



ANNALES

DE

L'INSTITUT FOURIER

Annette WERNER

Non-archimedean intersection indices on projective spaces and the Bruhat-Tits building for PGL

Tome 51, n° 6 (2001), p. 1483-1505.

http://aif.cedram.org/item?id=AIF_2001__51_6_1483_0

© Association des Annales de l'institut Fourier, 2001, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>

NON-ARCHIMEDEAN INTERSECTION INDICES ON PROJECTIVE SPACES AND THE BRUHAT-TITS BUILDING FOR PGL

by Annette WERNER

1. Introduction.

In this paper we interpret non-Archimedean local intersection numbers of linear cycles in \mathbb{P}^{n-1} with the combinatorial geometry of the Bruhat-Tits building associated to $PGL(n)$.

The ultimate motivation behind these results is to associate to a projective space a differential geometric object playing the role of a model at infinity in the sense of Manin, who constructed in [Ma] such an object for curves.

A first step in this direction is to look for a geometric interpretation of non-Archimedean intersection numbers which has an Archimedean analogue. It turns out that the Bruhat-Tits building for PGL is a good candidate for such a geometric framework.

The goal of the present paper is to express non-Archimedean intersection numbers in terms of the building. In another work (see [We]) we construct the desired “model at infinity” for projective spaces as an Archimedean analogue of the building. Besides, we use results of the present paper to derive parallel geometric formulas for Archimedean and non-Archimedean Arakelov intersection numbers.

Keywords: Intersection theory – Projective spaces – Bruhat-Tits building.
Math. classification: 14C17 – 14M15 – 20E42.

Let us now describe our main results. We denote by X the Bruhat-Tits building associated to the group $G = PGL(V)$, where V is an n -dimensional vector space over a non-Archimedean local field K of characteristic 0. The vertices in X correspond to the homothety classes $\{M\}$ of R -lattices M in V , where R is the ring of integers in K .

We fix a lattice M in V , which induces a projective space $\mathbb{P}(M)$ over R , and consider d linear cycles on $\mathbb{P}(M)$ intersecting properly in a cycle of dimension 0. These cycles are equal to projective spaces $\mathbb{P}(N_i)$ for split R -submodules N_i of M . We put $L_j = \cap_{i \neq j} N_i$, and define F as the following set of vertices in X :

$$F = \{ \{ \pi^{k_1} L_1 + \dots + \pi^{k_d} L_d \} : k_1, \dots, k_d \in \mathbb{Z} \}.$$

In Theorem 5.1 we express Serre's intersection number $\langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle$ of our linear cycles in $\mathbb{P}(M)$ as

$$\langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle = \text{dist}(\{M\}, F),$$

where dist is what we call the combinatorial distance function in X , *i.e.* the minimal length of a path consisting of 1-simplices connecting $\{M\}$ with a vertex in F .

In the case of \mathbb{P}^1 , this result specializes to a formula in [Ma].

Moreover we investigate the case of several linear cycles meeting properly in a cycle of higher dimension. In this case, of course, we no longer have an intersection number, but we can nevertheless describe the intersection cycle (see Theorem 5.2). It consists of one component coming from the generic fibre, which appears with multiplicity 1, and one component coming from the special fibre, appearing with multiplicity $\text{dist}(\{M\}, F)$, where F is defined in a similar way as in the previous result.

Acknowledgements. — I thank Ch. Deninger, G. Kings, K. Künnemann, E. Landvogt, Y.I. Manin, P. Schneider, E. de Shalit and M. Strauch for useful and inspiring discussions. I am also grateful to the Max-Planck-Institut für Mathematik in Bonn for financial support and the stimulating atmosphere during the early stages of this project.

2. Intersection theory.

In this section we will list the definitions and results from intersection theory which are needed later on, thereby fixing our notation.

Let Ω be a scheme, of finite type and separated over a regular ring. By $Z^p(\Omega)$ we denote the codimension p cycles on Ω , i.e. the free abelian group on the set of integral (i.e. irreducible and reduced) closed subschemes of codimension p . We write $CH^p(\Omega)$ for the quotient of $Z^p(\Omega)$ after the group generated by the principal cycles $\text{div}(f)$ for rational functions $f \neq 0$ on a codimension $p - 1$ integral closed subscheme, see [Fu], 1.3.

If $T \subset \Omega$ is a closed subset, let $Z_T^p(\Omega)$ denote the free abelian group on the set of codimension p integral closed subschemes of Ω , which are contained in T , and $CH_T^p(\Omega)$ the Chow group of cycles supported on T , i.e. $Z_T^p(\Omega)$ modulo the subgroup generated by all $\text{div}(f)$ for rational functions $f \neq 0$ on some codimension $p - 1$ integral closed subscheme of Ω which is contained in T . If $T = \emptyset$, we put $CH_T^p(\Omega) = 0$.

Now we will briefly recall Serre's intersection pairing. It is defined on any smooth scheme Ω over a discrete valuation ring by [Se], V-32. Two closed, integral subschemes X and Y of Ω meet properly if for every irreducible component W of $X \cap Y$ we have $\text{codim}(X) + \text{codim}(Y) = \text{codim}(W)$. By [Se] we always have the inequality " \geq ". If X and Y meet properly, then Serre defines an intersection index $i_W(X, Y)$ of X and Y along W by higher Tor functors, see [Se], V-21.

Let W be an irreducible component of $X \cap Y$, let $\mathcal{O}_{\Omega, w}$ be the local ring at the generic point w of W , and let p_X, p_Y be the ideals in $\mathcal{O}_{\Omega, w}$ corresponding to X and Y . If X and Y are locally Cohen-Macaulay, then

$$i_W(X, Y) = l_{\mathcal{O}_{\Omega, w}}(\mathcal{O}_{\Omega, w}/(p_X + p_Y))$$

by [Se], p. V-20.

We define the intersection cycle of properly intersecting X and Y by $X \cdot Y = \sum_W i_W(X, Y)W$ where the sum runs over all irreducible components of $X \cap Y$. We can continue this product linearly to arbitrary cycles X and Y meeting properly on Ω , which means that any irreducible component of X meets all the irreducible components of Y properly. If X_1, \dots, X_r are r closed integral subschemes meeting properly, i.e. so that every irreducible component W of $\cap_i X_i$ satisfies $\text{codim } W = \sum \text{codim } X_i$, then the intersection of X_1, \dots, X_r is defined inductively: $X_1 \cdot \dots \cdot X_r = (\dots((X_1 \cdot X_2) \cdot X_3) \dots) \cdot X_r$. Again we can extend this product linearly to arbitrary cycles X_1, \dots, X_r meeting properly.

Gillet and Soulé have defined an arithmetic intersection pairing for arithmetic Chow groups in [GiSo]. The local contributions of this pairing at the finite places (in the smooth case) can be described as follows (see [GiSo], 4.5.1):

Assume that $\pi : \Omega \rightarrow S = \text{Spec } R$ is smooth, separated and of finite type over the discrete valuation ring R and that Ω is irreducible. Let $Z_p(\Omega/S)$ be the free abelian group on the closed integral subschemes $Y \subset \Omega$ of relative dimension p over S . Here the relative dimension $\dim_S(Y)$ of Y over S is defined as

$$\dim_S Y = \text{transcendence degree of } k(Y) \text{ over } k(T) - \text{codim}_S(T),$$

where T is the closure of $\pi(Y)$ in S and $k(Y), k(T)$ are the function fields. This relative dimension has the property that

$$\dim_S(\Omega) = \dim_S(Y) + \text{codim}_\Omega(Y)$$

for all closed integral subschemes Y of Ω (see [Fu], Lemma 20.1).

Let $CH_p(\Omega/S)$ be $Z_p(\Omega/S)$ modulo rational equivalence. For all closed subschemes $T \subset \Omega$ we have $CH_T^p(\Omega) = CH_{d-p}(T/S)$, where d is the relative dimension of Ω over S .

For closed subschemes Y and Z we can define a pairing

$$CH_Y^p(\Omega) \times CH_Z^q(\Omega) \longrightarrow CH_{Y \cap Z}^{p+q}(\Omega)$$

as follows: It suffices to define a pairing

$$CH_{d-p}(Y/S) \times CH_{d-q}(Z/S) \longrightarrow CH_{d-p-q}(Y \cap Z/S).$$

Let $V \subset Y$ and $W \subset Z$ be integral closed subschemes. Then we define a cycle in $Y \times_S Z$ as follows:

$$V \otimes W = \begin{cases} 0, & \text{if } V \text{ and } W \text{ are contained in the closed fibre,} \\ [V \times_S W], & \text{otherwise.} \end{cases}$$

By [Fu], Proposition 20.2, this induces a pairing

$$CH_{d-p}(Y/S) \times CH_{d-q}(Z/S) \longrightarrow CH_{2d-p-q}(Y \times_S Z/S).$$

Since Ω is smooth over S , the diagonal embedding $\Delta : \Omega \rightarrow \Omega \times_S \Omega$ is a regular embedding, and we have Fulton's Gysin map (see [Fu], §6 and §20):

$$\Delta^! : CH_{2d-p-q}(Y \times_S Z/S) \longrightarrow CH_{d-p-q}(Y \cap Z/S).$$

Hence we get the desired pairing.

If V and W meet properly, their image under this pairing coincides with the image of Serre's intersection pairing in $CH_{Y \cap Z}^{p+q}(\Omega)$ by [Fu], 7.1.2 and 20.2.2.

In particular, if Y and Z are irreducible with $p = \text{codim}_\Omega(Y)$ and $q = \text{codim}_\Omega(Z)$, the image of (Y, Z) via $CH_Y^p(\Omega) \times CH_Z^q(\Omega) \rightarrow CH_{Y \cap Z}^{p+q}(\Omega)$ yields an intersection class

$$Y \cdot Z = \Delta^!(Y \otimes Z) \in CH_{Y \cap Z}^{p+q}(\Omega).$$

If one cycle is given by a Cartier divisor, say $Y = D$, then we have a different description of the image of $D \cdot Z$ in $CH_Z^{p+q}(\Omega)$ (see [Fu], 8.1.1, 20.2.1 and 6.1c): It is equal to the class of E , where E is any Weil divisor on Z induced by a Cartier divisor whose line bundle is isomorphic to $j^* \mathcal{O}(D)$. Here $j : Z \hookrightarrow \Omega$ is the embedding of Z into Ω and $\mathcal{O}(D)$ is the line bundle on Ω corresponding to the class of D . In particular, if D and D' are linear equivalent divisors on Ω , the images of $D \cdot Z$ and $D' \cdot Z$ in $CH_Z^{p+q}(\Omega)$ coincide.

If we have several irreducible cycles Y_1, Y_2, \dots, Y_r in Ω of codimensions p_1, p_2, \dots, p_r we can define inductively an intersection class $Y_1 \cdot \dots \cdot Y_r \in CH_{\bigcap Y_i}^{p_1 + \dots + p_r}(\Omega)$.

Let k be the residue field of R . We denote by \deg the degree map for 0-cycles in the special fibre Ω_k of Ω , i.e. for all $z = \sum n_P P \in Z^d(\Omega_k)$ we put $\deg z = \sum n_P [k(P) : k]$, where $k(P)$ is the residue field of P .

Assume additionally that Ω is proper over S , and let $Y \in Z^p(\Omega)$ and $Z \in Z^q(\Omega)$ be two irreducible closed subschemes such that $p + q = d + 1$ which intersect properly on the generic fibre of Ω . This means that their generic fibres are disjoint, so that $Y \cap Z$ is contained in the special fibre Ω_k of Ω . Hence we can define an intersection number

$$\langle Y, Z \rangle = \deg(Y \cdot Z),$$

where we take the degree of the image of $Y \cdot Z \in CH_{Y \cap Z}^{d+1}(\Omega)$ in $CH^d(\Omega_k)$. Similarly, if $Y_i \in Z^{p_i}(\Omega)$ for $i = 1, \dots, r$ are prime cycles with $\sum p_i = d + 1$ which meet properly on the generic fibre, we put $\langle Y_1, \dots, Y_r \rangle = \deg(Y_1 \cdot \dots \cdot Y_r)$.

3. Hyperplanes.

Throughout this paper we denote by K a finite extension of \mathbb{Q}_p , by R its valuation ring and by k the residue class field. Besides, v is the valuation map, normalized so that it maps a prime element to 1. We write q for the cardinality of the residue class field, and we normalize the absolute value on K so that $|x| = q^{-v(x)}$.

Besides, we fix an n -dimensional vector space V over K .

Let $\mathbb{P}(V) = \text{Proj Sym } V^*$ be the projective space corresponding to V , where V^* is the linear dual of V . Every non-zero linear subspace W

of V defines an integral (i.e. irreducible and reduced) closed subscheme $\mathbb{P}(W) = \text{Proj Sym } W^* \hookrightarrow \mathbb{P}(V)$ of codimension $n - \dim W$.

By “ R -lattice in V ” we always mean an R -lattice in V of full rank. Every R -lattice M in V defines a model $\mathbb{P}(M) = \text{Proj Sym}_R(M^*)$ of $\mathbb{P}(V)$ over R , where M^* is the R -linear dual of M . If the lattices M and N differ by multiplication by some $\lambda \in K^\times$ then the corresponding isomorphism $\mathbb{P}(M) \xrightarrow{\sim} \mathbb{P}(N)$ induces the identity on the generic fibre.

Throughout this paper we call a submodule N of M split, if the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ is split, i.e. if M/N is free (or, equivalently, torsion free). Every split R -submodule N of M defines a closed subscheme $\mathbb{P}(N) = \text{Proj Sym } N^* \hookrightarrow \mathbb{P}(M)$.

LEMMA 3.1. — *For every split R -submodule N of M , the closed subscheme $\mathbb{P}(N) = \text{Proj Sym } N^*$ of $\mathbb{P}(M)$ is integral, and has codimension $n - \text{rk } N$.*

Proof. — This follows from dualizing the sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$. \square

The cycles in $\mathbb{P}(M)$ induced by split submodules are called linear, and linear cycles of codimension 1 are called hyperplanes. The homogeneous prime ideal corresponding to the linear cycle $\mathbb{P}(N) \hookrightarrow \mathbb{P}(M)$ is generated by a base of $(M/N)^*$ (regarded in M^*). In particular, it is generated by homogeneous elements of degree one.

Now fix a lattice M in V and an R -basis x_1, \dots, x_n of M . Let B be a matrix in $GL(n, R)$ which we regard as an endomorphism of M via our fixed basis. Then B induces an automorphism (which we also denote by B) of $\mathbb{P}(M)$. The following lemma can be proven easily:

LEMMA 3.2. — *If the hyperplane H in $\mathbb{P}(M)$ is given by the linear homogeneous element*

$$f = \sum_{j=1}^n a_j x_j^* \in M^*,$$

where x_1^*, \dots, x_n^* is the dual basis of x_1, \dots, x_n , then $B(H)$ is given by the homogeneous element $\sum_{j=1}^n b_j x_j^*$ where

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = {}^t B^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Now we need an easy matrix lemma. We call a quadratic matrix a permutation matrix if it contains exactly one entry 1 in every line and column, and if all other entries are equal to zero.

LEMMA 3.3. — *Let $A = (a_{ij})$ be an $(n \times n)$ -matrix over R . Then there exist elements C and D in $GL(n, R)$, where D is a permutation matrix, such that the matrix $CAD = (b_{ij})_{i,j}$ is upper triangular with*

$$v(b_{11}) \leq \dots \leq v(b_{nn}) \quad \text{and} \quad v(b_{ii}) \leq v(b_{ij}) \quad \text{for all } i \leq j.$$

Proof. — We move a coefficient with minimal valuation in the upper left corner and eliminate the other entries in the first column. This can be repeated until our matrix is upper triangular. \square

A crucial step for our geometric formulas for intersection indices (to be proven in Section 5) is the expression of the intersection number of n hyperplanes in terms of their equations. We can do this for any n hyperplanes H_1, \dots, H_n in $\mathbb{P}(M)$ such that their generic fibres H_{1K}, \dots, H_{nK} meet properly on $\mathbb{P}(V)$.

THEOREM 3.4. — *Let M be a lattice in V . We fix a basis x_1, \dots, x_n of M , and denote by $x_1^*, \dots, x_n^* \in M^*$ the dual basis. Let H_1, \dots, H_n be hyperplanes in $\mathbb{P}(M)$ which intersect properly on the generic fibre. Let $f_i = \sum_j a_{ji} x_j^* \in M^*$ be a linear homogeneous element generating the ideal corresponding to H_i and put $A = (a_{kl})_{k,l}$. Then we have the following formula for the intersection number of H_1, \dots, H_n :*

$$\langle H_1, \dots, H_n \rangle = v(\det A).$$

Proof. — By Lemma 3.3 we find some $C \in GL(n, R)$ and a permutation matrix D such that $CAD = B = (b_{ij})$ is upper triangular and satisfies the inequalities

$$v(b_{11}) \leq \dots \leq v(b_{nn}) \quad \text{and} \quad v(b_{ii}) \leq v(b_{ij}) \quad \text{for all } i \leq j.$$

There is a permutation σ of $\{1, \dots, n\}$ such that AD is the coefficient matrix for the hyperplanes $H_{\sigma(1)}, \dots, H_{\sigma(n)}$. By Lemma 3.2, the linear element $\sum_{j \leq i} b_{ji} x_j^*$ corresponds to the hyperplane ${}^t C^{-1}(H_{\sigma(i)})$ for all $i = 1, \dots, n$. Now $\langle H_1, \dots, H_n \rangle = \langle {}^t C^{-1}(H_{\sigma(1)}), \dots, {}^t C^{-1}(H_{\sigma(n)}) \rangle$ and $v(\det A) = v(\det B)$. Hence we can assume that A is upper triangular with $v(a_{11}) \leq \dots \leq v(a_{nn})$ and $v(a_{ii}) \leq v(a_{ij})$ for $i \leq j$.

We can assume that $\cap H_i \neq \emptyset$, since otherwise our claim is trivial.

For all $a \in R$ we denote by \bar{a} its image in k . The reduction $(H_i)_k$ of H_i corresponds to the homogeneous ideal generated by $\bar{f}_i = \sum \bar{a}_{ji} x_j^*$ in $\mathbb{P}(M_k^*)$. Let \bar{A} be the matrix (\bar{a}_{ij}) . Now $\cap(H_i)_k$ is the linear cycle corresponding to the subspace $L_k \subset M_k$ which is equal to $\ker^t \bar{A}$ via the identification of M_k with k^n given by the reductions of x_1, \dots, x_n .

We will first assume that H_1, \dots, H_n meet properly on the whole of $\mathbb{P}(M)$. Hence their intersection consists of one point in the special fibre, and $v(a_{11}) = \dots = v(a_{n-1n-1}) = 0$. We can therefore assume that $a_{11} = \dots = a_{n-1n-1} = 1$. Hence for all $k \leq n-1$ the homogeneous ideal in $\text{Sym } M^*$ generated by $f_1 = x_1^*, f_2 = a_{12}x_1^* + x_2^*, \dots, f_k = \sum_{j=1}^{k-1} a_{jk}x_j^* + x_k^*$ is equal to the homogeneous ideal generated by x_1^*, \dots, x_k^* . Now we can compute Serre's intersection index as follows: Note that all H_i are isomorphic to \mathbb{P}_R^{n-2} , hence they are locally Cohen-Macaulay (even regular). Assume that $n > 2$. The closed subset $H_1 \cap H_2$ of $\mathbb{P}(M)$ is given by the homogeneous ideal $(f_1, f_2) = (x_1^*, x_2^*)$; the corresponding reduced closed subscheme is the linear cycle $\mathbb{P}(N) \hookrightarrow \mathbb{P}(M)$ for $N = (M^*/Rx_1^* + Rx_2^*)^*$. Hence $W = H_1 \cap H_2$ is a prime cycle. We have $i_W(H_1, H_2) = 1$, hence $H_1 \cdot H_2 = W$. Besides, W is a projective space over R , hence also locally Cohen-Macaulay.

The same argument (if $n > 3$) implies that $H_1 \cdot H_2 \cdot H_3$ is equal to the cycle given by the irreducible subset $H_1 \cap H_2 \cap H_3$. Finally we find that $H_1 \cdot H_2 \cdot \dots \cdot H_{n-1} = W$ where W is the prime cycle corresponding to the homogeneous ideal $(x_1^*, \dots, x_{n-1}^*)$. Now it is easy to calculate $\langle H_1, \dots, H_n \rangle = v(a_{nn}) = v(\det A)$, which proves our claim in the case of proper intersection.

Hence we can now assume that there is an $s < n-1$ such that $v(a_{11}) = \dots = v(a_{ss}) = 0$ and $v(a_{kk}) > 0$ if $k > s$. We write $l_i = v(a_{ii})$, and we can again assume that $a_{11} = \dots = a_{ss} = 1$. For all $m = 1, \dots, n-1$ let Y_m be the cycle corresponding to the integral subscheme given by the homogeneous ideal (x_1^*, \dots, x_m^*) of $\text{Sym } M^*$, and let Z_m be the cycle corresponding to the integral subscheme given by $(x_1^*, \dots, x_{m-1}^*, \pi)$, where π is a fixed prime element in R .

Assume that m is a number with $s < m < n$. Then the intersection of Y_{m-1} and H_m has two irreducible components, namely Y_m and Z_m . Since both have codimension m , the cycles Y_{m-1} and H_m meet properly. We want to calculate $i_{Y_m}(Y_{m-1}, H_m)$ and $i_{Z_m}(Y_{m-1}, H_m)$. Let y respectively z be the generic points of Y_m respectively Z_m . Since $m < n$, they are both contained in $U = \{x_n^* \neq 0\}$. We write $A = \mathcal{O}_{\mathbb{P}(M), y}$. Then $A = R[y_1, \dots, y_{n-1}]_{(y_1, \dots, y_m)}$ with $y_i = x_i^*/x_n^*$. Since Y_{m-1} and H_m are

locally Cohen-Macaulay, we have

$$i_{Y_m}(Y_{m-1}, H_m) = l_A(A/(y_1, \dots, y_{m-1}, \pi^{l_m} y_m)).$$

As π is a unit in A , this is equal to $l_A(A/(y_1, \dots, y_m)) = 1$.

Similarly, we put $B = \mathcal{O}_{\mathbb{P}(M), z}$, hence $B = R[y_1, \dots, y_{n-1}]_{(y_1, \dots, y_{m-1}, \pi)}$, and we get

$$i_{Z_m}(Y_{m-1}, H_m) = l_B(B/(y_1, \dots, y_{m-1}, \pi^{l_m} y_m)).$$

Here y_m is a unit in B , hence this length is equal to

$$\begin{aligned} l_B(B/(y_1, \dots, y_{m-1}, \pi^{l_m})) \\ = l_{(R[y_m, \dots, y_{n-1}]/(\pi^{l_m}))_{(\pi)}}((R[y_m, \dots, y_{n-1}]/(\pi^{l_m}))_{(\pi)}) = l_m, \end{aligned}$$

since the only ideals in $(R[y_m, \dots, y_{n-1}]/(\pi^{l_m}))_{(\pi)}$ are 0 and (π^k) for $0 < k < l_m$ and these are all distinct.

Now we will prove by induction that $H_1 \cdot \dots \cdot H_m$ is equal to the class of $Y_m + (\sum_{i \leq m} l_i)Z_m$ in $CH_{\cap_{i \leq m} H_i}^m(\mathbb{P}(M))$ for all $m \leq n - 1$. Since H_1 is irreducible, we have $l_1 = 0$ and $H_1 = Y_1$, which is our claim for $m = 1$.

Now we come to the induction step. Assume that our claim holds for some m with $1 \leq m \leq n - 2$. If $m \leq s$, we can move to $m + 1$ using the above calculations of intersection indices. Let us now assume that $m > s$, hence that l_m and l_{m+1} are strictly positive. Then H_{m+1} meets Y_m properly in the components Y_{m+1} and Z_{m+1} and we can calculate $Y_m \cdot H_{m+1}$ via Serre's intersection: $Y_m \cdot H_{m+1}$ is induced by the cycle $Y_{m+1} + l_{m+1}Z_{m+1}$ by our previous calculations.

Note that f_{m+1} is contained in $(x_1^*, \dots, x_{m-1}^*, \pi)$, since both l_m and l_{m+1} are strictly positive. Hence Z_m is contained in H_{m+1} . Now we can determine the intersection

$$Z_m \cdot H_{m+1} \in CH_{Z_m \cap H_{m+1}}^{m+1}(\mathbb{P}(M)) = CH_{Z_m}^{m+1}(\mathbb{P}(M))$$

by the recipe we described in Section 2 for intersections with divisors. Since H_{m+1} is linearly equivalent to the hyperplane H' given by the ideal (x_m^*) , we find that $Z_m \cdot H_{m+1}$ is equal to the image of $Z_m \cdot H'$ in $CH_{Z_m}^{m+1}(\mathbb{P}(M))$. Now $Z_m \cap H' = Z_{m+1}$ and Z_m and H' meet properly in this irreducible set with $i_{Z_{m+1}}(Z_m, H') = 1$, which implies that $Z_m \cdot H'$ is induced by the cycle Z_{m+1} .

Altogether we find that $H_1 \cdot \dots \cdot H_{m+1}$ is the image of $Y_{m+1} + (l_1 + \dots + l_{m+1})Z_{m+1}$ in $CH_{\cap_{i \leq m+1} H_i}^{m+1}(\mathbb{P}(M))$, which finishes the proof of our claim.

We know now that $H_1 \cdot \dots \cdot H_{n-1}$ is the image of $Y_{n-1} + (l_1 + \dots + l_{n-1})Z_{n-1}$ in $CH_{\cap_{i \leq n-1} H_i}^{n-1}(\mathbb{P}(M))$. Since $s < n - 1$, we have $l_{n-1} > 0$ and $l_n > 0$. Now Y_{n-1} meets H_n properly, hence we can apply our result in the case of hyperplanes meeting properly on the whole of $\mathbb{P}(M)$ and find $\deg(Y_{n-1} \cdot H_n) = l_n$.

Besides, we have $Z_{n-1} \cap H_n = Z_{n-1}$, and as in our induction step we can show that $Z_{n-1} \cdot H_n$ is the image of Z_n in $CH_{Z_{n-1}}^n(\mathbb{P}(M))$. Since Z_n is a k -rational point in the special fibre, we get $\deg(Z_{n-1} \cdot H_n) = 1$.

Altogether we find that

$$\langle H_1, \dots, H_n \rangle = l_1 + \dots + l_n = v(\det A),$$

whence our claim. □

4. The Bruhat-Tits building for PGL .

We denote by X the Bruhat-Tits building corresponding to the group $G = PGL(V)$ (see [BrTi]). X is a metric space with a continuous G -action and a simplicial structure. For our purposes, we can think of it as the geometric realization of the following simplicial complex: We call two lattices in V equivalent, if they differ by a factor in K^\times , and we write $\{M\}$ for the equivalence class of the lattice M . Two different lattice classes $\{M'\}$ and $\{N'\}$ are called adjacent, if there are representatives M and N of $\{M'\}$ and $\{N'\}$ such that

$$\pi N \subset M \subset N.$$

This relation defines a flag complex, namely the simplicial complex whose vertex set is the set of all classes $\{M\}$, and whose simplices are the sets of pairwise adjacent lattice classes. Note that it carries a natural G -action.

If $n = 2$, then X is an infinite regular tree, with $q + 1$ edges meeting in every vertex.

The building X is the union of its apartments, which correspond to the maximal split tori in G . We can describe them as follows: For every decomposition $V = \bigoplus_{1 \leq i \leq n} L_i$ of V in one-dimensional subspaces L_i generated by some vector v_i we define an apartment as the subcomplex of X given by all lattices M which can be diagonalized with respect to our decomposition, i.e. $M = \sum_{i=1}^n R\pi^{k_i} v_i$ for some integers k_i .

DEFINITION 4.1. — *The combinatorial distance $\text{dist}(x, y)$ of two vertices x and y in X is defined as*

$$\text{dist}(x, y) = \min\{k : \text{there are vertices } x = x_0, x_1, \dots, x_k = y, \\ \text{so that } x_i \text{ and } x_{i+1} \text{ are adjacent for all } i = 0, \dots, k-1\}.$$

Hence dist is the minimal number of 1-simplices forming a path between x and y . Note that dist is in general not proportional to the metric on X .

LEMMA 4.2. — *Let $x = \{M\}$ and $y = \{L\}$ be two vertices in X , and define*

$$s = \min\{k : \pi^k L \subset M\} \quad \text{and} \quad r = \max\{k : M \subset \pi^k L\}.$$

Then we have $\text{dist}(x, y) = s - r$.

Proof. — Note that the term on the right hand side is independent of the choice of a representative of the lattice classes.

Put $d = \text{dist}(x, y)$. Then we find lattices $M = M_0, M_1, \dots, M_d$ such that $M_d = \alpha L$ for some $\alpha \in K^\times$ and such that

$$\pi^d M_d \subset \pi^{d-1} M_{d-1} \subset \dots \subset \pi M_1 \subset M_0 \subset M_1 \subset \dots \subset M_d.$$

Hence $\pi^d \alpha L \subset M$, which implies $s \leq d + v(\alpha)$, and $M \subset \alpha L$, which implies $r \geq v(\alpha)$. Altogether we find that $s - r \leq d$.

Let us now show that also $s - r \geq d$ is true. We have by definition $\pi^s L \subset M$ and $M \subset \pi^r L$. Put $L' = \pi^r L$. By the invariant factor theorem, we find an R -basis w_1, \dots, w_n of L' , such that $M = \sum \pi^{k_i} R w_i$ for some integers k_i . Since $\pi^{s-r} L' \subset M \subset L'$, all k_i are between 0 and $s - r$.

Now put $l_{ij} = \max\{0, k_j - i\}$ for all $i \in \{0, \dots, s - r\}$ and all $j \in \{1, \dots, n\}$. Then we define for $i = 0, \dots, s - r$

$$M_i = \pi^{l_{i1}} R w_1 + \dots + \pi^{l_{in}} R w_n.$$

Note that $M_0 = M$, and $M_{s-r} = L'$. We have for all $i = 0, \dots, s - r - 1$ the inclusions $\pi M_{i+1} \subset M_i \subset M_{i+1}$. Since either $\{M_i\} = \{M_{i+1}\}$, or $\{M_i\}$ and $\{M_{i+1}\}$ are adjacent, we found a chain of adjacent lattices of length $\leq s - r$ connecting $\{M\}$ and $\{L\}$, which implies our claim. \square

5. Intersection indices via combinatorial geometry.

Let us fix a lattice M in V . We will now interpret non-Archimedean intersection numbers of linear cycles on $\mathbb{P}(M)$ with the combinatorial geometry of X .

Take some $d \geq 2$ and let $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ be d linear cycles on $\mathbb{P}(M)$ with

$$\sum_{i=1}^d \text{codim } \mathbb{P}(N_i) = n,$$

which meet properly on $\mathbb{P}(M)$. Then N_1, \dots, N_d are split submodules of M of rank r_1, \dots, r_d satisfying $\sum_{i=1}^d r_i = (d - 1)n$ by 3.1. We will always assume that $r_i \neq n$. For all $j = 1, \dots, d$ we put $L_j = \cap_{i \neq j} N_i \subset M$. Since all M/N_i are torsion free, the same holds for M/L_j , so that L_j is a split submodule of M .

Now let F be the set of all vertices in X of the form

$$\{\pi^{k_1} L_1 \oplus \dots \oplus \pi^{k_d} L_d\} \text{ for some } k_1, \dots, k_d \in \mathbb{Z}.$$

(Alternatively, we can also work with the convex hull of F in X .) Note that the intersection $\cap_{i=1}^d N_i$ is zero, as $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ do not meet on the generic fibre. Therefore the sum $\pi^{k_1} L_1 \oplus \dots \oplus \pi^{k_d} L_d$ is direct. On the other hand, an easy calculation shows that $\dim L_{jK} \geq n - r_j$. Since $\sum_j \dim L_{jK} \geq \sum_j (n - r_j) = n$, we must have $\dim L_{jK} = n - r_j$ and $V = \bigoplus_j L_{jK}$, so that $\pi^{k_1} L_1 \oplus \dots \oplus \pi^{k_d} L_d$ is indeed a lattice of full rank in V .

Obviously, F is the set of vertices contained in either a full apartment or an intersection of affine hyperplanes in some apartment. (Hence its convex hull is either an apartment of an intersection of affine hyperplanes.)

Let us describe F in two special cases:

i) The case of n hyperplanes.

First note that n hyperplanes H_{1K}, \dots, H_{nK} in the generic fibre $\mathbb{P}(V)$ intersecting properly, *i.e.* not at all, in $\mathbb{P}(V)$, define an apartment $A(H_{1K}, \dots, H_{nK})$ in X as follows: For all i the hyperplane H_{iK} is the linear cycle corresponding to an $(n - 1)$ -dimensional subspace W_i of V . Since the H_{iK} intersect properly, $\cap W_i$ is equal to 0. For all $j = 1, \dots, n$ put $U_j = \cap_{i \neq j} W_i$. Then all U_j are one-dimensional and $V = \bigoplus U_j$. We denote the apartment corresponding to this decomposition (see Section 4) by $A(H_{1K}, \dots, H_{nK})$.

If H_1, \dots, H_n are hyperplanes in $\mathbb{P}(M)$ which intersect properly on the whole of $\mathbb{P}(M)$, then the subset F is just the apartment $A(H_{1K}, \dots, H_{nK})$ corresponding to the generic fibres.

ii) The case of two cycles.

Let N_1 and N_2 be two non-trivial split submodules of M of rank p respectively $q = n - p$, such that the corresponding linear cycles $Z_1 = \mathbb{P}(N_1)$ and $Z_2 = \mathbb{P}(N_2)$ meet properly on $\mathbb{P}(M)$. Then the corresponding subset F is

$$\{\{N_1 + \pi^k N_2\} : k \in \mathbb{Z}\}.$$

In fact, one can show that F defines a doubly infinite geodesic in the building X whose boundary points (with respect to the Borel-Serre compactification of X) are the parabolics induced by the vector spaces $W_i = N_i \otimes_R K$ for $i = 1$ and 2 .

We will now show that the intersection number of d linear cycles meeting properly on $\mathbb{P}(M)$ is the combinatorial distance of the set F to the lattice M .

If n is equal to 2 (i.e. X is the Bruhat-Tits tree associated to $PGL(2, K)$), and we consider the case of two hyperplanes in $\mathbb{P}(M)$, then the ensuing formula is due to Manin (see [Ma], p. 232). Note that in this setting our special cases i) and ii) from above coincide, and the subset F we are dealing with is a geodesic in the Bruhat-Tits tree.

THEOREM 5.1. — *The intersection number of $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ on $\mathbb{P}(M)$ can be expressed as follows with the combinatorial geometry of X :*

$$\langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle = \text{dist}(\{M\}, F),$$

where dist denotes the combinatorial distance in X , and $\{M\}$ is the vertex in X defined by our fixed lattice M .

Proof. — Recall that r_i is the rank of N_i and that the numbers $m_i = n - r_i$ satisfy $\sum_{i=1}^d m_i = n$. Besides, put $n_0 = 0$ and $n_i = m_1 + \dots + m_i$ for $i = 1, \dots, d$. Then $n = n_d$.

For all $i = 1, \dots, d$ let $g_{n_{i-1}+1}, \dots, g_{n_i}$ be a R -basis of $(M/N_i)^* \hookrightarrow M^*$. The elements $g_{n_{i-1}+1}, \dots, g_{n_i}$ generate the homogeneous prime ideal corresponding to $\mathbb{P}(N_i) \hookrightarrow \mathbb{P}(M)$. Besides, fix a basis y_1, \dots, y_n of M , and let $A' = (a'_{ij})$ be the coordinate matrix of g_1, \dots, g_n with respect to the

dual basis y_1^*, \dots, y_n^* , i.e.

$$g_j = \sum_{i=1}^n a'_{ij} y_i^*.$$

Now choose an element a'_{ij} with $j \leq n_1$ such that $v(a'_{ij})$ is minimal among the entries of the first n_1 columns. We remove the j -th column and insert it before the first one. Then we switch rows so that this element sits in the upper left corner, and we perform some elementary row operations to eliminate $a'_{21}, \dots, a'_{n_1 1}$. Among the j between 2 and n_1 and among the $i \geq 2$ we choose an entry of minimal valuation. Again, after removing the corresponding column and putting it between the first and the second one, and after elementary row operations involving only the last $n - 1$ rows we can assume that $a'_{i2} = 0$ for all $i > 2$. We continue in this way until we reach the n_1 -th column. Then a'_{ij} is zero for $j \leq n_1$ and $i > j$, and the upper left corner satisfies the following divisibility conditions:

$$v(a'_{ii}) \leq v(a'_{i+1i+1}) \quad \text{for } i \leq n_1 - 1$$

and
$$v(a'_{ii}) \leq v(a'_{ij}) \quad \text{for } i < j \leq n_1.$$

Now choose an element a'_{ij} with $i, j \geq n_1 + 1$ and $j \leq n_2$ such that $v(a'_{ij})$ is minimal among the entries a'_{ij} with $i \geq n_1 + 1$ and $n_1 + 1 \leq j \leq n_2$. As before, we permute the columns corresponding to $g_{n_1+1}, \dots, g_{n_2}$ and perform elementary row operations involving only the rows with index bigger or equal to $n_1 + 1$ to achieve $a'_{ij} = 0$ for $j \leq n_2$ and $i > j$.

We continue this process until we worked our way through the whole matrix. We see that after permuting the equations $g_{n_{i-1}+1}, \dots, g_{n_i}$ corresponding to each of the N_1, \dots, N_d , and after switching to another basis of M (in order to take care of the row operations) we can assume that our coordinate matrix A' is upper triangular.

Let us denote the reduction of elements in R or M or of R -matrices by overlining. Besides we use the following notation for submatrices: For any $n \times n$ -matrix $D = (d_{kl})$ and any $i \leq j \leq n$ we write $D(i \rightarrow j)$ for the $n \times (j - i + 1)$ -submatrix consisting of the columns $i, i + 1, \dots, j$. Similarly we write $D(i \downarrow j)$ for the $(j - i + 1) \times n$ -submatrix consisting of the rows $i, i + 1, \dots, j$. By $D(i_1 \rightarrow j_1, i_2 \rightarrow j_2)$ we mean the submatrix where we take columns i_1, \dots, j_1 followed by columns j_1, \dots, j_2 . Besides, $D(i_1 \downarrow j_1)(i_2 \rightarrow j_2)$ is the submatrix consisting of all entries d_{kl} with $i_1 \leq k \leq j_1$ and $i_2 \leq l \leq j_2$.

For $n \times n$ -matrices we will furthermore abbreviate $D_{ij} = D(n_{i-1} + 1 \downarrow n_i)(n_{j-1} + 1 \rightarrow n_j)$, if i and j are $\leq d$. If we divide D into rectangular

submatrices according to our partition $n = m_1 + \dots + m_d$, then D_{ij} is the rectangle in position (ij) .

Since $g_{n_{i-1}+1}, \dots, g_{n_i}$ form a basis of the split R -submodule $(M/N_i)^*$ of M^* , their reductions generate a vector space of rank m_i over k , so that the coordinate matrix $\overline{A}'(n_{i-1} + 1 \rightarrow n_i)$ has full rank m_i .

Since $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ meet properly, their intersection is empty or zero-dimensional and contained in the special fibre. Hence $\cap_i \mathbb{P}(N_{ik})$ is empty or has dimension zero, so that $\cap_i N_{ik}$ has dimension ≤ 1 . Since N_{ik} is equal to the kernel of $(\overline{g}_{n_{i-1}+1}, \dots, \overline{g}_{n_i})$, this means that $\text{rk } \overline{A}' \geq n - 1$.

The intersection of $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ is empty iff $\text{rk } \overline{A}' = n$. In this case all elements on the diagonal of A' are units. After switching to another basis of M , we can therefore assume that A' is diagonal. Then $M = L_1 + \dots + L_d$, so that $\{M\}$ is actually contained in F , and our formula holds.

Hence we only have to deal with the case $\text{rk } \overline{A}' = n - 1$. Here the intersection of $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ is not empty. As $v(\det A') > 0$, we find indices $p \leq q \leq d$ with

$$v(\det A'_{11}) = \dots = v(\det A'_{p-1p-1}) = 0,$$

$$v(\det A'_{pp}) > 0, \quad v(\det A'_{qq}) > 0$$

and
$$v(\det A'_{q+1q+1}) = \dots = v(\det A'_{dd}) = 0.$$

Note that $p \geq 2$, since $\overline{A}'(1 \rightarrow n_1)$, and thus \overline{A}'_{11} has full rank. All elements on the diagonal of $A'_{11}, \dots, A'_{p-1p-1}, A'_{q+1q+1}, \dots, A'_{dd}$ are units. After performing some elementary row operations we can assume that $A'_{ij} = 0$, if $i < j \leq p - 1$ or if $j \geq q + 1$ and $i < j$. Note that the divisibility conditions in A'_{qq} imply that all elements on the diagonal except possibly the last one are units. Therefore we can eliminate all entries a'_{ij} in A' such that $1 \leq i \leq n_{q-1}$ and $n_{q-1} + 1 \leq j \leq n_q - 1$ by elementary row operations. Hence after switching to another basis y_1, \dots, y_n of M we may assume that our coordinate matrix A' contains zeros above $A'_{22}, \dots, A'_{p-1p-1}$ and $A'_{q+1,q+1}, \dots, A'_{dd}$, and that all columns above A'_{qq} are zero except possibly the last one.

Now recall that for $j = 1, \dots, d$ the module $L_j = \cap_{i \neq j} N_i$ is a split submodule of M of rank m_j . For each $j = 1, \dots, d$ choose an R -basis

$$w_{n_{j-1}+1}, \dots, w_{n_j}$$

of L_j . We denote by B' the transpose of the coordinate matrix of w_1, \dots, w_n

with respect to y_1, \dots, y_n , i.e. B' is the matrix $B' = (b'_{ij})_{ij}$ so that

$$w_i = \sum_{j=1}^n b'_{ij} y_j.$$

For all $j = 1, \dots, d$ there are matrices $C_j, D_j \in GL(m_j, R)$ such that $C_j B'_{jj} D_j$ is a diagonal matrix with entries β_{ii} such that $v(\beta_{11}) \leq \dots \leq v(\beta_{m_j m_j})$. Let C (respectively D) be the matrix with diagonal components C_1, \dots, C_d (respectively D_1, \dots, D_d), and define $B = CB'D$.

Then D is the transpose of the transition matrix from a base x_1, \dots, x_n of M to our base y_1, \dots, y_n . If we put

$$v_i = \sum_{j=1}^n b_{ij} x_j,$$

then for all $h \leq d$ the elements $v_{n_{h-1}+1}, \dots, v_{n_h}$ form a basis of L_h . The matrix B is by definition the transpose of the transition matrix from x_1, \dots, x_n to v_1, \dots, v_n .

Now D^{-1} is the transition matrix from the dual basis x_1^*, \dots, x_n^* of M^* to y_1^*, \dots, y_n^* . Hence the coordinate matrix of g_1, \dots, g_n with respect to x_1^*, \dots, x_n^* is equal to $A'' = D^{-1}A'$. The matrix A'' has the property that if a block A'_{ij} is zero or zero up to the last column, then the same holds for A''_{ij} . Besides, $v(\det A''_{jj}) = v(\det A'_{jj})$. However, the diagonal blocks $A''_{11}, \dots, A''_{dd}$ may not be upper triangular any longer.

After permuting the first $n_1 = m_1$ columns we may assume that $v(a''_{n_1 n_1}) \leq v(a''_{n_1 j})$ for all $j = 1, \dots, n_1$. By a series of elementary column operations we can eliminate $a''_{n_1 1}, \dots, a''_{n_1 n_1-1}$. Now we permute the first $n_1 - 1$ columns to achieve $v(a''_{n_1-1 j}) \leq v(a''_{n_1-1 n_1-1})$ for all $j = 1, \dots, n_1 - 1$, and we clear out $a''_{n_1-1 1}, \dots, a''_{n_1-1 n_1-2}$. Note that these column operations affect only the first $n_1 - 1$ columns, hence the first $n_1 - 1$ elements in the n_1 -th row remain zero. We repeat this process until A''_{11} is upper triangular. These column operations amount to passing to another basis f_1, \dots, f_{n_1} of $(M/N_1)^*$

Now we work on $A''(n_1 + 1 \rightarrow n_2)$. First we switch columns inside this block to achieve $v(a''_{n_2 j}) \leq v(a''_{n_2 n_2})$ for all $j = n_1 + 1, \dots, n_2 - 1$, and we clear out $a''_{n_2 n_1+1}, \dots, a''_{n_2 n_2-1}$, then we eliminate $a''_{n_2-1 n_1+1}, \dots, a''_{n_2-1 n_2-2}$, and so on, until A''_{22} is upper triangular. We do this with block after block until A'' is upper triangular.

Since the columns were transformed block by block, we can find for each $i = 1, \dots, d$ a new R -basis

$$f_{n_{i-1}+1}, \dots, f_{n_i}$$

of $(M/N_i)^*$, whose coordinate matrix A with respect to the basis x_1^*, \dots, x_n^* is upper triangular and has the property that the columns above $A_{22}, \dots, A_{p-1p-1}$ and above $A_{q+1q+1}, \dots, A_{dd}$ are zero. Besides, we still have $v(\det A_{pp}) > 0$ and $v(\det A_{qq}) > 0$. Since there exists a matrix $D \in GL(m_q, R)$ such that

$$A(n_{q-1} + 1 \rightarrow n_q)(1 \downarrow n_{q-1}) = A''(n_{q-1} + 1 \rightarrow n_q)(1 \downarrow n_{q-1})D,$$

we still know that $\text{rk } \overline{A}(n_{q-1} + 1 \rightarrow n_q)(1 \downarrow n_{q-1}) \leq 1$.

Let H_j be the hyperplane given by the linear homogeneous element f_j . Then $H_{n_{i-1}+1}, \dots, H_{n_i}$ intersect properly and $\mathbb{P}(N_i) = H_{n_{i-1}+1} \cdot \dots \cdot H_{n_i}$. Hence by 3.4 we have

$$\langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle = v(\det A).$$

Besides note that

$$f_j(v_i) = \sum_k a_{kj} b_{ik},$$

so that the entries of BA are equal to $(f_j(v_i))_{ij}$.

Now by definition, N_i lies in the kernel of all $f_{n_{i-1}+1}, \dots, f_{n_i}$. Since L_j is contained in N_i for all $i \neq j$, this implies that the blocks $(BA)_{ij}$ are equal to 0 for $i \neq j$. Since all B_{jj} are diagonal matrices, B is therefore upper triangular. Besides we have $B_{ij} = 0$, if $i < j \leq p - 1$, and if $i < j$ and $j > q$.

Since $v_{n_{j-1}+1}, \dots, v_{n_j}$ are a basis of the split submodule $L_j \subset M$, their reductions are still linear independent, hence the rank of $\overline{B}(n_{j-1} + 1 \downarrow n_j)$ is equal to n_j . We will now show that only the last diagonal element $b_{n_j n_j}$ in the block B_{jj} may not be a unit. Indeed, assume that $v(b_{ii}) > 0$ for some $n_{j-1} + 1 \leq i \leq n_j - 1$. By the divisibility conditions along the diagonal of B_{jj} we have $v(b_{n_j-1 n_j-1}) > 0$ and $v(b_{n_j n_j}) > 0$. For all $l = n_{j-1} + 1, \dots, n_j$ we find $f_l(v_{n_j-1}) = b_{n_j-1 n_j-1} a_{n_j-1 l}$, so that $\overline{f_l}(v_{n_j-1}) = 0$. Similarly, $\overline{f_l}(v_{n_j}) = 0$, so that $\overline{v_{n_j-1}}$ and $\overline{v_{n_j}}$ are contained in N_{jk} . Since L_j is contained in all N_i for $i \neq j$, the elements $\overline{v_{n_j-1}}$ and $\overline{v_{n_j}}$ lie also in N_{ik} for all $i \neq j$. Hence we found two linear independent vectors in $\cap_i N_{ik}$, which contradicts our assumption that $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ meet properly.

We will now show the following claim:

- (1) There exists an index $t \leq p - 1$ such that $\overline{A}(1 \rightarrow n_{t-1}, n_t + 1 \rightarrow n)$ has full rank.

Let us first assume that $p < q$. Note that $\text{rk } \overline{A}(n_{q-1} + 1 \downarrow n_q)$ is strictly smaller than m_q , since $v(\det(A_{qq})) > 0$. Since $\text{rk } \overline{A} = n - 1$, the matrix $\overline{A}(n_{p-1} + 1 \downarrow n_{q-1})$ must have full rank $(n_{q-1} - n_{p-1})$. Disregarding the zeros, we find that also the $(n_{q-1} - n_{p-1}) \times (n_q - n_{p-1})$ -matrix

$$\overline{A}(n_{p-1} + 1 \downarrow n_{q-1})(n_{p-1} + 1 \rightarrow n_q)$$

must have full rank $(n_{q-1} - n_{p-1})$. This matrix consists of the two vertical chunks

$$\overline{A}(n_{p-1} + 1 \downarrow n_{q-1})(n_{p-1} + 1 \rightarrow n_{q-1})$$

and

$$\overline{A}(n_{p-1} + 1 \downarrow n_{q-1})(n_{q-1} + 1 \rightarrow n_q).$$

We know that the second chunk $\overline{A}(n_{p-1} + 1 \downarrow n_{q-1})(n_{q-1} + 1 \rightarrow n_q)$ has rank ≤ 1 . Hence the rank of the first chunk

$$\Omega := \overline{A}(n_{p-1} + 1 \downarrow n_{q-1})(n_{p-1} + 1 \rightarrow n_{q-1})$$

must be $\geq (n_{q-1} - n_{p-1} - 1)$.

Besides, as $v(\det A_{pp}) > 0$, there must be an element on the diagonal of A_{pp} which has positive valuation, i.e. there exists an index $n_{p-1} + 1 \leq i \leq n_p$ such that $\overline{a}_{ii} = 0$. Since Ω has rank $\geq (n_{q-1} - n_{p-1} - 1)$, the upper left corner $\overline{A}(n_{p-1} + 1 \downarrow i - 1)(n_{p-1} + 1 \rightarrow i)$ must have full rank $(i - 1 - n_{p-1})$. Now recall that $\overline{A}(n_{p-1} + 1 \rightarrow n_p)$ has full rank, hence the first $i - n_{p-1}$ columns of this matrix, namely $\overline{A}(n_{p-1} + 1 \rightarrow i)$, also have full rank $i - n_{p-1}$. Put

$$\lambda_j = \overline{A}(n_{p-1} + 1 \rightarrow i)(j \downarrow j),$$

which is just the j -th row of this matrix. We have seen that $\lambda_{n_{p-1}+1}, \dots, \lambda_{i-1}$ are linear independent, hence there exists a row λ_{j_0} with $j_0 \leq n_{p-1}$ so that $\lambda_{j_0}, \lambda_{n_{p-1}+1}, \dots, \lambda_{i-1}$ is a full linear independent set of rows in $\overline{A}(n_{p-1} + 1 \rightarrow i)$.

Now let $t \leq p - 1$ be the index of the block in which λ_{j_0} lies, i.e. $n_{t-1} + 1 \leq j_0 \leq n_t$. We want to show that $\overline{A}(1 \rightarrow n_{t-1}, n_t + 1 \rightarrow n)$ has full rank. It obviously suffices to show that $\overline{A}(n_t + 1 \rightarrow n)(n_{t-1} + 1 \downarrow n)$ has full rank. Removing the first block of rows $\overline{A}(n_t + 1 \rightarrow n)(n_{t-1} + 1 \downarrow n_t)$ and putting it before the block $\overline{A}(n_t + 1 \rightarrow n)(n_{p-1} + 1 \downarrow n_p)$, we see that it remains to show that the matrix

$$\Lambda = \begin{pmatrix} \overline{A}_{tp} & \dots & \overline{A}_{td} \\ \overline{A}_{pp} & \dots & \overline{A}_{pd} \\ & \ddots & \vdots \\ 0 & & \overline{A}_{dd} \end{pmatrix}$$

has full rank $n - n_{p-1}$. Recall that we fixed an element a_{ii} on the diagonal of A_{pp} with positive valuation. Now

$$\Lambda = \left(\begin{array}{ccc|ccc} \bar{a}_{n_{t-1}+1, n_{p-1}+1} & \cdots & \bar{a}_{n_{t-1}+1, i} & & & \\ & & \vdots & & & * \\ \bar{a}_{n_t, n_{p-1}+1} & \cdots & \bar{a}_{n_t, i} & & & \\ \bar{a}_{n_{p-1}+1, n_{p-1}+1} & \cdots & \bar{a}_{n_{p-1}+1, i} & & & \\ & & \vdots & & & \\ \bar{a}_{i-1, n_{p-1}+1} & \cdots & \bar{a}_{i-1, i} & & & \\ \hline & & & & \bar{a}_{ii+1} & \cdots & \bar{a}_{in} \\ & & & & \bar{a}_{i+1i+1} & \cdots & \bar{a}_{i+1n} \\ & & 0 & & & \ddots & \vdots \\ & & & & & & \bar{a}_{nn} \end{array} \right)$$

Hence in the upper left corner of Λ we find the rows λ_j for $j = n_{t-1} + 1, \dots, n_t$ and $j = n_{p-1} + 1, \dots, i - 1$, and we know that there are $(i - n_{p-1})$ linear independent ones among them. So the upper left corner of Λ has full rank $i - n_{p-1}$. The lower right corner (which is a $(n - i + 1) \times (n - i)$ -matrix) has the same rank as the $(n - i + 1) \times n$ -matrix $\bar{A}(i \downarrow n)$. Since $\text{rk}(\bar{A}) = n - 1$, the rank of the lower right corner must therefore be bigger or equal to $n - i$ which means that this corner has also full rank. Hence Λ has full rank. This finishes the proof of claim (1) in the case $p < q$.

If p is equal to q , similar arguments can be used to prove (1).

Now we claim that we can calculate our intersection number as follows: For the index t we found in (1) we have

$$(2) \quad \langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle = v(b_{n_t n_t}).$$

Put $w_1 = v_{n_{t-1}+1}, \dots, w_{m_t} = v_{n_t}$, which is a basis of $L_t = \cap_{i \neq t} N_i$. Since L_t is a split submodule of M , we can complete this basis to a basis w_1, \dots, w_n of M . Since $f_j = \sum_{i=1}^n f_j(w_i)w_i^*$, the coordinate matrix of f_1, \dots, f_n with respect to the dual basis w_1^*, \dots, w_n^* is equal to $(f_j(w_i))_{i,j}$. Using 3.4, we can therefore calculate the intersection number as follows:

$$\langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle = v(\det(f_j(w_i))_{i,j=1, \dots, n}).$$

Now $f_j(w_i) = 0$ for $i = 1, \dots, m_t$ and $j \notin J_t := \{n_{t-1} + 1, \dots, n_t\}$. Hence after permuting columns the matrix $(f_j(w_i))_{i,j=1, \dots, n}$ looks like this $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, and we get

$$\begin{aligned} v(\det(f_j(w_i))_{i,j=1, \dots, n}) \\ = v(\det(f_j(w_i))_{i=1, \dots, m_t, j \in J_t}) + v(\det(f_j(w_i))_{i=m_t+1, \dots, n, j \notin J_t}). \end{aligned}$$

Note that $(\overline{f_j(w_i)})_{i=1, \dots, m_t, j \in J_t} = B_{tt}A_{tt}$. As t is strictly smaller than p , the determinant of A_{tt} is a unit. Besides, as we have shown above, only the last element on the diagonal of B_{tt} may not be a unit, so that we can calculate the first term as follows:

$$v(\det(f_j(w_i)_{i=1, \dots, m_t, j \in J_t})) = v(b_{n_t n_t}).$$

It remains to show that the second term is zero, hence that $(\overline{f_j(w_i)})_{i=m_t+1, \dots, n, j \notin J_t}$ is an invertible $(n - m_t) \times (n - m_t)$ -matrix over k . Since $f_j(w_i) = 0$ for $i = 1, \dots, m_t$ and $j \notin J_t$ we can as well show that the matrix $(f_j(w_i))_{i=1, \dots, n, j \notin J_t}$ has full rank $n - m_t$. In order to prove this we may change the base of M and show that $\overline{A}(1 \rightarrow n_{t-1}, n_t + 1 \rightarrow n) = (\overline{f_j(x_i)})_{i=1, \dots, n, j \notin J_t}$ has full rank $n - m_t$, which was done in (1).

Let us fix some $h = 1, \dots, d$. We will first show by induction that for all $i \geq n_h$ we have

$$(3) \quad b_{n_h n_h} \text{ divides } b_{n_h i} \prod_{j=n_h+1}^i a_{jj},$$

where the empty product is equal to 1. For $i = n_h$ this is trivial. So let us suppose our claim is true for all i with $n_h \leq i < i_0$ for some $i_0 \geq n_h + 1$. Then

$$0 = f_{i_0}(v_{n_h}) = \sum_{i=n_h}^{i_0} b_{n_h i} a_{i i_0}.$$

We multiply by $\prod_{j=n_h+1}^{i_0-1} a_{jj}$ and get

$$0 = \sum_{i=n_h}^{i_0-1} b_{n_h i} \left(\prod_{j=n_h+1}^{i_0-1} a_{jj} \right) a_{i i_0} + b_{n_h i_0} \left(\prod_{j=n_h+1}^{i_0-1} a_{jj} \right) a_{i_0 i_0},$$

hence the induction hypothesis implies our claim for i_0 .

Note that (3) implies that $v(b_{n_t n_t}) = \max_{i=1, \dots, n} v(b_{ii})$. Indeed, if i is not equal to n_h for some $h = 1, \dots, d$, then b_{ii} is a unit. Since \overline{B} contains no zero lines, for every n_h there must be an index $i \geq n_h$ such that $v(b_{n_h i}) = 0$. By (3) we find that

$$v(b_{n_h n_h}) \leq \sum_{j=n_h+1}^i v(a_{jj}) \leq v(\det A).$$

But $v(\det A)$ is equal to the intersection number $\langle \mathbb{P}(N_1), \dots, \mathbb{P}(N_d) \rangle$, hence equal to $v(b_{n_t n_t})$ by (2).

Now we can prove

$$(4) \quad \text{dist}(\{M\}, F) = v(b_{n_t n_t}).$$

We take a lattice class $\{L\}$ corresponding to a vertex in F , i.e. $L = \pi^{k_1}L_1 + \dots + \pi^{k_{d-1}}L_{d-1} + L_d$ for some integers k_1, \dots, k_{d-1} . Then $\min\{k : \pi^k L \subset M\} \geq -k_t$ and $\max\{k : M \subset \pi^k L\} \leq -k_t - v(b_{n_t n_t})$, so that 4.2 implies $\text{dist}(\{M\}, \{L\}) \geq v(b_{n_t n_t})$. Hence we are done if we show that

$$\text{dist}(\{M\}, \{L_1 + \dots + L_d\}) \leq v(b_{n_t n_t}).$$

Recall that BA is the matrix consisting of the diagonal blocks $B_{11}A_{11}, \dots, B_{dd}A_{dd}$. Now for all $h = 1, \dots, d$ the determinant of A_{hh} is $\prod_{j=n_{h-1}+1}^{n_h} a_{jj}$, and $v(\det B_{hh}) = v(b_{n_h n_h})$. Applying again (3) and the fact that in every line in B there must be a unit, we find

$$v(b_{n_h n_h}) \leq v\left(\prod_{j=n_h+1}^n a_{jj}\right),$$

so that

$$\begin{aligned} v(\det B_{hh}A_{hh}) &\leq v\left(\prod_{j=n_h+1}^n a_{jj}\right) + v\left(\prod_{j=n_{h-1}+1}^{n_h} a_{jj}\right) \\ &\leq v(\det A) = v(b_{n_t n_t}), \end{aligned}$$

hence $\pi^{v(b_{n_t n_t})}(A_{hh}^{-1}B_{hh}^{-1})$ has R -coefficients for all h . Therefore $\pi^{v(b_{n_t n_t})}B^{-1}$ has R -coefficients, which implies $M \subset \pi^{-v(b_{n_t n_t})}(L_1 + \dots + L_d)$, as ${}^tBM = L_1 + \dots + L_d$. Now it is easy to see that indeed $\text{dist}(\{M\}, \{L_1 + \dots + L_d\}) \leq v(b_{n_t n_t})$. \square

One may wonder if our assumption that the linear cycles meet properly on the whole of $\mathbb{P}(M)$ is really necessary. In fact, in 3.4 we have proven a formula for the intersection number of n hyperplanes meeting properly only on $\mathbb{P}(V)$. Hence it is tempting to try and use this as a starting point to generalize Theorem 5.1 at least to the case of n hyperplanes H_1, \dots, H_n in $\mathbb{P}(M)$ meeting properly only on the generic fibre $\mathbb{P}(V)$. Unfortunately, the geometric expression $\text{dist}(\{M\}, A(H_{1K}, \dots, H_{nK}))$ does in general no longer coincide with the intersection number $\langle H_1, \dots, H_n \rangle$ in this case. In fact, one can show as in the proof of 5.1 that for H_1, \dots, H_n with coordinate matrix $A = (a_{ij})$ (upper triangular and subject to $v(a_{ii}) \leq v(a_{i+1i+1})$ and $v(a_{ii}) \leq v(a_{ij})$ for $i < j$) we have $\text{dist}(\{M\}, A(H_{1K}, \dots, H_{nK})) = v(a_{nn})$, whereas the intersection number $\langle H_1, \dots, H_n \rangle = \sum_{i=1}^n v(a_{ii})$ (see 3.4) may be bigger.

We will conclude this paper by generalizing Theorem 5.1 in another direction. Namely, let us consider the case of $d \geq 2$ linear cycles on $\mathbb{P}(M)$ meeting properly in a cycle of dimension bigger than zero. So let N_1, \dots, N_d

be split submodules of M of rank r_1, \dots, r_d , so that $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ are linear cycles meeting properly on $\mathbb{P}(M)$, i.e. any irreducible component in the intersection $\cap_i \mathbb{P}(N_i)$ has codimension

$$\sum_i \text{codim } \mathbb{P}(N_i) = dn - \sum_{i=1}^d r_i.$$

Note that only the case $\sum_i \text{codim } \mathbb{P}(N_i) \leq n$ is interesting, since otherwise $\cap \mathbb{P}(N_i)$ must be empty. Since we already dealt with the case $\sum_i \text{codim } \mathbb{P}(N_i) = n$ in 5.1, let us now assume that $\sum_i \text{codim } \mathbb{P}(N_i) < n$. Put $r_0 = n - (dn - \sum_i r_i) = \sum_i r_i - (d - 1)n > 0$, and let L_0 be the intersection $L_0 = \cap_{i=1}^d N_i$. Since all M/N_i are torsion free, the same holds for M/L_0 , so that L_0 is a split submodule of M . Hence $\mathbb{P}(L_0)$ is a linear cycle contained in the intersection $\cap_i \mathbb{P}(N_i)$, so it has codimension $\geq n - r_0$, which implies that $\text{rk } L_0 \leq r_0$. On the other hand, we can calculate

$$\text{rk } L_0 = \dim(\cap N_{iK}) \geq \sum_{i=1}^d \dim(N_{iK}) - (d - 1)n = \sum_{i=1}^d r_i - (d - 1)n = r_0,$$

so that we find $\text{rk } L_0 = r_0$.

For $j = 1, \dots, d$ we put $L_j = \cap_{i \neq j} N_i \subset M$. Then L_0 is also a split submodule of L_j , hence there exists a free R -module L'_j with $L_j = L_0 \oplus L'_j$. Let F be the set of vertices in X of the form

$$\{\pi^{k_1} L'_1 \oplus \dots \oplus \pi^{k_d} L'_d \oplus \pi^{k_{d+1}} L_0\} \text{ for some } k_1, \dots, k_{d+1} \in \mathbb{Z}.$$

Of course, F depends on the choice of L'_1, \dots, L'_d . It is easy to see that the sum $\pi^{k_1} L'_1 + \dots + \pi^{k_d} L'_d + \pi^{k_{d+1}} L_0$ is indeed direct. Since

$$\dim L_{iK} \geq \sum_{j \neq i} \dim N_{jK} - (d - 2)n = \sum_{j \neq i} r_j - (d - 2)n = n - r_i + r_0$$

we find that

$$\sum_{i=1}^d \dim L'_{iK} + \dim L_{0K} = \sum_{i=1}^d \dim L_{iK} - (d - 1)r_0 \geq dn - \sum_{i=1}^d r_i + r_0 = n,$$

so that $\pi^{k_1} L'_1 \oplus \dots \oplus \pi^{k_d} L'_d \oplus \pi^{k_{d+1}} L_0$ is indeed a lattice of full rank in V . Besides we see that $\text{rk } L'_i = n - r_i$.

Then the strategy of the proof of 5.1 can be modified to prove the following result:

THEOREM 5.2. — *Let $\mathbb{P}(N_1), \dots, \mathbb{P}(N_d)$ be linear cycles meeting properly on $\mathbb{P}(M)$ such that $\sum_{i=1}^d \text{codim } \mathbb{P}(N_i) < n$. For any choice of F*

as above, we can describe the Serre intersection cycle $\mathbb{P}(N_1) \cdot \dots \cdot \mathbb{P}(N_d)$ as follows:

$$\mathbb{P}(N_1) \cdot \dots \cdot \mathbb{P}(N_d) = \overline{\mathbb{P}(\cap_{i=1}^d N_{iK})} + \text{dist}(\{M\}, F) \mathbb{P}(\cap_{i=1}^d N_{ik}).$$

Here the first cycle $\overline{\mathbb{P}(\cap_{i=1}^d N_{iK})}$ is the closure in $\mathbb{P}(M)$ of the linear cycle $\mathbb{P}(\cap_{i=1}^d N_{iK})$ on $\mathbb{P}(V)$, which is just the Serre intersection cycle $\mathbb{P}(N_1)_K \cdot \dots \cdot \mathbb{P}(N_d)_K$ of the generic fibres. The second cycle $\mathbb{P}(\cap_{i=1}^d N_{ik})$ is the linear cycle on the special fibre corresponding to the subspace $\cap_{i=1}^d N_{ik}$ of M_k , which we regard as a cycle on $\mathbb{P}(M)$.

Both our results 5.1 and 5.2 are proven by a complicated series of direct computations once the correct formulas are found. It would be desirable to give a more conceptual proof providing deeper insights in the nature of our formulas and allowing generalizations.

BIBLIOGRAPHY

- [BrTi] F. BRUHAT, J. TITS, Groupes réductifs sur un corps local. I. Données radicielles valuées, Publ. Math. IHES, 41 (1972), 5–252.
- [Fu] W. FULTON, Intersection theory, Second edition, Springer 1998.
- [GiSo] H. GILLET, C. SOULÉ, Arithmetic intersection theory, Publ. Math. IHES, 72 (1990), 93–174.
- [Ma] Y.I. MANIN, Three-dimensional hyperbolic geometry as ∞ -adic Arakelov geometry, Invent. Math., 104 (1991), 223–244.
- [Se] J.-P. SERRE, Algèbre locale. Multiplicités, Lecture Notes in Mathematics, 11, Springer, 1965.
- [We] A. WERNER, Arakelov intersection indices of linear cycles and the geometry of buildings and symmetric spaces, To appear in Duke Math. J.

Manuscrit reçu le 3 janvier 2001,
accepté le 9 mars 2001.

Annette WERNER,
Universität Münster
Mathematisches Institut
Einsteinstr. 62
48149 Münster (Allemagne).
werner@math.uni-muenster.de