ANNALES

DE

L'INSTITUT FOURIER

Hua CHEN, Zhuangchu LUO & Hidetoshi TAHARA

Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity

Tome 51, n° 6 (2001), p. 1599-1620.

<http://aif.cedram.org/item?id=AIF_2001__51_6_1599_0>

Ann. Inst. Fourier, Grenoble
51, 6 (2001), 1599–1620

FORMAL SOLUTIONS OF NONLINEAR FIRST ORDER TOTALLY CHARACTERISTIC TYPE PDE WITH IRREGULAR SINGULARITY

by H. CHEN, Z. LUO and H. TAHARA

1. Introduction.

Let $(t,x) \in \mathbb{C}_t \times \mathbb{C}_x$, $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{Z}_+ = \{0,1,2,\ldots\}$, and denote by $\mathbb{C}[[t,x]]$ (resp. by $\mathbb{C}[[x]]$) the ring of formal power series in the variables (t,x) (resp. in the variable x).

Let us consider the following nonlinear singular first order partial differential equation:

$$t\frac{\partial u}{\partial t} = F\left(t,x,u,\frac{\partial u}{\partial x}\right), \tag{1.1}$$

where $u = u(t,x)$ is an unknown function, and $F(t,x,u,v)$ is a function defined in an open polydisc Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$. Set $\Delta_0 = \Delta \cap \{t=0, u=0 \text{ and } v=0\}$. We impose the following condition on $F(t,x,u,v)$:

(F1) $F(t,x,u,v)$ is a holomorphic function on Δ;

(F2) $F(0,x,0,0) \equiv 0$ on Δ_0.

Then by the Taylor expansion in (t,u,v) we can express $F(t,x,u,v)$ in the form

$$F(t,x,u,v) = a(x)t + b(x)u + \gamma(x)v + \sum_{i+j+\alpha\geqslant 2} a_{i,j,\alpha}(x)t^i u^j v^\alpha,$$

Keywords: Formal solution – Totally characteristic PDE – Gevrey index.
Math. classification: 35A07 – 35A10 – 35A20.

and $a(x), b(x), \gamma(x), a_{i,j,\alpha}(x)$ are all holomorphic functions on Δ_0.

If $\gamma(x) \equiv 0$ on Δ_0, the equation (1.1) is called a non-linear Fuchsian type PDE (or is called a "Briot-Bouquet type PDE" in [4], [5]); this situation has been discussed by [4]–[7]. If $\gamma(0) \neq 0$, we can solve $\partial u/\partial x$ from the equation (1.1) and then we can apply the Cauchy-Kowalewski theorem. If $\gamma(x) \not\equiv 0$ and $\gamma(0) = 0$, the indicial operator $C(\lambda, x, \partial/\partial x) = \lambda - b(x) - \gamma(x)\partial/\partial x$ is a singular differential operator; in this situation the equation (1.1) has been called a totally characteristic type PDE by [1], [2] and [3]. Thus, in this paper we assume:

(F3) $\gamma(x) = x^p c(x)$ for $p \in \mathbb{N}$ and $c(0) \neq 0$.

In the case $p = 1$ we already have the following result.

THEOREM 1.1 (Chen-Tahara [2]). — *Assume $p = 1$ and $|i - nb(0) - jc(0)| \neq 0$ for any $(i, j) \in \mathbb{N} \times \mathbb{Z}_+$. Then we have*

(1) *The equation (1.1) has a unique formal solution $u(t, x) \in \mathbb{C}[[t, x]]$ with $u(0, x) \equiv 0$.*

(2) *Moreover, if $c(0) \in \mathbb{C} \setminus [0, \infty)$ holds the unique formal solution in (1) is convergent in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.*

In this paper we shall discuss the case $p \geqslant 2$. In this case the indicial operator $C(\lambda, x, \partial/\partial x) = \lambda - b(x) - x^p c(x)\partial/\partial x$ has an irregular singularity at $x = 0 \in \mathbb{C}$ and the formal power series solution of (1.1) is not convergent in general; but still it belongs to a formal Gevrey class.

DEFINITION. — *Let $s \geqslant 1$ and $\sigma \geqslant 1$. We say that a formal power series $f(t, x) = \sum_{i \geqslant 0, j \geqslant 0} f_{i,j} t^i x^j \in \mathbb{C}[[t, x]]$ belongs to the formal Gevrey class $G\{t, x\}_{(s,\sigma)}$ if the power series*

$$\sum_{i \geqslant 0, j \geqslant 0} \frac{f_{i,j}}{(i!)^{s-1}(j!)^{\sigma-1}} t^i x^j$$

is convergent in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x$.

The following result is a consequence of the main theorem (Theorem 2.1) of this paper.

THEOREM 1.2. — *Assume $p \geqslant 2$ and $b(0) \notin \mathbb{N}$. Then*

(1) *The equation (1.1) has a unique formal solution $u(t, x) \in \mathbb{C}[[t, x]]$ with $u(0, x) \equiv 0$.*

(2) *Moreover, it belongs to the formal Gevrey class* $G\{t,x\}_{(s,\sigma)}$ *for any* $s \geqslant p/(p-1)$ *and* $\sigma \geqslant p/(p-1)$.

The result of this type is often called a Maillet's type theorem (see [6], [7], [9]).

In this paper, we have confined ourselves to the study of formal power series solutions $u(t,x) \in \mathbb{C}[[t,x]]$ of (1.1). The relation between true solutions of (1.1) and the formal solution obtained in this paper will be discussed in a forthcoming paper.

2. Main results.

We discuss the same equation (1.1) as in §1 under the conditions (F1),(F2),(F3), and $p \geqslant 2$.

Our equation is written as

$$\left(t\frac{\partial}{\partial t} - b(x) - x^p c(x)\frac{\partial}{\partial x}\right) u = a(x)t + \sum_{i+j+\alpha\geqslant 2} a_{i,j,\alpha}(x) t^i u^j \left(\frac{\partial u}{\partial x}\right)^{\alpha} \tag{2.1}$$

where $a(x), b(x), c(x), a_{i,j,\alpha}(x)$ are all holomorphic functions on Δ_0, $c(0) \neq 0$, and the right hand side is a holomorphic function on Δ with $v = \partial u/\partial x$.

Set

$$J = \Big\{(i,j,\alpha); i+j+\alpha \geqslant 2, \alpha > 0, \text{ and } a_{i,j,\alpha}(0) \neq 0\Big\}.$$

We have

THEOREM 2.1. — *Assume (F1),(F2),(F3),* $p \geqslant 2$ *and* $b(0) \notin \mathbb{N}$. *Then, the equation (2.1) has a unique formal solution* $u(t,x) \in \mathbb{C}[[t,x]]$ *with* $u(0,x) \equiv 0$ *and it belongs to the formal Gevrey class* $G\{t,x\}_{(s,\sigma)}$ *for any* (s,σ) *satisfying*

$$s \geqslant 1 + \max\left[0, \sup_{(i,j,\alpha)\in J}\left(\frac{1}{(p-1)(i+j+\alpha-1)}\right)\right] \tag{2.2}$$

and $\sigma \geqslant p/(p-1)$.

The proof of this theorem will be given in §4. Note that

$$1 + \frac{1}{(p-1)(i+j+\alpha-1)} \leqslant 1 + \frac{1}{(p-1)(2-1)} = \frac{p}{p-1}$$

and therefore $s \geqslant p/(p-1)$ implies the condition (2.2). Thus, Theorem 1.2 follows from Theorem 2.1.

As a particular case, we have

COROLLARY 2.2. — *If $J = \emptyset$, the unique formal solution $u(t,x)$ belongs to the class $G\{t,x\}_{(1,p/(p-1))}$.*

This implies that the formal solution is holomorphic in the variable t.

For $f(x) = \sum_{j\geqslant 0} f_j x^j \in \mathbb{C}[[x]]$ we write $f(x) \gg 0$ if $f_j \geqslant 0$ holds for all $j \geqslant 0$. The following proposition asserts that our condition (2.2) is the best possible result in a generic case.

PROPOSITION 2.3. — *Assume (F1),(F2),(F3), $p \geqslant 2$ and $b(0) \notin \mathbb{N}$. Moreover, assume the following additional conditions:*

c1) $a(0) > 0$, $(\partial a/\partial x)(0) > 0$ and $a(x) \gg 0$;

c2) $b(0) < 1$ and $(b(x) - b(0)) \gg 0$;

c3) $c(0) > 0$ and $c(x) \gg 0$;

c4) $a_{i,j,\alpha}(x) \gg 0$ (for $i + j + \alpha \geqslant 2$).

Then, the unique formal solution $u(t,x)$ in Theorem 2.1 belongs to the class $G\{t,x\}_{(s,\sigma)}$ if and only if (s,σ) satisfies (2.2) and $\sigma \geqslant p/(p-1)$.

The proof of this proposition will be given in §5.

Thus, we may say that the index (s_0, σ_0) defined by

$$s_0 = 1 + \max\left[0, \sup_{(i,j,\alpha)\in J}\left(\frac{1}{(p-1)(i+j+\alpha-1)}\right)\right], \quad \sigma_0 = \frac{p}{p-1} \tag{2.3}$$

is *the formal Gevrey index of the equation (2.1).*

For other types of partial differential equations, the formal Gevrey index is calculated by [6], [7], [8], [9].

Example 2.4. — Let $p, q, l, m, n \in \mathbb{Z}_+$ satisfying $p \geqslant 2$, $n \geqslant 1$ and $l + m + n \geqslant 2$. Let us consider

$$t\frac{\partial u}{\partial t} = (1+x)t + x^p\frac{\partial u}{\partial x} + x^q t^l u^m \left(\frac{\partial u}{\partial x}\right)^n. \tag{2.4}$$

We have

1) (2.4) has a unique formal solution $u(t,x) \in \mathbb{C}[[t,x]]$ with $u(0,x) \equiv 0$.

2) When $q \geqslant 1$, $u(t,x)$ belongs to the class $G\{t,x\}_{(s,\sigma)}$ if and only if

$$s \geqslant 1 \quad \text{and} \quad \sigma \geqslant \frac{p}{p-1}.$$

3) When $q = 0$, $u(t,x)$ belongs to the class $G\{t,x\}_{(s,\sigma)}$ if and only if

$$s \geqslant 1 + \frac{1}{(p-1)(l+m+n-1)} \quad \text{and} \quad \sigma \geqslant \frac{p}{p-1}.$$

3. Preparatory discussions.

Before the proof of Theorem 2.1 we shall present some preparatory lemmas.

For $f(x) = \sum_{j\geqslant 0} f_j x^j \in \mathbb{C}[[x]]$, we write

$$|f|(x) = \sum_{j\geqslant 0} |f_j| x^j,$$

$$S(f)(x) = \sum_{j\geqslant 0} f_{j+1} x^j = \frac{f(x)-f(0)}{x},$$

$$B_\sigma(f)(x) = \sum_{j\geqslant 0} \frac{f_j}{(j!)^{\sigma-1}} x^j, \quad \sigma > 1.$$

$B_\sigma(f)(x)$ is a variation of the Borel transform of $f(x)$. For $f(x) = \sum_{j\geqslant 0} f_j x^j$, $g(x) = \sum_{j\geqslant 0} g_j x^j$ we write $f(x) \ll g(x)$ if $|f_j| \leqslant g_j$ holds for all $j \geqslant 0$.

It is easy to show (see also [7]):

LEMMA 3.1. — *For $\sigma > 1$, $a(x)$, $\phi(x)$, $f(x) \in \mathbb{C}[[x]]$ we have*

1) $|a\phi|(x) \ll |a|(x)|\phi|(x)$;

2) $B_\sigma(a\phi)(x) \ll B_\sigma(|a|)(x) B_\sigma(|\phi|)(x)$;

3) *if $c \neq 0$ and $\phi(0) = 0$ then*

$$B_\sigma\left(\left|\frac{1}{c+\phi}\right|\right)(x) \ll \frac{1}{|c| - B_\sigma(|\phi|)(x)};$$

4) $B_\sigma\left(x\frac{\partial f}{\partial x}\right)(x) = x\frac{\partial}{\partial x}B_\sigma(f)(x) \ll x\frac{\partial}{\partial x}B_\sigma(|f|)(x)$;

5) *if $p \geqslant 2$ and $\sigma \geqslant p/(p-1)$ then*

$$B_\sigma\left(x^p\frac{\partial f}{\partial x}\right)(x) \ll x^{p-1} B_\sigma(|f|)(x);$$

6) $S(f)(x) \ll \frac{\partial}{\partial x}|f|(x)$ *and* $B_\sigma(S(f))(x) \ll B_\sigma\left(\frac{\partial}{\partial x}|f|\right)(x)$.

We say that $f(x) \in \mathbb{C}[[x]]$ belongs to the formal Gevrey class $G\{x\}_\sigma$ if $B_\sigma(f)(x)$ is convergent in a neighborhood of $x = 0$. The following lemma is used to construct a formal solution of (2.1).

Lemma 3.2. — *Let $b(x)$, $c(x) \in \mathbb{C}[[x]]$, $p \geqslant 2$, $k \in \mathbb{N}$ and assume that $b(0) \neq k$. We have*

1) *For any $g(x) \in \mathbb{C}[[x]]$, the equation*

$$\left(k - b(x) - x^p c(x)\frac{\partial}{\partial x}\right) w = g(x) \tag{3.1}$$

has a unique solution $w(x) \in \mathbb{C}[[x]]$.

2) *If $b(x)$, $c(x)$, $g(x) \in G\{x\}_\sigma$ for some $\sigma \geqslant p/(p-1)$ we have $w(x) \in G\{x\}_\sigma$ and moreover if $|k - b(0)| \geqslant \rho k$ with $\rho > 0$ we have*

$$B_\sigma(|w|)(x) \ll \frac{1}{k}\frac{1}{\rho - \Phi(x)} B_\sigma(|g|)(x) \tag{3.2}$$

where $\Phi(x) = xB_\sigma(|S(b)|)(x) + x^{p-1}B_\sigma(|c|)(x) \gg 0$. Note that $\Phi(0) = 0$ holds.

Proof. — 1) is verified by a calculation. Since (3.1) is written as

$$(k - b(0))\, w = xS(b)(x)w + x^p c(x)\frac{\partial w}{\partial x} + g(x),$$

by using the B_σ-transformation and 5) of Lemma 3.1 we have

$$\begin{aligned} &\rho k B_\sigma(|w|)(x) \\ &\quad \ll xB_\sigma(|S(b)|)(x)B_\sigma(|w|)(x) + x^{p-1}B_\sigma(|c|)(x)B_\sigma(|w|)(x) + B_\sigma(|g|)(x) \\ &\quad = \Phi(x)B_\sigma(|w|)(x) + B_\sigma(|g|)(x) \\ &\quad \ll k\Phi(x)B_\sigma(|w|)(x) + B_\sigma(|g|)(x) \end{aligned}$$

which leads us to the conclusion of 2). Lemma 3.2 is proved.

In order to estimate the term $B_\sigma(\partial u/\partial x)$ we need the following lemma.

Lemma 3.3. — *Let $\sigma > 1$ and $0 < R < 1$. If $f(x) \in G\{x\}_\sigma$ satisfies*

$$B_\sigma(f)(x) \ll \frac{C}{(R-x)^a} \tag{3.3}$$

for some $C > 0$ and $a \geqslant 1$, we have

$$B_\sigma\left(x\frac{\partial f}{\partial x}\right)(x) \ll \frac{aC}{(R-x)^{a+1}} \ll \frac{aC}{(R-x)^{a+\sigma}}, \tag{3.4}$$

$$B_\sigma\left(\frac{\partial f}{\partial x}\right)(x) \ll \frac{e^\sigma(a+\sigma)^\sigma C}{(R-x)^{a+\sigma}}. \tag{3.5}$$

Proof. — Assume that $f(x) \in G\{x\}_\sigma$ satisfies (3.3). Then

$$B_\sigma\left(x\frac{\partial f}{\partial x}\right)(x) = x\frac{\partial}{\partial x}B_\sigma(f)(x) \ll x\frac{\partial}{\partial x}\frac{C}{(R-x)^a} = \frac{xaC}{(R-x)^{a+1}}.$$

Combining this with

$$\frac{x}{R-x} \ll \frac{R}{R-x} \ll \frac{1}{R-x} \ll \frac{R^{\sigma-1}}{(R-x)^\sigma} \ll \frac{1}{(R-x)^\sigma}$$

(since $0 < R < 1$) we obtain (3.4). Note that the function $1/(R-x)^a$ is expressed as

$$\frac{1}{(R-x)^a} = \sum_{j\geqslant 0}\frac{1}{R^{a+j}}\frac{\Gamma(a+j)}{\Gamma(a)\Gamma(j+1)}x^j.$$

Therefore, if we prove the inequality

$$\sup_{a\geqslant 1, j\geqslant 1}\left(\frac{j^{\sigma-1}}{(a+\sigma)^\sigma}\frac{\Gamma(a+\sigma)\Gamma(a+j)}{\Gamma(a)\Gamma(a+j+\sigma-1)}\right) \leqslant e^\sigma, \tag{3.6}$$

a simple calculation shows that (3.5) follows easily from (3.3).

Since a sharp form of the Stirling's formula for the Γ-function guarantees

$$1 < \frac{\Gamma(x)}{\sqrt{2\pi}x^{x-1/2}e^{-x}} < \exp\left(\frac{1}{12x}\right) < \sqrt{e} \quad \text{for } x \geqslant 1 \tag{3.7}$$

(see [10]), the inequality (3.6) is verified as follows:

$$\begin{aligned}
&\frac{j^{\sigma-1}}{(a+\sigma)^\sigma}\frac{\Gamma(a+\sigma)\Gamma(a+j)}{\Gamma(a)\Gamma(a+j+\sigma-1)} \\
&\leqslant \frac{j^{\sigma-1}}{(a+\sigma)^\sigma}\frac{\sqrt{2\pi}(a+\sigma)^{a+\sigma-1/2}e^{-a-\sigma}\sqrt{e}\sqrt{2\pi}(a+j)^{a+j-1/2}e^{-a-j}\sqrt{e}}{\sqrt{2\pi}a^{a-1/2}e^{-a}\sqrt{2\pi}(a+j+\sigma-1)^{a+j+\sigma-1-1/2}e^{-a-j-\sigma+1}} \\
&= \left(\frac{a}{a+\sigma}\right)^{1/2}\left(1+\frac{\sigma}{a}\right)^a\frac{j^{\sigma-1}(a+j)^{a+j-1/2}}{(a+j+\sigma-1)^{(\sigma-1)+(a+j-1/2)}} \\
&\leqslant \left(1+\frac{\sigma}{a}\right)^a \leqslant e^\sigma.
\end{aligned}$$

LEMMA 3.4. — *Let $k \geqslant 2$, $i, j, \alpha \in \mathbb{Z}_+$, $m_1, \dots, m_j \in \mathbb{N}$, and $n_1, \dots, n_\alpha \in \mathbb{N}$. Assume $2 \leqslant i + j + \alpha \leqslant k$ and $i + |m| + |n| = k$,* *where* $|m| = m_1 + \cdots + m_j$ *and* $|n| = n_1 + \cdots + n_\alpha$. *Then we have* 1) $(m_1-1)!\cdots(m_j-1)!(n_1-1)!\cdots(n_\alpha-1)! \leqslant (k-2)! \leqslant (k-1)!$;

2) $(m_1-1)!\cdots(m_j-1)!(n_1-1)!\cdots(n_\alpha-1)! \leqslant \frac{e^{i+j+\alpha}}{k^{i+j+\alpha-1}}(k-1)!$;

3) $\frac{1}{m_1\cdots m_j n_1\cdots n_\alpha} \leqslant \frac{i+j+\alpha}{k}$.

Proof. — 1) is verified by

$$\begin{aligned}&(m_1-1)!\cdots(m_j-1)!(n_1-1)!\cdots(n_\alpha-1)!\\ &\quad\leqslant (|m|+|n|-j-\alpha)! = (i+|m|+|n|-i-j-\alpha)!\\ &\quad= (k-i-j-\alpha)!\\ &\quad\leqslant (k-2)! \leqslant (k-1)!.\end{aligned}$$

By using the Stirling's formula (3.7) we have

$$\begin{aligned}&\frac{(m_1-1)!\cdots(m_j-1)!(n_1-1)!\cdots(n_\alpha-1)!}{(k-1)!}\\ &\quad\leqslant \frac{(k-i-j-\alpha)!}{(k-1)!} = \frac{\Gamma(k-i-j-\alpha+1)}{\Gamma(k)}\\ &\quad\leqslant \frac{\sqrt{2\pi}(k-i-j-\alpha+1)^{k-i-j-\alpha+1-1/2}e^{-k+i+j+\alpha-1}e}{\sqrt{2\pi}k^{k-1/2}e^{-k}}\\ &\quad= \left(\frac{k-i-j-\alpha+1}{k}\right)^{k-i-j-\alpha+1-1/2}\frac{e^{i+j+\alpha}}{k^{i+j+\alpha-1}}\\ &\quad\leqslant \frac{e^{i+j+\alpha}}{k^{i+j+\alpha-1}}\end{aligned}$$

which proves 2). Since $m_p \geqslant 1$ and $n_q \geqslant 1$, we have

$$(m_1+\cdots+m_j+n_1+\cdots+n_\alpha) \leqslant (j+\alpha)\,(m_1\cdots m_j n_1\cdots n_\alpha)$$

and therefore

$$\begin{aligned}k = i+|m|+|n| &\leqslant i+(j+\alpha)\,(m_1\cdots m_j n_1\cdots n_\alpha)\\ &\leqslant (i+j+\alpha)\,(m_1\cdots m_j n_1\cdots n_\alpha)\end{aligned}$$

which proves 3). Thus Lemma 3.4 is proved.

4. Proof of Theorem 2.1.

Now, by using Lemmas 3.1 $\sim$ 3.4 we shall give here a proof of Theorem 2.1.

In this section we set $\sigma = p/(p-1)$; then the condition (2.2) is written as

$$s \geqslant 1 + \max\left[0, \sup_{(i,j,\alpha)\in J}\left(\frac{\sigma - 1}{i+j+\alpha-1}\right)\right]. \tag{4.1}$$

Since $b(0) \notin \mathbb{N}$ is assumed, we can find a $\rho > 0$ such that $|k - b(0)| \geqslant \rho k$ holds for all $k \in \mathbb{N}$.

First, let us look for a formal solution $u(t,x)$ of the form

$$u(t,x) = \sum_{k\geqslant 1} u_k(x)t^k, \quad u_k(x) \in G\{x\}_\sigma \text{ (for } k \geqslant 1). \tag{4.2}$$

Under (4.2) the equation (2.1) is decomposed into the following recurrent family:

$$\left(1 - b(x) - x^p c(x)\frac{\partial}{\partial x}\right) u_1 = a(x), \tag{4.3}$$

and for $k \geqslant 2$

$$\left(k - b(x) - x^p c(x)\frac{\partial}{\partial x}\right) u_k = \sum_{2\leqslant i+j+\alpha\leqslant k} a_{i,j,\alpha}(x)\left[\sum_{i+|m|+|n|=k} u_{m_1}\cdots u_{m_j} \times \frac{\partial u_{n_1}}{\partial x}\cdots\frac{\partial u_{n_\alpha}}{\partial x}\right], \tag{4.4}$$

where $|m| = m_1 + \cdots + m_j$ and $|n| = n_1 + \cdots + n_\alpha$. Therefore, if $b(0) \notin \mathbb{N}$ by Lemma 3.2 we can determine $u_k(x) \in G\{x\}_\sigma$ $(k = 1, 2, \ldots)$ inductively on k. Thus, we have obtained a unique formal solution $u(t,x)$ in (4.2).

Next, let us prove that this formal solution $u(t,x)$ belongs to the formal Gevrey class $G\{t,x\}_{(s,\sigma)}$ if s satisfies the condition (4.1). To do so, we set

$$w_k(x) = S(u_k)(x) \in G\{x\}_\sigma, \quad k = 1, 2, \ldots.$$

Then we have $u_k(x) = u_k(0) + xw_k(x)$ and by (4.3),(4.4) we have

$$(1 - b(0))\, u_1(0) = a(0), \tag{4.5}$$

$$\left(1 - b(x) - x^{p-1}c(x) - x^p c(x)\frac{\partial}{\partial x}\right) w_1 = S(b)(x)u_1(0) + S(a)(x), \tag{4.6}$$

and for $k \geqslant 2$

$$(k - b(0))\, u_k(0) = \sum_{2\leqslant i+j+\alpha\leqslant k} a_{i,j,\alpha}(0)\left[\sum_{i+|m|+|n|=k} u_{m_1}(0)\cdots u_{m_j}(0) w_{n_1}(0)\cdots w_{n_\alpha}(0)\right], \tag{4.7}$$

$$(4.8)\quad \Big(k - b(x) - x^{p-1}c(x) - x^p c(x)\frac{\partial}{\partial x}\Big) w_k$$
$$= S(b)(x)u_k(0)$$
$$+ \sum_{2\leqslant i+j+\alpha\leqslant k} S(a_{i,j,\alpha})(x)\Bigg[\sum_{i+|m|+|n|=k}\Big(u_{m_1}(0)+xw_{m_1}\Big)$$
$$\times\cdots\times\big(u_{m_j}(0)+xw_{m_j}\big)\times\Big(w_{n_1}+x\frac{\partial w_{n_1}}{\partial x}\Big)\cdots\Big(w_{n_\alpha}+x\frac{\partial w_{n_\alpha}}{\partial x}\Big)\Bigg]$$
$$+ \sum_{2\leqslant i+j+\alpha\leqslant k} a_{i,j,\alpha}(0)\Bigg[\sum_{i+|m|+|n|=k}\Bigg(\frac{1}{x}\Big\{\Big(u_{m_1}(0)+xw_{m_1}\Big)$$
$$\times\cdots\times\big(u_{m_j}(0)+xw_{m_j}\big)\times\Big(w_{n_1}+x\frac{\partial w_{n_1}}{\partial x}\Big)\cdots\Big(w_{n_\alpha}+x\frac{\partial w_{n_\alpha}}{\partial x}\Big)$$
$$- u_{m_1}(0)\cdots u_{m_j}(0)w_{n_1}\cdots w_{n_\alpha}\Big\}+u_{m_1}(0)\cdots u_{m_j}(0)S\left(w_{n_1}\cdots w_{n_\alpha}\right)\Bigg)\Bigg].$$

Choose $0 < R < 1$ and $A > 0$ so that $|u_1(0)| \leqslant A$ and

$$(4.9)\qquad B_\sigma\left(w_1\right)(x) \ll \frac{A}{(R-x)^\sigma}.$$

Put $\Phi(x) = xB_\sigma(|S(b)|)(x) + 2x^{p-1}B_\sigma(|c|)(x)$ and take $B > 0$ such that

$$\frac{B_\sigma(|S(b)|)(x)}{\rho - \Phi(x)} \ll \frac{B}{(R-x)^\sigma}.$$

Similarly, choose $A^{(0)}_{i,j,\alpha} \geqslant 0$ and $A_{i,j,\alpha} \geqslant 0$ so that $|a_{i,j,\alpha}(0)| \leqslant A^{(0)}_{i,j,\alpha}$,

$$\frac{|a_{i,j,\alpha}(0)|}{\rho-\Phi(x)} \ll \frac{A^{(0)}_{i,j,\alpha}}{(R-x)^\sigma}, \qquad \frac{B_\sigma(|S(a_{i,j,\alpha})|)(x)}{\rho-\Phi(x)} \ll \frac{A_{i,j,\alpha}}{(R-x)^\sigma}$$

and that

$$\sum_{i+j+\alpha\geqslant 2} A^{(0)}_{i,j,\alpha} t^i u^j v^\alpha \quad\text{and}\quad \sum_{i+j+\alpha\geqslant 2} A_{i,j,\alpha} t^i u^j v^\alpha$$

are convergent in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_u \times \mathbb{C}_v$. We may assume that $A^{(0)}_{i,j,\alpha} = 0$ if $a_{i,j,\alpha}(0) = 0$.

Using these constants, let us consider the following functional equation with respect to Y:

$$(4.10)\quad Y = \frac{A}{(R-x)^{2\sigma}}t + \frac{1}{(R-x)^\sigma}\sum_{i+j+\alpha\geqslant 2}\frac{C_{i,j,\alpha}}{(R-x)^{\sigma(4i+2j+2\alpha-3)}}t^i(2Y)^j(2\beta Y)^\alpha$$

where $\beta = (4e\sigma)^\sigma$ and

(4.11)
$$C_{i,j,\alpha} = \left((1+B/\rho)A^{(0)}_{i,j,\alpha}+A_{i,j,\alpha}\right)(i+j+\alpha)^{\sigma-1}+A^{(0)}_{i,j,\alpha}\left(e^{i+j+\alpha}\right)^{s-1}.$$

Note that by $i+j+\alpha \geqslant 2$ we have $4i+2j+2\alpha-3 \geqslant 1$.

Since (4.10) is an analytic functional equation with respect to Y, by the implicit function theorem we see that (4.10) has a unique holomorphic solution $Y = Y(t,x)$ in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x$ with $Y(0,x) \equiv 0$. If we expand Y into the form

$$Y(t,x) = \sum_{k\geqslant 1} Y_k(x)t^k$$

we see that the coefficients $Y_k(x)$ $(k \geqslant 1)$ are determined by the following recurrent formula:

$$Y_1 = \frac{A}{(R-x)^{2\sigma}}, \tag{4.12}$$

and for $k \geqslant 2$

$$(4.13)\ Y_k = \frac{1}{(R-x)^\sigma} \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{C_{i,j,\alpha}}{(R-x)^{\sigma(4i+2j+2\alpha-3)}} \left[\sum_{i+|m|+|n|=k} \left(2Y_{m_1}\right)\times\cdots\times\left(2Y_{m_j}\right)\left(2\beta Y_{n_1}\right)\cdots\left(2\beta Y_{n_\alpha}\right)\right].$$

Moreover we can prove by induction on k that $Y_k(x)$ has the form

$$Y_k(x) = \frac{M_k}{(R-x)^{\sigma(4k-2)}} \quad (\text{for } k \geqslant 1) \tag{4.14}$$

with constants $M_1 = A$ and $M_k \geqslant 0$ (for $k \geqslant 2$).

In addition, we have the following lemma.

LEMMA 4.1. — *Let $\beta = (4e\sigma)^\sigma$, and let $u(t,x)$ be the unique formal solution in (4.2). If s satisfies the condition (4.1) we have the following estimates for all $k \in \mathbb{N}$:*

$$|u_k(0)| \ll \frac{(k-1)!^{s-1}}{k^\sigma} Y_k(x), \tag*{$(4.15)_k$}$$

$$B_\sigma\left(|w_k|\right)(x) \ll \frac{(k-1)!^{s-1}}{k^\sigma} Y_k(x), \tag*{$(4.16)_k$}$$

$$B_\sigma\left(x\frac{\partial}{\partial x}|w_k|\right)(x) \ll \frac{(k-1)!^{s-1}}{k^{\sigma-1}} \beta Y_k(x), \tag*{$(4.17)_k$}$$

$$(4.18)_k \qquad B_\sigma\left(\frac{\partial}{\partial x}|w_k|\right)(x) \ll \frac{(k-1)!^{s-1}}{1}\beta Y_k(x).$$

We admit this lemma for a while. Then, by (4.15) and (4.16) we have

$$\begin{aligned}
\sum_{k\geqslant 1}\frac{B_\sigma(|u_k|)(x)}{(k-1)!^{s-1}}t^k &\ll \sum_{k\geqslant 1}\frac{|u_k(0)|}{(k-1)!^{s-1}}t^k + x\sum_{k\geqslant 1}\frac{B_\sigma(|w_k|)(x)}{(k-1)!^{s-1}}t^k\\
&\ll \sum_{k\geqslant 1}\frac{1}{k^\sigma}Y_k(x)t^k + x\sum_{k\geqslant 1}\frac{1}{k^\sigma}Y_k(x)t^k\\
&\ll (1+x)\sum_{k\geqslant 1}Y_k(x)t^k = (1+x)Y(t,x).
\end{aligned}$$

This implies that our formal solution $u(t,x)$ in (4.2) belongs to the class $G\{t,x\}_{(s,\sigma)}$.

Thus, to complete the proof of Theorem 2.1 it is sufficient to give a proof of Lemma 4.1.

Proof of Lemma 4.1. — Assume that s satisfies the condition (4.1). We have

$$(4.19) \qquad (i+j+\alpha-1)(s-1) \geqslant \sigma - 1 \quad \text{for any } (i,j,\alpha)\in J.$$

First let us prove the case $k=1$. Since $|u_1(0)| \leqslant A$ is assumed, we have

$$|u_1(0)| \leqslant A \ll \frac{A}{(R-x)^{2\sigma}} = Y_1(x)$$

which is $(4.15)_1$. Using (4.9) and Lemma 3.3 we can verify $(4.16)_1$, $(4.17)_1$, $(4.18)_1$ as follows:

$$\begin{aligned}
B_\sigma(|w_1|)(x) &\ll \frac{A}{(R-x)^\sigma} \ll \frac{A}{(R-x)^{2\sigma}} = Y_1(x),\\
B_\sigma\left(x\frac{\partial}{\partial x}|w_1|\right)(x) &\ll \frac{\sigma A}{(R-x)^{2\sigma}} = \sigma Y_1(x) \ll \beta Y_1(x),\\
B_\sigma\left(\frac{\partial}{\partial x}|w_1|\right)(x) &\ll \frac{e^\sigma(\sigma+\sigma)^\sigma A}{(R-x)^{2\sigma}} = (2e\sigma)^\sigma Y_1(x) \ll \beta Y_1(x).
\end{aligned}$$

Here we used the conditions $1 \ll 1/(R-x)^\sigma$ (since $0<R<1$) and $\beta = (4e\sigma)^\sigma$.

Next, let us show the general case $k \geqslant 2$ by induction on k.

Let $k \geqslant 2$ and suppose that $(4.15)_i \sim (4.18)_i$ are already proved for all $i \leqslant k-1$. Then by (4.7) and the induction hypotheses we have

$$|u_k(0)| \ll \frac{1}{k\rho} \sum_{2\leqslant i+j+\alpha\leqslant k} |a_{i,j,\alpha}(0)| \times \left[\sum_{i+|m|+|n|=k} \left(\frac{(m_1-1)!^{s-1}}{{m_1}^{\sigma}} Y_{m_1} \right) \cdots \left(\frac{(m_j-1)!^{s-1}}{{m_j}^{\sigma}} Y_{m_j} \right) \times \left(\frac{(n_1-1)!^{s-1}}{{n_1}^{\sigma}} Y_{n_1} \right) \cdots \left(\frac{(n_\alpha-1)!^{s-1}}{{n_\alpha}^{\sigma}} Y_{n_\alpha} \right) \right].$$

Therefore, by 1), 3) of Lemma 3.4 and by using the inequality $(i+j+\alpha)/k \leqslant 1$ we have

$$(4.20) \qquad |u_k(0)| \ll \frac{1}{k}\frac{1}{\rho} \sum_{2\leqslant i+j+\alpha\leqslant k} |a_{i,j,\alpha}(0)| \left[\sum_{i+|m|+|n|=k} (k-1)!^{s-1} \times \left(\frac{i+j+\alpha}{k}\right)^{\sigma-1} \left(\frac{i+j+\alpha}{k}\right) \times Y_{m_1}\cdots Y_{m_j} \times Y_{n_1}\cdots Y_{n_\alpha} \right]$$
$$\ll \frac{(k-1)!^{s-1}}{k^{\sigma}} \frac{1}{\rho} \sum_{2\leqslant i+j+\alpha\leqslant k} |a_{i,j,\alpha}(0)|\,(i+j+\alpha)^{\sigma-1} \times \left[\sum_{i+|m|+|n|=k} Y_{m_1}\cdots Y_{m_j} \times Y_{n_1}\cdots Y_{n_\alpha} \right].$$

Hence, if we note that

$$\frac{1}{\rho}\,|a_{i,j,\alpha}(0)| \ll \frac{|a_{i,j,\alpha}(0)|}{\rho-\Phi(x)} \ll \frac{A^{(0)}_{i,j,\alpha}}{(R-x)^{\sigma}},$$

we have

$$|u_k(0)| \ll \frac{(k-1)!^{s-1}}{k^{\sigma}} \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{A^{(0)}_{i,j,\alpha}(i+j+\alpha)^{\sigma-1}}{(R-x)^{\sigma}} \times \left[\sum_{i+|m|+|n|=k} Y_{m_1}\cdots Y_{m_j} \times Y_{n_1}\cdots Y_{n_\alpha} \right].$$

By comparing this with (4.13) and by using $A^{(0)}_{i,j,\alpha}(i+j+\alpha)^{\sigma-1} \leqslant C_{i,j,\alpha}$, $4i+2j+2\alpha-3 \geqslant 1$ and $1 \ll 1/(R-x)^{\sigma}$ we can easily obtain $(4.15)_k$.

Let us show $(4.16)_k$, $(4.17)_k$ and $(4.18)_k$. To do so, it is sufficient to prove

$$
\begin{aligned}
B_\sigma(|w_k|)(x) &\ll \frac{(k-1)!^{s-1}}{k^\sigma}(R-x)^\sigma Y_k(x) \\
&= \frac{(k-1)!^{s-1}}{k^\sigma}\frac{M_k}{(R-x)^{\sigma(4k-3)}}
\end{aligned}
\tag{4.21}
$$

(see (4.14)). In fact, if we know this, by using $1 \ll 1/(R-x)^\sigma$ and (4.14) we have $(4.16)_k$, and by applying Lemma 3.4 we can obtain $(4.17)_k$ and $(4.18)_k$.

Let us prove (4.21) from now. By applying 2) of Lemma 3.2 to (4.8) we have

$$B_\sigma\left(|w_k|\right)(x) \ll I_1 + I_2 + I_3$$

with

$$
\begin{aligned}
I_1 &= \frac{1}{k}\frac{B_\sigma(|S(b)|)(x)}{\rho-\Phi(x)}\,|u_k(0)|, \\
I_2 &= \frac{1}{k}\sum_{2\leqslant i+j+\alpha\leqslant k}\frac{B_\sigma(|S(a_{i,j,\alpha})|)(x)}{\rho-\Phi(x)}\Bigg[\sum_{i+|m|+|n|=k}\Big(|u_{m_1}(0)| + xB_\sigma\left(|w_{m_1}|\right)\Big) \\
&\quad \times\cdots\times\Big(\left|u_{m_j}(0)\right| + xB_\sigma\left(|w_{m_j}|\right)\Big) \\
&\quad \times\Big(B_\sigma\left(|w_{n_1}|\right) + B_\sigma\Big(x\frac{\partial}{\partial x}|w_{n_1}|\Big)\Big)\cdots\Big(B_\sigma\left(|w_{n_\alpha}|\right) + B_\sigma\Big(x\frac{\partial}{\partial x}|w_{n_\alpha}|\Big)\Big)\Bigg], \\
I_3 &= \frac{1}{k}\sum_{2\leqslant i+j+\alpha\leqslant k}\frac{|a_{i,j,\alpha}(0)|}{\rho-\Phi(x)}\Bigg[\sum_{i+|m|+|n|=k}\Bigg(\frac{1}{x}\Bigg\{\Big(|u_{m_1}(0)| + xB_\sigma\left(|w_{m_1}|\right)\Big) \\
&\quad \times\cdots\times\Big(\left|u_{m_j}(0)\right| + xB_\sigma\left(|w_{m_j}|\right)\Big) \\
&\quad \times\Big(B_\sigma\left(|w_{n_1}|\right) + xB_\sigma\Big(\frac{\partial}{\partial x}|w_{n_1}|\Big)\Big)\cdots\Big(B_\sigma\left(|w_{n_\alpha}|\right) + xB_\sigma\Big(\frac{\partial}{\partial x}|w_{n_\alpha}|\Big)\Big) \\
&\qquad - |u_{m_1}(0)|\cdots\left|u_{m_j}(0)\right| B_\sigma\left(|w_{n_1}|\right)\cdots B_\sigma\left(|w_{n_\alpha}|\right)\Bigg\} \\
&\qquad + |u_{m_1}(0)|\cdots\left|u_{m_j}(0)\right| B_\sigma\Big(\left|S\left(w_{n_1}\cdots w_{n_\alpha}\right)\right|\Big)\Bigg)\Bigg].
\end{aligned}
$$

I_1 is estimated by (4.20):

$$
\begin{aligned}
(4.22)\quad I_1 \ll \frac{1}{k}\frac{(k-1)!^{s-1}}{k^\sigma}\frac{1}{\rho} & \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{B_\sigma(|S(b)|)}{\rho-\Phi(x)}\, |a_{i,j,\alpha}(0)|\,(i+j+\alpha)^{\sigma-1} \\
& \times \left[\sum_{i+|m|+|n|=k} Y_{m_1}\cdots Y_{m_j} \times Y_{n_1}\cdots Y_{n_\alpha}\right] \\
\ll \frac{(k-1)!^{s-1}}{k^\sigma} & \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{(B/\rho)A^{(0)}_{i,j,\alpha}(0)(i+j+\alpha)^{\sigma-1}}{(R-x)^\sigma} \\
& \times \left[\sum_{i+|m|+|n|=k} Y_{m_1}\cdots Y_{m_j} \times Y_{n_1}\cdots Y_{n_\alpha}\right].
\end{aligned}
$$

Since $Y_l(x)$ has the form (4.14) and $0 < R < 1$ is assumed, we have

$$
xY_l(x) \ll RY_l(x) \ll Y_l(x).
$$

By using this and the induction hypotheses, we see

$$
\begin{aligned}
I_2 \ll \frac{1}{k} \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{A_{i,j,\alpha}}{(R-x)^\sigma} & \left[\sum_{i+|m|+|n|=k} \left(\frac{(m_1-1)!^{s-1}}{{m_1}^\sigma} 2Y_{m_1}\right)\right. \\
& \times\cdots\times \left(\frac{(m_j-1)!^{s-1}}{{m_j}^\sigma} 2Y_{m_j}\right)\left(\frac{(n_1-1)!^{s-1}}{{n_1}^{\sigma-1}}\left(\frac{1}{n_1}+\beta\right)Y_{n_1}\right) \\
& \left.\times\cdots\times \left(\frac{(n_\alpha-1)!^{s-1}}{{n_\alpha}^{\sigma-1}}\left(\frac{1}{n_\alpha}+\beta\right)Y_{n_\alpha}\right)\right].
\end{aligned}
$$

Therefore, by 1), 3) of Lemma 3.4 and by the same argument as in (4.20) we obtain

$$
\begin{aligned}
(4.23)\quad I_2 \ll \frac{(k-1)!^{s-1}}{k^\sigma} \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{A_{i,j,\alpha}(i+j+\alpha)^{\sigma-1}}{(R-x)^\sigma} & \left[\sum_{i+|m|+|n|=k} \left(2Y_{m_1}\right)\right. \\
\times\cdots\times \left(2Y_{m_j}\right) \times \left(\left(\frac{1}{n_1}+\beta\right)Y_{n_1}\right) & \left.\cdots\left(\left(\frac{1}{n_\alpha}+\beta\right)Y_{n_\alpha}\right)\right].
\end{aligned}
$$

In order to estimate I_3 we note

$$\begin{aligned}
&B_\sigma\Big(|S(w_{n_1}\cdots w_{n_\alpha})|\Big)(x)\\
&\quad\ll B_\sigma\Big(\frac{\partial}{\partial x}|w_{n_1}\cdots w_{n_\alpha}|\Big)(x)\\
&\quad\ll \sum_{i=1}^{\alpha} B_\sigma\Big(|w_{n_1}|\Big)\cdots B_\sigma\Big(\frac{\partial}{\partial x}|w_{n_i}|\Big)\cdots B_\sigma\Big(|w_{n_\alpha}|\Big)\\
&\quad\ll \alpha\beta\left(\frac{(n_1-1)!^{s-1}}{1}Y_{n_1}\right)\cdots\left(\frac{(n_\alpha-1)!^{s-1}}{1}Y_{n_\alpha}\right)\\
&\quad\ll \left(\frac{(n_1-1)!^{s-1}}{1}\beta Y_{n_1}\right)\cdots\left(\frac{(n_\alpha-1)!^{s-1}}{1}\beta Y_{n_\alpha}\right);
\end{aligned}$$

here we used 6) of Lemma 3.1, the induction hypotheses, the inequality $\alpha\beta \leqslant \beta^\alpha$, and

$$B_\sigma\Big(|w_n|\Big) \ll \frac{(n-1)!^{s-1}}{n^\sigma}Y_n \ll \frac{(n_1-1)!^{s-1}}{1}Y_n.$$

Therefore, using this and $xY_l(x) \ll Y_l(x)$ we can estimate I_3 in the following way:

$$\begin{aligned}
(4.24)\quad I_3 \ll \frac{1}{k}\sum_{2\leqslant i+j+\alpha\leqslant k}\frac{A^{(0)}_{i,j,\alpha}}{(R-x)^\sigma}\Bigg[\sum_{i+|m|+|n|=k}&\left(\frac{(m_1-1)!^{s-1}}{m_1{}^\sigma}2Y_{m_1}\right)\\
&\times\cdots\times\left(\frac{(m_j-1)!^{s-1}}{m_j{}^\sigma}2Y_{m_j}\right)\\
&\times\left(\frac{(n_1-1)!^{s-1}}{1}2\beta Y_{n_1}\right)\cdots\left(\frac{(n_\alpha-1)!^{s-1}}{1}2\beta Y_{n_\alpha}\right)\Bigg].
\end{aligned}$$

If $\alpha = 0$, then by 1), 3) of Lemma 3.4 we have

$$(4.25)\qquad \frac{(m_1-1)!^{s-1}}{m_1{}^\sigma}\cdots\frac{(m_j-1)!^{s-1}}{m_j{}^\sigma} \leqslant \frac{(k-1)!^{s-1}}{k^{\sigma-1}}(i+j+\alpha)^{\sigma-1}$$

as in the proof of (4.20). If $\alpha > 0$ and $a_{i,j,\alpha}(0) = 0$, we have $A^{(0)}_{i,j,\alpha} = 0$ and nothing to do. If $\alpha > 0$ and $a_{i,j,\alpha}(0) \neq 0$, we know that s satisfies the condition (4.19); in this case by 2) of Lemma 3.4 we have

$$\begin{aligned}
(4.26)\quad &\frac{(m_1-1)!^{s-1}}{m_1{}^\sigma}\cdots\frac{(m_j-1)!^{s-1}}{m_j{}^\sigma}\frac{(n_1-1)!^{s-1}}{1}\cdots\frac{(n_\alpha-1)!^{s-1}}{1}\\
&\leqslant (m_1-1)!^{s-1}\cdots(m_j-1)!^{s-1}(n_1-1)!^{s-1}\cdots(n_\alpha-1)!^{s-1}\\
&\leqslant \left(\frac{e^{i+j+\alpha}}{k^{i+j+\alpha-1}}\right)^{s-1}(k-1)!^{s-1} = \frac{\left(e^{i+j+\alpha}\right)^{s-1}}{k^{(i+j+\alpha-1)(s-1)}}(k-1)!^{s-1}\\
&\leqslant \frac{\left(e^{i+j+\alpha}\right)^{s-1}}{k^{\sigma-1}}(k-1)!^{s-1}.
\end{aligned}$$

Hence, applying (4.25) and (4.26) to (4.24) we obtain

$$(4.27)\qquad I_3 \ll \frac{(k-1)!^{s-1}}{k^\sigma} \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{A^{(0)}_{i,j,\alpha}}{(R-x)^\sigma} \Big((i+j+\alpha)^{\sigma-1} + \big(e^{i+j+\alpha}\big)^{s-1}\Big) \times \left[\sum_{i+|m|+|n|=k} \Big(2Y_{m_1}\Big)\cdots\Big(2Y_{m_1}\Big) \times \Big(2\beta Y_{n_1}\Big)\cdots\Big(2\beta Y_{n_\alpha}\Big)\right].$$

Thus, by (4.22), (4.23) and (4.27) we have

$$B_\sigma(|w_k|)(x) \ll I_1 + I_2 + I_3 \ll \frac{(k-1)!^{s-1}}{k^\sigma} \sum_{2\leqslant i+j+\alpha\leqslant k} \frac{C_{i,j,\alpha}}{(R-x)^\sigma} \left[\sum_{i+|m|+|n|=k} \Big(2Y_{m_1}\Big) \times\cdots\times \Big(2Y_{m_j}\Big)\Big(2\beta Y_{n_1}\Big)\cdots\Big(2\beta Y_{n_\alpha}\Big)\right]$$

and by comparing this with (4.13) we obtain

$$B_\sigma(|w_k|)(x) \ll \frac{(k-1)!^{s-1}}{k^\sigma}(R-x)^\sigma Y_k(x)$$

which proves (4.21).

Thus, the proof of Lemma 4.1 is completed.

The proof of Theorem 2.1 is also completed.

By the above proof, we can say more. Let $p \geqslant 2$ and $\sigma \geqslant p/(p-1)$. Assume the conditions: (i) $\hat a(x)$, $\hat b(x)$, $\hat c(x)$ and $\hat a_{i,j,\alpha}(x)$ are all formal power series in x belonging to the class $G\{x\}_\sigma$; and (ii) the series

$$\sum_{i+j+\alpha\geqslant 2} B_\sigma(\hat a_{i,j,\alpha})(x) t^i u^j v^\alpha$$

is convergent in a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_u \times \mathbb{C}_v$.

Let us consider the following formal equation:

$$(4.28)\qquad \left(t\frac{\partial}{\partial t} - \hat b(x) - x^p \hat c(x)\frac{\partial}{\partial x}\right)\hat u = \hat a(x)t + \sum_{i+j+\alpha\geqslant 2} \hat a_{i,j,\alpha}(x) t^i \hat u^j \left(\frac{\partial \hat u}{\partial x}\right)^\alpha.$$

Then we have

THEOREM 4.2. — *Let $p \geqslant 2$ and $\sigma \geqslant p/(p-1)$. Assume the above conditions (i) and (ii). Then, if $\hat b(0) \notin \mathbb{N}$, the formal equation (4.28) has a* *unique formal power series solution* $\hat{u}(t,x) \in \mathbb{C}[[t,x]]$ *with* $\hat{u}(0,x) \equiv 0$ *and it belongs to the formal Gevrey class* $G\{t,x\}_{(s,\sigma)}$ *for any* s *satisfying*

$$s \geqslant 1 + \max\left[0, \sup_{(i,j,\alpha)\in J}\left(\frac{\sigma-1}{i+j+\alpha-1}\right)\right],$$

where $J = \left\{(i,j,\alpha); i+j+\alpha \geqslant 2, \alpha > 0, \text{ and } \hat{a}_{i,j,\alpha}(0) \neq 0\right\}$.

5. Proof of Proposition 2.3.

Before the proof of Proposition 2.3 we shall show the following lemma.

LEMMA 5.1. — *Let* $p \geqslant 2$ *and* $q \geqslant 1$ *be integers, let* $A > 0$, $C > 0$, $K > 0$, *and let us consider*

$$t\frac{\partial u}{\partial t} = Axt + Cx^p\frac{\partial u}{\partial x} + Kt^q\frac{\partial u}{\partial x}. \tag{5.1}$$

We have

1) *(5.1) has a unique formal solution* $u(t,x) \in \mathbb{C}[[t,x]]$ *with* $u(0,x) \equiv 0$.

2) $u(t,x)$ *belongs to the class* $G\{t,x\}_{(s,\sigma)}$ *if and only if*

$$s \geqslant 1 + \frac{1}{(p-1)q} \quad \text{and} \quad \sigma \geqslant \frac{p}{p-1}. \tag{5.2}$$

Proof. — Let $u(t,x)$ be the formal solution of (5.1) in 1). Since Theorem 2.1 is already proved, we have only to show that $u(t,x) \in G\{t,x\}_{(s,\sigma)}$ implies the condition (5.2). Note that in case $p = 2$ the condition (5.2) is given in [11].

Suppose that $u(t,x) \in G\{t,x\}_{(s,\sigma)}$ holds. Without loss of generality we may assume $A \geqslant 1$, $C \geqslant 1$ and $K \geqslant 1$; if otherwise, we apply the change of variables $t \longrightarrow h_1 t$, $x \longrightarrow h_2 x$ for sufficiently large h_1, h_2 and we can reduce the equation to the case where $A \geqslant 1$, $C \geqslant 1$ and $K \geqslant 1$ hold. Then, the formal solution $w(t,x) \in \mathbb{C}[[t,x]]$ of

$$t\frac{\partial w}{\partial t} = xt + x^p\frac{\partial w}{\partial x} + t^q\frac{\partial w}{\partial x} \tag{5.3}$$

with $w(0,x) \equiv 0$ satisfies $0 \ll w(t,x) \ll u(t,x)$ and therefore we have $w(t,x) \in G\{t,x\}_{(s,\sigma)}$; in particular, we have $w(t,0) \in G\{t\}_s$ and $(\partial w/\partial t)(0,x) \in G\{x\}_\sigma$.

It is easy to see that $w(t,x)$ has the form

$$w(t,x) = \sum_{k\geqslant 0} w_{1+kq}(x)t^{1+kq}, \quad w_{1+kq}(x) \in \mathbb{C}[[x]](\text{for } k \geqslant 0)$$

and the coefficients are determined by the following recurrent formula:

$$w_1 = x + x^p \frac{\partial w_1}{\partial x}, \tag{5.4}$$

and for $k \geqslant 1$

$$(1+kq)w_{1+kq} = x^p \frac{\partial w_{1+kq}}{\partial x} + \frac{\partial w_{1+(k-1)q}}{\partial x}. \tag{5.5}$$

By solving the equation (5.4) we have

$$\begin{aligned} w_1(x) &= x + x^p + \sum_{l\geqslant 1}\Big((1+(p-1))(1+2(p-1))\cdots(1+l(p-1))\Big)x^{p+l(p-1)} \\ &\gg x^p \sum_{l\geqslant 1} (p-1)^l l! x^{l(p-1)}. \end{aligned} \tag{5.6}$$

Since $w_1(x) = (\partial w/\partial t)(0,x) \in G\{x\}_\sigma$ is known, we have

$$\sum_{l\geqslant 1} (p-1)^l l! x^{l(p-1)} \in G\{x\}_\sigma,$$

which immediately leads us to the condition $\sigma \geqslant p/(p-1)$.

Since $w_{1+kq}(x) \gg 0$ is known, by (5.5) we have

$$\begin{aligned} w_{1+kq}(x) &= \frac{1}{1+kq}\left(x^p \frac{\partial}{\partial x} w_{1+kq}(x) + \frac{\partial}{\partial x} w_{1+(k-1)q}(x)\right) \\ &\gg \frac{1}{1+kq}\frac{\partial}{\partial x} w_{1+(k-1)q}(x) \end{aligned}$$

and by repeating this k-times we have

$$w_{1+kq}(x) \gg \frac{1}{(1+q)(1+2q)\cdots(1+kq)}\left(\frac{\partial}{\partial x}\right)^k w_1(x).$$

Since $w_1(x)$ is given explicitly in the equality (5.6), by putting $k = p + l(p-1)$ and $x = 0$ we have

$$\begin{aligned} & w_{1+(p+l(p-1))q}(0) \\ &\quad \geqslant \frac{(p+l(p-1))! \times (1+(p-1))(1+2(p-1))\cdots(1+l(p-1))}{(1+q)(1+2q)\cdots(1+(p+l(p-1))q)} \\ &\quad \geqslant \frac{\Gamma(1/q)}{\Gamma(1/(p-1))} q^{-p-l(p-1)}(p-1)^l l! \end{aligned}$$

and therefore

$$u(t,0) \gg \sum_{l\geqslant 1} w_{1+(p+l(p-1))q}(0)t^{1+(p+l(p-1))q}$$
$$\gg t^{1+pq}\sum_{l\geqslant 1}\frac{\Gamma(1/q)}{\Gamma(1/(p-1))}q^{-p-l(p-1)}(p-1)^l l! t^{l(p-1)q}.$$

Thus, by the condition $u(t,0) \in G\{t\}_s$ we obtain

$$\sum_{l\geqslant 1}(p-1)^l l! t^{l(p-1)q} \in G\{t\}_s$$

which immediately leads us to the condition $s \geqslant 1 + (1/(p-1)q)$.

Thus, we have proved that $u(t,x) \in G\{t,x\}_{(s,\sigma)}$ implies the condition (5.2).

Proof of Proposition 2.3. — Let $u(t,x)$ be the unique formal power series solution of (2.1) with $u(0,x) \equiv 0$. Since Theorem 2.1 is already proved, to complete the proof of Proposition 2.3 it is sufficient to show that $u(t,x) \in G\{t,x\}_{(s,\sigma)}$ implies the condition (2.2) and $\sigma \geqslant p/(p-1)$. If $J = \emptyset$ we have nothing to do; hence from now we assume that $J \neq \emptyset$ holds.

By the conditions c1) $\sim$ c4) we see that $u(t,x) \gg 0$ and we can choose $M > 0$ so that $0 < k - b(0) \leqslant Mk$ for all $k \in \mathbb{N}$. Put $a_0 = a(0) > 0$ and $a_1 = (\partial a/\partial x)(0) > 0$. Take any $(i,j,\alpha) \in J$. Then,

$$Mt\frac{\partial u}{\partial t} \gg \left(t\frac{\partial}{\partial t} - b(0)\right)u$$
$$= xS(b)(x)u + x^p c(x)\frac{\partial u}{\partial x} + a(x)t + \sum_{k+l+m\geqslant 2} a_{k,l,m}(x)t^k u^l\left(\frac{\partial u}{\partial x}\right)^m$$
$$\gg x^p c(0)\frac{\partial u}{\partial x} + (a_0 + a_1 x)t + a_{i,j,\alpha}(0)t^i u^j\left(\frac{\partial u}{\partial x}\right)^\alpha.$$

Therefore, we can see that the unique formal solution $w(t,x) \in \mathbb{C}[[t,x]]$ of

$$t\frac{\partial w}{\partial t} = \frac{1}{M}\left[(a_0 + a_1 x)t + x^p c(0)\frac{\partial w}{\partial x} + a_{i,j,\alpha}(0)t^i w^j\left(\frac{\partial w}{\partial x}\right)^\alpha\right] \tag{5.7}$$

with $w(0,x) \equiv 0$ satisfies $0 \ll w(t,x) \ll u(t,x)$ and therefore we have $w(t,x) \in G\{t,x\}_{(s,\sigma)}$.

Moreover, $w(t,x)$ has the form

$$w(t,x) = \left(\frac{a_0}{M} + \frac{a_1}{M}x\right)t + O(t^2)$$

and by (5.7) we have

$$t\frac{\partial w}{\partial t} = \frac{1}{M}\Bigg[(a_0 + a_1 x)t + x^p c(0)\frac{\partial w}{\partial x} + a_{i,j,\alpha}(0)t^i\left(\left(\frac{a_0}{M}+\frac{a_1}{M}x\right)t + O(t^2)\right)^j \left(\left(\frac{a_1}{M}\right)t + O(t^2)\right)^{\alpha-1}\frac{\partial w}{\partial x}\Bigg]$$
$$\gg \frac{1}{M}\left[a_1 xt + x^p c(0)\frac{\partial w}{\partial x} + a_{i,j,\alpha}(0)\left(\frac{a_0}{M}\right)^j\left(\frac{a_1}{M}\right)^{\alpha-1} t^{i+j+\alpha-1}\frac{\partial w}{\partial x}\right].$$

Thus we can see also that the unique formal solution $W(t,x) \in \mathbb{C}[[t,x]]$ of (5.8)

$$t\frac{\partial W}{\partial t} = \frac{1}{M}\left[a_1 xt + x^p c(0)\frac{\partial W}{\partial x} + a_{i,j,\alpha}(0)\left(\frac{a_0}{M}\right)^j\left(\frac{a_1}{M}\right)^{\alpha-1} t^{i+j+\alpha-1}\frac{\partial W}{\partial x}\right]$$

with $W(0,x) \equiv 0$ satisfies $0 \ll W(t,x) \ll w(t,x)$ and $W(t,x) \in G\{t,x\}_{(s,\sigma)}$.

Now, let us apply Lemma 5.1 to (5.8). Since $W(t,x) \in G\{t,x\}_{(s,\sigma)}$ is known, we can conclude that (s,σ) satisfies

$$s \geqslant 1 + \frac{1}{(p-1)(i+j+\alpha-1)} \quad \text{and} \quad \sigma \geqslant \frac{p}{(p-1)}.$$

Since $(i,j,\alpha) \in J$ is taken arbitrarily, we obtain

$$s \geqslant 1 + \sup_{(i,j,\alpha)\in J}\left(\frac{1}{(p-1)(i+j+\alpha-1)}\right)$$

which implies the condition (2.2).

Thus, the proof of Proposition 2.3 is completed.

Remark. — By the above proof we can see the following: if the equation (2.1) satisfies

$$(i,j,\alpha) \in J \Longrightarrow j = 0, \tag{5.9}$$

we can remove the assumption $a(0) > 0$ from the condition c1) in Proposition 2.3.

Example 5.2. — Let $p,l,n \in \mathbb{Z}_+$ satisfying $p \geqslant 2$, $n \geqslant 1$ and $l+n \geqslant 2$. Let us consider

$$t\frac{\partial u}{\partial t} = xt + x^p\frac{\partial u}{\partial x} + t^l\left(\frac{\partial u}{\partial x}\right)^n. \tag{5.10}$$

Then, the unique formal solution $u(t,x) \in \mathbb{C}[[t,x]]$ with $u(0,x) \equiv 0$ belongs to the class $G\{t,x\}_{(s,\sigma)}$ if and only if

$$s \geqslant 1 + \frac{1}{(p-1)(l+n-1)} \quad \text{and} \quad \sigma \geqslant \frac{p}{p-1}.$$

Acknowledgement. — A part of this work was done while H. Tahara was staying at the Institute of Mathematics of Wuhan University in March, 2000; he would like to thank the Institute for the invitation and its hospitality. The work was supported by the National Natural Science Foundation of China. H. Chen and Z. Luo would like to thank the Foundation for the consistent support. H. Chen also thank the LI Foundation for the support.

BIBLIOGRAPHY

[1] H. CHEN and H. TAHARA, On the holomorphic solution of non-linear totally characteristic equations, to appear in Mathematische Nachrichten, Germany.

[2] H. CHEN and H. TAHARA, On totally characteristic type non-linear partial differential equations in the complex domain, Publ. RIMS, Kyoto Univ., 26 (1999), 621–636.

[3] H. CHEN and Z. LUO, On the holomorphic solution of nonlinear totally characteristic equations with several space variables, Preprint.

[4] R. GÉRARD and H. TAHARA, Nonlinear singular first order partial differential equations of Briot-Bouquet type, Proc. Japan Acad., 66 (1990), 72–74.

[5] R. GÉRARD and H. TAHARA, Holomorphic and singular solution of nonlinear singular first order partial differential equations, Publ. RIMS, Kyoto Univ., 26 (1990), 979–1000.

[6] R. GÉRARD and H. TAHARA, Singular nonlinear partial differential equations, Aspects of Mathematics, E 28, Vieweg, 1996.

[7] R. GÉRARD and H. TAHARA, Formal power series solutions of nonlinear first order partial differential equations, Funkcial. Ekvac., 41 (1998), 133–166.

[8] S. OUCHI, Formal solutions with Gevrey type estimates of nonlinear partial differential equations, J. Math. Sci. Univ. Tokyo, 1 (1994), 205–237.

[9] A. SHIRAI, Maillet type theorems for nonlinear partial differential equations and the Newton polygons, Preprint.

[10] E. T. WHITTAKER and G. N. WATSON, A course of modern analysis (4th edition, reprinted), Cambridge Univ. Press, 1958.

[11] H. YAMAZAWA, Newton polyhedrons and a formal Gevrey space of double indices for linear partial differential operators, Funkcial. Ekvac., 41 (1998), 337–345.

Manuscrit reçu le 4 septembre 2000,
accepté le 9 mars 2001.

Hua CHEN, Zhuangchu LUO,
Wuhan University
Institute of Mathematics
Wuhan (P.R. China).
chenhua@whu.edu.cn
&
Hidetoshi TAHARA,
Sophia University
Department of Mathematics
Tokyo (Japan).
h-tahara@hoffman.cc.sophia.ac.jp