ANNALES

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Tome 51, n° 4 (2001), p. 1071-1087.

<http://aif.cedram.org/item?id=AIF_2001__51_4_1071_0>

Ann. Inst. Fourier, Grenoble
51, 4 (2001), 1071–1087

CONSTRUCTING BLOW-ANALYTIC ISOMORPHISMS

by T. FUKUI, T.-C. KUO and L. PAUNESCU

1. Introduction.

In the study of real analytic singularities, a very important objective is to find a natural equivalence relation, leading to better classification theorems.

H. Whitney's family, $f_t(x,y)=f(x,y;t) = xy(x-y)(x-ty)$, $t>2$, shows that the classification obtained by using C^1-diffeomorphisms is far from satisfactory. Indeed, classifying by C^1-diffeomorphisms gives that each f_t belongs to a different C^1-class, which is something we certainly do not want.

As an alternative, one can use the notion of blow-analytic homeomorphism to define the so-called blow-analytic equivalence relation ([9], [8], see also the survey article [3]). These are important notions and using them, several authors have proved important classification theorems (see for instance [3]). Hence it is interesting in itself to see what these blow-analytic homeomorphisms look like and how one can decide whether or not a given mapping is a blow-analytic homeomorphism. We expect a kind of an inverse mapping theorem in this category and we offer a version of it (Theorem 6.1), via toric modifications.

In this paper we consider only blow-analytic homeomorphisms which admit desingularizations that also preserve the analytic structure of the exceptional loci.

The authors thank the referee for useful comments.

Keywords: Blow-analytic – Arc-analytic.
Math. classification: 32B20 – 14Pxx.

2. Blow-analytic functions.

Let U be a neighbourhood of the origin of $\mathbb{R}^n$, M a real analytic manifold and $\pi : M \to U$ be a proper analytic real modification whose complexification is also a proper modification near its real part (we often simply say that "π is a modification"). Let $f : U \dashrightarrow \mathbb{R}$ denote a function defined on U except possibly some thin subset of U. In this case, we shall simply say that f is defined almost everywhere.

We say that f is *blow-analytic via* π if $f \circ \pi$ has an analytic extension on M.

We say that f is *blow-analytic* if it does so via some modification.

Let $g : U \dashrightarrow \mathbb{R}$ be a blow-analytic function via a modification $\pi : M \to U$. For an irreducible divisor E of M, we denote by $\mathrm{ord}_E(g)$ the vanishing order of $g \circ \pi$ at a generic point of E. For an analytic arc $\alpha : (\mathbb{R}, 0) \to U$ such that $g \circ \alpha : (\mathbb{R}, 0) \to \mathbb{R}$ extends to an analytic function, we denote by $\mathrm{ord}_\alpha(g)$, the order of $g \circ \alpha : (\mathbb{R}, 0) \to \mathbb{R}$.

Let P be a function defined almost everywhere on U.

We say that P is a *blow-analytic unit* via a modification $\pi : M \to U$, if $P \circ \pi$ extends to an analytic function on M, which is a unit as an analytic function.

LEMMA 2.1. — *Let $f, g : U \dashrightarrow \mathbb{R}$ be two blow-analytic functions. If along any analytic arc $\alpha : (\mathbb{R}, 0) \to (U, 0)$ we have $\mathrm{ord}_\alpha(f) \geqslant \mathrm{ord}_\alpha(g)$ (in particular we assume that they can be defined simultaneously) then there exist a small neighbourhood of the origin and a positive constant c, such that $|g| \geqslant c|f|$.*

Proof. — One can find a modification $\pi : M \to U$ such that both $f \circ \pi$ and $g \circ \pi$ are locally monomials, see [4] for instance. The assumption will therefore imply that their quotient $|g/f|$ is bounded away from zero.□

COROLLARY 2.2. — *Let $\pi : M \to U$ be a modification and P a function defined almost everywhere on U. Then the following conditions are equivalent:*

(i) *P is a blow-analytic unit via π, in a possibly smaller neighbourhood of the origin.*

(ii) *P is blow-analytic via π, and* $\operatorname{ord}_\alpha(P) = 0$ *for any analytic arc* $\alpha : (\mathbb{R}, 0) \to (U, 0)$ *(wherever defined).*

Proof. — Since π is a real modification, trivially (i) implies (ii) (one can lift analytic arcs). For the other implication apply twice the previous lemma to P and 1 to get that P and $1/P$ are bounded away from zero. □

3. Blow-analytic isomorphisms.

Let U_1, U_2 be two neighbourhoods of the origin of $\mathbb{R}^n$. Let $h : U_1 \to U_2$ be a map.

We say that h is a *blow-analytic isomorphism* if $h : U_1 \to U_2$ is a homeomorphism and there is an analytic isomorphism of pairs $H : (M_1, E_1) \to (M_2, E_2)$ so that $h \circ \pi_1 = \pi_2 \circ H$ for some modifications $\pi_i : M_i \to U_i$, $i = 1, 2$, where E_i (as analytic spaces) denote the critical loci of π_i, $i = 1, 2$.

If there is a modification-germ $\pi : (M, \pi^{-1}(0)) \to (\mathbb{R}^n, 0)$ such that π_1, π_2 are its representatives, we call h a *blow-analytic isomorphism(-germ)* via π.

Remark 3.1. — The notion introduced here is slightly different from that of blow-analytic homeomorphism in [3]. A blow-analytic isomorphism requires not only that $H : M_1 \to M_2$ is an analytic isomorphism but also that H induces an analytic isomorphism between the critical loci of π_i, $i = 1, 2$, regarded as analytic spaces. Clearly a blow-analytic isomorphism is a blow-analytic homeomorphism, but the converse is not true. Examples of blow-analytic homeomorphisms which are not blow-analytic isomorphisms can be found in [5].

One of the advantages of our definition is that allows us to easily prove the following.

THEOREM 3.2. — *If h is a blow-analytic isomorphism then the jacobian* $\det\left(\frac{\partial h_j}{\partial x_i}\right)$ *is a blow-analytic unit.*

Proof. — Let $\pi_i : M_i \to U_i$, $i = 1, 2$ be modifications and E_i denote the critical loci of π_i, $i = 1, 2$. We assume that $h \circ \pi_1 = \pi_2 \circ H$ for some analytic isomorphism $H : (M_1, E_1) \to (M_2, E_2)$. By considering compositions of blowing-ups with nonsingular centres we can find a modification $\pi_1' : M \to M_1$, so that the critical locus E_1' of $\pi_1 \circ \pi_1'$ is in normal crossing. It is clear that $\pi_2' = H \circ \pi_1'$ is also a modification and that the critical locus E_2' of $\pi_2 \circ \pi_2'$ is just E_1', so we may assume that the critical loci of π_i, $i = 1, 2$, are the same (normal crossing divisors), and H is the identity map.

Let $P \in M_1$ be a point and $y = (y_1, \dots, y_n)$ a local coordinate system around P. We set $\psi_i(y) = x_i \circ \pi_1(y)$ and $\phi_i(y) = x_i \circ \pi_2(y)$, $i = 1, \dots, n$. We may assume that

$$\det\left(\frac{\partial \psi_i}{\partial y_j}\right) = y_1^{e_1} \cdots y_k^{e_k} U$$

where U is an analytic unit near P, and we may also assume that

$$\det\left(\frac{\partial \phi_i}{\partial y_j}\right) = y_1^{e_1} \cdots y_k^{e_k} \overline{U}$$

where $\overline{U}$ is an analytic unit near P as well. Since

$$d\pi_2(Q) = dh(\pi_1(Q)) d\pi_1(Q)$$

for generic points Q near P, we obtain

$$\frac{\overline{U}(Q)}{U(Q)} = \det\left(\frac{\partial h_j}{\partial x_i}(\pi_1(Q))\right),$$

clearly showing that $\det\left(\frac{\partial h_j}{\partial x_i}\right)$ is a blow-analytic unit. □

The following question is natural to ask.

Question 3.3. — If $\det\left(\frac{\partial h_j}{\partial x_i}\right)$ is a blow-analytic unit, is h a blow-analytic isomorphism?

Surprisingly enough, the answer is **No**, as the following example shows.

Example 3.4. — Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a map-germ defined by

$$(x, y) \mapsto \left(\frac{x^3 + x^2 y - 4xy^2 + y^3}{x^2 + y^2}, \frac{x^3 - 4x^2 y + xy^2 + y^3}{x^2 + y^2}\right).$$

The jacobian matrix is given by the following:

$$\begin{pmatrix} \dfrac{x^4+7x^2y^2-4y^4}{(x^2+y^2)^2} & \dfrac{x^4-10x^3y+2x^2y^2+y^4}{(x^2+y^2)^2} \\ \dfrac{x^4+2x^2y^2-10xy^3+y^4}{(x^2+y^2)^2} & \dfrac{-4x^4+7x^2y^2+y^4}{(x^2+y^2)^2} \end{pmatrix},$$

and its determinant is

$$-\frac{5(x^2-xy+y^2)^2}{(x^2+y^2)^2},$$

which is a blow-analytic unit by the blow-up at the origin. However this map is not injective as we have $h(a,0)=h(0,a)$, so h is not a homeomorphism-germ. Actually setting $x=r\cos\theta$, $y=r\sin\theta$, we have that

$$h(x,y)=\frac{r}{4}(-\cos\theta+4\sin\theta+5\cos 3\theta, 4\cos\theta-\sin\theta-5\sin 3\theta)$$

and the mapping degree of h at 0 is -3.

However, this kind of behaviour is not possible in a higher codimension.

THEOREM 3.5. — *Let $h:(\mathbb{R}^n,C)\to(\mathbb{R}^n,C')$ be a continuous blow-analytic map, analytic everywhere except possibly on a proper subvariety C, such that its jacobian* $\det\left(\frac{\partial h_j}{\partial x_i}\right)$ *is a blow-analytic unit via a modification whose critical locus lies above C. If $h|C$ is injective and the codimension of C is at least 3, then h is a homeomorphism between two neighbourhoods of the origin of $\mathbb{R}^n$.*

Proof. — Using the curve selection lemma and the assumption we can easily show that $h^{-1}(C')=C$ in a small neighbourhood of C. This in turn will imply that h is a proper continuous map between two neighbourhoods of C and C' respectively (suitably chosen, see for instance §3.2 [2]). This fact shows that in those neighbourhoods, $h|(\mathbb{R}^n-C)$ is actually a covering map. However if the codimension of C is at least 3 this implies that h is a homeomorphism between the corresponding neighbourhoods. □

The following result is implicit in [10].

THEOREM 3.6. — *Let U be a neighbourhood of the origin of $\mathbb{R}^n$ and $f,g:U\dashrightarrow\mathbb{R}$ be two blow-analytic functions such that $f(x)/g(x)$ is a blow-analytic unit (or equivalently $\mathrm{ord}_\alpha(f)=\mathrm{ord}_\alpha(g)$ for any analytic arc at the origin). Then f is blow-analytic equivalent to either g or $-g$. More precisely, if f/g is positive then f is blow-analytic equivalent to g.*

Proof. — The proof is the same as the one given in [10]. The new observation is that one can choose the analytic isomorphism, constructed there, to be the identity on the exceptional set, which in turn allows factorization to a homeomorphism in the diagram on page 472, [10]. □

Now we can state the following immediate consequence.

COROLLARY 3.7. — *Let P be a blow-analytic unit and f a blow-analytic function in a neighbourhood of the origin in $\mathbb{R}^n$. Then Pf is blow-analytic equivalent to* $\operatorname{sign}(P)f$.

4. Toric modifications.

Let U be a neighbourhood of the origin of $\mathbb{R}^n$. We assume some knowledge about toric modifications. See §1.5 in [11], or §5 in [1], for definitions and fundamental properties in the complex algebraic case. In this paper we consider the real part of a complex toric modification $\pi : M \to \mathbb{C}^n$, which means its restriction to the real point set $\pi|_{M(\mathbb{R})} : M(\mathbb{R}) \to \mathbb{R}^n$. We abbreviate $\pi|_{M(\mathbb{R})}$ to π and $M(\mathbb{R})$ to M, and we just say "$\pi : M \to \mathbb{R}^n$ is a toric modification".

Let $\pi : M \to \mathbb{R}^n$ be a toric modification. Then M admits coordinate patches (U_α, y_α), $\alpha \in A$ (i.e., an open covering $M = \bigcup_{\alpha \in A} U_\alpha$), such that $y_\alpha : U_\alpha \to \mathbb{R}^n$ is an isomorphism for $\alpha \in A$, and $\pi|_{U_\alpha}$ is expressed by $x = {}^A y_\alpha$ for some unimodular matrix $A = (a_i^j)$. Here we use the following notation:

$$^A y_\alpha = (y_{\alpha,1}^{a_1^1} \cdots y_{\alpha,n}^{a_1^n}, \dots, y_{\alpha,1}^{a_n^1} \cdots y_{\alpha,n}^{a_n^n}) \quad \text{for} \quad y_\alpha = (y_{\alpha,1}, \dots, y_{\alpha,n}).$$

LEMMA 4.1. — *Let $\pi : M \to \mathbb{R}^n$ be a toric modification. The vector fields $x_i \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, admit analytic lifts on U_α, tangent to each irreducible component of the critical locus of π.*

The proof is trivial.

5. Constructing examples of blow-analytic isomorphisms.

We present an idea on how to show that a given continuous map $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a blow-analytic isomorphism via a modification

π. Here $\pi : M \to \mathbb{R}^n$ is a modification whose critical locus is normal crossing. Let us consider $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $0 \leqslant t \leqslant 1$, a continuous family of maps with well defined partial derivatives at each point. We set $F(x;t) = f_t(x) = (F_1(x;t), \dots, F_n(x;t))$, and

$$\xi = -\frac{A_1}{B}\frac{\partial}{\partial x_1} - \dots - \frac{A_n}{B}\frac{\partial}{\partial x_n} + \frac{\partial}{\partial t} \tag{5.1}$$

where

$$A_i = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_{i-1}} & \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial x_{i+1}} & \dots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \dots & \frac{\partial F_n}{\partial x_{i-1}} & \frac{\partial F_n}{\partial t} & \frac{\partial F_n}{\partial x_{i+1}} & \dots & \frac{\partial F_n}{\partial x_n} \end{vmatrix}, \quad B = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \dots & \frac{\partial F_n}{\partial x_n} \end{vmatrix}.$$

Since $\xi F \equiv 0$, by Cramaer's rule, this ξ trivializes the family f_t wherever defined. Then h is a blow-analytic isomorphism via π, if the following conditions are satisfied:

(i) $f_1 = h$ and f_0 is a blow analytic isomorphism via π.

(ii) ξ extends continuously on $U \times [0,1]$.

(iii) ξ admits an analytic lift $\tilde{\xi}$ via $\pi \times \mathrm{id}_{[0,1]}$.

(iv) $\tilde{\xi}$ is tangent to each irreducible component of the critical locus of π.

LEMMA 5.1. — *Let P be a blow-analytic function via some toric modification $\pi : M \to \mathbb{R}^n$. Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the continuous map defined by*

$$(x_1, \dots, x_n) \mapsto (x_1 P(x), x_2, \dots, x_n).$$

If $P + x_1 \frac{\partial P}{\partial x_1}$ is a blow-analytic unit via $\pi : M \to \mathbb{R}^n$, has a continuous extension on $(\mathbb{R}^n - 0)$, then h is a blow-analytic isomorphism via π.

Proof. — We may assume that $P + x_1 \frac{\partial P}{\partial x_1}$ is a positive blow-analytic unit via π. We set

$$F(x;t) = (x_1\{(1-t) + tP\}, x_2, \dots, x_n), \quad 0 \leqslant t \leqslant 1.$$

By (5.1) we have

$$\xi = -\frac{P-1}{1-t+t(P+x_1\frac{\partial P}{\partial x_1})} x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial t},$$

and this admits an analytic lift via the toric modification π. Thus this vector field is bounded near 0. The denominator is a blow-analytic unit, therefore bounded away from zero, and by assumption is continuously extendable on $(\mathbb{R}^n - 0)$, which shows that ξ admits a continuous extension on $(\mathbb{R}^n, 0)$. □

Example 5.2. — Let $h_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a map-germ defined by

$$(x, y) \mapsto (Px, y) \quad \text{where} \quad P = \frac{x^2 + ty^2}{x^2 + y^2}.$$

Then, by elementary computation, we have

$$P + x\frac{\partial P}{\partial x} = \frac{x^4 + (3-t)x^2y^2 + ty^4}{(x^2 + y^2)^2}.$$

This is a blow-analytic unit if and only if $0 < t < 9$. So despite the fact that P is a blow-analytic unit for all $t > 0$, h_t is a blow-analytic isomorphism only if $0 < t < 9$ (by Theorem 3.2).

Remark 5.3. — Let P_i, $i = 1, \dots, n$, be blow-analytic functions via some toric modification π. We assume that π is an isomorphism except over the origin. In the same way as above, we can show that the map $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ defined by

$$(x_1, \dots, x_n) \mapsto (x_1 P_1(x), \dots, x_n P_n(x)),$$

is a blow analytic isomorphism via π, if all A_i in (5.1) are continuously extendable on $(\mathbb{R}^n - 0)$ and

$$B = \det\begin{pmatrix} 1-t+t(P_1+x_1\frac{\partial P_1}{\partial x_1}) & tx_1\frac{\partial P_1}{\partial x_2} & \cdots & tx_1\frac{\partial P_1}{\partial x_n} \\ tx_2\frac{\partial P_2}{\partial x_1} & 1-t+t(P_2+x_2\frac{\partial P_2}{\partial x_2}) & \cdots & tx_2\frac{\partial P_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ tx_n\frac{\partial P_n}{\partial x_1} & tx_n\frac{\partial P_n}{\partial x_2} & \cdots & 1-t+t(P_n+x_n\frac{\partial P_n}{\partial x_n}) \end{pmatrix}$$

is a blow-analytic unit via π for $0 \leqslant t \leqslant 1$, and continuously extendable on $(\mathbb{R}^n - 0)$.

Lemma 5.1 can be generalized as follows.

PROPOSITION 5.4. — *Let P be a blow-analytic function via some toric modification $\pi : M \to \mathbb{R}^n$. We assume that π is an isomorphism except over the origin. Let $\omega_1, \dots, \omega_n$ be some positive integers. If $P + \sum_{i=1}^n \omega_i x_i \frac{\partial P}{\partial x_i}$ is a blow-analytic unit via π, and P and $\sum_{i=1}^n \omega_i x_i \frac{\partial P}{\partial x_i}$ are continuously extendable on $(\mathbb{R}^n - 0)$, then the map*

$$(5.2) \qquad h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \quad (x_1, \dots, x_n) \mapsto (x_1 P^{\omega_1}, \dots, x_n P^{\omega_n})$$

is a blow-analytic isomorphism via π.

Proof. — We define $F : (\mathbb{R}^n, 0) \times [0,1] \to (\mathbb{R}^n, 0)$ by $(x_1, \dots, x_n; t) \mapsto (x_1 \widetilde{P}^{\omega_1}, \dots, x_n \widetilde{P}^{\omega_n})$ where $\widetilde{P} = \widetilde{P}(x,t) = (1-t) + tP(x)$. We denote by F_j the j-th component function of F. Then we obtain

$$\begin{aligned}
B &= \det\left(\frac{\partial F_j}{\partial x_i}\right) = \det\left(\delta_{ij}\widetilde{P}^{\omega_j} + \omega_j x_j \frac{\partial \widetilde{P}}{\partial x_i}\widetilde{P}^{\omega_j - 1}\right)_{1\leqslant i,j\leqslant n} \\
&= \widetilde{P}^{\omega_1+\cdots+\omega_n-n}\det\left(\delta_{ij}\widetilde{P} + \omega_j x_j \frac{\partial \widetilde{P}}{\partial x_i}\right)_{1\leqslant i,j\leqslant n} \\
&= \widetilde{P}^{\omega_1+\cdots+\omega_n-n}\sum_{k=0}^{n} \text{sum of } k\text{-th principal minors of} \\
&\qquad\qquad \left(\omega_j x_j \frac{\partial \widetilde{P}}{\partial x_i}\right)_{1\leqslant i,j\leqslant n} \widetilde{P}^{n-k} \\
&= \widetilde{P}^{\omega_1+\cdots+\omega_n-1}\left(\widetilde{P} + \sum_{j=1}^{n}\omega_j x_j \frac{\partial \widetilde{P}}{\partial x_j}\right).
\end{aligned}$$

For $i = 1, \dots, n$, we thus obtain

$$\begin{aligned}
A_i &= \frac{\partial \widetilde{P}}{\partial t}\widetilde{P}^{\omega_1+\cdots\omega_n-n}\det\begin{pmatrix}
\widetilde{P} + \omega_1 x_1 \frac{\partial \widetilde{P}}{\partial x_1} & \cdots & \omega_1 x_1 & \cdots & \omega_1 x_1 \frac{\partial \widetilde{P}}{\partial x_n} \\
\vdots & & \vdots & & \vdots \\
\omega_i x_i \frac{\partial \widetilde{P}}{\partial x_1} & \cdots & \omega_i x_i & \cdots & \omega_i x_i \frac{\partial \widetilde{P}}{\partial x_n} \\
\vdots & & \vdots & & \vdots \\
\omega_n x_n \frac{\partial \widetilde{P}}{\partial x_1} & \cdots & \omega_n x_n & \cdots & \widetilde{P} + \omega_n x_n \frac{\partial \widetilde{P}}{\partial x_n}
\end{pmatrix} \\
&= \frac{\partial \widetilde{P}}{\partial t}\widetilde{P}^{\omega_1+\cdots+\omega_n-1}\omega_i x_i.
\end{aligned}$$

Therefore we conclude

$$\begin{aligned}
\xi &= -\frac{\frac{\partial \widetilde{P}}{\partial t}\sum_{i=1}^{n}\omega_i x_i \frac{\partial}{\partial x_i}}{\widetilde{P} + \sum_{j=1}^{n}\omega_j x_j \frac{\partial \widetilde{P}}{\partial x_j}} + \frac{\partial}{\partial t} \\
&= -\frac{(P-1)\sum_{i=1}^{n}\omega_i x_i \frac{\partial}{\partial x_i}}{(1-t) + t(P + \sum_{j=1}^{n}\omega_j x_j \frac{\partial P}{\partial x_j}} + \frac{\partial}{\partial t},
\end{aligned}$$

and by our assumptions this admits an analytic lift via π. C^0-extendability implies also that ξ has a continuous extension on $(\mathbb{R}^n, 0)$. □

The following has a similar proof :

PROPOSITION 5.5. — *Let P be a non-negative blow-analytic function via some toric modification $\pi : M \to \mathbb{R}^n$. Let $\omega_1, \dots, \omega_n$ be some positive real numbers. If $P + \sum_{i=1}^n \omega_i x_i \frac{\partial P}{\partial x_i}$ is a blow-analytic unit via the modification π, and P and $\sum_{i=1}^n \omega_i \frac{\partial P}{\partial x_i}$ are continuously extendable on $(\mathbb{R}^n - 0)$, then the map defined by (5.2) is a blow-analytic isomorphism via π.*

Example 5.6. — Let $h_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a map-germ defined by

$$(x, y) \mapsto (Px, Py) \quad \text{where} \quad P = \frac{x^2 + ty^4}{x^2 + y^4}.$$

Then, by elementary computation, we have

$$P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = \frac{x^4 + (3t-1)x^2y^4 + ty^8}{(x^2 + y^4)^2}.$$

This is a blow-analytic unit if and only if $t > 1/9$. So, by 5.4, h_t is a blow-analytic isomorphism, if $t > 1/9$. This shows that the multiplication of the identity by a blow-analytic unit, does not necessarily give a blow-analytic isomorphism (by Theorem 3.2, the jacobian must be a blow-analytic unit).

Example 5.7. — Let P be a function defined by

$$P = 2 + \frac{x^2y^2}{x^2 + y^2} + \frac{z^2w^2}{z^2 + w^2}.$$

Let $\pi : M \to \mathbb{R}^4$ be the direct product of the blow-ups of $\mathbb{R}^2$'s at the origins. Then, we have

$$P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} + w\frac{\partial P}{\partial w} \quad \text{and} \quad P$$

are blow-analytic units via π. We also remark that these are continuously extendable on $(\mathbb{R}^4, 0)$. So the map defined by

$$(x, y, z, w) \mapsto (xP, yP, zP, wP)$$

is a blow-analytic isomorphism via π (by 5.4).

LEMMA 5.8. — *Suppose that $Q : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a continuous function which does not depend on x_1 (we write $Q = Q(x_2, \dots, x_n)$). Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the continuous map defined by*

$$(x_1, \dots, x_n) \mapsto (x_1 + Q(x_2, \dots, x_n), x_2, \dots, x_n).$$

If Q is blow-analytic, then h is a blow-analytic isomorphism.

Proof. — Since Q is blow-analytic and does not depend on x_1, there is a modification $\pi_1 : M_1 \to \mathbb{R}^{n-1}$ so that $Q \circ (\mathrm{id}_{\mathbb{R}} \times \pi_1)$ is analytic. It follows that h and h^{-1} are both blow-analytic via $\mathrm{id}_{\mathbb{R}} \times \pi_1$. □

Example 5.9 ([12]). — The map $h : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ defined by

$$(x, y, z) \mapsto \left(x + \frac{yz^5}{y^4 + z^6}, y, z \right)$$

is a blow analytic isomorphism via some toric modification. But $\frac{\partial h_1}{\partial y}$, $\frac{\partial h_1}{\partial z}$ are not blow-analytic. So the entries of the jacobian matrix of a blow-analytic isomorphism may not be blow-analytic.

Let U be a neighborhood of the origin of $\mathbb{R}^{n-1}$ and $f : U \to \mathbb{R}$ denote a continuous blow-analytic function defined almost everywhere on U. Assume that for some analytic arc $\alpha : (\mathbb{R}, 0) \to U$, we have that $\mathrm{ord}_\alpha(f) < \mathrm{ord}(\alpha)$ (here by $\mathrm{ord}(\alpha)$ we understand the minimum of the orders of its components). Then in [12] an "associated" blow-analytic isomorphism is constructed between two neighborhoods of the origin of $\mathbb{R}^n$ which in particular drops the order along the arc $(\alpha, 0)$. Example 5.9 shows that via this kind of blow-analytic isomorphism, a curve like $(0, t^3, t^2)$ goes to a smooth one, i.e., that blow-analytic isomorphism makes the cusp smooth.

In the sequel we will show that for any irreducible curve $\alpha(t)$ in $\mathbb{R}^2$ with a parameterization $\alpha(t) = (ct^p + \cdots + \text{h.o.t.}, t^n)$, $n \leqslant p$, we can construct an explicit rational blow-analytic function $f : U \to \mathbb{R}$, U a neighborhood of the origin of $\mathbb{R}^2$, such that $f(\alpha(t))$ is a smooth analytic arc. This will give us a lot of explicit examples for which we can use the general construction in [12] to provide interesting blow-analytic isomorphisms (see for instance Corollary 5.11 below).

THEOREM 5.10. — *For any irreducible curve $\alpha : (\mathbb{R}, 0) \to U$, U a neighborhood of the origin of $\mathbb{R}^2$, there is a continuous rational (blow-analytic) function $f : U \to \mathbb{R}$, defined constructively, such that* $\mathrm{ord}_\alpha(f) = 1$.

Proof. — The proof is based on [6] and [7].

Let us consider S a finitely generated, additive semigroup of the positive rationals, $S = S(w_0, w_1, \ldots, w_N)$, with generators $w_0 < w_1 < \cdots < w_N$ and $w_i \notin S(w_0, \ldots, w_{i-1})$, $i \geqslant 1$. We put $d_0 = 1$ and let d_i be the smallest integer such that $d_i w_i \in S(w_0, \ldots, w_{i-1})$, $i \geqslant 1$.

S is called a Newton-Puiseux semigroup if $w_i > d_{i-1}w_{i-1}$, $i \geqslant 1$. We denote by A, the abelian group generated by w_i, $0 \leqslant i \leqslant N$. Clearly this group is generated by some positive rational number. Note that S coincides with A beyond $d_N w_N$ ([7], Lemma 2). A typical element in S is written as

$$\sum_{i=0}^{N} m_i w_i$$

and this expression is called *admissible* if $0 \leqslant m_i < d_i$, $1 \leqslant i \leqslant N$. Note that m_0 can be any non-negative integer.

In the case of a Newton-Puiseux semigroup S, the corollary in [7] tells us that every element in S has a unique admissible expression.

In [6], starting with a sequence of Newton-Puiseux pairs one can construct a sequence of monic polynomials in x, G_i, $i = 1, 2, \ldots$, with coefficients in $\mathbb{R}[[y]]$, a special case of G-adic expansions defined by Abhyankar and Moh. We refer to [6], page 2, for the dictionary relating the Newton-Puiseux pairs and the weights w_i, $i = 1, 2, \ldots$.

By associating to our initial curve $\alpha : (\mathbb{R}, 0) \to U$, its Newton-Puiseux series, we obtain a Newton-Puiseux semigroup S as above, and therefore we can perform all the previous constructions. Then we have

$$d_N w_N = \sum_{i=0}^{N-1} r_i w_i, \quad 0 \leqslant r_i < d_i$$

for $i = 1, \ldots, N-1$ and $r_0 \geqslant 0$, integers. If A is generated by the positive rational number r, then

$$2d_N w_N + r \in S$$

so it can be written in a unique way as

$$\sum_{i=0}^{N} q_i w_i, \quad 0 \leqslant q_i < d_i, \ 1 \leqslant i \leqslant N$$

(no restriction on q_0).

In our case for the given parameterization of $\alpha(t) = (ct^p + \cdots + \text{h.o.t.}, t^n)$, $n \leqslant p$, we find that the generator of the corresponding group A is just $r = \frac{1}{n}$.

After all this preparation we can introduce the following rational function:

$$f(x, y) = \frac{\prod_{i=0}^{N} G_i^{q_i}}{G_N^{2d_N} + \prod_{i=0}^{N-1} G_i^{2r_i}}.$$

Note that G_i, $i \geqslant 0$, define distinct irreducible curve germs, and therefore we may assume that in a small neighborhood of the origin, the denominator of f vanishes only at the origin. As a consequence of the fact that all polynomials G_i have weight w_i, $i \geqslant 0$ with respect to $y = t^n$, we have that the weight of f is positive and actually it is just $r = \frac{1}{n}$, i.e., this means exactly that both $f(\alpha(t))$ is smooth and that f is continuous (therefore blow-analytic), so this finishes the proof of our theorem. □

COROLLARY 5.11. — *Any analytic curve in $\mathbb{R}^n$, $n \geqslant 3$, can be deformed via a rational blow-analytic isomorphism of $\mathbb{R}^n$, to a smooth analytic arc.*

Remark 5.12. — The new thing here is both that the homeomorphism is rational and that it is given constructively in terms of the given arc. If we forget the restriction imposed on our blow-analytic isomorphisms, we can use [5] to obtain a similar result (without rationality and constructiveness).

Proof. — Indeed one can assume that the initial curve has a parameterization given by

$$\alpha(t) = (c_1 t^{p_1} + \text{h.o.t.}, c_2 t^{p_2} + \text{h.o.t.}, c_{n-1} t^{p_{n-1}} + \text{h.o.t.}, t^n),$$
$$n \leqslant p_i,\ \ i = 1, 2, \ldots, n-1.$$

Applying the construction above for the semigroup corresponding to the last two components, we find a continuous rational blow-analytic function f such that $f(c_{n-1} t^{p_{n-1}} + \text{h.o.t.}, t^n)$ is a smooth analytic curve. Then we define the desired homeomorphism by the following formula (see also Lemma 5.8 above):

$$h(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_1, x_2, \ldots, x_{n-2} + cf(x_{n-1}, x_n), x_{n-1}, x_n)$$

(here c is a suitable constant). □

6. An inverse mapping theorem.

The goal of this section is to prove the following

THEOREM 6.1. — *Let $h = (h_1, \ldots, h_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a continuous blow-analytic map. If there are some permutations $i_1, \ldots, i_n$,*

$j_1, \dots, j_n$ *so that the jacobians*

$$\frac{\partial h_{j_1}}{\partial x_{i_1}},\ \det\left(\frac{\partial(h_{j_1}, h_{j_2})}{\partial(x_{i_1}, x_{i_2})}\right), \dots, \det\left(\frac{\partial(h_{j_1}, \dots, h_{j_n})}{\partial(x_{i_1}, \dots, x_{i_n})}\right)$$

are blow-analytic units via some toric modification $\pi : M \to \mathbb{R}^n$, *and they are continuously extendable on* $(\mathbb{R}^n - 0)$, *then* h *is a blow-analytic isomorphism via some toric modification.*

LEMMA 6.2. — *Let* $f(x)$ *be a continuous function on* U. *Assume that there is a thin subset* N *of* $\mathbb{R}^{n-1}$ *so that* $\frac{\partial f}{\partial x_1}$ *is defined on* $(\mathbb{R} - \{0\}) \times (\mathbb{R}^{n-1} - N)$, *and* $\frac{\partial f}{\partial x_1}$ *is continuously extendable on* $(\mathbb{R}^n - 0)$. *If* f *and* $\frac{\partial f}{\partial x_1}$ *are blow-analytic via* π, *then there are unique blow-analytic functions* $P = P(x)$ *and* $Q = Q(x)$ *via* π *so that* $f(x) = x_1 P + Q$, *and* Q *does not depend on* x_1. *Moreover, if* $\frac{\partial f}{\partial x_1}$ *is a blow-analytic unit via* π, *then* $P = P(x)$ *can be chosen as a blow-analytic unit via* π.

Proof. — We set $g(t) = f(tx_1, x_2, \dots, x_n)$. We have

$$g(1) - g(0) = \int_0^1 x_1 \frac{\partial f}{\partial x_1}(tx_1, x_2, \dots, x_n)dt \quad \text{for} \quad (x_2, \dots, x_n) \notin N.$$

Since $g(1) = f(x)$, and $g(0) = f(0, x_2, \dots, x_n)$, we obtain $f(x) = x_1 P + Q$, where $P = \int_0^1 \frac{\partial f}{\partial x_1}(tx_1, x_2, \dots, x_n)dt$, $Q = f(0, x_2, \dots, x_n)$. Obviously P, Q are blow-analytic via π, and Q does not depend on x_1. Moreover, if $\frac{\partial f}{\partial x_1}$ is a blow-analytic unit, then so is P. □

LEMMA 6.3. — *Let* $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a continuous map defined by*

$$(x_1, \dots, x_n) \mapsto (f(x), x_2, \dots, x_n).$$

Assume that there is a thin subset N *of* $\mathbb{R}^{n-1}$ *so that* $\frac{\partial f}{\partial x_1}$ *is defined on* $(\mathbb{R} - \{0\}) \times (\mathbb{R}^{n-1} - N)$ *near* 0 *and that* $\frac{\partial f}{\partial x_1}$ *is continuously extendable on* $(\mathbb{R}^n - 0)$. *If* f *is blow-analytic and* $\frac{\partial f}{\partial x_1}$ *is a blow-analytic unit via some toric modification* π, *then* h *is a blow-analytic isomorphism via some toric modification.*

Proof. — We may assume that $\frac{\partial f}{\partial x_1}$ is a positive blow-analytic unit via π. By Lemma 6.2, there are blow-analytic functions P, Q so that $f(x) = x_1 P + Q$. By supposition, $P + x_1 \frac{\partial P}{\partial x_1} = \frac{\partial f}{\partial x_1}$ is a blow-analytic unit via π. Using Lemmas 5.1 and 5.8 we get that the following composition is the desired blow-analytic isomorphism:

$$\begin{array}{ccccc} (\mathbb{R}^n,0) & \longrightarrow & (\mathbb{R}^n,0) & \longrightarrow & (\mathbb{R}^n,0) \\ (x_1,\dots,x_n) & \longrightarrow & (x_1P,x_2,..,x_n)=(x'_1,\dots,x'_n) & \longrightarrow & (x'_1+Q,x'_2,\dots,x'_n). \end{array}$$

□

Proof of Theorem 6.1. — It is enough to show Theorem 6.1 in the case $i_s = j_s = s$, $s = 1,\dots,n$.

Since $\frac{\partial h_1}{\partial x_1}$ is a blow-analytic unit, there are blow-analytic functions P_1 and Q_1 via π so that P_1 is a blow-analytic unit via π, and that $Q_1 = Q_1(x'_2,\dots,x'_n)$ does not depend on x_1. Since $\frac{\partial h_1}{\partial x_1}$ is continuously extendable on $(\mathbb{R}^n - 0)$, so are P and Q.

We define a map $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ by

$$(x_1,\dots,x_n) \mapsto (x'_1,\dots,x'_n) = (x_1P_1,x_2,\dots,x_n),$$

which is a blow-analytic isomorphism via π, and a map $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ by

$$(x'_1,\dots,x'_n) \mapsto (\overline{x}_1,\dots,\overline{x}_n) = (x'_1 + Q_1(x'_2,\dots,x'_n),x'_2,\dots,x'_n),$$

which is a blow-analytic isomorphism via some toric modification (by Lemmas 5.1 and 5.8). Since $\frac{\partial h}{\partial \overline{x}} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial x'}\frac{\partial x'}{\partial \overline{x}}$, we obtain that $\det\big(\frac{\partial(h_2,\dots,h_i)}{\partial(\overline{x}_2,\dots,\overline{x}_i)}\big)$, $i = 2,\dots,n$, are blow-analytic units via π, and continuously extendable on $(\mathbb{R}^n - 0)$.

So in the same way as above (by 6.2), we can write $\frac{\partial h_2}{\partial x_2} = x_2P_2 + Q_2$ for some blow-analytic unit P_2 via π and a blow-analytic function Q_2 which does not depend on x_2.

We define a map $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ by

$$(\overline{x}_1,\dots,\overline{x}_n) \mapsto (\overline{x}'_1,\dots,\overline{x}'_n) = (\overline{x}_1,\overline{x}_2P_2,\overline{x}_3,\dots,\overline{x}_n),$$

which is a blow-analytic isomorphism via π, and a map $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ by

$$(\overline{x}'_1,\dots,\overline{x}'_n) \mapsto (\overline{\overline{x}}_1,\dots,\overline{\overline{x}}_n) = (\overline{x}'_1,\overline{x}'_2 + Q_2(\overline{x}'_1,\overline{x}'_3\cdots,\overline{x}'_n),\overline{x}'_3,\dots,\overline{x}'_n),$$

which is a blow-analytic isomorphism via some toric modification (by Lemmas 5.1 and 5.8). Since $\frac{\partial h}{\partial \overline{\overline{x}}} = \frac{\partial h}{\partial \overline{x}}\frac{\partial \overline{x}}{\partial \overline{x}'}\frac{\partial \overline{x}'}{\partial \overline{\overline{x}}}$, we obtain $\det\big(\frac{\partial(h_3,\dots,h_i)}{\partial(\overline{\overline{x}}_3,\dots,\overline{\overline{x}}_i)}\big)$, $i = 3,\dots,n$, are blow-analytic units via π, and continuously extendable on $(\mathbb{R}^n - 0)$.

Using this process repeatedly, Theorem 6.1 is reduced to the following assertion which follows from Lemma 6.3: A map $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ defined by

$$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_{n-1},h_n(x))$$

is a blow-analytic isomorphism via some toric modification, if $\frac{\partial h_n}{\partial x_n}$ is a blow-analytic unit via π, and is continuously extendable on $(\mathbb{R}^n - 0)$. □

Remark 6.4. — From the proof one can see that the blow-analytic isomorphisms from our Theorem 6.1 are generated by those appearing in Lemmas 5.1 and 5.8. An interesting question to ask is whether or not all blow-analytic isomorphisms are of the above type.

Note that our results above should be understood modulo analytic isomorphisms, for instance:

Example 6.5. — If $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ is a map-germ defined by

$$(x,y) \mapsto (Px,Py) \quad \text{where} \quad P = \frac{(x+y)^2+2y^4}{(x+y)^2+y^4},$$

then all of the entries of its jacobian matrix are not bounded so we cannot apply our Theorem 6.1 to h to conclude that it is a blow-analytic isomorphism. However we can apply Theorem 6.1 to

$$(x,y) \mapsto (Px,Py) \quad \text{where} \quad P = \frac{x^2+2y^4}{x^2+y^4},$$

and this shows that our initial h is a blow-analytic isomorphism.

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Manuscrit reçu le 14 avril 2000,
accepté le 16 novembre 2000.

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