_P under the map $P = N_{\mathbf{P}}M_{\mathbf{P}}A_{\mathbf{P}} \to M_{\mathbf{P}}$.

By Lemma 11.4, under this identification, $\delta(\,.\,,.)$ is the distance function on $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}} \times \text{Span}(H)^{\perp}$ induced from the Riemannian metric of $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}} \times A_{\mathbf{P}}$ and hence complete. □

14.8. Definition. — *For any EDM geodesic γ in $\Gamma\backslash X$, the rank of γ is defined as*

$$r(\gamma) = \max\big\{k \in \mathbb{Z} \mid \textit{ there exists a faithful isometric action of } \mathbb{R}^{k-1} \textit{ on C-Bundle}(\gamma)\big\}.$$

An EDM geodesic γ' in C-Bundle(γ) is defined L-related (linearly) to γ if γ, γ' belong to one $\mathbb{R}^{r-1}$-orbit of a congruence bundle C-Bundle(γ) of the isometric action of $\mathbb{R}^{r-1}$, where $r = r(\gamma)$.

14.9. Lemma. — *The L-relation is an equivalence relation on the set of EDM geodesics in $\Gamma\backslash X$.*

Proof. — If $\gamma(t)$ is the image of $\tilde{\gamma}(t) = (u, z, a\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ with $H \in \mathfrak{a}^+$, then the rank of γ is equal to $\dim A_{\mathbf{P}}$, i.e., the $\mathbb{Q}$-rank of $\mathbf{P}$. By definition, if two EDM geodesics are L-related, then they belong to the same congruence bundle. Two geodesics $\widetilde{\gamma_i} = (u_i, z_i, a_i \exp tH)$, $i = 1, 2$, in C-Bundle(γ) project to two L-related EDM geodesics in $\Gamma\backslash X$ if and only if u_1, u_2 project to the same point in $\Gamma_{N_{\mathbf{P}}}\backslash N_{\mathbf{P}}$, and z_1, z_2 project to the same point in $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$. Then it is clear that the L-relation defines an equivalence relation. □

14.10. Definition.

1) *For any EDM geodesic in $\Gamma\backslash X$, the finite bundle of EDM geodesics containing γ is defined as*

$$\textit{F-Bundle}(\gamma) = \big\{\gamma' \mid \lim_{t\to+\infty} \sup d(\gamma(t), \gamma'(t)) < +\infty\big\}.$$

2) *A finite bundle F-Bundle(γ) of EDM geodesics corresponds to an equivalence class defined in* (1.1, 9.1) *and hence a point* $q = [\gamma] \in \Gamma\backslash X(\infty)$. *The set of the* N*-equivalence classes in this finite bundle, F-Bundle(γ)$/N$, is called the metric link of the finite bundle F-Bundle(γ), and denoted by* $S(\gamma)$. *It is also called the metric link of the point* $q \in \Gamma\backslash X(\infty)$, *and denoted by* $S(q)$ *as in* 1.6.

3) *The L-relation restricts to an equivalence relation on a F-Bundle(γ), and the dimension of the quotient F-Bundle(γ)$/L$ is called the mobility degree of the EDM ray* γ.

4) *The L-relation defines an equivalence relation on the metric link* $S(\gamma)$, *and the quotient* $S(\gamma)/L$ *is called the reduced metric link and denoted by* $\widetilde{S}(\gamma)$. (*See Remark* 14.20 *for another definition of the reduced metric link.*)

Remarks. — 1) The F-Bundle(γ) is just the equivalence class of EDM rays containing γ in the sense (see 1.1, 9.1). Such a set of geodesics in the symmetric space X is called a finite bundle by Karpelevic in [KA]. Since we have several relations in this section, we use the finite bundle to distinguish it from other relations.

2) The metric link $S(\gamma)$ can be identified with the congruence bundle C-Bundle(γ) in 14.5.

14.11. Recall from 7.3 that for any rational parabolic subgroup $\mathbf{P}$, $e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}$ is the boundary component associated with $\mathbf{P}$ in the partial compactification $\overline{X}^{BS}$. Then the image of $e(\mathbf{P})$ in $\overline{\Gamma\backslash X}^{BS}$ is $\Gamma_P\backslash e(\mathbf{P})$, where $\Gamma_P = \Gamma \cap P$. Let $\mathbf{P}_1, \dots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of proper rational parabolic subgroups in $\mathbf{G}$. Then

$$\overline{\Gamma\backslash X}^{BS} = \Gamma\backslash X \cup \coprod_{i=1}^{n} \Gamma_i\backslash e(\mathbf{P}_i),$$

where $\Gamma_i = \Gamma \cap P_i$, and $\Gamma_i\backslash e(\mathbf{P}_i)$'s are called the boundary components of $\overline{\Gamma\backslash X}^{BS}$. Similarly,

$$\overline{\Gamma\backslash X}^{RBS} = \Gamma\backslash X \cup \coprod_{i=1}^{n} \Gamma_{M_i}\backslash \hat{e}(\mathbf{P}_i),$$

where $\hat{e}(\mathbf{P}_i) = X_{\mathbf{P}_i}$ and $\Gamma_{M_i} \cong \Gamma_i \cap N_{\mathbf{P}_i}\backslash\Gamma_i$ is the image of Γ_i under the projection $P_i \to M_{P_i}$, and $\Gamma_{M_i}\backslash\hat{e}(\mathbf{P}_i)$ are called the boundary components of $\overline{\Gamma\backslash X}^{RBS}$.

14.12. PROPOSITION. — *Every EDM geodesic γ converges to a boundary point in $\overline{\Gamma\backslash X}^{BS}$. Let $\Gamma_i\backslash e(\mathbf{P}_i)$ be the unique boundary component containing $\lim_{t\to+\infty}\gamma(t)$. Then $\Gamma_i\backslash e(\mathbf{P}_i)$ can be identified with F-Bundle(γ)/L, the set of L-classes in F-Bundle(γ).*

Proof. — By Theorem 10.18, γ is the projection in $\Gamma\backslash X$ of a geodesic in X of the form $\tilde{\gamma}(t) = (u, z, a\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $u \in N_{\mathbf{P}}, z \in X_{\mathbf{P}}$, $a \in A_{\mathbf{P}}$, and $H \in A^+_{\mathbf{P}}(\infty)$. Then $\tilde{\gamma}(t)$ converges to $(u, z) \in e(\mathbf{P})$ in $\overline{X}^{BS}$ as $t \to +\infty$, and hence $\gamma(t)$ converges in $\overline{\Gamma\backslash X}^{BS}$ to the image of (u, z) in $\Gamma_i\backslash e(\mathbf{P}_i)$, where $\mathbf{P}_i$ is the unique representative which is Γ-conjugate to $\mathbf{P}$. By Proposition 11.3, all EDM geodesics in F-Bundle(γ) have lifts in X of the form $(u', z', a'\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $u' \in N_{\mathbf{P}}, z' \in X_{\mathbf{P}}$, $a' \in A_{\mathbf{P}}$. Then it is clear from the definition of the L-relation that the quotient F-Bundle(γ)/L can be canonically identified with $\Gamma_i\backslash e(\mathbf{P}_i)$ by mapping the geodesic $(u', z', a'\exp(tH))$ to the image of (u', z') in $\Gamma_i\backslash e(\mathbf{P}_i)$. □

Similarly, we have the following description of the boundary components of $\overline{\Gamma\backslash X}^{RBS}$.

14.13. PROPOSITION (see 1.6). — *Every EDM geodesic γ converges to a boundary point in $\overline{\Gamma\backslash X}^{RBS}$. Let $\Gamma_{M_i}\backslash\hat{e}(\mathbf{P}_i)$ be the unique boundary component containing the limit point $\lim_{t\to+\infty}\gamma(t)$. Then $\Gamma_{M_i}\backslash\hat{e}(\mathbf{P}_i)$ can be identified with the reduced metric link $\widetilde{S}(\gamma)$, i.e., the boundary component $\Gamma_{M_i}\backslash\hat{e}(\mathbf{P}_i)$ can be identified with the reduced metric link $\widetilde{S}(q)$, where $q = [\gamma] \in \Gamma\backslash X(\infty)$.*

In the following, we introduce more equivalence relations on EDM geodesics and identify the boundaries $\partial(\overline{\Gamma\backslash X}^{BS})$ and $\partial(\overline{\Gamma\backslash X}^{RBS})$ with certain equivalence classes of EDM geodesics.

14.14. DEFINITION.

1) *Two L-equivalence classes $[\gamma_0]_L$ and $[\gamma_1]_L$ are R-related (rotationally) if there exist representatives $\gamma_0(t)$ and $\gamma_1(t)$ and a family of EDM geodesics $\gamma_s(t)$ connecting them such that the mobility degree of $\gamma_s(t)$ does not change, and $d(\gamma_{s_1}(t), \gamma_{s_2}(t)) = c|s_1 - s_2|t$ when $t \geq 0$, where c is some constant.*

2) *Two geodesics $\gamma_0(t)$ and $\gamma_1(t)$ are RL-related if their L-classes $[\gamma_0(t)]_L$ and $[\gamma_1(t)]_L$ are R-related.*

14.15. LEMMA. — *The RL-relation is an equivalence relation on EDM geodesics on $\Gamma\backslash X$.*

Proof. — From the proof of Lemma 14.9, we see that any EDM γ geodesic in $\Gamma\backslash X$ which is the projection of a geodesic in X of the form $\tilde{\gamma}(t) = (u, z, a\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ has mobility degree equal to $\dim N_{\mathbf{P}} + \dim X_{\mathbf{P}}$.

We claim that another EDM geodesic γ' is RL-related to γ if and only if γ' has a lift in X of the form $\widetilde{\gamma'}(t) = (u, z, a'\exp(tH')) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $a' \in A_{\mathbf{P}}$, $H' \in \mathfrak{a}_{\mathbf{P}}^{+}$ are arbitrary. Then the lemma follows easily from the claim.

Suppose first that γ' has such a lift. Recall the disjoint decomposition $X = \coprod_0^n A_{\mathbf{P}_i,T}x_0$ from Proposition 4.6. By the proof of Lemma 14.9, the L-equivalence class of γ does not depend on a. We can choose $a = a'$ and $\log a \gg 0$ such that for some i and $t \geq 0$, $\gamma(t), \gamma'(t) \in \omega_i A_{\mathbf{P}_i,T}$. Connect γ and γ' by the family of DM rays which are projections of $\tilde{\gamma}_s(t) = (u, z, a\exp(tH_s)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$, where $H_s = sH + (1-s)H'$, $s \in [0,1]$. Then this family satisfies the property in Definition 14.14. In fact, let w be the component of $\gamma(0) = \gamma'(0)$ in ω_i. Then the family of rays $\gamma_s(t), t \geq 0$, are contained in $wA_{\mathbf{P}_i,T}$. By the same proof of Proposition 5.12, we can show that the Riemann distance of $\Gamma\backslash X$ restricts to the metric d_S on $wA_{\mathbf{P}_i,T} \cong \mathfrak{a}_{\mathbf{P}_i,T}$ induced from the Killing form. Then the property in Definition 14.14 is clear.

On the other hand, suppose that γ and γ' are RL-related. Then $\gamma(0) = \gamma'(0)$. Let $\widetilde{\gamma'}(t) = (u', z', a'\exp(tH')) \in N_{\mathbf{P}'} \times X_{\mathbf{P}'} \times A_{\mathbf{P}'}$ be the lift of $\gamma'(0)$ with $\tilde{\gamma}'(0) = \tilde{\gamma}(0) \in X$. we claim that $\mathbf{P}' = \mathbf{P}$. If not, the connecting family of EDM geodesics $\gamma_s(t)$ also lifts to a family $\tilde{\gamma}_s(t) = (u(s), z(s), a(s)\exp(tH(s))) \in N_{\mathbf{P}(s)} \times X_{\mathbf{P}(s)} \times A_{\mathbf{P}(s)}$ with $\tilde{\gamma}_s(0) = \tilde{\gamma}(0)$, where $\mathbf{P}(s)$ is a parabolic subgroup. Since $\mathbf{P} \neq \mathbf{P}'$, the dimension of $A_{\mathbf{P}(s)}$ will change as s changes from 0 to 1. By the first paragraph, this implies that the mobility degree of γ_s will change also. This contradicts the assumption on the family γ_s. So $\mathbf{P}' = \mathbf{P}$. Since $\tilde{\gamma}(0) = \tilde{\gamma}'(0)$, $uaz = u'a'z'$. By the uniqueness of the horoshperical decomposition (3.5.2), $u = u', z = z'$, and hence the claim is proved. □

14.16. PROPOSITION. — *All the EDM geodesics in a RL-equivalence class converge to the same point in $\partial(\overline{\Gamma\backslash X}^{BS})$, and the set of the RL-equivalence classes corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{BS})$ through the map $\gamma(t) \mapsto \lim_{t\to+\infty} \gamma(t)$.*

Proof. — From the definition of $\overline{\Gamma\backslash X}^{BS}$ that an EDM geodesic which is the projection of a geodesic in X of the form $(u, z, a\exp(tH)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ converges to the image of uz in $\partial(\overline{\Gamma\backslash X}^{BS})$. Then Proposition 14.16 follows from the claim in the proof of Lemma 14.15. □

14.17. Definition. — *Two EDM geodesics γ_1 and γ_2 in $\Gamma\backslash X$ are NRL-related if there exists a EDM geodesic γ' such that γ' is RL-related to γ_1, and γ' is N-related to γ_2.*

14.18. Lemma. — *The NRL-relation defines an equivalence relation.*

Proof. — By Lemma 11.4 and Proposition 11.3, two EDM rays γ_1, γ_2 in $\Gamma\backslash X$ are N-related if and only if they have lifts of the form $\widetilde{\gamma}_i(t) = (u_i, z_i, a_i\exp(tH_i)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ with $z_1 = z_2$, $a_1 = a_2$, and $H_1 = H_2$ for some rational parabolic subgroup. Combining with the proof of Lemma 14.15 which identifies RL-classes, we conclude that two EDM rays γ_1, γ_2 are NRL-related if and only if they have lifts of the form $\widetilde{\gamma}_i(t) = (u_i, z_i, a_i\exp(tH_i)) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ with $z_1 = z_2$. □

14.19. Proposition. — *The set of the NRL-equivalence classes corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{RBS})$ through the map $\gamma \longrightarrow \lim_{t\to+\infty}\gamma(t)$.*

Proof. — It follows from Proposition 14.13 that every EDM geodesic in $\Gamma\backslash X$ converges to a boundary point of $\overline{\Gamma\backslash X}^{RBS}$. Then the proposition follows from the conclusion in the proof of Lemma 14.18. □

14.20. *Remark.* — Proposition 14.13 shows that $\overline{\Gamma\backslash X}^{RBS}$ is analogous to the Karpelevic compactification of the symmetric space in [KA], §13. In fact, as mentioned above, the definitions of finite bundles and N-relation are adapted from [KA].

Before divided out by the L-relation, the congruence bundles C-Bundle(γ) have infinite volume in general (see the proof of Lemma 14.6). So the L-relation is a renormalization procedure.

If we want to recover only the boundary components of $\overline{\Gamma\backslash X}^{RBS}$ from EDM geodesics converging to them, there is a better alternative way as mentioned in 1.6, which can be generalized to other manifolds.

Given an EDM geodesic γ in $\Gamma\backslash X$, consider the finite bundle of EDM geodesics F-Bundle(γ). Identifying any two geodesics which are N-related, we get the metric link $S(\gamma)$, which can also be identified with C-Bundle(γ).

By the proof of Lemma 14.7, this metric link $S(\gamma)$ is the product of a boundary locally symmetric space $\Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$ of finite volume and an Euclidean space. Then we renormalize the metric link $S(\gamma)$ by dropping the Euclidean factor to get the reduced metric link $\widetilde{S}(\gamma)$ that has finite volume.(11)

The advantage of this definition of the reduced metric link is that it can be generalized to other manifolds in the next remark.

14.21. *Remark.* — It seems that the approach in 14.20 can be generalized to a "geometrically finite" complete noncompact Riemannian manifold to yield boundary components of reductive Borel-Serre type. Then these boundary components may form the boundary of a compactification of the manifold which is dual to the geodesic compactification, whose topology satisfies the property that an EDM ray converges to a point in its boundary component. And this pair of dual compactifications can be used to describe the continuous spectrum of the manifold (see the next remark).

More precisely, by a geometric finite manifold M, we mean that Assumptions 9.11 and 9.16 are satisfied. The finite bundles and the N-relation on them can clearly be defined, and so the metric link can be defined. By normalizing the metric link in a suitable way, we get the reduced metric link of finite volume. The reduced metric links are analogues of the boundary components in the reductive Borel-Serre compactification. Clearly, two equivalent EDM geodesics define the same reduced metric link.

14.22. *Remark.* — Using this construction of boundary components, we propose the following description of the continuous spectrum of a "geometrically finite" complete manifold M. Any equivalence class of EDM rays $[\gamma] \in M(\infty)$ and a function (or an eigenfunction) on the boundary component of γ produce one dimensional continuous spectrum (a half line); and the union of these spectra is the continuous spectrum of M.

The results in §13 (see Proposition 13.15) show that this picture is true for $\Gamma\backslash X$. If $M = X$ is a Riemannian symmetric space of noncompact type, then the above picture is also true. In fact, for every geodesic γ in X, its metric link is a product of a Riemannian symmetric space of noncompact type of lower dimension and an Euclidean space [KA], §7. Since both factors

(11) We need this normalization in order to get square integrable eigenfunctions on the reduced metric link.

have infinite volume, the reduced metric link is a point, and hence the dual compactification is the one point compactification $M \cup \{\infty\}$. Then Harish-Chandra's theory of Plancherel formula shows that the above description of the continuous spectrum holds for X (see [J1] for more details).

If M is a manifold with corners and the metric is an exact b-metric as in [ME, 7.9], then Assumptions 9.11 and 9.16 are easily seen to be satisfied, because the metric has product structure near every point at infinity. The boundary components of EDM geodesics are the boundary faces of the manifold, and the dual compactification is homeomorphic to the manifold with the corners. And the above picture is consistent with a conjecture of Melrose [ME], Conj. 7.1. In fact, for boundary face of M of codimension p, it seems that the set of EDM classes in $M(\infty)$ whose boundary component is the given face form a simplex of dimension $p-1$. This should be the "(pseudo-)manifold" as remarked by Melrose in [ME], Footnote 38, p. 93.

15. The Martin compactification of $\Gamma\backslash X$.

15.1. In this section, we construct a minimal function in $C_\lambda(\Gamma\backslash X) = \{u \in C^\infty(\Gamma\backslash X) \mid \Delta u = \lambda u, u > 0\}$, $\lambda < 0$, for every point in the geodesic boundary $\Gamma\backslash X(\infty)$ (see 15.7). When the $\mathbb{Q}$-rank of $\Gamma\backslash X$ is equal to 1, we show that every minimal function in $C_\lambda(\Gamma\backslash X)$ is one of those constructed above and hence identify the minimal Martin boundary $\partial_{\lambda,\min}\Gamma\backslash X$ with $\Gamma\backslash X(\infty)$ (see 15.15). Suggested by these results, we conjecture that the geodesic compactification $\Gamma\backslash X \cup \Gamma\backslash X(\infty)$ is the Martin compactification of $\Gamma\backslash X$ (see 15.14). Since the geodesic boundary $\Gamma\backslash X(\infty)$ is the same as the Tits complex $\Delta(\Gamma\backslash X)$ (see 11.3, 11.8), in the following, we use the Tits compactification $\overline{\Gamma\backslash X}^T$ and its boundary $\Delta(\Gamma\backslash X)$ instead of the geodesic compactification for technical convenience.

In 15.2–15.4, we introduce the Martin compactification of a Riemannian manifold and related concepts. For every point in $\Delta(\Gamma\backslash X)$, we define an Eisenstein series in 15.5. To prove that these Eisenstein series are minimal 15.7, we recall the Martin compactification of X in 15.8–15.10. Then the minimal property follows from the ergodicity of the Γ-action on the maximal Furstenberg boundary of X. To identify minimal functions in $C_\lambda(\Gamma\backslash X)$ (see 15.15), we use the Harnack inequality to show that every element in $C_\lambda(\Gamma\backslash X)$ has moderate growth (see 15.17) and a minimal one is an automorphic form 15.18–15.19. Then the $\mathbb{Q}$-rank 1 assumption allows us to identify these automorphic forms.

15.2. Let M be a complete non-compact Riemannian manifold, Δ its Laplace operator which is normalized to be non-negative, and $\lambda_0(M)$ the bottom of the spectrum of Δ,

$$\lambda_0(M) = \inf_{u \in C_0^\infty(M)} \frac{\int_M \Delta(u) u}{\int_M u^2}.$$

Then it is known that the cone $C_\lambda(M) = \{u \in C^\infty(M) \mid \Delta u = \lambda u, u > 0\}$ is non-empty if and only if $\lambda \leq \lambda_0(M)$ (see [SU2]). Clearly, $C_\lambda(M)$ is a convex cone. A basic question in potential theory is to understand this cone, in particular, its generators.

We recall several basic facts about the Martin compactification. For more detailed discussions, see [GJT]. For $\lambda < \lambda_0(M)$, let $G_\lambda(x, y)$ be the Green function of $\Delta - \lambda$. Then $\Delta G_\lambda(x, y) = \lambda G_\lambda(x, y)$, $G_\lambda(x, y) > 0$ for every y. Fix a basepoint x_0 and define $K_\lambda(x, y) = G_\lambda(x, y)/G_\lambda(x_0, y)$. Then $K_\lambda(x_0, y) = 1$ for any $y \neq x_0$.

A sequence y_n in M going to infinity is called a fundamental sequence for the Martin compactification for λ if $K_\lambda(x, y_n)$ converges uniformly over compact subsets to a function on M. Two fundamental sequences are equivalent if they give rise to the same limit function. Then the Martin boundary $\partial_\lambda M$ is defined to be the set of such equivalence classes of fundamental sequences. For each equivalence class $\xi \in \partial_\lambda M$, we denote the corresponding limit function by $K_\lambda(x, \xi)$, which clearly satisfies $(\Delta - \lambda)K_\lambda(x, \xi) = 0$, $K_\lambda(x_0, \xi) = 1$, $K_\lambda(x, \xi) > 0$ for all $x \in M$. The topology of the Martin compactification $M \cup \partial_\lambda M$ is defined as follows: a sequence y_n in $M \cup \partial_\lambda M$ converges to a boundary point ξ if and only if $K_\lambda(x, y_n)$ converges to $K_\lambda(x, \xi)$ uniformly for x in compact subsets of M.

For $\lambda_0 = \lambda_0(M)$, if $G_{\lambda_0}(x, y)$ exists and is positive, we define the compactification $M \cup \partial_{\lambda_0} M$ as above. Otherwise, the cone C_{λ_0} is one dimensional, and hence $M \cup \partial_{\lambda_0} M$ is defined to be the one point compactification.

15.3. Definition. — *A function $u \in C_\lambda(M)$ is called minimal if any function $v \in C_\lambda(M)$ which is bounded by u is a multiple of u. A boundary point $\xi \in \partial_\lambda M$ is called minimal if the function $K_\lambda(x, \xi)$ is minimal.*

Clearly the minimal functions are the extremal elements of the cone $C_\lambda(M)$. Denote the set of these minimal points in $\partial_\lambda M$ by $\partial_{\lambda,\min} M$. Then $\partial_{\lambda,\min} M$ is a G_δ set, i.e., the union of a countable open sets. In some cases, for example, symmetric spaces X of rank greater than or equal to 2, it is a proper subset of $\partial_\lambda M$ (see [GJT] and Proposition 15.8 below).

The motivation for the introduction of the minimal Martin boundary is the following integral representation formula.

15.4. PROPOSITION. — *Given any positive solution u of $\Delta u = \lambda u$, i.e., $u \in C_\lambda(M)$, there exists a unique positive measure μ on $\partial_{\lambda,\min} M$ such that*

$$u(x) = \int_{\partial_{\lambda,\min} M} K_\lambda(x,\xi) d\mu(\xi).$$

This measure is called the representing measure of u. If v is another positive solution bounded by u, then the representing measure of v is absolutely continuous with respect to μ. In particular, all the minimal functions in $C_\lambda(M)$ are multiples of $K_\lambda(x,\xi)$, $\xi \in \partial_{\lambda,\min} M$.

Proof. — For domains in $\mathbb{R}^n$ and $\lambda = 0$, this result was proved by Martin in [MA], Thm. III, p. 160. The same argument also works in the more general situation here. □

The existence of $\partial_{\lambda,\min} M$ and the integral representation formula can also be obtained from Choquet Theorem in abstract potential theory [BR]. But the approach of Martin compactification is constructible.

In general, among this family of compactifications, the most interesting one corresponds to $\lambda = 0$, which is the classical Martin compactification and describes positive harmonic functions.

In our situation, $M = \Gamma\backslash X$, $\lambda_0(\Gamma\backslash X) = 0$, and all positive harmonic functions are constant, and hence $\Gamma\backslash X \cup \partial_0 \Gamma\backslash X$ is the one point compactification. Therefore, in the rest of this section, λ is assumed to be negative unless otherwise specified.

15.5. We now construct functions of $C_\lambda(\Gamma\backslash X)$ using Eisenstein series. For any rational parabolic subgroup $\mathbf{P}$, let $\rho_{\mathbf{P}}$ be the half of sum (with multiplicity) of the positive roots in $\Phi^+(P, A_{\mathbf{P}})$. Identifying the dual $\mathfrak{a}_{\mathbf{P}}^*$ with $\mathfrak{a}_{\mathbf{P}}$ using the Killing form $\langle\, .\, ,\, .\, \rangle$, we get $\rho_{\mathbf{P}} \in \mathfrak{a}_{\mathbf{P}}^+$.

For $\xi \in A_{\mathbf{P}}^+(\infty)$, i.e., $\xi \in \mathfrak{a}_{\mathbf{P}}^+$ and $\|\xi\| = 1$, choose a positive number $c = c(\xi)$ such that

$$\langle \rho_{\mathbf{P}} + c\xi, \rho_{\mathbf{P}} + c\xi \rangle = \langle \rho_{\mathbf{P}}, \rho_{\mathbf{P}} \rangle - \lambda$$

(this choice of $c(\xi)$ is explained in the proof of Lemma 15.6 below). Since $\lambda < 0$, such $c(\xi)$ exists and is unique. For any $x \in X$, write

$x = (u_{\mathbf{P}}(x), z_{\mathbf{P}}(x), \exp(H_{\mathbf{P}}(x))) \in N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ (3.5.2). Define the Eisenstein series $E_\xi(x, \lambda)$ for ξ by

$$(15.5.1) \qquad E_\xi(x,\lambda) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \exp(\langle 2\rho_{\mathbf{P}} + c\xi, H_{\mathbf{P}}(\gamma x)\rangle).$$

This Eisenstein series is a special case of $E(\mathbf{P}|\varphi, \Lambda)$ in 13.3 when $\varphi = 1$.

15.6. Lemma. — *The series defining $E_\xi(x, \lambda)$ converges absolutely, and $E_\xi(x, \lambda)$ is a positive solution of $\Delta u = \lambda u$.*

Proof. — By a result of Godement [HC], p. 31, Remark 1, this series converges uniformly for x in compact subsets, and hence defines a positive smooth function on X. Since $\Gamma_P = \Gamma \cap P$ leaves $H_{\mathbf{P}}(x)$ invariant, $E_\xi(x, \lambda)$ is Γ-invariant and hence descends to a function on $\Gamma\backslash X$.

From the expression for the horospherical part of the Beltrami-Laplace operator Δ of X (see also the proof of Lemma 13.4), we get

$$\Delta E_\xi(x,\lambda) = (\langle \rho_{\mathbf{P}} + c\xi, \rho_{\mathbf{P}} + c\xi\rangle - \langle \rho_{\mathbf{P}}, \rho_{\mathbf{P}}\rangle) E_\xi(x,\lambda) = \lambda E_\xi(x,\lambda).$$

(This equation explains the choice of $c = c(\xi)$ above.) □

Normalize the value of $E_\xi(x, \lambda)$ at x_0 by setting

$$\widetilde{E}_\xi(x,\lambda) = E_\xi(x,\lambda)/E_\xi(x_0,\lambda).$$

Then $\widetilde{E}_\xi(x, \lambda) > 0$, $\widetilde{E}_\xi(x_0, \lambda) = 1$, and $(\Delta - \lambda)\widetilde{E}_\xi(x, \lambda) = 0$. If $\xi_1, \xi_2 \in \Delta_{\mathbb{Q}}(X) = \coprod_{\mathbf{P}} A^+_{\mathbf{P}}(\infty)$ are conjugate under Γ, then $\widetilde{E}_{\xi_1}(x, \lambda) = \widetilde{E}_{\xi_2}(x, \lambda)$. Therefore, we can define a unique function $\widetilde{E}_\xi(x, \lambda)$ for every boundary point in $\xi \in \Delta(\Gamma\backslash X)$.

15.7. Proposition. — *For any $\xi \in \Delta(\Gamma\backslash X)$, the function $\widetilde{E}_\xi(x, \lambda)$ is minimal. And if $\xi' \in \partial(\overline{\Gamma\backslash X}^T)$ is a different point, then $\widetilde{E}_\xi(x, \lambda) \neq \widetilde{E}_\xi(x, \lambda)$.*

To prove this proposition, we need to recall several results on the Martin compactification of X. Since X uses the real Langlands decomposition while $\Gamma\backslash X$ uses the rational Langlands decomposition, we need to point out the difference between these two decompositions.

For every real parabolic subgroup P of the Lie group $G = \mathbf{G}(\mathbb{R})$, let A_P be the maximal split torus in P stable under the Cartan involution of the fixed basepoint x_0, and M_P the complement of A_P in the Levi factor of P. Then P admits a real Langlands decomposition $P = N_P M_P A_P$ as in (3.5.1). Define $X_P = M_P / K \cap M_P$. Then X_P is a symmetric space of noncompact type and called the boundary symmetric space for P. The real Langlands decomposition of P induces an horospherical decomposition $X = N_P \times X_P \times A_P$ as in (3.5.2). For any $x \in X$, denote its component in A_P by $\exp(H_P(x))$, and its component in X_P by $z_P(x)$.

When $\mathbf{P}$ is a rational parabolic subgroup of $\mathbf{G}$, its real locus P is a real parabolic subgroup of G. Then $N_{\mathbf{P}} = N_P$, but the maximal $\mathbb{Q}$-split torus $A_{\mathbf{P}}$ is contained in the maximal real split torus A_P, and $M_{\mathbf{P}}$ contains M_P. Unless $A_{\mathbf{P}} = A_P$, the boundary symmetric space $X_{\mathbf{P}}$ defined through the rational Langlands decomposition of P differs from the boundary symmetric space X_P defined for the real parabolic subgroup. In fact, let $A_P^{\mathbf{P}}$ be the orthogonal complement of $A_{\mathbf{P}}$ in A_P with respect to the Killing form. Then $X_{\mathbf{P}} = A_P^{\mathbf{P}} \times X_P$. Let $\rho_{\mathbf{P}}$ be the half sum of the (positive) roots of the adjoint action of $A_{\mathbf{P}}$ on $N_{\mathbf{P}}$ with multiplicity as in 13.3, and ρ_P be the half sum of the (positive) roots of the adjoint action of A_P on N_P. Then $\rho_{\mathbf{P}} = \rho_P$, and $\rho_{\mathbf{P}}$ is perpendicular to $\mathfrak{a}_P^{\mathbf{P}}$, the Lie algebra of $A_P^{\mathbf{P}}$. Denote by $\exp H_P(x)$, $\exp H_{\mathbf{P}}(x)$ the A_P, $A_{\mathbf{P}}$-components of $x \in X$ with respect to the two horospherical decompositions $X = N_P \times X_P \times A_P$, $X = N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$. Then for any $\xi \in \mathfrak{a}_{\mathbf{P}}^*$,

$$e^{\langle 2\rho_{\mathbf{P}} + \xi, H_{\mathbf{P}}(x) \rangle} = e^{\langle 2\rho_P + \xi, H_P(x) \rangle}. \tag{15.7.1}$$

Denote the maximal Satake-Furstenberg compactification of X by $\overline{X}^{SF}$ [SA1], [FU]. Then one of the results in [GJT], Chap. 1, §1, is the following.

15.8. PROPOSITION. — *For $\lambda < \lambda_0(X)$ as above,*

$$\partial_\lambda X = \coprod_P \overline{X_P}^{SF} \times A_P^+(\infty),$$

where P is over all the proper parabolic subgroups of the Lie group G. And the minimal Martin boundary is given by

$$\partial_{\lambda,\min} X = \coprod_{P \text{ is minimal}} \overline{A_P^+}(\infty),$$

where P is over all minimal real parabolic subgroups of G, and for such P, X_P is a point.

Let P be a minimal parabolic subgroup of G. For any $\xi \in \overline{A_P^+}(\infty)$, the corresponding minimal $K_\lambda(x, \xi)$ function is given as follows: Let ρ_P be half sum of the (positive) roots of the adjoint action of A_P on N_P. Then for any $x \in X$,

$$K_\lambda(x,\xi) = \mathrm{e}^{\langle 2\rho_P + c\xi, H_P(x)\rangle},$$

where $c = c(\xi)$ is chosen as in 15.5 above (see [GJT], Thm. 8.2, [KA], Thm. 17.2.1). The Martin kernel $K_\lambda(x,\xi)$ for other boundary points can also be written down explicitly.

If P is not a minimal real parabolic subgroup, then the corresponding functions $e^{\langle 2\rho_P + c\xi, H_P(x)\rangle}$ are not minimal. This implies that unless the real locus P of the rational parabolic subgroup $\mathbf{P}$ is a minimal real parabolic subgroup, every term of the Eisenstein series $E_\xi(x,\lambda)$ in Equation (15.5.1) for points in $\xi \in A^+_{\mathbf{P}}(\infty)$ is not a minimal function in $C_\lambda(X)$, and hence it is not obvious that $E_\xi(x,\lambda)$ is a minimal function in $C_\lambda(\Gamma\backslash X)$. For this purpose, we need to represent $e^{\langle 2\rho_P + c\xi, H_P(x)\rangle}$ as superpositions of the minimal ones.

15.9. Lemma. — *For any minimal parabolic subgroup P of G, $\rho_P/|\rho_P|$ defines a point in $A^+_P(\infty)$, which is the barycenter of the simplex $\overline{A^+_P}(\infty)$. Then the union*

$$\coprod_{\mathbf{P}\text{ is minimal}} \rho_P/|\rho_P| \subset \partial_{\lambda,\min} X$$

can be identified with the maximal Furstenberg boundary $G/P \cong K/K\cap P$, denoted by $\mathfrak{F}(X)$. Furthermore, the representing measure of the constant function 1 on X is the Haar measure on the maximal Furstenberg boundary $\mathfrak{F}(X)$ when identified with $K/K \cap P$.

Proof. — For definition of the maximal Furstenberg boundary and the determination of the representing measure of 1, see [FU]. For the the identification of this union with the maximal Furstenberg boundary, see [KA], Thm. 18.1.1. □

For the boundary symmetric space X_P of noncompact type for a real parabolic subgroup P, denote its maximal Furstenberg boundary by $\mathfrak{F}(X_P)$.

15.10. Lemma. — *Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$. If the real locus P is not a minimal real parabolic subgroup of G, then for any $\xi \in A^+_{\mathbf{P}}(\infty)$, the representing measure of the function $\mathrm{e}^{\langle 2\rho_{\mathbf{P}} + c(\xi)\xi, H_{\mathbf{P}}(x)\rangle}$ is supported on $\mathfrak{F}(X_P) \times \{\xi\} \subset (\overline{X_P}^{SF} \times A^+_P(\infty)) \cap \partial_{\lambda,\min} X$ and is equal to the Haar measure under the identification $\mathfrak{F}(X_P) \times \{\xi\} \cong \mathfrak{F}(X_P)$, where X_P is the noncompact type factor of $X_{\mathbf{P}}$, i.e., the boundary symmetric space associated with the real parabolic subgroup P.*

Proof. — By Equation (15.7.1),

$$e^{\langle 2\rho_{\mathbf{P}}+\xi, H_{\mathbf{P}}(x)\rangle} = e^{\langle 2\rho_P+\xi, H_P(x)\rangle}.$$

Hence we can use the horospherical decomposition $X = N_P \times X_P \times A_P$ for the real Langlands decomposition of P.

For any minimal positive solution b of $\Delta b = 0$ on the boundary symmetric space X_P of noncompact type, define a function on X by $x \mapsto e^{\langle \rho_P+c(\xi)\xi, H_P(x)\rangle} b(z_P(x))$, where $z_P(x)$ is the component of x in X_P in the horospherical decomposition $X = N_P \times X_P \times A_P$. Then from the above description of the minimal functions on X and hence on X_P, it follows that the function $e^{\langle \rho_P+c(\xi)\xi, H_P(x)\rangle} b(z_P(x))$ above is a minimal solution of $\Delta u = \lambda u$ on X. Treating the function $e^{\langle \rho_P+c(\xi)\xi, H_P(x)\rangle}$ as the product of $e^{\langle \rho_P+c(\xi)\xi, H_P(x)\rangle}$ with the constant function 1 on X_P, we get from Lemma 15.9 that the representing measure of $e^{\langle \rho_P+c(\xi)\xi, H_P(x)\rangle}$ is supported on $\mathfrak{F}(X_P) \times \{\xi\}$ and is equal to transplantation of the Haar measure on $\mathfrak{F}(X_P) \subset \partial_{\lambda,\min} X$. □

To prove Proposition 15.7, we also need the following result from ergodic theory.

15.11. Lemma. — *For a rational parabolic subgroup* $\mathbf{P}$, *let* Γ_{M_P} *be the image of* $\Gamma_P = \Gamma \cap P$ *in* M_P *under the projection* $P = N_P A_P M_P \to M_P$ *for the real Langlands decomposition. Then* Γ_{M_P} *acts ergodically on the maximal Furstenberg boundary* $\mathfrak{F}(X_P)$.

Proof. — It is known that for the rational Langlands decomposition $P = N_{\mathbf{P}} A_{\mathbf{P}} M_{\mathbf{P}}$, the projection of Γ_P in $M_{\mathbf{P}}$ is a cofinite lattice (see [BJ], Prop. 2.6). Then Γ_{M_P} is the image of $\Gamma_{M_{\mathbf{P}}}$ in the quotient $M_P = M_{\mathbf{P}}/A_P^{\mathbf{P}}$ and hence a cofinite lattice in M_P also. Thus it suffices to prove that Γ acts ergodically on the maximal boundary G/P, where P is a minimal parabolic subgroup of G. Assume first that Γ is an irreducible lattice in G. Then [ZI], Thm. 2.2.6, shows that P acts ergodically on G/Γ, and [ZI], Cor. 2.2.3, implies that Γ acts on G/P.

If Γ is reducible, say $\Gamma \cong \Gamma_1 \times \Gamma_2$, up to finite index, where Γ_i is an irreducible lattice in G_i, $i = 1, 2$, and $G = G_1 \times G_2$. Since $G/P = G_1/P_1 \times G_2/P_2$, where P_i is a minimal parabolic subgroup of G_i, and Γ_i acts ergodically on G_i/P_i, it follows that Γ acts ergodically on G/P. □

15.12. *Proof of Proposition 15.7.* — Consider $\widetilde{E}_\xi(x,\lambda)$ as a Γ-invariant function on X. Since the representing measure of the sum of two functions in $C_\lambda(X)$ is the sum of the representating measures, by adding the representing measures of the summands of the Eisenstein series $\widetilde{E}_\xi(x,\lambda)$, we get from Lemma 15.10 that the support of the representing measure $d\mu_\xi$ of $\widetilde{E}_\xi(x,\lambda)$ on $\partial_{\lambda,\min}X$ is equal to the closure of the following disjoint union:

$$\coprod_{\gamma\in\Gamma/\Gamma_{\mathbf{P}}} \gamma(\mathfrak{F}(X_P)\times\{\xi\}),$$

where we have used the fact that $\Gamma_{\mathbf{P}}$ leaves the set $\mathfrak{F}(X_P)\times\{\xi\}$ invariant, and that the restriction of $d\mu_\xi$ to each subset $\mathfrak{F}(X_P)\times\{\xi\}\cong\mathfrak{F}(X_P)$ is a multiple of the Haar measure on $\mathfrak{F}(X_P)$ defined in Lemma 15.10. It should be pointed out that since the Martin kernel functions are not Γ-invariant, but rather Γ-quasi-invariant, this representing measure $d\mu_\xi$ is also Γ-quasi-invariant, but not Γ-invariant.

If $u\in C_\lambda(\Gamma\backslash X)$ is bounded by $\widetilde{E}_\xi(x,\lambda)$, then, by Proposition 15.4, the representing measure $d\mu_u$ of u is of the form $f d\mu_\xi$, where f is an integrable function on the support of $d\mu_\xi$. Since $d\mu_u$ and $d\mu_\xi$ are Γ-quasi-invariant with the same module of quasi-invariance, f is invariant under Γ, and hence its restriction to each subset $\gamma(\mathfrak{F}(X_P)\times\{\xi\})$ is invariant under $\gamma\Gamma_{M_P}\gamma^{-1}$. Then Lemma 15.11 implies that the restriction of f to each subset $\gamma(\mathfrak{F}(X_P)\times\{\xi\})$ is constant. The Γ-invariance of f implies that f takes the same value on different subsets $\gamma(\mathfrak{F}(X_P)\times\{\xi\})$. Therefore, the representing measure $d\mu_u$ of u is a multiple of $d\mu_\xi$, the representating measure of $\widetilde{E}_\xi(x,\lambda)$, and u is a multiple of $\widetilde{E}_\xi(x,\lambda)$. This proves $\widetilde{E}_\xi(x,\lambda)$ is minimal.

If ξ' is a different point in $\partial(\overline{\Gamma\backslash X}^T)$, then the support of the representing measure of $\widetilde{E}_{\xi'}(x,\lambda)$ is disjoint from that of $\widetilde{E}_\xi(x,\lambda)$. Therefore $\widetilde{E}_{\xi'}(x,\lambda)$ is not equal to $\widetilde{E}_\xi(x,\lambda)$. This completes the proof of Proposition 15.7. □

15.13. Proposition. — *For every $\lambda<0$, there is a canonical injective map $\iota:\overline{\Gamma\backslash X}^T\to\Gamma\backslash X\cup\partial_\lambda\Gamma\backslash X$ given by $\xi\to E_\xi(x,\lambda)$ on the boundary and equal to the identity map in the interior.*

Proof. — By Proposition 15.7, for every $\xi\in\Delta(\Gamma\backslash X)=\partial\overline{\Gamma\backslash X}^T$, the function $\widetilde{E}_\xi(x,\lambda)$ is a minimal function of $C_\lambda(\Gamma\backslash X)$. Then Proposition 15.4 implies that $\widetilde{E}_\xi(x,\lambda)$ is equal to $K_\lambda(x,\eta)$ for a unique point $\eta\in\partial_\lambda\Gamma\backslash X$.

This defines a map $\iota : \partial\overline{\Gamma\backslash X}^T \to \partial_\lambda\Gamma\backslash X$, $\iota(\xi) = \eta$. Proposition 15.7 also shows that this map ι is injective. Combined with the identity map on $\Gamma\backslash X$, it gives the required map $\iota : \overline{\Gamma\backslash X}^T \to \Gamma\backslash X \cup \partial_\lambda\Gamma\backslash X$. □

15.14. Conjecture (see 2.1). — *The map $\iota : \overline{\Gamma\backslash X}^T \to \Gamma\backslash X \cup \partial_\lambda\Gamma\backslash X$ is continuous, and hence ι is a homeomorphism. In particular, the Martin boundary $\partial_\lambda\Gamma\backslash X$ can be identified with the Tits complex $\Delta(\Gamma\backslash X) = \partial(\overline{\Gamma\backslash X}^T)$, in particular, in particular every boundary point in $\partial_\lambda\Gamma\backslash X$ is minimal.*

The evidence for this conjecture is as follows: The map ι has a dense image. If it is continuous, then it is surjective; since both $\overline{\Gamma\backslash X}^T$ and $\Gamma\backslash X \cup \partial_\lambda\Gamma\backslash X$ are compact and Hausdorff, it follows that ι is a homeomorphism.

It can be shown that the restriction to the boundary $\iota_{|\Delta(\Gamma\backslash X)}$ continuous. The problem is the continuity from the interior to the boundary. From the definition of $\Gamma\backslash X \cup \partial_\lambda\Gamma\backslash X$, the continuity of ι depends on the asymptotic behavior at infinity of the Green function $G_\lambda(x, y)$ of $\Gamma\backslash X$. But such asymptotics are not known except for the case of Riemann surfaces. For Riemann surfaces with hyperbolic metric, this conjecture is true (see 16.2 below).

In the rest of this section, we prove the following weaker result.

15.15. Proposition (see 2.1). — *When the $\mathbb{Q}$-rank of $\Gamma\backslash X$ is equal to 1, every minimal function in $C_\lambda(\Gamma\backslash X)$ is a multiple of $\widetilde{E}_\xi(x, \lambda)$ for some $\xi \in \Delta(\Gamma\backslash X)$, and hence the minimal Martin boundary $\partial_{\lambda,\min}\Gamma\backslash X$ can be identified with the Tits complex $\Delta(\Gamma\backslash X) = \partial(\overline{\Gamma\backslash X}^T)$.*

To prove this theorem, we need to recall some results from theory of automorphic forms. In the following, we identify a function on $\Gamma\backslash X$ with a Γ invariant function on X.

15.16. Definition. — *A function u on $\Gamma\backslash X$ has uniform moderate growth if for every rational parabolic subgroup $\mathbf{P}$ and any Siegel set $\omega A_{\mathbf{P},t}x_0$, there exists $\Lambda \in \mathfrak{a}_{\mathbf{P}}^+$ such that for any invariant differential operator D on X,*

$$\big|Du(w\exp(H))\big| \leq c(D)\exp\big(\Lambda(H)\big),$$

where $w \in \omega$, $\exp(H) \in A_{\mathbf{P},t}$, and $c(D)$ is a constant depending on D.

15.17. Lemma. — *If $u \in C_\lambda(\Gamma\backslash X)$, then u has uniform moderate growth.*

Proof. — Consider u as a positive solution of $\Delta u = \lambda u$ on X. By Harnack inequality [GT], Thm. 8.20, there exists a positive constant c such that for any two points $x_1, x_2 \in X$ with $d(x_1, x_2) \leq 1$, $u(x_1) \leq cu(x_2)$. By iteration, there exists another positive constant c_1 such that for any $x \in X$, $u(x) \leq u(x_0)\exp(c_1 d(x, x_0))$.

By Lemma 10.3, there exists $\Lambda \in \mathfrak{a}_{\mathbf{P}}^+$ such that for any $x = w\exp(H)x_0 \in \omega A_{\mathbf{P},t}x_0$, $d(x, x_0) \leq \Lambda(H)$. Combining with the above inequality, we get that

$$u(x) \leq u(x_0)\exp(c_1\Lambda(H)).$$

To get the bound for other invariant differential operators D, we notice that $\Delta Du = \lambda Du$. Then the bounds follow from the Schauder interior estimates [GT], §6.1, the above bound for u, and the induction on the degree of D. □

15.18. Lemma. — *If $u \in C_\lambda(\Gamma\backslash X)$ is minimal, then for any invariant differential operator D on X, there exists a constant $\chi(D)$ such that $Du = \chi(D)u$.*

Proof. — A function $\Phi(x, y)$ on $X \times X$ defines a G-invariant integral operator with compact support,

$$\Phi u(x) = \int_X \Phi(x, y)u(y)\,\mathrm{d}y, \quad u \in C^\infty(X)$$

if the following conditions are satisfied:

1) For any $g \in G$, $\Phi(gx, gy) = \Phi(x, y)$.

2) There exists a positive constant c such that $\Phi(x, y) = 0$ if $d(x, y) > c$.

Then it follows from [SE2], p. 51, that $\Delta\Phi u = \lambda\Phi u$. Since Φ is G-invariant, we can see easily that Φu is Γ-invariant, and hence defines a function on $\Gamma\backslash X$.

We first prove that for any such integral operator Φ, Φu is a multiple of u.

If Φ is positive, then Φu is positive and hence belongs to $C_\lambda(\Gamma\backslash X)$. It follows from the compactness of the support of Φ and the Harnack inequality [GT], Thm. 8.20, as in the proof of Lemma 15.17 that Φu is bounded by u. Since u is minimal, Φu is a multiple of u.

In general, any such integral operator Φ can be written as the difference of two positive G invariant integral operators with compact supports, and hence Φu is also a multiple of u.

Next we reduce invariant differential operators to the integral operators. Let P be a minimal parabolic subgroup of G. Let ϕ be a positive smooth function with compact support on the open chamber A_P^+. For any $\varepsilon > 0$, define ϕ_ε on A_P^+ by $\phi_\varepsilon(\exp H) = \phi(\exp H/\varepsilon)$. By the Cartan decomposition of G: $G = K\overline{A_P^+}K$ [HE], p. 402, ϕ_ε defines a smooth positive K biinvariant function on G, still denoted by ϕ_ε. The support of ϕ_ε shrinks to the identity element of G as $\varepsilon \to 0$. Choose a constant c such that $\int_G c\phi_\varepsilon(g)\,\mathrm{d}g = 1$. Denote $c\phi_\varepsilon$ by $\tilde{\phi}_\varepsilon$.

The function $\tilde{\phi}_\varepsilon$ defines a G-invariant integral operator on X with compact support as follows: For $x = g_1K, y = g_2K \in X = G/K$,

$$\widetilde{\Phi}_\varepsilon(x, y) = \tilde{\phi}_\varepsilon(g_1^{-1}g_2).$$

By the choice of $\tilde{\phi}_\varepsilon$ above, it is clear that $\widetilde{\Phi}_\varepsilon u(x) \to u(x)$ uniformly for x in compact subsets as $\varepsilon \to 0$.

Let D be any invariant differential operator on X. Then $D\widetilde{\Phi}_\varepsilon(x, y)$, differentiated on x, is still a G-invariant integral operator with compact support. By the above discussion, there exists a constant χ_ε such that

$$D\widetilde{\Phi}_\varepsilon u = \chi_\varepsilon u.$$

Let $C_0^\infty(X)$ be the space of smooth functions on X with compact support. We claim that for any $\varphi \in C_0^\infty(X)$, if $(u, \varphi) = \int_X u(x)\varphi(x)\,\mathrm{d}x = 0$, then $(Du, \varphi) = 0$.

Clearly,

$$(D\widetilde{\Phi}_\varepsilon u, \varphi) = \chi_\varepsilon(u, \varphi) = 0.$$

On the other hand, as $\varepsilon \to 0$,

$$(D\widetilde{\Phi}_\varepsilon u, \varphi) = (\widetilde{\Phi}_\varepsilon u, D^t\varphi) \to (u, D^t\varphi) = (Du, \varphi),$$

where D^t is the transpose of D. This implies $(Du, \varphi) = 0$.

From the claim it is clear that Du is a multiple of u as distribution. Since both Du and u are analytic, $Du = \chi(D)u$ for some constant $\chi(D)$. This completes the proof of this proposition. □

Lemma 15.17 and Proposition 15.18 imply the following.

15.19. Proposition. — *Any minimal function u in $C_\lambda(\Gamma\backslash X)$ is an automorphic function on $\Gamma\backslash X$ in the following sense:*

1) *u has moderate growth.*

2) *Let $\mathfrak{Z}$ be the algebra of invariant differential operators on X. Then there exists a character $\chi : \mathfrak{Z} \to \mathbb{C}$ such that for all $Z \in \mathfrak{Z}$, $Zu = \chi(Z)u$.*

15.20. *Proof of Proposition 15.15.* — By assumption, the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 1. Let $\mathbf{P}_1, \dots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of parabolic subgroups. For $i = 1, \cdots, n$, $\dim A_{\mathbf{P}_i} = 1$, and $A^+_{\mathbf{P}_i}(\infty)$ consists of one point, denoted by ξ_i. Then $\rho_{\mathbf{P}_i} = |\rho_{\mathbf{P}_i}|\xi_i$. For $\lambda < 0$, if $u \in C_\lambda(\Gamma\backslash X)$ is a minimal function, then by Proposition 15.19, u is an automorphic function on $\Gamma\backslash X$. We claim that for some positive constants $c_1, \dots, c_n$,

$$u(x) \leq c_1 E_{\xi_1}(x, \lambda) + \cdots + c_n E_{\xi_n}(x, \lambda).$$

In fact, since each $E_{\xi_i}(x, \lambda)$ is positive, it suffices to prove that on Siegel sets $\omega_i \times A_{\mathbf{P}_i, t} x_0$ associated P_i,

$$u(x) \leq c_i \, e^{\langle 2\rho_{\mathbf{P}_i} + c(\xi_i)\xi_i, H_{\mathbf{P}_i}(x)\rangle} \leq c_i E_{\xi_i}(x, \lambda).$$

Since u is an automorphic function in the sense of Proposition 15.19 with $\Delta u = \lambda u$, the constant term u_{P_i} of u along the parabolic subgroup $\mathbf{P}_i$ is of the form

$$u_P = e^{\langle \rho_{\mathbf{P}_i} + d(\xi)\xi, H_{\mathbf{P}_i}(x)\rangle} f_1 + e^{\langle \rho_{\mathbf{P}_i} - d(\xi)\xi, H_{\mathbf{P}_i}(x)\rangle} f_2,$$

where f_i, $i = 1, 2$, are automorphic forms on $\Gamma_{M_{\mathbf{P}_i}} \backslash X_{\mathbf{P}_i}$ and $\Delta f_i = \nu f_i$, $\nu \geq 0$, and $d(\xi) = \sqrt{\lambda - |\rho_{\mathbf{P}_i}|^2 - \nu}$. (We emphasize that it is the fact that u is a joint eigenfunction of all invariant differential operators which allows us to conclude the above equation for u_P.) Since $d(\xi) \leq \sqrt{\lambda - |\rho_{\mathbf{P}_i}|^2} = |\rho_{\mathbf{P}_i}| + c(\xi)$, there exists a constant c'_i such that on the Siegel sets $\omega_i A_{\mathbf{P}_i, t} x_0$,

$$u_{P_i} \leq c_i \, e^{\langle \rho_{\mathbf{P}_i} + |\rho_{\mathbf{P}_i}| + c(\xi_i)\xi_i, H_{\mathbf{P}_i}(x)\rangle} = c_i \, e^{\langle 2\rho_{\mathbf{P}_i} + c(\xi_i)\xi_i, H_{\mathbf{P}_i}(x)\rangle}.$$

Since the $\mathbb{Q}$-rank of $\mathbf{G}$ is equal to 1, the difference $u - u_{P_i}$ decays rapidly on the Siegel sets $\omega_i A_{\mathbf{P}_i, t} x_0$, and hence the above bound on u_{P_i} implies that

$$u \leq c_i \, e^{\langle 2\rho_{\mathbf{P}_i} + c(\xi_i)\xi_i, H_{\mathbf{P}_i}(x)\rangle}$$

on the Siegel sets $\omega_i A_{\mathbf{P}_i, t} x_0$, for some constant c_i. Hence the claim is proved. □

By Proposition 15.7, each $E_{\xi_i}(x,\lambda)$ is a minimal function of $C_\lambda(\Gamma\backslash X)$. By Proposition 15.4, the claim implies that the representing measure of u on $\partial_\lambda\Gamma\backslash X$ is supported on $\{\xi_1,\cdots,\xi_n\}$ and hence of the form $b_1\delta_{\xi_1}+\cdots+b_n\delta_{\xi_n}$ for some nonnegative numbers b_i. By Proposition 15.4 again, this implies that

$$u = b_1 E_{\xi_1}(x,\lambda)+\cdots E_{\xi_n}(x,\lambda).$$

Since u is a minimal function and hence an extremal element of the convex cone $C_\lambda(\Gamma\backslash X)$, there exists a unique i such that $u = b_i E_{\xi_i}(x,\lambda)$. This completes the proof.

15.21. *Remark.* — It was claimed in the first version of this paper that Proposition 15.15 also holds in the higher $\mathbb{Q}$-rank case, but there was some gap in the proof. For $\Gamma\backslash X$ of arbitrary $\mathbb{Q}$-rank, let $u\in C_\lambda(\Gamma\backslash X)$ be a minimal function. By Proposition 5.19, u is an automorphic function. Then a result of Franke [FR], Cor. 1 in §6, says that u is the derivative of an Eisenstein series, say $E(\mathbf{P}|\varphi,\Lambda)$. It is conceivable that the positivity of u should imply that φ is a constant, u is a multiple of $E(\mathbf{P}|\varphi,\Lambda)$ instead of its derivative, and $\Lambda\in\rho_{\mathbf{P}}+\mathfrak{a}_{\mathbf{P}}^+$, i.e., u is a multiple of $E_\xi(x,\Lambda)$ defined in 15.5. These assertions can be proved for the $\mathbb{Q}$-rank one case, but are not known for the higher $\mathbb{Q}$-rank case. The difficulty is that when $E(\mathbf{P}|\varphi,\Lambda)$ is not a cuspidal Eisenstein series, its constant terms are not known.

16. Examples.

16.1. *Euclidean space.* — This is not an example of $\Gamma\backslash X$, but many analogous constructions mentioned above can be written down explicitly.

The Euclidean space $\mathbb{R}^n$ is homeomorphic to the unit ball $B=\{x\in\mathbb{R}^n;\ \|x\|<1\}$ through the map $x\mapsto x/(1+\|x\|)$ and hence admits a natural compactification $B\cup S$, where S is the unit sphere in $\mathbb{R}^n$. This is an analogue of the Tits compactification and hence denoted by $\overline{\mathbb{R}^n}^T$.

To identify the Gromov compactification $\overline{\mathbb{R}^n}^G$, denote the standard metric on $\mathbb{R}^n$ by $d_0(\,.\,,.)$. We note that for any unbounded sequence y_k in $\mathbb{R}^n$, the normalized distance function $d_0(y_k,x)-|y_k|$ is asymptotic to $-2\langle x, y_k/|y_k|\rangle$ when $k\to+\infty$ and hence converges uniformly for x in compact subsets if and only if $y_k/|y_k|$ is convergent in S. Therefore, $\overline{\mathbb{R}^n}^G$ is homeomorphic to $\overline{\mathbb{R}^n}^T$.

For any $t>0$, $(\mathbb{R}^n,\frac{1}{t}d_0)$ is clearly isometric to $(\mathbb{R}^n,d_0)$. Therefore $T_\infty\mathbb{R}^n=(\mathbb{R}^n,d_0)$ and is a metric cone over the unit sphere S.

All geodesics in $\mathbb{R}^n$ are distance minimizing and the set $\mathbb{R}^n(\infty)$ of their equivalence classes can naturally be identified with the unit sphere S. Assumptions 9.11 and 9.16 are obviously satisfied, and hence $\mathbb{R}^n \cup \mathbb{R}^n(\infty)$ is homeomorphic to $\overline{\mathbb{R}^n}^T$.

For any $\lambda > 0$, Fourier analysis on $\mathbb{R}^n$ shows that the generalized eigenspace of eigenvalue λ has a basis $\exp i\sqrt{\lambda}\langle x, \omega\rangle$, where $\omega \in S$. This example is the starting point for the geometric scattering theory in [ME].

For the Martin compactification, we assume that $n \geq 3$ and $\lambda < 0$. Then the Green function of $\Delta - \lambda$ has the following asymptotics at infinity:

$$G_\lambda(x, 0) \sim \text{const. } \|x\|^{\frac{1}{2}(1-n)} \, \mathrm{e}^{-\sqrt{|\lambda|}\,\|x\|}, \quad \text{as } x \to \infty$$

(see [ME], p. 7 or [KT], §2). From this, we can show easily that the Martin compactification $\mathbb{R}^n \cup \partial_\lambda \mathbb{R}^n$ for $\lambda < 0$ is homeomorphic to $\overline{\mathbb{R}^n}^T$. For any boundary point $\omega \in S$, the corresponding function is $\exp \sqrt{|\lambda|}\langle x, \omega\rangle$ and is minimal.

16.2. *Riemann surfaces.* — We want to use this example to show some ideas and methods in the proofs of the results stated above. For more details of this example, see [J1].

Assume $G = \mathrm{SL}(2, \mathbb{R})$, and $X = \mathbf{H}^2$, the upper half plane. Then $\Gamma\backslash\mathbf{H}^2$ is a finite area hyperbolic surface. This is the simplest example of locally symmetric spaces.

Suppose $\Gamma\backslash\mathbf{H}^2$ has m cusps. Then $\Delta(\Gamma\backslash\mathbf{H}^2)$ consists of m points. The Tits compactification $\overline{\Gamma\backslash\mathbf{H}^2}^T$ is obtained by adding one point to each cusp.

Using reduction theory for Γ, we can show that for any point $z = x + iy \in \mathbf{H}^2$, there exists a point in the orbit Γz with maximum imaginary part. Then we can show easily that $\overline{\Gamma\backslash\mathbf{H}^2}^T$ dominates the Gromov compactification $\overline{\Gamma\backslash\mathbf{H}^2}^G$. By examining the asymptotic behavior of the horofunctions of the ideal points in $\overline{\Gamma\backslash\mathbf{H}^2}^T$, we can show that $\overline{\Gamma\backslash\mathbf{H}^2}^T$ is homeomorphic to $\overline{\Gamma\backslash\mathbf{H}^2}^G$. This argument is similar to the general case of $\Gamma\backslash X$ (see 12.10–12.11), but the reduction theory in 4.6 is more complicated.

To show that $T_\infty\Gamma\backslash\mathbf{H}^2$ is a cone over $\Delta(\Gamma\backslash\mathbf{H}^2)$, we decompose $\Gamma\backslash\mathbf{H}^2$ into disjoint union of m cusp neighborhoods and a compact region. We notice that each cusp neighborhood can be approximated by a DM ray (see Figure 5.11). Then the result follows easily. This disjoint decomposition is crucial and follows from precise reduction theory, whose generalization to higher rank case is given in 4.6.

Each equivalence class of DM rays corresponds to one cusp. This follows from study of the Dirichlet fundamental domain. Using this, one can check easily that the Assumptions 9.11 and 9.16 hold and hence the geometric compactification $\Gamma\backslash\mathbf{H}^2 \cup \Gamma\backslash\mathbf{H}^2(\infty)$ is homeomorphic to the Tits compactification $\overline{\Gamma\backslash\mathbf{H}^2}^T$. In this example, a stronger statement holds: For a generic basepoint $x_0 \in \Gamma\backslash\mathbf{H}^2$, the metric cone over $\Delta(\Gamma\backslash\mathbf{H}^2)$ can isometrically embedded in $\Gamma\backslash\mathbf{H}^2$ with vertex at x_0. This result also holds for general rank one locally symmetric spaces.

Selberg's spectral theory of $\Gamma\backslash\mathbf{H}^2$ [SE2], [HEJ] shows that for each cusp C_j, there is an Eisenstein series $E_j(z,s)$ which is a generalized eigenfunction when $\mathrm{Re}(s) = \frac{1}{2}$. The Eisenstein series $E_j(z, \frac{1}{2} \pm it)$, $t \in \mathbb{R}$, have the same eigenvalue $\frac{1}{4} + t^2$. The functional equation for Eisenstein series shows that there are exactly m linearly independent generalized eigenfunctions $E_j(z, \frac{1}{2} + it), j+1, \dots, m$, of eigenvalue $\frac{1}{4} + t^2$. The spectral theory for general locally symmetric spaces $\Gamma\backslash X$ is established by Langlands [LA], [AR2] (see also [MW] and [OW2]) and is more complicated.

For $r > 0$, the m Eisenstein series $E_j(z, 1 + r)$ form a basis of the cone $C_\lambda(\Gamma\backslash\mathbf{H}^2)$, where $\lambda = -r(1+r)$. In fact, the Martin compactification $\Gamma\backslash\mathbf{H}^2 \cup \partial_\lambda\Gamma\backslash\mathbf{H}^2$ is homeomorphic to $\overline{\Gamma\backslash\mathbf{H}^2}^T$. It follows from the asymptotic behavior of the Green function near the cusps in [HEJ], Eq. 6.4, p. 47. Unfortunately, such asymptotics are not available for general $\Gamma\backslash X$.

In this example, the Borel-Serre compactification $\overline{\Gamma\backslash\mathbf{H}^2}^{BS}$ is obtained by adding a circle to each cusp. Each such circle parametrizes DM rays going out to infinity in that cusp.

16.3. *Products of Riemann surfaces* $\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2$. — This is the simplest example of higher rank locally symmetric spaces. We use it to show the necessity of the DM condition on rays.

For simplicity, we assume that both $\Gamma_1\backslash\mathbf{H}^2$ and $\Gamma_2\backslash\mathbf{H}^2$ have only one cusp. Then $\Delta(\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2)$ is a simplex of dimension 1.

A ray in $\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2$ is DM if and only if its projections on both factor spaces are DM. It then follows from this that the set of equivalence classes of DM rays in $\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2$ corresponds to $\Delta(\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2)$.

An interesting phenomenon here is that there exist infinitely many rays which leave any compact subset eventually but are not DM. One such geodesic can be constructed as follows: Take a DM ray $\gamma_1(t)$ in $\Gamma_1\backslash\mathbf{H}^2$ and a closed geodesic $\gamma_2(t)$ in $\Gamma_2\backslash\mathbf{H}^2$, then $\gamma(t) = (\gamma_1(\frac{1}{\sqrt{2}}t), \gamma_2(\frac{1}{\sqrt{2}}t))$ is such a ray. This ray does converge in $\overline{\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2}^T$, but it does not converge in the reductive Borel-Serre compactification $\overline{\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2}^{RBS}$, whose boundary is $\Gamma_1\backslash\mathbf{H}^2 \cup \Gamma_2\backslash\mathbf{H}^2 \cup \{\infty\}$, where the boundary point ∞ fills in both cusps. This is one reason why we impose the DM condition on rays.

Another reason is the difficulty to define a relation on rays that go to infinity but are *not necessarily* DM such that the set of the equivalence classes corresponds to the Tits boundary $\Delta(\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2)$. Specifically, there exist two rays $\gamma'(t)$ and $\gamma''(t)$ converging to the same point in $\Delta(\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2)$, but $\lim_{t\to+\infty} d(\gamma'(t), \gamma''(t)) = +\infty$. For example, take $\gamma'(t)$ to be the ray $\gamma(t)$ constructed in the previous paragraph, and $\gamma''(t)$ be a ray constructed similarly by replacing $\gamma_2(t)$ by a geodesic which has deeper and deeper penetration into the cusp but returns to the compact core infinitely many times as in [SU1].

In $\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2$, any ray that leaves any compact subset eventually always converges in the Tits compactification. This might be true for general $\Gamma\backslash X$. The problem is that we do not understand the structure of such rays. It seems that the Tits compactification is the only nontrivial compactification that might have this property (the trivial one point compactification clearly enjoys this property). This could be another justification for the Tits compactification $\overline{\Gamma\backslash X}^T$.

Finally, we would like to mention that a possible relaxation of the DM condition on rays is the almost DM condition. A ray $\gamma(t)$ is almost DM if there exists a positive constant C such that $d(\gamma(t), \gamma(0)) \geq t - C$ for any $t > 0$. It seems conceivable that a ray in $\Gamma\backslash X$ is almost DM if and only if it is DM eventually. This is true for $\Gamma\backslash\mathbf{H}^2$ and $\Gamma_1\backslash\mathbf{H}^2 \times \Gamma_2\backslash\mathbf{H}^2$. The problem here is also that we do not understand the structure of such almost DM rays in general.

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