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Ami FISCHMAN

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ON THE IMAGE OF Λ-ADIC GALOIS REPRESENTATIONS

by **Ami FISCHMAN**

1. Introduction.

F. Momose has proved that the image of a restricted l-adic Galois representation attached to an appropriately generic (normalized, ordinary, no CM) modular form is full in the sense that it contains the special linear group (SL_2) for all but finitely many l [Mom81]. In this paper, we generalize that result to the Λ-adic setting developed by Hida in [Hid86a], [Hid86b], and [Hid86c]. We show that the image of the subgroup determined by the twists of the form is full (Theorem 4.8) and further determine the exact image of the Galois group of $\mathbb{Q}$ under the Λ-adic representation.

We then show that in some sense, of all generic Λ-adic Galois representations, all but a density 0 subset have full image as in Theorem 4.8 (Theorem 5.5).

In order to obtain analogs of the classical results in the Λ-adic situation, we determine the exact structure of the Λ-adic Hecke algebra and coefficient ring. We then lift the classical results of Momose to the Λ-adic setting using a proposition of N. Boston from the appendix to [MW86].

Keywords: Modular form – p-adic family – Galois representation – p-adic modular form.
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1.1. Layout.

In Section 2, we discuss the case of weight 2, and the results already known for it. In particular, we quote the result of Momose which guarantees that the restricted l-adic representation attached to a generic weight 2 modular form is full for all but a finite set of primes l. Then in Section 3, we lift the weight 2 modular form to a Λ-adic modular form and prove that in fact the Λ-adic Hecke algebra is of the form of a power series ring over the classical Hecke algebra. This allows us to compute the coefficient ring of the lifted form, Λ, explicitly, and subsequently to set up the proposition of N. Boston. In Section 4, we prove that the restricted Λ-adic Galois representation is full, and in Section 5 we show that for all but a density 0 subset of generic modular forms, the attached representations are full.

1.2. Notation.

Throughout, A_m will denote the completion of A at m and $A_{(m)}$ will denote the localization of A at m, for a ring A and a prime ideal m.

We recall the following standard definitions and notation for an odd prime l. Let $\mu_n := \left\{\zeta \in \overline{\mathbb{Q}} \mid \zeta^n = 1\right\}$ be the set of n^{th} roots of unity. Define the Teichmüller character at l, ω_l, to be the first component of the canonical isomorphism

$$\mathbb{Z}_l^\times \xrightarrow{\sim} \mu_{l-1} \times \overbrace{(1 + l\mathbb{Z}_l)}^{\Gamma}$$

and define $\langle\rangle : \mathbb{Z}_l^\times \to 1 + l\mathbb{Z}_l$ via $\langle x\rangle := \omega_l(x)^{-1}x$. Then $x \mapsto (\omega_l(x), \langle x\rangle)$ is the canonical isomorphism above.

The cyclotomic character at l is defined by letting n approach ∞ in the following canonical sequence:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \stackrel{\mathrm{res}}{\to} \mathrm{Gal}(\mathbb{Q}(\mu_{l^n})/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/l^n\mathbb{Z})^\times$$

to get the character

$$\nu_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \stackrel{\mathrm{res}}{\to} \mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_l^\times.$$

We define $\iota : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \stackrel{\mathrm{res}}{\to} \mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}) \stackrel{\mathrm{res}}{\to} \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow{\sim} 1 + l\mathbb{Z}_l \stackrel{\kappa}{\hookrightarrow} \Lambda^\times := \mathbb{Z}_l[[X]]^\times$ to be the canonical character (so ι is onto $1+l\mathbb{Z}_l$, and κ maps $(1+l) \mapsto (1+X)$). We note that $\iota(\sigma) = \kappa(\langle\nu_l(\sigma)\rangle)$ by definition. In particular, for a prime $q \neq l$, $\iota(\mathrm{Frob}_q) = \kappa(\langle q\rangle)$.

Most of these facts are summarized in the following diagram:

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[dr, "\nu_l"] \arrow[drrrr, bend right] & \mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}) \arrow[rr] \arrow[d, equal] & & \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \arrow[d, equal] & \\
& \mathbb{Z}_l^\times \arrow[r, "\sim"] & \mu_{l-1} \times (1 + l\mathbb{Z}_l) \arrow[r, "\langle\rangle"] & 1 + l\mathbb{Z}_l \arrow[r, hook, "\kappa"] & \Lambda^\times = \mathbb{Z}_l[[X]]^\times
\end{tikzcd}$$

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2. Weight 2 situation.

2.1. Notation.

Throughout l will denote a (variable) prime number greater than 3. Let $f = \sum_{n=1}^{\infty} a(n,f)q^n \in S_2(\Gamma_0(N), \varepsilon)$ be a weight 2 newform without complex multiplication. Fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$ once and for all. Let $E_f := \mathbb{Q}(a(n,f) | n \in \mathbb{N})$ be the subfield of $\overline{\mathbb{Q}} \subset \mathbb{C}$ generated by the coefficients of f, and note that E_f is finite dimensional over $\mathbb{Q}$ (since it is generated by eigenvalues of a finite dimensional algebra). Let $O_f := O_{E_f}$ be the integer ring of E_f, and $O_{f,l} := O_f \otimes_{\mathbb{Z}} \mathbb{Z}_l$ be the completion at l (for each l). Similarly for each l, complete E_f to $E_{f,l} := E_f \otimes_{\mathbb{Q}} \mathbb{Q}_l$.

Let $\rho_{f,l} : G = G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(O_{f,l}) \subset GL_2(E_{f,l})$ be the continuous, unramified outside lN Galois representation attached to f such that $\mathrm{Tr}(\rho_{f,l}(\mathrm{Frob}_p)) = a(p,f)$ and $\det(\rho_{f,l}(\mathrm{Frob}_p)) = \varepsilon(p)p$ for all $p \nmid lN$, where Frob_p is a Frobenius element at p. Note that this determines $\rho_{f,l}$ uniquely up to isomorphism as an $E_{f,l}$-representation, but not necessarily as an $O_{f,l}$-representation. This representation comes from the action of G on l-power division points of an abelian variety over $\mathbb{Q}$. For more information on the construction and properties of $\rho_{f,l}$ the reader should consult [Shi71].

For an automorphism $\gamma \in \mathrm{Aut}(E_f)$, a Dirichlet character χ may exist such that

$$\gamma(a(p, f)) = \chi(p)a(p, f)$$

for all but finitely many p. If it exists, call this character χ_γ. Let Γ_f be the set of $\gamma \in \mathrm{Aut}(E_f)$ for which there exists a χ_γ.

2.2. Known results.

Work of Momose and Ribet allows us to make the following definitions with implicit claims:

(1) Set $H_f := \cap_{\gamma\in\Gamma_f} \ker \chi_\gamma$;

(2) Γ_f is an abelian subgroup of $\mathrm{Aut}(E_f)$;

(3) Set $F_f := {E_f}^{\Gamma_f}$ to be the fixed subfield;

(4) The index $[E_f : F_f]$ is finite. Set $n := [E_f : F_f]$;

(5) Set $R_f := O_{F_f}$ to be the integer ring of the fixed subfield;

(6) Set $R_{f,l} := R_f \otimes_{\mathbb{Z}} \mathbb{Z}_l$ to be the completion at l, for each prime l; and

(7) Set $A_{f,l} := \{x \in GL_2(R_{f,l}) | \det(x) \in \mathbb{Z}_l^\times\}$.

These definitions come from [Rib85] and allow us to state [Re85, Th. 3.1]:

THEOREM 2.1 (Momose). — *For all but finitely many primes l, $\rho_{f,l}(H_f) = A_{f,l}$, and in particular, $\rho_{f,l}(H_f) \supset SL_2(R_{f,l})$.*

2.3. l-ordinarity.

We will be dealing with two similar, but not equivalent, notions of "ordinarity." The first notion is for rational primes (say $l \in \mathbb{Z}$) where we say that f is l-ordinary if $l \nmid a(l, f)$. The second notion is for primes lying over l, say $\mathfrak{l}$. Then we say that f is $\mathfrak{l}$-ordinary if $\mathfrak{l} \nmid a(l, f)$.

When do the two notions clash? Clearly if $l \mid a(l, f)$ then $\mathfrak{l}|a(l, f)$ for any prime $\mathfrak{l} \mid l$, so $\mathfrak{l}$-ordinarity implies l-ordinarity. However, it is possible that $\mathfrak{l} \mid a(l, f)$ (i.e. f is not $\mathfrak{l}$-ordinary) and that $l \nmid a(l, f)$ (i.e. f is l-ordinary). We will see shortly that this will not be a problem for us.

We recall the bound on the size of the coefficients of an eigenform of weight k: $|a(l,f)| \leqslant 2l^{(k-1)/2}$, which in weight 2 gives: $|a(l,f)| \leqslant 2\sqrt{l}$. This implies that f is l-ordinary if and only if $a(l,f) \neq 0$ for $l > 3$ (since if it is non-zero, it is between 0 and $2\sqrt{l} < l$ in absolute value and not l-divisible).

Serre shows that for a real number x, the number of primes l less than x not dividing the level N such that $a(l,f) = 0$ is $P_{f,0}(x) = O(x/(\log(x)^{3/2-\delta}))$ for any $\delta > 0$ [Ser81, Theorem 15, p.174]. Using the prime number theorem, one sees that the density of non-ordinary (rational) primes is $d_f := \lim_{x\to\infty} \frac{P_{f,0}(x)}{x/\log(x)} = 0$, since δ can be taken to be less than a half. Thus the density of ordinary primes must be 1. We summarize this as:

PROPOSITION 2.2 (Serre). — *Any form f as above of weight 2 is l-ordinary for a set of primes $\{l\}$ of density* 1.

For any form f as above, let Σ_f be the set of primes l for which f is l-ordinary and which avoid the union of the following finite sets of primes:

(1) the finitely many primes excluded by Theorem 2.1;

(2) the primes dividing the discriminant of the reduced Hecke algebra

$$h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})^{\mathrm{red}}$$

where

$$h_2(\Gamma_0(N), \varepsilon, \mathbb{Z}) := \mathbb{Z}[T(n) \mid n = 1\ldots] \subset \mathrm{End}(S_2(\Gamma_0(N), \varepsilon)).$$

(3) the primes dividing $30N$.

Note that the last set of primes excludes all l such that $l \leqslant 2\sqrt{l} + 1$, so in particular excludes the possibility that $l | (a(l,f)^2 - \varepsilon(l))$.

Then Serre's result (Proposition 2.2) shows that Σ_f is a density 1 set of primes.

The reason we exclude the primes dividing the discriminant of the Hecke algebra is that those might ramify in O_f. Now, for a prime $l \in \Sigma_f$, we are guaranteed that there is an $\mathfrak{l}$ lying over l such that f is $\mathfrak{l}$-ordinary. This follows since if all of the primes lying over l divided $a(l,f)$, then so would l, because it is unramified. The notion of $\mathfrak{l}$-ordinarity will be the key to lifting f to a Λ-adic eigenform F later on.

LEMMA 2.3. — *For a prime l, $l \nmid \mathrm{disc}(h_2^{\mathrm{red}}(\Gamma_0(N), \varepsilon, \mathbb{Z}))$ implies that l doesn't ramify in O_f.*

Proof. — We have the map $\lambda : h_2(\Gamma_0(N), \varepsilon, \mathbb{Z}) \to O_f$ mapping a Hecke operator $T(n)$ to the eigenvalue of f at n. A prime l ramifying in O_f must divide the discriminant of $\mathrm{Im}(\lambda)$, since $\mathrm{disc}(\mathrm{Im}(\lambda)) = \mathrm{disc}(O_f)(O_f : \mathrm{Im}(\lambda))^2$ by [FT93, (2.3), p.121]. Let l be such a prime. Then the pre-image of the different of $\mathrm{Im}(\lambda)$ must contain l and thus l divides the discriminant of $h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})$ which shows the lemma. $\square$

Note that by excluding the primes dividing $30N$, we exclude the primes dividing the conductors of the χ_γ by [Mom81, Rmk. 1.6].

3. Machinery.

3.1. Lifting the eigenform.

Throughout this section, fix f_2 as in the previous section, a prime $l \in \Sigma_{f_2}$, and an $\mathfrak{l} \mid l$ for which f_2 is $\mathfrak{l}$-ordinary. Then there is a unique $\mathfrak{l}$-ordinary eigenform $f \in S_2(\Gamma_0(N) \cap \Gamma_1(l), \varepsilon, \mathbb{Z})$ such that f and f_2 have the same eigenvalues for all $T(p)$, $p \nmid Nl$. We note that $O_{f,l}$ has $O_{f,\mathfrak{l}}$ (the completion of O_f at $\mathfrak{l}$) as a direct summand and we will be mostly concerned with $O_{f,\mathfrak{l}}$ in the sequel. Similarly, we let $R_{f,\mathfrak{l}}$ be the corresponding direct summand of the fixed subring. We use here (and elsewhere) the convention that a comma-seperated list of subscripts is associative, so for instance $O_{f,\mathfrak{l}} = (O_f)_\mathfrak{l}$ is the completion at $\mathfrak{l}$ of O_f.

Let $\Lambda := \mathbb{Z}_l[[X]]$, $\mathbf{L}$ be its quotient field, and let $\overline{\mathbf{L}}$ be an algebraic closure of $\mathbf{L}$. From [Hid89, Thms. 4.5 and 4.6] we have that there is a finite extension of $\mathbf{L}$, call it $\mathbf{K} \subset \overline{\mathbf{L}}$, with integral closure $\mathbf{I}$ of Λ in $\mathbf{K}$ such that there is an $\mathbf{I}$-adic normalized Hecke eigenform $F \in \mathbf{S}^{\mathrm{ord}}(Nl^\infty, \chi)$ that specializes to f at weight 2 of character χ, where $\chi := \varepsilon\omega_l^2$. We use here the notation $\mathbf{S}^{\mathrm{ord}}(Nl^\infty, \chi)$ to denote the space of (ordinary) Λ-adic modular cusp forms of outside level N with character χ (see [Hid93, Sec. 7.6] for definitions and examples). For the proof of the existence of a Λ-adic lift the reader should refer to the proof of [Hid86c, Cor. 3.7].

Let $F = \sum_{n=1}^{\infty} a(n, F)(X)q^n$ be the q-expansion of F.

We adapt here an argument of F. Gouvêa's from [Gou92] to the weight 2 situation to show that

THEOREM 3.1. — *The coefficient ring* $\mathbf{I}$ *is a power series ring in one variable. Specifically,* $\mathbf{I} \cong O_{f,\mathfrak{l}} \hat{\otimes}_{\mathbb{Z}_l} \mathbb{Z}_l[[X]] \cong O_{f,\mathfrak{l}}[[X]]$.

To prove this theorem, we set up some notation. Because f and F are eigenforms for their respective Hecke algebras, we can define maps

$$\lambda_f : h := h_2(\Gamma_0(N) \cap \Gamma_1(l), \varepsilon, \mathbb{Z}) \to O_f$$

and

$$\lambda_F : \mathbf{h} := \mathbf{h}^{\mathrm{ord}}(Nl^\infty, \chi) \to \mathbf{I}$$

mapping a Hecke operator to its eigenvalue on f and F, respectively. Let $m_f := \lambda_f^{-1}(\mathfrak{l})$ be the inverse image of $\mathfrak{l}$.

Lemma 3.2. — *m_f is a maximal ideal of $h_2(\Gamma_0(N) \cap \Gamma_1(l), \varepsilon, \mathbb{Z})$.*

Proof. — Since $(\mathfrak{l}) \subset O_f$ is a maximal ideal and $O_f/(\mathfrak{l})$ is a finite field, we have that $h_2(\Gamma_0(N) \cap \Gamma_1(l), \varepsilon, \mathbb{Z})/m_f$ is a finite ring. But m_f is prime because $xy \in m_f$ implies $\lambda_f(xy) = \lambda_f(x)\lambda_f(y) \in (\mathfrak{l})$ and $(\mathfrak{l})$ is prime, so $\lambda_f(x) \in (\mathfrak{l})$ or $\lambda_f(y) \in (\mathfrak{l})$. This means that $h_2(\Gamma_0(N) \cap \Gamma_1(l), \varepsilon, \mathbb{Z})/m_f$ is a finite integral domain and this a finite field, so m_f is maximal. □

Then m_f is a maximal ideal of h and contains $\ker(\lambda_f)$. We will see shortly that there is a unique maximal ideal containing $\ker(\lambda_F)$, and we will call it m_F. Since the m's contain the kernels, λ_f factors through h_{m_f} and λ_F factors through $\mathbf{h}_{m_F}$. When no confusion can arise, we will write h_m for h_{m_f} and $\mathbf{h}_m$ for $\mathbf{h}_{m_F}$.

Lemma 3.3. — *Extend the scalars of h to $\mathbb{Z}_l$ by $h' := h \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Then $m' := m_f \otimes_{\mathbb{Z}} \mathbb{Z}_l$ is the unique maximal ideal of h' which contains* $\ker(\lambda_f)$.

Proof. — Since h' is an algebra of finite rank over the local ring $\mathbb{Z}_l$, it is semi-local, and in particular has only finitely many prime ideals. Write $h' = \prod_{n \subset h'} h'_n$ where the product ranges over the maximal ideals of h'. Then note that in general, $\mathrm{Spec}(A \oplus B) = \mathrm{Spec}(A) \sqcup \mathrm{Spec}(B)$ (see, for instance, [Sha94, p. 12, Example 1]), so a prime ideal in the product must only actually show up in one of the components, proving the lemma. □

Since $\mathbf{h}$ is already semi-local, the same proof yields:

Corollary 3.4. — *There is a unique maximal ideal m_F containing* $\ker(\lambda_F)$.

Claim 3.5. — *m_f does not contain primes dividing the discriminant of*

$$h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})^{\mathrm{red}}.$$

Proof. — Note that $m_f \cap \mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, so $m_f \cap \mathbb{Z} = (q)$ for some rational prime q. This shows that m_f can contain only one rational prime. But by definition $l \in m_f$, so the claim follows. $\square$

LEMMA 3.6. — *For $l \nmid N$, $k \geqslant 2$, and $m \subset h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})$ the maximal ideal associated to the primitive form f_2, the completion $h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})_m$ of the Hecke algebra is an unramified discrete valuation ring if m doesn't contain a prime dividing the discriminant of the reduced part of the Hecke algebra.*

Proof. — For simplicity of notation, let $\underline{h} := h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})$ and let K be its total fraction ring, and regard K as a finite dimensional algebra over $\mathbb{Q}$. Then let $U \subset K^{\mathrm{red}}$ be the integral closure of $\mathbb{Z}$ in K^{red} and $\underline{h}^{\mathrm{red}} = \mathbb{Z}[T(n) \mid n = 1, \cdots]^{\mathrm{red}} \subset U$. Note that $\mathbb{Q}(T(n)) = K$, so $\underline{h}^{\mathrm{red}}$ is a lattice in U and $(U : \underline{h}^{\mathrm{red}})$ is finite. Further, if $q \nmid (U : \underline{h}^{\mathrm{red}})$ then $U_q = \underline{h}_q^{\mathrm{red}}$. Now K^{red} is a commutative semi-simple finite-dimensional algebra over $\mathbb{Q}$, so we can write $K^{\mathrm{red}} = K_1 \times \cdots \times K_r$ for some r, with each K_i a number field, and $U = U_1 \times \cdots \times U_r$, with each U_i an order in a number field. Then we have that $\underline{h}_q^{\mathrm{red}} \cong U_{1,q} \times \cdots \times U_{r,q}$, and each component is a discrete valuation ring.

Since the discriminant of $\underline{h}^{\mathrm{red}}$ is $\mathrm{disc}(\underline{h}^{\mathrm{red}}) = \mathrm{disc}(U)(U : \underline{h}^{\mathrm{red}})^2$ ([FT93, (2.3) on p.121]), the assumption on m implies that it contains no primes dividing $(U : \underline{h}^{\mathrm{red}})$. Specifically, let q be a rational prime in m and obtain $\underline{h}_m^{\mathrm{red}} = (\underline{h}_q^{\mathrm{red}})_m$ is a completion at one of the components of $U_{1,q} \times \cdots \times U_{r,q}$, and so a discrete valuation ring. Since the q we chose is in fact outside $\mathrm{disc}(\underline{h}^{\mathrm{red}})$, it does not divide $\mathrm{disc}(U)$ and so each $U_{i,q}$ is unramified, so $\underline{h}_m^{\mathrm{red}}$ is unramified. Since f is primitive, $\underline{h}_m$ is reduced ([Hid00, p. 106]) and the lemma is shown. $\square$

LEMMA 3.7. — *If $a(l, f) \not\equiv \pm\sqrt{\varepsilon(l)} \bmod m$ (which is always the case if $l \nmid N$ and f_2 is of level N) then $h_2(\Gamma_0(N) \cap \Gamma_1(l), \varepsilon, \mathbb{Z})_m \cong h_2(\Gamma_0(N), \varepsilon, \mathbb{Z})_m$, and the former is an unramified discrete valuation ring.*

Proof. — For the proof of this lemma we refer the reader to [Hid00, p. 106]. $\square$

PROPOSITION 3.8. — $\mathbf{h}_m \cong h_m \hat{\otimes}_{\mathbb{Z}_l} \mathbb{Z}_l[[X]]$. *In particular, we have*

$$\mathrm{Aut}_\Lambda(\mathbf{h}_m) \cong \mathrm{Aut}_{\mathbb{Z}_l}(h_m).$$

Proof. — From [Hid86c, CXor. 3.2], we have that $\mathbf{h}_m/P_2\mathbf{h}_m \cong h_m$. Since h_m is an unramified discrete valuation ring over $\mathbb{Z}_l$, $\mathbf{h}_m$ is a regular local ring of dimension 2 with a regular sequence (P_2, l). Now, $\mathbf{h}_m/m_F \cong (\mathbf{h}_m/P_2)/(l) \cong h_m/(l) \cong \mathbf{F}$, and h_m is unramified over $\mathbb{Z}_l$, so we get $h_m \cong W(\mathbf{F}) \hookrightarrow \mathbf{h}_m$ (by the universal properties of Witt vectors). Then together with the map given by the diamond action $\mathbb{Z}_l[[X]] \to \mathbf{h}_m$ we get a map $h_m \hat{\otimes}_{\mathbb{Z}_l} \mathbb{Z}_l[[X]] \to \mathbf{h}_m$ which induces an isomorphism on residue fields and maps the regular sequence (P_2, l) for $h_m \hat{\otimes}_{\mathbb{Z}_l} \mathbb{Z}_l[[X]]$ into a regular sequence for $\mathbf{h}_m$ and is thus an isomorphism.

In order to finish the proof of Theorem 3.1, it suffices to show that the local Hecke algebras are isomorphic to the coefficient rings:

LEMMA 3.9. — *We can identify the local Hecke algebras with coefficient rings as: $h_m \cong O_{f,\mathfrak{l}}$ and $\mathbf{h}_m \cong \mathbf{I}$.*

Proof. — By the definition of m_f, for any $b \in h \setminus m$, $\lambda_f(b) \neq l$ and more generally, $\lambda_f(b) \in O_{f,\mathfrak{l}}^\times$. This allows us to define a map $h_{(m)} \to O_{f,\mathfrak{l}}$ by mapping

$$\frac{a}{b} \mapsto \frac{\lambda_f(a)}{\lambda_f(b)}$$

which is now well-defined and a homomorphism. By the proof of Lemma 3.6 we know that this map is surjective because λ_f maps onto $\mathbb{Z}[a(n,f)]$ and l (the prime at which we are localizing and then completing) is outside the index of $(O_f : \mathbb{Z}[a(n,f)])$. Taking the completion of the left-hand-side gives us a surjection $\tilde{\lambda} : h_m \to O_{f,\mathfrak{l}}$ of discrete valuation rings. Furthermore h_m is of dimension 1 because h is a finite rank $\mathbb{Z}$-module, h and $\mathbb{Z}$ have the same dimension, and localizing and completing doesn't affect the dimension. $O_{f,\mathfrak{l}}$ is also of dimension 1, so we have a surjection of discrete valuation rings of equal dimension, hence an isomorphism.

Similarly, the map $\mathbf{h}_m \to \mathbf{I}$ is surjective, and both are finite rank Λ-modules, so they have the same dimension (2). But both are integral domains, so there must be no kernel and the rings are isomorphic. □

Proof of Theorem 3.1. — Given the results above, the theorem follows trivially by noting that $\mathbf{I} \cong \mathbf{h}_m \cong h_m \hat{\otimes}_{\mathbb{Z}_l} \mathbb{Z}_l[[X]] \cong O_{f,\mathfrak{l}} \hat{\otimes}_{\mathbb{Z}_l} \mathbb{Z}_l[[X]] \cong O_{f,\mathfrak{l}}[[X]]$. □

For each $k \geqslant 2$ set $P_k := ((1+X) - (1+l)^k)$ to be the prime ideal of "weight k" in $\mathbf{I}$.

3.2. Setting up notation.

We now define Γ_F and H_F, similarly to the weight 2 analogues Γ_f, H_f. Let

$$\begin{aligned}\Gamma_F &:= \{\gamma \in \mathrm{Aut}(\mathbf{K}) \mid \gamma(\mathfrak{l}) = \mathfrak{l} \text{ and } \exists \text{ a Dirichlet character } \chi_\gamma \text{ such that} \\ &\quad \gamma(a(p,F)) = \chi_\gamma(p)a(p,F) \text{ for all but finitely many } p\} \\ \Gamma'_f &:= \{\gamma \in \Gamma_f \mid \gamma(\mathfrak{l}) = \mathfrak{l}\} \\ R_F &:= \mathbf{I}^{\Gamma_F} \\ H_F &:= \cap_{\gamma\in\Gamma_F} \ker(\chi_\gamma).\end{aligned}$$

Further, as we will see shortly, more important than Γ_f for us will be Γ'_f.

We will show that in fact as far as the fixed rings R_F and $R_{f,\mathfrak{l}}$ go, there is no new information in the Λ-adic setting over the weight 2 setting: $R_F \bmod P_2 \cong R_{f,\mathfrak{l}}$. To prove this we record the following lemmas:

LEMMA 3.10. — *Recall that we defined $h' = h \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Then let h^{ord} be the l-ordinary part of h', i.e. the product of local rings of h' in which the image of $T(l)$ is a unit. Then the set of local rings of h^{ord} is in bijection with the set of local rings of* $\mathbf{h}$.

Proof. — For brevity in the proof we use h to denote h^{ord}. Write the decompositions as $\mathbf{h} = \prod_{i=1}^{s} \mathbf{h}_{\underline{m}_i}$ and $h = \prod_{j=1}^{r} h_{m_j}$. Clearly the map $\mathbf{h} \to h$ modulo the prime ideal of weight 2 is surjective, and each local component of $\mathbf{h}$ gets mapped onto a local component of h, so the only thing to show is that no two local components of $\mathbf{h}$ get mapped to the same component of h. Let $\overline{e_i} : h \to h_{m_i}$ be the canonical projections, and note that the $\{\overline{e_i}\}$ form a complete set of orthogonal idempotents. Then by Hensel's lemma (for instance [Eis95, Cor. 7.5 on p.187]), we can lift this set of idempotents to $\{e_i\} : \mathbf{h} \to \mathbf{h}$, and $\mathbf{h} = \prod_{i=1}^{r} e_i\mathbf{h}$. Then the lemma will follow if each $e_i\mathbf{h}$ is local. Suppose $n_1 \neq n_2 \subset e_i\mathbf{h}$ are both maximal ideals. Then we may decompose $e_i\mathbf{h} = R_1 \times R_2$, and both R_j's are Λ-algebras. Since $e_i\mathbf{h}$ is semi-local, $P_2e_i\mathbf{h}$ is contained in the Jacobson radical of $e_i\mathbf{h}$, and so by Nakayama's lemma neither of the factors in $\overline{e_i}h = e_i\mathbf{h}/P_2e_i\mathbf{h} = R_1/P_2R_1 \times R_2/P_2R_2$ is trivial, contradicting the fact that $\overline{e_i}h = h_{m_i}$ is local. $\square$

Recall that $\mathrm{Aut}_\Lambda(\mathbf{h}_m) \cong \mathrm{Aut}_{\mathbb{Z}_l}(h_m)$ from Proposition 3.8. If $\gamma \in \mathrm{Aut}_{\mathbb{Z}_l}(h_m)$ then we will denote by $\tilde{\gamma} \in \mathrm{Aut}_\Lambda(\mathbf{h}_m)$ the isomorphic image of γ.

LEMMA 3.11.— *Let $\gamma \in \Gamma_f \subset \mathrm{Aut}_{\mathbb{Z}_l}(h_m)$ and let $\tilde{\gamma} \in \mathrm{Aut}_\Lambda(\mathbf{h}_m)$ be the isomorphic automorphism. Then the following diagram commutes for $P = P_2$:*

$$\begin{tikzcd} O_{f,\mathfrak{l}}[[X]] \cong \mathbf{I} \cong \mathbf{h}_m \arrow[r, "\mathrm{mod} P"] \arrow[d, "\tilde{\gamma}"] & O_{f,\mathfrak{l}} \cong h_m \arrow[d, "\gamma"] \\ O_{f,\mathfrak{l}}[[X]] \cong \mathbf{I} \cong \mathbf{h}_m \arrow[r, "\mathrm{mod} P"] & O_{f,\mathfrak{l}} \cong h_m. \end{tikzcd}$$

Proof.— Note that for $P = (X)$ the diagram obviously commutes. Now consider $t := 1 + X - (1+l)^2 \in O_{f,\mathfrak{l}}[[X]]$. Clearly $O_{f,\mathfrak{l}}[[X]] \cong O_{f,\mathfrak{l}}[[t]]$, and substituting $O_{f,\mathfrak{l}}[[t]]$ for $O_{f,\mathfrak{l}}[[X]] = \mathbf{h}_m$ in the diagram shows that it does indeed commute for $P = (t) = P_2$, since the difference between X and t is $1 - (1+l)^2 \in \mathbb{Z}_l$ and thus fixed by both γ and $\tilde{\gamma}$. □

Now we make precise the notion that there are no twists in the Λ-adic setting that were not present in weight 2:

PROPOSITION 3.12.— *The Λ-adic twists are those weight 2 twists that fix $\mathfrak{l}$. More precisely,*

$$\Gamma_F \cong \Gamma'_f \subset \Gamma_f.$$

Proof.— We know that

$$\Gamma_F \subset \mathrm{Aut}_\Lambda(\mathbf{I}) \cong \mathrm{Aut}_{\mathbb{Z}_l}(O_{f,\mathfrak{l}}) \supset \Gamma'_f$$

so the proposition follows if we can show that the (fixed) isomorphism between the automorphism groups maps each Γ into the other.

Let $\gamma \in \Gamma'_f$, and let $\tilde{\gamma} \in \mathrm{Aut}_\Lambda(\mathbf{I})$ be its isomorphic image. Then to show that $\tilde{\gamma} \in \Gamma_F$ it suffices to show that there is a Dirichlet character that is compatible with it. By Lemma 3.11, $\tilde{\gamma}F \bmod P_2 \doteq f \otimes \chi_\gamma$, where the $\doteq$ symbol signifies that all but finitely many of the prime-index Fourier coefficients on both sides agree. But $F \otimes \chi_\gamma \bmod P_2 \doteq f \otimes \chi_\gamma$ as well. Since eigenforms that agree on all but finitely many prime-index coefficients are equal, $F \otimes \chi_\gamma \bmod P_2 \doteq \tilde{\gamma}F \bmod P_2$. But then the uniqueness of the Λ-adic lift yields that $F \otimes \chi_\gamma \doteq \tilde{\gamma}F$, so $\chi_{\tilde{\gamma}} = \chi_\gamma$, and $\tilde{\gamma} \in \Gamma_F$. So $\Gamma'_f \hookrightarrow \Gamma_F$.

Conversely, let $\tilde{\gamma} \in \Gamma_F$, and $\gamma \in \mathrm{Aut}_{\mathbb{Z}_l}(O_{f,\mathfrak{l}})$ be its isomorphic image. Then

$$f \otimes \chi_{\tilde{\gamma}} = F \otimes \chi_{\tilde{\gamma}} \bmod P_2 \doteq \tilde{\gamma} F \bmod P_2 = \gamma(F \bmod P_2) = \gamma f$$

and $\gamma \in \Gamma'_f$, so $\Gamma_F \hookrightarrow \Gamma'_f$. □

The last proposition gives us an injection $\varphi : \Gamma_F \hookrightarrow \Gamma_f$, so we expect the corresponding Galois groups to have the opposite relationship:

Proposition 3.13. — *Under the above assumptions, $H_f \subset H_F$.*

Proof. — Let $g \in H_f = \bigcap_{\gamma \in \Gamma_f} \ker(\chi_\gamma)$. Then to prove the proposition, it suffices to show that for any $\gamma' \in \Gamma_F$, χ'_γ is trivial on g. This is equivalent to saying that $\gamma'(a(p,F)) = a(p,F)$ for almost all p. $g \in H_f$ means that $\gamma'(a(p,F)) \bmod P_2 = \varphi(\gamma')(a(p,f)) = a(p,f)$ for almost all p. So $\frac{\gamma'(a(p,F))}{a(p,F)} - 1 \in P_2$ for almost all p. But $\chi_{\gamma'}(p)$ is a finite order character, so $\frac{\gamma'(a(p,F))}{a(p,F)}$ is an element of finite order in $1 + P_2$, and thus must be 1 for almost all p. □

Now we are ready to prove:

Theorem 3.14. — $R_F \bmod P_2 \cong R_{f,\mathfrak{l}}$.

Proof. — Given the previous results, this follows immediately:

$$(R_F \bmod P_2) = \left(\mathbf{I}^{\Gamma_F} \bmod P_2\right)$$
$$\cong \left(O_{f,\mathfrak{l}}[[X]]^{\Gamma_F} \bmod P_2\right) \cong O_{f,\mathfrak{l}}^{\Gamma'_f} = O_{f,\mathfrak{l}}^{\Gamma_f} = R_{f,\mathfrak{l}}.$$

Note that we are abusing notation slightly here, where Γ_f doesn't act on $O_{f,\mathfrak{l}}$ (an element of Γ_f might permute the components of $O_{f,l}$). Instead we take $O_{f,\mathfrak{l}}^{\Gamma_f}$ to mean that subset of $O_{f,\mathfrak{l}}$ which is fixed by Γ_f when embedded in $O_{f,l}$. □

3.3. A proposition of N. Boston.

In the appendix to [MW86], N. Boston gives a criterion for lifting the property of being full from a residual representation to the original one. We suspend for this section our notation from above and use Boston's.

If I is an ideal of a ring A, define $\Gamma(I) := \ker(SL_n(A) \to SL_n(A/I))$ (n will be clear from the context). Also, for any group D, let $Z(D)$ denote the center of D.

Call an element $T \in GL_n(W)$ a transvection if $Td = d$ for all $d \in D$ for some hyperplane D, and for all $x \in V = W^n$, $T(x) = x + d_x$, for some $d_x \in D$.

Let R be a complete, Noetherian, local ring, with maximal ideal m, and residue characteristic $p \geqslant 5$. We further assume that R/m is finite, that R is regular and of Krull dimension 2, and that $m = (p,t)$. Suppose $p \nmid n$. Let $I_1, \ldots, I_d$ be minimal ideals of R/m^2 that generate m/m^2, where $d := \dim_{R/m}(m/m^2)$.

We use the following proposition:

PROPOSITION 3.15 [MW86, Cor. in appendix]. — *Let D be a closed subgroup of $SL_n(R)$ projecting onto $SL_n(R/m)$, such that for $1 \leqslant i \leqslant d$, there exists $x_i \in \Gamma(I_i) \setminus Z(SL_n(R/m^2))$ normalizing the image of D in $SL_n(R/m^2)$. Then $D = SL_n(R)$.*

Then a slightly modified form of Boston's result is:

PROPOSITION 3.16 (Boston). — *Let $\rho : G \to GL_2(R)$ be a continuous representation, inducing $\overline{\rho} : G \to GL_2(R/m)$. Let $L \subset G$ be a subgroup. Suppose $\overline{\rho}$ is full. Then if*

(1) $\rho(L) \subset \left\{\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}\right\}$;

(2) *there exists a matrix of the form* $\begin{pmatrix} 1 & * \\ 0 & (1+p)^{-1}(1+t) \end{pmatrix}$ *in $\rho(L)$; and*

(3) *for each $b \in \mathbf{F}_p^\times \subset (R/m)^\times$, there exists a matrix of the form* $\begin{pmatrix} 1 & * \\ 0 & b \end{pmatrix} \in \overline{\rho}(L)$,

then ρ is full.

Proof. — The proof here basically follows the one given in the appendix to [MW86], with the exception that we make it more explicit here, and use the condition 2 above instead of Boston's condition that there exists a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & (1+t) \end{pmatrix}$ in $\rho(L)$. Also, we use the more general setting of the groups $L \subset G$ instead of Boston's specific use of the inertia group at p: $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\mathbf{F} := \mathbf{F}_{p^r} \cong R/m$ be the residue field of R. Let $D := \rho(G) \cap SL_2(R)$ and $D_2 := D \bmod m^2 \subset SL_2(R/m^2)$ be its projection. Then we take as the ideals for Proposition 3.15 above $J_1 := (p, m^2)/m^2 \subset R/m^2$ and $J_2 := (t, m^2)/m^2 \subset R/m^2$. These obviously generate m/m^2 and are minimal in R/m^2.

If we can find elements $x_i \in \Gamma(J_i) \setminus Z(SL_2(R/m^2))$ for $i := 1, 2$ that normalize D_2, we will be done by Proposition 3.15.

Claim 3.17. — $\Gamma(J_1) \subset D_2$.

Proof. — Let $1 + u \in D_2$ be a lift of a non-scalar transvection to $D_2 \subset SL_2(R/m^2)$ from $SL_2(R/m)$ (possible since $p \geqslant 5$ and D surjects onto $SL_2(R/m)$). Let $x_1 := (1+u)^p = 1 + pu \in D_2$. Note that $x_1 \bmod J_1 = 1$, so that $x_1 \in D_2 \cap \Gamma(J_1)$. But $\Gamma(J_1)$ is minimal as an $SL_2(R/m)$-module (since J_1 is) so $D_2 \cap \Gamma(J_1) = \Gamma(J_1)$, i.e. $\Gamma(J_1) \subset D_2$. So the claim is done. □

This gives us the x_1 we were looking for (being **in** D_2 guarantees normalizing it). So we only need to find an x_2 now:

By the second hypothesis, there exists at least one matrix of the form $\begin{pmatrix} 1 & r \\ 0 & (1+p)^{-1}(1+t) \end{pmatrix}$ in $\rho(L)$. Two cases arise:

(1) one of these r's satisfies $r \in m$. Then choose $a, b \in R$ so that $r = pa + tb$ and consider the matrix

$$\begin{aligned} A' &:= (1+t)^{-1/2} \begin{pmatrix} 1 & pa+tb \\ 0 & (1+p)^{-1}(1+t) \end{pmatrix} \begin{pmatrix} 1 & -pa(1+p) \\ 0 & 1+p \end{pmatrix} \\ &= (1+t)^{-1/2} \begin{pmatrix} 1 & tb(1+p) \\ 0 & 1+t \end{pmatrix} = \begin{pmatrix} (1+t)^{-1/2} & tb(1+p) \\ 0 & (1+t)^{1/2} \end{pmatrix}. \end{aligned}$$

Note that A' has determinant 1 and mod t is the identity matrix (so $A' \in \Gamma(J_2)$), and non-scalar (so non-central in $SL_2(R/m^2)$). But A' is just a product of a matrix in $\rho(L)$ mod m^2 and a matrix in $\Gamma(J_1) \subset D_2$ with a scalar, so it certainly normalizes D_2, and we have our desired $x_2 := A'$, so the proof of the proposition is done in this case.

(2) None of the r's that arise in this manner are inside m. Thus we get a matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \bar{\rho}(\mathrm{L})$ such that $a \neq 0$ ($r \equiv a \bmod m$). By assumption, $p \geqslant 5$, so we can choose $n \in \{2, \ldots, p-2\}$ such that when viewed as an element of $\mathbf{F}_p \subset R/m$, n is a unit and $n^{-1} \neq -1$. Let $c := n^{-1} + 1 \neq 0$ ($c \in \mathbf{F}_p$). Then by the third assumption of the proposition, there exists $s \in R/m$ such that $\begin{pmatrix} 1 & s \\ 0 & c \end{pmatrix} \in \bar{\rho}(\mathrm{L})$. Then (letting L' denote the

commutator of L):

$$\begin{aligned}\overline{\rho}(L') \ni (1/c)\begin{pmatrix}1 & na\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & s\\ 0 & c\end{pmatrix}\begin{pmatrix}1 & -na\\ 0 & 1\end{pmatrix}\begin{pmatrix}c & -s\\ 0 & 1\end{pmatrix}\\ = (1/c)\begin{pmatrix}1 & s+nac\\ 0 & c\end{pmatrix}\begin{pmatrix}c & -s-na\\ 0 & 1\end{pmatrix}\\ = \begin{pmatrix}1 & s-s+(c-1)na\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} =: \overline{B}.\end{aligned}$$

Now we consider what $\overline{B}$ might lift to: $\det(\rho(L')) = \{1\}$, so by the first hypothesis, $\overline{B}$ must be the reduction of some $B = \begin{pmatrix}1 & r'\\ 0 & 1\end{pmatrix} \in \rho(L')$ such that $r' \equiv a \bmod m$. In particular, we get that

$$\begin{pmatrix}1 & r-r'\\ 0 & (1+p^{-1}(1+t)\end{pmatrix} = \begin{pmatrix}1 & r\\ 0 & (1+p^{-1}(1+t)\end{pmatrix}\begin{pmatrix}1 & r'\\ 0 & 1\end{pmatrix} \in \rho(L)$$

and $r - r' \in m$. This is a contradiction to the assumption of this case, so the proof of the proposition is done. □

Having set up this machinery, we are ready to lift the weight 2 results to the Λ-adic setting.

4. Λ-adic situation.

4.1. Lifting the representation fullness.

Let $\rho := \rho_F : G_{\mathbb{Q}} \to GL_2(\mathbf{I})$ be the Galois representation attached to F from Section 3. Let $\overline{\rho_F}$ be the reduction $\rho \bmod \mathfrak{m}$. Then we set $\rho_k := \rho_F \bmod P_k$ for each weight k to be the weight k specialization of ρ. Note that this ρ_2 coincides with the $\rho_{2,l}$ from Section 2.

Proposition 4.1.— *Up to conjugation $\rho(H_F) \subset GL_2(R_F)$.*

Outline of Proof. — By the Cebotarev density theorem, it is in fact enough to show that for any $\mathrm{Frob}_q \in H_F$, $\rho(\mathrm{Frob}_q) \in GL_2(R_F)$. From the properties of ρ, we see that $\mathrm{tr}(\rho(\mathrm{Frob}_q)) = a(q, F)$ and $\mathrm{Frob}_q \in H_F$ implies that $\chi_\gamma(q) = 1$ for all $\gamma \in \Gamma_F$ so $\mathrm{tr}(\rho(\mathrm{Frob}_q)) \in R_F$. Using Wiles' theory of pseudo-representations (for instance, [Hid00, Prop. 2.16]), there exists a representation $\pi : H_F \to GL_2(R_F)$ whose trace agrees with that of $\rho|_{H_F}$. But $(\rho \bmod \mathfrak{m})|_{H_F}$ is irreducible, so using a result of Carayol and Serre ([Hid00, Prop. 2.13]) we see that $\rho|_{H_F}$ and π are conjugate, and the proposition follows. $\square$

Hereafter we restrict our attention to the image of $H_?$. Thus we let $\rho'_f := \rho_f|_{H_f}$ and $\rho'_F = \rho_F|_{H_F}$. In this section we let $\rho := \rho'_F$ to simplify notation where no confusion can arise. Note also that because we have shown that $H_f \subset H_F$, the image of $\rho \bmod P_2$ will contain the image of ρ'_f.

In order to apply Boston's proposition to our situation we record the following observations:

Lemma 4.2. — *Specializing to the weight* 2 *representation and then reducing* mod (l) *is equivalent to reducing the* $\mathbf{I}$*-adic representation* mod $\mathfrak{m}$.

Note that the former is what Corollary 2.1 gives us, and the latter is what Proposition 3.16 requires.

Proof. — This is trivial by noting that the following diagram commutes:

$$\begin{tikzcd}
H_F \arrow[r, "\rho"] \arrow[dr, "\overline{\rho}"'] & GL_2(R_F) \arrow[r, "(1+X)\mapsto(1+l)^2", "\bmod P_2"'] \arrow[d, "\bmod \mathfrak{m}"'] & GL_2(R_{f,\mathfrak{l}}) \arrow[dl, "\bmod (l)"] \\
 & GL_2(\mathbf{F}_{l^r}) &
\end{tikzcd}$$

Combining Lemma 4.2 with Corollary 2.1 we get:

Lemma 4.3. — *$\overline{\rho}$ is full:* $\mathrm{Im}(\overline{\rho}) \supset SL_2(\mathbf{F}_{l^r})$.

Thus, to apply Proposition 3.16, we only need to verify the three technical conditions of Proposition 3.16. We make the following substitutions for the proposition (where I_l is the inertia at l in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$): $L = I_l \cap H_F$, $p = l$, $R = \mathbf{I}$, $\mathfrak{m} = (l, X)$, and $R/\mathfrak{m} = \mathbf{F}_{l^r}$. We will let $J := I_l \cap H_F$.

Lemma 4.4. — *The first condition of Proposition 3.16 is satisfied:*

$$\rho(J) \subset \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}.$$

Proof. — The definition of ρ_F being l-ordinary is that $\rho_F \mid_{D_l} \cong \begin{pmatrix} \delta & * \\ 0 & \psi \end{pmatrix}$ where δ is unramified at l (i.e., $\delta \mid_{I_l} = 1$). So $\rho_F \mid_{I_l} \cong \begin{pmatrix} 1 & * \\ 0 & \psi \end{pmatrix}$ for some character ψ. An $\mathbf{I}$-adic representation is always ordinary at the prime over which $\mathbf{I}$ is defined, so we are done. □

In order to verify the other two criteria for Proposition 3.16, we make the following observation:

LEMMA 4.5. — *$J \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ maps onto*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{l^\infty})) \cong \mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}).$$

Call this latter Galois group L.

Proof. — As noted in Section 2.3, $l \nmid \mathrm{cond}(\chi_\gamma)$, so in particular, $l \nmid \mathrm{ord}(\chi_\gamma)$, and thus

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\ker(\chi_\gamma)$$

is prime-to-l (contains no non-trivial homomorphic image of $\mathbb{Z}/l\mathbb{Z}$). Fix for the moment γ. Let $K := \ker(\chi_\gamma)$. Then $K' := K/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{l^\infty})) \subset L$ must be pro-l because L is, so L/K' is pro-l. But $L/K' \cong \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/K$ which is prime-to-l as shown above, so L/K' must be trivial, i.e. $K' = L$. Varying γ now, we see that $H_F/\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{l^\infty}))$ must also be equal to L. Thus the lemma follows since I_l surjects onto L. □

LEMMA 4.6. — *There exists a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & (1+l)^{-1}(1+X) \end{pmatrix}$ in $\rho(J)$.*

Proof. — The defining properties of ρ as an $\mathbf{I}$-adic representation are laid out in [Hid86a, Thm. 2.1 and following remarks]. In particular, $\det(\rho) \mid_{I_l} = \chi\nu_l^{-1}\iota$. As shown above, J surjects onto L, so ι (which factors through L) must have a surjective image on $J \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, so that $\iota(J) = 1 + l\mathbb{Z}_l \hookrightarrow \Lambda$. For the same reason, $\nu_l(J) = \mathbb{Z}_l^\times$.

Let $\sigma \in J \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $a_\sigma := \omega_l(\nu_l(\sigma)) \in (\mathbb{Z}/l\mathbb{Z})^\times$, $b_\sigma := \langle \nu_l(\sigma) \rangle \in 1 + l\mathbb{Z}_l$, and $s \in \mathbb{Z}_l$ such that $b_\sigma = (1+l)^s$. Then

$$\begin{aligned} \det(\rho_F(\sigma)) &= \chi(\sigma)\nu_l(\sigma)^{-1}\iota(\sigma) = \chi(\sigma)\nu_l(\sigma)^{-1}\kappa(\langle \nu_l(\sigma) \rangle) = \chi(\sigma)b_\sigma^{-1}a_\sigma^{-1}\kappa(b_\sigma) \\ &= \chi(\sigma)(1+l)^{-s}a_\sigma^{-1}(1+X)^s = \varepsilon(\sigma)\omega_l^2(\sigma)(1+l)^{-s}a_\sigma^{-1}(1+X)^s \\ &= \varepsilon(\sigma)(1+l)^{-s}a_\sigma(1+X)^s. \end{aligned}$$

By the surjectivity of ν_l and ι on J it follows that we can choose an element $\sigma \in J$ to hit any given a_σ and any $s \in \mathbb{Z}_l$ (independently). Let

$M := \mathrm{cond}(\varepsilon)$, and notice that by [Mom81, Rmk. 1.6], $M|N$. Note that since $l \nmid N$, $\mathbb{Q}(\mu_{l^\infty})$ and $\mathbb{Q}(\mu_M)$ are linearly disjoint inside $\mathbb{Q}(\mu_{l^\infty}, \mu_M) \subset \overline{\mathbb{Q}}$, we can choose a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to yield any given $s \in \mathbb{Z}_l, a_\sigma \in (\mathbb{Z}/l\mathbb{Z})^\times$ and keeping $\varepsilon(\sigma) = 1$. So we can choose $\sigma \in J$ so that $s = 1$ and $\varepsilon(\sigma) = a_\sigma = 1$, yielding the desired element of J. □

LEMMA 4.7. — *For each $b \in \mathbf{F}_l^\times \subset (R/\mathfrak{m})^\times$, there exists a matrix of the form $\begin{pmatrix} 1 & * \\ 0 & b \end{pmatrix}$ in $\overline{\rho}(J)$.*

Proof. — Using the proof of the previous lemma, one notes that the choice of a_σ is independent of the choice of s, so one may choose $\sigma \in J$ so that $s = 0$ and $\chi(\sigma)a_\sigma$ hits any desired $b \in \overline{\rho}(J)$. The lemma follows. □

Putting Lemmas 4.3, 4.4, 4.6, and 4.7 all together with Proposition 3.16, we obtain:

THEOREM 4.8. — *The restricted Galois representation attached to F is full, i.e. $\rho_F(H_F) \supset SL_2(R_F)$.*

4.2. The exact image of H_F.

We now know that $SL_2(R_F) \subset \rho_F(H_F) \subset GL_2(R_F)$, and are interested in finding out just where between the two matrix groups the image of H_F under ρ_F lies.

We record here a simple lemma for use later:

LEMMA 4.9. — *If*

$$SL_2(R) \subset B \subset GL_2(R)$$

is an inclusion of groups and

$$\det(B) = A^\times$$

for some ring A, then

$$B = \{X \in GL_2(R) \mid \det(X) \in A^\times\}.$$

As shown in the proof of Lemma 4.6,

$$\det(\rho_F(\sigma)) = \chi(\sigma)\nu_l(\sigma)^{-1}\iota(\sigma) = \varepsilon(\sigma)(1+l)^{-s}a_\sigma(1+X)^s$$

and the $\sigma \in J = I_l \cap H_F \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can be chosen to produce any (independently) chosen $\varepsilon(\sigma) \in O_{f,\mathfrak{l}}, a_\sigma \in (\mathbb{Z}/l\mathbb{Z})^\times$, and $s \in \mathbb{Z}_l$.

Let

$$D := \left\{ca(1+l)^{-s}(1+X)^s \in O_{f,\mathfrak{l}}[[X]] \mid c^M = a^l = 1, s \in \mathbb{Z}_l\right\}.$$

Then we get

Proposition 4.10. — *Recalling that M is the conductor of the weight 2 nebentype and $l \in \Sigma_f$ for the weight 2 specialization of F, we have*

$$\det(\rho_F(H_F)) = D \cong \mu_M \times \mu_l \times \Gamma'$$

where

$$\Gamma' := \left\{(1+l)^{-s}(1+X)^s \mid s \in \mathbb{Z}_l\right\} \cong \Gamma \subset \mathbb{Z}_l[[X]].$$

Imitating the weight 2 situation, we let

$$A_F := \{x \in GL_2(R_F) \mid \det(x) \in D\}$$

and putting the previous two lemmas together we get

Corollary 4.11. — *The image of $\rho_F(H_F)$ is as large as it can be given the determinant condition on it, i.e., $\rho_F(H_F) = A_F$.*

This is the analogous result to (Momose's) Theorem 2.1 in weight 2.

Recalling that we made arbitrary choices for f_2 and for $l \in \Sigma_{f_2}$, we reformulate Theorem 4.8:

Corollary 4.12. — *Let $f \in S_2^{\mathrm{new}}(\Gamma_0(N), \varepsilon)$ be a normalized eigenform without complex multiplication. Let Σ'_f be the set of ordinary primes for f (which is of density 1 by Proposition 2.2). Let*

$$\mathbf{B}_f := \left\{F_l \in \mathbf{S}^{\mathrm{ord}}(Nl^\infty, \varepsilon\omega_l^2) \mid F_l \textit{ specializes to } f \textit{ at weight } 2 \textit{ and } l \in \Sigma'_f\right\}.$$

Then the restricted Galois representations (in the sense of Theorem 4.8) attached to all but finitely many of the elements of $\mathbf{B}_f$ are full.

4.3. The exact image of $G(\mathbb{Q})$.

The computation of the exact image of $G(\mathbb{Q})$ under ρ_F follows almost word for word the argument of E. Papier as presented in [Rib85, Sec. 4]. We include the proof here for the reader's convenience.

The basic result of this section is that the difference between $\rho_F(G)$ and $\rho_F(H_F)$ is similar to the difference between G and H_F.

LEMMA 4.13. — *Given $\gamma \in \Gamma_F$, $\gamma\rho_F$ and $\rho_F\otimes\chi_\gamma$ are irreducible.*

Proof. — Suppose $0 \neq V \subsetneq \mathbf{K}^2$ is a non-trivial stable subspace under $\gamma\rho_F(G)$. Note that $\gamma\rho_F(g)\binom{x}{y} = \gamma\left(\rho_F(g)\gamma^{-1}\binom{x}{y}\right)$, so if $(\gamma\rho_F(G))V = V$ then $\gamma(\rho_F(G)\gamma^{-1}(V)) = V$, and $\rho_F(G)\gamma^{-1}(V) = \gamma^{-1}(V)$, contradicting the irreducibility of ρ_F. The second assertion follows from the fact that tensoring two irreducible representations yields a third irreducible representation. □

By the definition of χ_γ, it is clear that $\gamma\rho_F$ and $\rho_F\otimes\chi_\gamma$ have the same trace on Frob_p for any $p \nmid lN$, so they have equal traces everywhere (by the Cebotarev density theorem). The two representations must then be equivalent (being semi-simple and of equal trace). So there is a matrix $X \in GL_2(\mathbf{K})$ such that $X\gamma\rho_F X^{-1} = \gamma\otimes\chi_\gamma$. Considering the restriction to H_F, we see that X commutes with $\rho_F(H_F) \supset SL_2(R_F)$, so X is a scalar matrix and in fact $\gamma\rho_F(g) = \rho_F(g)\chi_\gamma(g)$ for any $g \in G$.

Fix for the moment $\gamma \in \Gamma_F$ and $g \in G$. From the structure of $\mathbf{I}$ and R_F which we have already computed, it follows that $\mathbf{I}/m_{\mathbf{I}}$ and R_F/m_{R_F} are finite. Call these fields $\mathbf{E}$ and $\mathbf{F}$ respectively. Then we note that $\Gamma_F \cong \mathrm{Gal}(\mathbf{E}/\mathbf{F})$ and we can view γ as acting on elements of $\mathbf{E}$. Notice that being a root of unity, $\chi_\gamma(g)$ has the same order when projected into $\mathbf{E}$, so we view it as being both an element of $\mathbf{I}^\times$ and $\mathbf{E}^\times$. Using Hilbert's theorem 90 we can find an element $\overline{\alpha(g)} \in \mathbf{E}$ such that $\gamma\left(\overline{\alpha(g)}\right)/\overline{\alpha(g)} = \chi_\gamma(g)$. $\mathbf{I}$ is unramified, so we can lift $\overline{\alpha(g)}$ to $W(\mathbf{E}) \hookrightarrow \mathbf{I}$ and call this lift $\alpha(g) \in \mathbf{I}$. Since $\mathbf{E}$ is finite, $\overline{\alpha(g)}$ has finite order, so $\alpha(g)$ has finite order, and $\alpha(g) \in \mathbf{I}^\times$. Note that the choice of g modulo H_F is irrelevant, since χ_γ is trivial on H_F, so there are only finitely many $\alpha(g)$'s.

LEMMA 4.14. — *When χ is the nebentype of F, we have*

$$\gamma(\chi(g)) = \chi_\gamma(g)\chi(g)^2.$$

Proof. — It suffices to prove the lemma for $g = \mathrm{Frob}_p$ for all $p \nmid lN$. Let p be a prime outside lN. We consider the difference of Hecke operators: $T(p)^2 - T(p^2)$. Specifically, we apply the two operators to F and take the

p'th Fourier coefficient in the q-expansion of the resulting eigenform:

$$\begin{aligned}
a(p, F|T(p)) &= a(p^2, F) + \kappa(\langle p\rangle)\chi(p)p^{-1} \\
a(p^2, F|T(p)) &= a(p^3, F) + \kappa(\langle p\rangle)\chi(p)p^{-1}a(p, F) \\
a(p, F|T(p)^2) &= a(p, (F|T(p))|T(p)) = a(p^2, f|T(p)) + \kappa(\langle p\rangle)\chi(p)p^{-1} \\
&= a(p^3, F) + \kappa(\langle p\rangle)\chi(p)p^{-1}a(p, F) + \kappa(\langle p\rangle)\chi(p)p^{-1} \\
a(p, F|T(p^2)) &= a(p^3, F) + \kappa(\langle p\rangle)\chi(p)p^{-1}a(p, F).
\end{aligned}$$

So

$$a(p, F|T(p)^2) - a(p, F|T(p^2)) = \kappa(\langle p\rangle)\chi(p)p^{-1}$$

but F is a normalized eigenform so $a(p, F)^2 = a(p, F|T(p)^2)$ and $a(p^2, F) = a(p, F|T(p^2))$ yielding

$$a(p, F)^2 - a(p^2, F) = \kappa(\langle p\rangle)\chi(p)p^{-1}.$$

Applying γ to both sides and using the fundamental relationship between γ and χ_γ we get:

$$\begin{aligned}
\chi_\gamma(p)^2a(p, F)^2 - \chi_\gamma(p)^2a(p^2, F) &= \gamma(\kappa(\langle p\rangle)\chi(p)p^{-1}) \\
\chi_\gamma(p)^2(a(p, F)^2 - a(p^2, F)) &= \gamma(\chi(p))\kappa(\langle p\rangle)p^{-1} \\
\chi_\gamma(p)^2\kappa(\langle p\rangle)\chi(p)p^{-1} &= \gamma(\chi(p))\kappa(\langle p\rangle)p^{-1} \\
\chi_\gamma(p)^2\chi(p) &= \gamma(\chi(p))
\end{aligned}$$

and the lemma is proven. □

The preceding lemma tells us that $\gamma(\frac{\alpha(g)^2}{\chi(g)}) = \frac{\chi_\gamma(g)^2\alpha(g)^2}{\chi_\gamma(g)^2\chi(g)} = \frac{\alpha(g)^2}{\chi(g)}$, so $\frac{\alpha(g)^2}{\chi(g)} \in R_F$.

Re-write $\rho_F(g)$ as

$$\rho_F(g) = \begin{pmatrix} \alpha(g) & 0 \\ 0 & \chi(g)/\alpha(g) \end{pmatrix} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \alpha(g)^2/\chi(g) \end{pmatrix} (\alpha(g)^{-1}\rho_F(g)) \right\}.$$

Then the product in curly brackets is in $GL_2(R_F)$ and has determinant in D, so is in A_F. This allows us to give a full characterization of the image of ρ_F:

THEOREM 4.15. — *The image of ρ_F is the subgroup of $GL_2(\mathbf{I})$ generated by A_F and the finite set of matrices:* $\begin{pmatrix} \alpha(g) & 0 \\ 0 & \frac{\chi(g)}{\alpha(g)} \end{pmatrix}$, *where $g \in G/H_F$.*

5. Density analysis.

Now that we have the general machinery for lifting the fullness of a weight-2 Galois representation, we wish to consider how often this machinery succeeds in telling us that an **I**-adic representation is full.

To avoid repetition, call an ordinary, normalized eigenform which has no complex multiplication a generic form.

A naive approach to the analysis problem might be to decompose the set of all generic **I**-adic forms into a disjoint union of lift families indexed over $f \in S_2(N,\chi)$, which is a finite set, and in each family only finitely many elements are non-full, so we would get that only finitely many **I**-adic forms are not full. This reasoning is not correct, since some generic **I**-adic forms **don't** come from a weight-2 form of level N, but rather of level Nl, for some prime l.

Instead, we note that an arbitrary generic **I**-adic form with level N outside its structure prime (call it l) will be a lift of some generic form in $S_2(Nl,\chi\omega_l^a)$. To put this observation into a usable form, we make the following definitions:

DEFINITION 5.1.— *Fix an integer $N \geqslant 1$, a prime $l \nmid N$, and a primitive character χ mod N. Then define*

$$C(N,l,\chi) := \bigsqcup_{\substack{a \in \{0\ldots l-1\} \\ g \in S_2(Nl,\chi\omega_l^a) \\ g \text{ is generic}}} B_g.$$

Further, varying l, define

$$B(N,\chi) := \bigcup_{l \nmid N} C(N,l,\chi).$$

In this setup, each of the B_g's is an infinite set (indexed by a density 1 set of primes), and the Galois representations attached to all but finitely many of the elements of each B_g are full. Moreover, the disjoint union is taken over a finite set. Thus we obtain:

COROLLARY 5.2.— *The Galois representations attached to all but finitely many of the elements of $C(N,l,\chi)$ is full.*

Now, to get a density theorem on the totality of $B(N,\chi)$, we need to define partial subsets of B and C as follows: for a positive integer M let

$$B_g^{(M)} := \{F \in B_g \mid \text{ the structure prime of } F < M\}$$

and notice that this set is finite. Similarly, define

$$C(N,l,\chi)^{(M)} := \bigsqcup_{\substack{a \in \{0\ldots l-1\} \\ g \in S_2(Nl,\chi\omega_l^a) \\ g \text{ is generic}}} B_g^{(M)} \text{ and}$$

$$B(N,\chi)^{(M,M')} := \bigcup_{\substack{l \nmid N \\ l < M'}} C(N,l,\chi)^{(M)}$$

and note that both of the above sets are finite.

For any of the preceding sets of modular forms (B, C, etc.), let the superscript $+$ on the letter denote the subset with full attached Galois representations. Then define

$$d_{N,l,\chi}^{(M)} := \frac{|C^+(N,l,\chi)^{(M)}|}{|C(N,l,\chi)^{(M)}|}.$$

Then Corollary 5.2 implies:

COROLLARY 5.3. — $d_{N,l,\chi}^{(M)} \to 1$ *as* $M \to \infty$.

Similarly, we may define

$$b_{N,\chi}^{(M,M')} := \frac{|B^+(N,\chi)^{(M,M')}|}{|B(N,\chi)^{(M,M')}|}.$$

Then the previous corollary gives us:

COROLLARY 5.4. — *For any* $M' > 1$

$$b_{N,\chi}^{(\infty,M')} := \lim_{M\to\infty} b_{N,\chi}^{(M,M')} = 1.$$

But the density of $B^+(N,\chi)$ in $B(N,\chi)$ can be thought of precisely as $b_{N,\chi}^{(\infty,\infty)}$, and by the preceding corollary, we get:

THEOREM 5.5. — *Keeping the outside level and character fixed, the set of generic* **I**-*adic forms that have full attached Galois representations* *has density* 1 *in the set of all generic* **I***-adic modular forms. Since this is true for any level and character, the same density statement holds for all generic* **I***-adic modular forms.*

Note that calculating "along the diagonal", i.e. $b_{N,\chi}^{(X,X)} \to 1$ as $X \to \infty$ works just as well by the above reasoning. Note also that this reasoning does **not** work if one attempts to calculate $b_{N,\chi}^{(M,\infty)}$ (since we do not have a good understanding of how $d_{N,l,\chi}^{(M)}$ behaves as $l \to \infty$ and M is fixed).

It should also be noted that the density results here are optimal in the sense that it is not the case that only a finite number of generic **I**-adic modular forms have non-full representations. Each irregular prime gives rise to a Λ-adic form whose representation is reducible modulo the maximal ideal of the coefficient ring, so clearly it cannot be full. Then the infinitude of irregular primes guarantees the infinitude of non-full representations attached to generic **I**-adic modular forms.

Appendix A. Complex multiplication.

A.1. Specialization and CM.

In this section we consider the relationship between a Λ-adic modular form F and its specializations f_k to weight $k \geqslant 2$ with respect to complex multiplication. In particular, we show (Proposition A.1) that in our setting, if F has no CM then neither do the f_k.

We say of a modular form $f = \sum_{n=0}^{\infty} a_n q^n$ (classical or Λ-adic) that it has complex multiplication (CM) if there exists a non-trivial Dirichlet character φ such that $\varphi(p)a_p = a_p$ for a density 1 set of primes p. If φ is defined mod D then $\varphi(p)a_p = a_p$ holds for all $p \nmid DN$ when f is a classical modular form of level N. Further, φ has to be a quadratic character. If the kernel of φ in $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a quadratic field then we say f has CM by this quadratic field.

In this section, $F \in \mathbf{S}_{\mathbf{I}}^{\mathrm{ord}}(Nl^{\infty}, \chi)$ will denote a normalized Hecke eigenform with no CM, and $f_k \in S_k(Nl, \varepsilon)$ will denote the specialization modulo the prime P_k of F (so $\chi = \varepsilon\omega_l^k$). Then f_k is a normalized Hecke eigenform (though not a newform for level Nl if $k > 2$).

PROPOSITION A.1. — *If F has no complex multiplication, then neither does f_k for any $k \geqslant 2$.*

Proof. — The key point is that the weight k is integral and greater than 1. Denote by the subscript $\cdot^0$ the primitive part of a Hecke algebra. Then we have $\mathbf{h}^0/P_k\mathbf{h}^0 \cong h_k^{0,\mathrm{ord}}$ and so the completions obey

$$\mathbf{h}^0_{P_k}/P_k\mathbf{h}^0_{P_k} \cong h_k^{0,\mathrm{ord}}\otimes\mathbb{Q} \cong U_1 \times \cdots \times U_v$$

for some fields U_i. But this means that

$$\mathbf{h}^0_{P_k} \cong \mathcal{U}_1 \times \cdots \times \mathcal{U}_v$$

where the $\mathcal{U}$'s and the U's are in bijection (by the proof of Lemma 3.10). Each of the $\mathcal{U}$'s is an irreducible component of the completed Hecke algebra.

Now suppose f_k has CM for integral $k \geqslant 2$. As proven by Hida in [Hid93, Sec. 7.6], a classical eigenform with CM gives rise to a Λ-adic CM eigenform, so there is a Λ-adic eigenform F' that specializes to f_k mod P_k. But by the bijection between the Λ-adic and classical local decompositions in the previous paragraph, this means that F' and F must belong to the same $\mathcal{U}_i$. But since F' has CM and F does not, they can't be conjugate and thus can't belong to the same irreducible component of the completed Hecke algebra. □

Note that further, in the "worst" possible case, F can only specialize to finitely many CM classical forms:

LEMMA A.2. — *f_k does not have CM for all but finitely many values of k (none of which are natural numbers greater than 1).*

Proof. — Fix a character θ and let $\mathfrak{a}_\theta$ be the ideal generated by $a(l,F) - \theta(l)a(l,F)$ for all primes $l \nmid DN$ (where D is the conductor of θ). Then for f_k to have CM by θ is equivalent to $P_k \supset \mathfrak{a}_\theta$. This implies that P_k is in $V := \mathrm{Spec}(\Lambda/\mathfrak{a}_\theta)(\overline{\mathbb{Q}_p})$, which is a proper closed subset of $\mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}_p})$. But the latter is 1-dimensional, so V must be 0-dimensional, i.e. only contain finitely many points. Thus only finitely many $P_k \in V$, and only finitely many f_k have CM by θ.

On the other hand, the level of $f\otimes\theta$ is the product of the level of f and the square of the conductor of θ. So if f is to have CM by θ, θ's conductor must be bounded, so only finitely many such θ's exist and the lemma is shown. □

A.2. Galois representations.

In this section we consider the effect CM has on the Galois representations attached to a modular form. For f classical or Λ-adic , let ρ_f be the attached l-adic or Λ-adic Galois representation.

PROPOSITION A.3. — *f has CM by a character θ if and only if there is a non-trivial quadratic character θ such that $\rho_f \cong \rho_f \otimes \theta$. Writing $F = \mathbb{Q}(\sqrt{-D})$ for the kernel of θ, f having CM by θ and F is equivalent to $\rho_f = \operatorname{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \psi$ for some character ψ.*

Proof. — Note that in the first assertion, the $(\Leftarrow)$ direction is trivial (since the traces of ρ_f on Frobenius elements are the $a(p, f)$).

The second assertion follows from the first by [DHI98, Lemma 3.2].

Notice that in [DHI98, Lemma 3.2] there is an apparent requirement that $\overline{\rho_f}$ be absolutely irreducible. This is not necessary. The only place this is used is in determining that C^r is a scalar matrix. But this can be obtained separately in the cases we deal with. In the classical case, Ribet has shown that ρ_f is irreducible, which is enough to give that C^r is scalar (see for instance [Hid00, p. 111]). In the Λ-adic case, the situation is only slightly more complicated, and comes from the classical case.

Letting $\Lambda_{(P)}$ be the localization at a prime P, consider the diagram:

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r, "\rho"] \arrow[dr, "\rho_P"'] & GL_2(\Lambda) \arrow[r] \arrow[d, "\bmod P"] & GL_2(\Lambda_{(P)}) \arrow[d, "\bmod P"] \\
& GL_2(\mathbb{Z}_p) \arrow[r] & GL_2(\mathbb{Q}_p)
\end{tikzcd}$$

Then ρ_P is irreducible from the classical case. Viewing $Z(\rho) \subset M_2(\Lambda_{(P)})$, we get a short exact sequence: $\Lambda_{(P)} \longrightarrow Z(\rho) \xrightarrow{\bmod P} \mathbb{Q}_p^\times$. Tensoring with $\mathbb{Q}_p$ over $\Lambda_{(P)}$ yields $\Lambda_{(P)} \otimes_{\Lambda_{(P)}} \mathbb{Q}_p = \mathbb{Q}_p \to Z(\rho) \otimes_{\Lambda_{(P)}} \mathbb{Q}_p = \mathbb{Q}_p$ so by Nakayama's lemma $Z(\rho) = \Lambda_{(P)}$, and we get that the centralizer of ρ is trivial again.

All that remains is to prove the direct direction of the first assertion. From the definition of CM, $\mathrm{tr}(\rho_f)(\mathrm{Frob}_q) = \mathrm{tr}(\rho_f \otimes \theta)(\mathrm{Frob}_q)$ for $q \nmid DN$ and Frob_q the Frobenius element at q. Thus by Cebotarev's density theorem, it follows that the traces of the two representations must be equal.

But then we can use the Brauer-Nesbitt theorem ([Hid00, Cor. 2.8]) to see that $\rho_f \cong \rho_f \otimes \theta$ (equivalence is over the field of fractions). $\square$

There is a simple corollary to Proposition A.3 which explicitly describes the shape of the image. We include it here for completeness.

COROLLARY A.4. — *If f has CM by θ and F ($F = \mathbb{Q}(\sqrt{-D})$ for $D > 0$ square-free), then* $\mathrm{Im}(\rho_f|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}) \subset \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$. *Further,* $\rho(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \subset \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right\}$.

Proof. — Consider the Galois tower:

$$\begin{tikzcd}
\overline{\mathbb{Q}} \arrow[d, dash, "H"] \arrow[dd, bend right=60, dash, "G"'] \\
\mathbb{Q}(\sqrt{-D}) = F \arrow[d, dash, "\langle\theta\rangle"] \\
\mathbb{Q}
\end{tikzcd}$$

By Proposition A.3 $\rho_f = \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}\psi$ for some character ψ of H. Writing the coset decomposition of G/H as $G = \sqcup_{i=1}^{[G:H]} H\sigma_i$ we get a matrix representation of ρ_f as: $\rho_f(g) = \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}\psi(g) = \left(\psi(\sigma_i g \sigma_j{}^{-1})\right)_{[G:H]}$ (where ψ takes the value 0 on any argument not in H). Concretely, we have that $[G:H] = 2$ and we take $\sigma_1 = \mathrm{id}$, $\sigma_2 \in G \setminus H$.

Consider now the image of Frobenius elements: let Frob_q be a Frobenius element at q. Then $\rho_f(\mathrm{Frob}_q) = \begin{pmatrix} \theta(g) & \theta(g\sigma_2{}^{-1}) \\ \theta(\sigma_2 g) & \theta(\sigma_2 g \sigma_2{}^{-1}) \end{pmatrix}$. If $\mathrm{Frob}_q \in H$, then $\theta(g)$ and $\theta(\sigma_2 g \sigma_2{}^{-1})$ are non-zero, and $\theta(g\sigma_2{}^{-1}) = \theta(\sigma_2 g) = 0$. If $\mathrm{Frob}_q \in G \setminus H$ then $\theta(g\sigma_2{}^{-1})$ and $\theta(\sigma_2 g)$ are non-zero, and $\theta(g)$ and $\theta(\sigma_2 g \sigma_2{}^{-1})$ are zero. Since Frobenius elements generate the Galois group, this shows the proposition. $\square$

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Ami FISCHMAN,
517 N 137th Street
Seattle WA 98133 (USA).
math@fischman.org